On the normal-form theorem for non-connected 2D-cobordism diagrams

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Fixes a flaw in the proof of 1.4.39 of [1].

0.1 Theorem. *By means of the relations listed, any cobordism diagram can be brought on the following normal form: first a permutation cobordism, then a disjoint union of connected cobordism diagrams on connected normal form, and finally another permutation cobordism.*

0.2 Theorem. *The normal form is unique in the following sense: the disjoint union of connected-normal forms in the middle is unique up to permutation of the components. The permutation cobordism diagrams can be chosen to be shuffles, with respect to the blocks defined by the connected components of the middle part. With this proviso, the in- and out-permutations are unique, so with a suitable choice of normal form for permutations as generated by twists, the overall normal form is unique.*

Proof of the first theorem. By Lemma 0.3 below, we reduce to the case where there are no caps involved. The main point is then (Proposition 0.11 below) that we can move the multiplication pieces west, and the comultiplication pieces east (except for handles). For this we need to refine the moving-multiplications-west and moving-comultiplication-east procedures. These pieces cannot quite move through all twists, so we push some of them ahead. A subtlety not met in the connected case is to show that twists cannot get stuck in between pair-of-pants wanting to move past each other. This is Lemma 0.8 below (which in turn depends on Lemmas 0.10).

The result of Proposition 0.11 is an in-part consisting only of multiplication pieces and twists, then a genus part with twists, and finally an out-part, consisting only of comultiplication pieces and twists.

The genus part consists of all the multiplications and comultiplication that could not be moved past each other. We need to show that this part can be made free of twists: they can be moved to either side. In the end the genus part is a disjoint union of strings of handles as in the connected case.

Finally in the in-part we can move all twists west, and in the out-part we can move all twists east.

Details follow below. □

0.3 Regarding caps. All birth-of-a-circle can be moved west by extending with cylinders, and by moving under any other tubes, using the caps relations in **??** (that's [1], 1.4.35) In this way, a first step is to move them out of the way like that, arriving at a three-part decomposition like this

caps	*nocaps*	*caps*

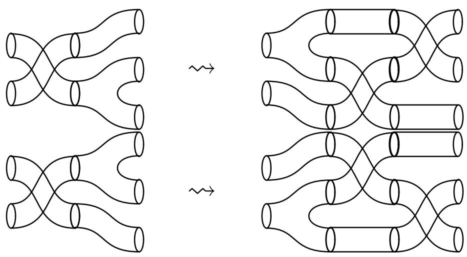
where the first part consists only of birth-of-circles (and cylinders) and the third part has only death-of-circles (and cylinders) while the middle part is free from caps. If the middle part is on normal form as in the theorem, then the caps can be moved back in again, where they will typically cancel some pair-of-pants.

FROM NOW ON WE ASSUME THERE ARE NO CAPS.

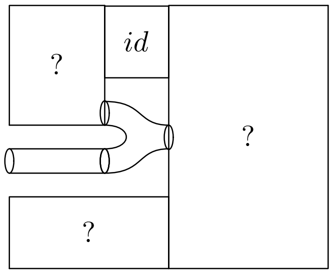
0.4 Handles can be moved around freely. Moving a handle through other pair-of-pants is already explained in the book, and moving through twists causes no further problems, by the naturality.

0.5 Sailing a comultiplication west. A comulitplication can be sailed west all the way to the boundary. Sailing through other pair-of-pants is already explained in the book: sailing through a multiplication, by the Frobenius relation, and through another comultiplication, using the coassociativity relation.

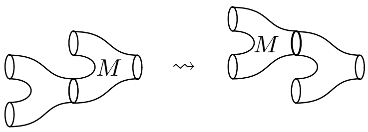
Regarding twists, the relevant relations are:



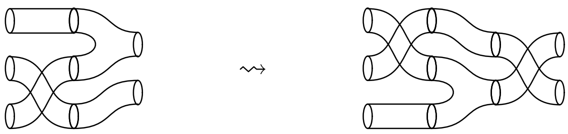
0.6 Lemma. *A multiplication M can be sailed southwest until either it meets a comultiplication in a double handshake, forming a handle, or it has only cylinders to the southwest of it, as in this picture:*



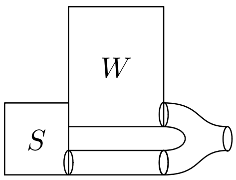
Proof. If to the immediate southwest (modulo cylinders) there is a handle, sail through it as already explained. If to the immediate southwest (modulo cylinders) there is another multiplication, pass through it by an application of the associativity relation



If to the immediate southwest (modulo cylinders) there is the northeast boundary of a comultiplication, pass through it using the Frobenius relation. If to the immediate southwest (modulo cylinders) there is the northeast boundary of a twist, sail through it using



If there is a sequence of cylinders and then a southeast boundary of either a comultiplication or a twist, then we are in the situation



where S is either a comultiplication or a twist. But then W can be brought on the form with at most one twist in the bottom row, by Lemma 0.8 below. If there are no twists in the bottom row, then M meets either a comultiplication in a double handshake or a twist that is eliminated by commutativity. If there is one twist in the bottom row of W then we are in the situation of Example **??** in the book (1.4.38), which allows us to move the multiplication past the comultiplication. Or we are in the situation of the twist relation.

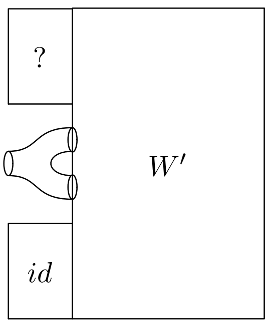
Inductively then, we can sail M southwest until it reaches the boundary (except for some cylinders). □

Similarly:

0.7 Lemma. *A comultiplication can be sailed southeast to the boundary, except for some cylinders.*

0.8 Lemma. *Any cobordism diagram W with at least one input and at least one output is equivalent to one in which the bottom row contains at most one twist.*

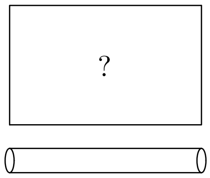
Proof. By induction on the number of pair-of-pants. If there are none, then W is a permutation diagram, handled by Lemma 0.10 below. For the general case, without loss of generality, assume there is a comultiplication in W, sail it west (0.5). Eventually it will be adjacent to the west boundary, and modulo some cylinders, we are in the situation



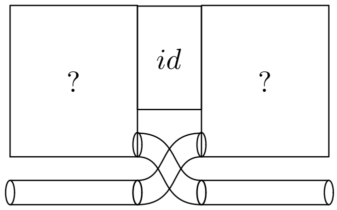
(where the id-block may be empty). Now W' has fewer pair-of-pants than W, so by induction, it can be brought on the desired form, and therefore W can be brought on the desired form. □

0.9 Remark. The proof uses sailing a comultiplication west. This is counter intuitive, as in the end we actually want to sail them east. The aim is just to get a pair-of-pants out to the boundary so as to use induction. We could also try to sail a multiplication west; this is possible by Lemma 0.6, which however depends on Lemma 0.8. The argumentation could be made non-circular by proving both lemmas inside a common induction and noting that Lemma 0.6 calls Lemma 0.8 only with strictly fewer pair-of-pants. The chosen solution is simpler logically, although more expensive in terms of total moves.

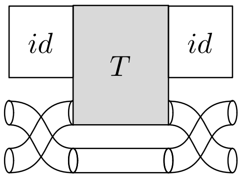
0.10 Lemma. *Any nonempty permutation diagram W is equivalent to one in which the bottom row has at most one twist. Precisely, either of the form*



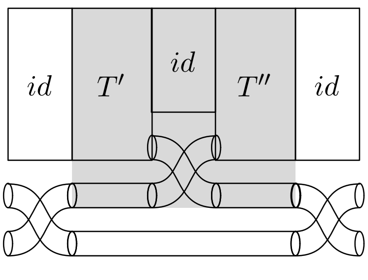
or of the form



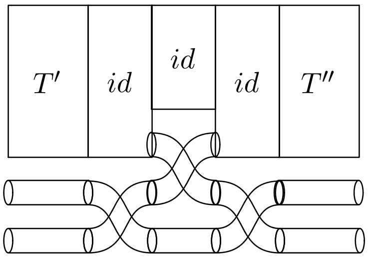
Proof. By induction on the number of twists, the case of zero or one twist being obvious. If there are two or more twists in the bottom row, locally at two adjacent twists A and B in the bottom row the situation is



Now by induction, T can be brought on the form with at most one twist on the bottom line. If there are zero, then the A and B cancel out, and we have reduced the number of twists in the bottom row of W. If there is one twist in the bottom line of T, then we are in the situation



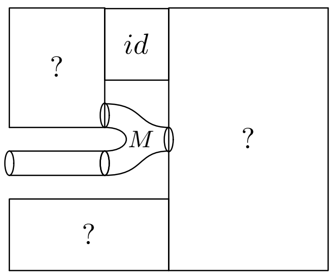
which, give and take some cylinders, is the same as



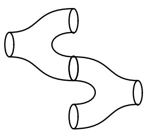
and now we can apply the 'symmetric-group relation' to reduce the number of twists in the bottom row of W by one. $\square$

0.11 Proposition. *A cobordism diagram without caps can be brought on a form AB where, except for handles, there are no comultiplications in A and no multiplications in B.*

Proof. Induction on the number of pair-of-pants (modulo handles). If there are no pair-of-pants, any decomposition will do. If there is a multiplication M, Lemma 0.6 says we can sail it southwest until the boundary like this:



By induction we can now assume that ? is of the pseudo-normal form, and from here we sail its comultiplications southeast one by one: either they go above M, and then into the east, or they meet M, then necessarily as



and by the Frobenius relation we can move it past M and argue now with a smaller cobordism diagram than the question mark. In any case, all the comultiplications in ? can be sent to the east, so in the end then we can slide the bottom block to the east, and the red line separates at least one multiplication (namely M) and no comultiplications to the west, and hence we can deal with the right-hand side of it by induction, and we are done. $\square$

Now we can finish the proof of the Theorem: we already have the pseudo-normal form. Now inside the part A, move the twists west and the handles east, and inside the part B, move the twists east and the handles west.

Finally, whenever the two tubes of some twist in the west part belong to the same connected component, use the symmetric-group relations to move that twist east until it is adjacent to the in-part of the normal form, where it will be eliminated by commutativity, after a suitable use of the associativity relation.

0.12 Corollary. *If the cobordism diagram is connected, there will be no twists left in the end.*

0.13 Remarks. The Corollary shows that the connected-normal-form theorem in the book is a consequence of the more general normal-form theorem. Furthermore, the proof of the general normal-form theorem is a refinement of the connected-normal-form proof in the book, not a refinement of the twist-elimination strategy outlined there. (I now think that the twist-elimination strategy is misguided.)

References

[1] JOACHIM KOCK. *Frobenius algebras and 2D topological quantum field theories.* No. 59 in London Mathematical Society Student Texts. Cambridge University Press, Cambridge, 2003.