PLANE MODEL-FIELDS OF DEFINITION, FIELDS OF DEFINITION, THE FIELD OF MODULI OF SMOOTH PLANE CURVES

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Abstract. Given a smooth plane curve $\overline{C}$ of genus $g \geq 3$ over an algebraically closed field $\overline{k}$, a field $L \subseteq \overline{k}$ is said to be a *plane model-field of definition for* $\overline{C}$ if L is a field of definition for $\overline{C}$, i.e. $\exists$ a smooth curve C' defined over L where $C' \times_L \overline{k} \cong \overline{C}$, and such that C' is L-isomorphic to a non-singular plane model $F(X, Y, Z) = 0$ in $\mathbb{P}^2_L$.

In this short note, we construct a smooth plane curve $\overline{C}$ over $\overline{\mathbb{Q}}$, such that the field of moduli of $\overline{C}$ is not a field of definition for $\overline{C}$, and also fields of definition do not coincide with plane model-fields of definition for $\overline{C}$. As far as we know, this is the first example in the literature with the above property, since this phenomenon does not occur for hyperelliptic curves, replacing plane model-fields of definition with the so-called hyperelliptic model-fields of definition.

1. Introduction

Consider F the base field for an algebraically closed field $\overline{k}$. Let $F \subseteq L \subseteq \overline{k}$ be fields, given a smooth projective curve $\overline{C}$ over $\overline{k}$, then $\overline{C}$ is *defined* over L if and only if there is a curve C' over L that is $\overline{k}$-isomorphic to $\overline{C}$, i.e. $C' \times_L \overline{k} \cong \overline{C}$. In such case, L is called a *field of definition* of $\overline{C}$. We say that $\overline{C}$ is *definable* over L if there is a curve C'/L such that $\overline{C}$ and $C' \times_L \overline{k}$ are $\overline{k}$-isomorphic.

Definition 1.1. The *field of moduli* of a smooth projective curve $\overline{C}$ defined over $\overline{k}$, denoted by $K_{\overline{C}}$, is the intersection of all fields of definition of $\overline{C}$.

It becomes very natural to ask when the field of moduli of a smooth projective curve $\overline{C}$ is also a field of definition. A necessary and sufficient condition (Weil's cocycle criterion of descent) for the field of moduli to be a field of definition was provided by Weil [12]. If $\mathrm{Aut}(\overline{C})$ is trivial, then this condition becomes trivially true and so the field of moduli needs to be a field of definition. It is also quite well known that a smooth curve $\overline{C}$ of genus $g = 0$ or 1 can be defined over its field of moduli, where g is the geometric genus of $\overline{C}$. However, if $g > 1$ and $\mathrm{Aut}(\overline{C})$ is non-trivial, then Weil's conditions are difficult to be checked and so there is no guarantee that the field of moduli is a field of definition for $\overline{C}$. This was first pointed out by Earle [4] and Shimura [11]. More precisely, in page 177 of [11], the first examples not definable over their field of moduli are introduced, which are hyperelliptic curves over $\mathbb{C}$ with two automorphisms. There are also examples of non-hyperelliptic curves not definable over their field of moduli given in [2, 5]. B. Huggins [6] studied this problem for hyperelliptic curves over a field $\overline{k}$ of characteristic $p \neq 2$, proving that a hyperelliptic curve $\overline{C}$ of genus $g \geq 2$ with hyperelliptic involution ι can be defined over $K_{\overline{C}}$ when $\mathrm{Aut}(\overline{C})/\langle\iota\rangle$ is not cyclic or is cyclic of order divisible by p.

On the other hand, one may define fields of definition of models of the same concrete type for a smooth projective curve $\overline{C}$. For example, if $\overline{C}$ is hyperelliptic, a field M is called a *hyperelliptic model-field of definition for* $\overline{C}$ if M, as a field of definition for $\overline{C}$, satisfies that $\overline{C}$ is M-isomorphic to a hyperelliptic model of the form $y^2 = f(x)$, for some polynomial $f(x)$ of degree $2g + 1$ or $2g + 2$.

By the work of Mestre [10], Huggins [5, 6], Lercier-Ritzenthaler [7], Lercier-Ritzenthaler-Sijsling [8] and Lombardo-Lorenzo in [9], one gets fair-enough characterizations for the interrelations between the three fields; the field of moduli, fields of definition and hyperelliptic model-fields of definition. For instance, if $\overline{C}$ is hyperelliptic, then there are always two of these fields, which are equal. Summing up, one obtains the next table issued from Lercier-Ritzenthaler-Sijsling [8], where $k = F$ is a perfect field of characteristic $char(F) \neq 2$:

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$H=\mathrm{Aut}(\overline{C})/\langle\iota\rangle$	Conditions	Fields of definition = Hyperelliptic model-fields	The field of moduli= A field of definition
Not tamely cyclic		Yes	Yes
Tamely cyclic with $\#H>1$	g odd,$\#H$odd	No	Yes
	g even or $\#H$ even	Yes	No
Tamely cyclic with $\#H=1$	g odd	No	Yes
	g even	Yes	No

By *tamely cyclic*, we mean that the group is cyclic of order not divisible by the $char(F)$.

Now, consider a smooth plane curve $\overline{C}$, i.e. $\overline{C}$ viewed as a smooth curve over $\overline{k}$ admits a non-singular plane model defined by an equation of the form $F(X,Y,Z)=0$ in $\mathbb{P}^2_{\overline{k}}$, where $F(X,Y,Z)$ is a homogenous polynomial of degree $d\geq 4$ over $\overline{k}$ with $g=\frac{1}{2}(d-1)(d-2)\geq 3$. Similarly, we define a so-called *plane model-fields of definition for* C:

Definition 1.2. Given a smooth plane curve $\overline{C}$ over $\overline{k}$, a subfield $M\subset\overline{k}$ is said to be a *plane model-field of definition for* C if and only if the following conditions holds

(i) M is a field of definition for $\overline{C}$.
(ii) $\exists$ a smooth curve C' defined over M, which is $\overline{k}$-isomorphic to $\overline{C}$, and M-isomorphic to a non-singular plane model $F(X,Y,Z)=0$, for some homogenous polynomial $F(X,Y,Z)\in M[X,Y,Z]$ of degree $d\geq 3$.

In this short note, we start with a smooth plane curve $\overline{C}$ over $\overline{\mathbb{Q}}$ where the field of moduli is not a field of definition by the work of B. Huggins in [5]. Next, we go further, following the techniques developed in [1], to construct a twist of $\overline{C}$, for which there is a field of definition for $\overline{C}$, which is not a plane model-field of definition.

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2. The example

Consider the *Hessian group of order* 18, denoted by Hess_{18}, which is $\mathrm{PGL}_3(\overline{\mathbb{Q}})$-conjugate to the group generated by

$$S:=\begin{pmatrix}1&0&0\\0&\zeta_3&0\\0&0&\zeta_3^2\end{pmatrix},\ T:=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix},\ \text{and } R:=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}.$$

First, we reproduce an example, by B. Huggins in [5, Chp. 7, §2], of a smooth $\overline{\mathbb{Q}}$-plane curve of genus 10 not definable over its field of moduli, and with full automorphism groups Hess_{18}.

Definition 2.1. A quaternion extension of a field K is a Galois extension K'/K such that $\mathrm{Gal}(K'/K)$ is isomorphic to the quaternion group of order 8.

Definition 2.2. ([5, Lemma 7.2.3]) A field K is of level 2 if -1 is not a square in K, but it is a sum of two squares in K.

Lemma 2.3. *([5, Lemma 7.2.3]) Let K be a field of level 2. Then, for $u,v\in K^*\setminus(K^*)^2$ such that $uv\notin(K^*)^2$, $K(\sqrt{u},\sqrt{v})$ is embeddable into a quaternion extension of K if and only if $-u$ is a norm from $K(\sqrt{-v})$ to K (i.e. $-u=x^2+vy^2$ for some $x,y\in K$).*

For instance, the field $K:=\mathbb{Q}(\zeta_3)$ is of level 2, since $(\zeta_3^2)^2+\zeta_3^2=-1$ and $\sqrt{-1}\notin K$. It is easily shown that ± 2 are not norms from $K(\sqrt{-13})$ to K. So neither $K(\sqrt{2},\sqrt{13})$ nor $K(\sqrt{-2},\sqrt{13})$ are embeddable into a quaternion extension of K.

Now fix K to be the field $\mathbb{Q}(\zeta_3)$, and define the following:

$$\begin{aligned}\phi &:= XYZ,\\ \psi &:= X^3+Y^3+Z^3,\\ \chi &:= (XY)^3+(YZ)^3+(XZ)^3.\end{aligned}$$

Suppose that $u, v \in \mathbb{Q}^*$, such that $L := K(\sqrt{u}, \sqrt{v})$ is a $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ extension of K that can not be embedded into a quaternion extension of K. Let

$$c_{\phi^2} := \zeta_3\sqrt{u} + \sqrt{v} + \zeta_3^2\sqrt{uv},$$
$$c_{\phi\psi} := \zeta_3^2\sqrt{u} + \sqrt{v} + \zeta_3\sqrt{uv},$$
$$c_{\psi^2} := \sqrt{u} + \sqrt{v} + \sqrt{uv} - \frac{1}{12}.$$

Fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ containing L as above.

Theorem 2.4. *(B. Huggins, [5, Lemma 7.2.5 and Proposition 7.2.6]) Following the above notations, let*

$$F_{\sqrt{u},\sqrt{v}}(X,Y,Z) := c_{\phi^2}\phi^2 - 6c_{\phi\psi}\phi\psi - 18c_{\psi^2}\psi^2 + \chi.$$

Then the equation $F_{\sqrt{u},\sqrt{v}}(X,Y,Z) = 0$ such that $F_{\sqrt{u},\sqrt{v}}(X,1,1)$ is square free, defines a smooth $\overline{\mathbb{Q}}$-plane curve $\overline{C}$ over $\overline{\mathbb{Q}}$, with automorphism group Hess_{18}. *The field of moduli $K_{\overline{C}}$ is $K = \mathbb{Q}(\zeta_3)$, but it is not a field of definition.*

Remark 2.5. *The condition that $F_{\sqrt{u},\sqrt{v}}(X,1,1)$ is square free is possible. For example, with $u = 2$ and $v = 13$, the resultant of $F_{\sqrt{2},\sqrt{13}}(X,1,1)$ and $\frac{\partial F}{\partial X}(X,1,1)$ is not zero.*

Lemma 2.6. *Let $\overline{C}$ be a smooth curve defined over an algebraically closed field $\overline{k}$, with $F = k$ and k perfect. An $\overline{k}$-isomorphism $\phi : \overline{C'} \to \overline{C}$ does not change the field of moduli or fields of definition, that is both $\overline{C}$ and $\overline{C'}$ have the same fields of moduli and fields of definitions.*

Proof. A field $L \subseteq \overline{k}$ is a field of definition for $\overline{C}$ if and only if there exists a smooth curve C'' over L, such that $C'' \times_L \overline{k}$ is $\overline{k}$-isomorphic to $\overline{C}$ through some $\psi : C'' \times_L \overline{k} \to \overline{C}$. Hence $\phi^{-1} \circ \psi : C'' \times_L \overline{k}$ is a $\overline{k}$-isomorphism, and L is a field of definition for $\overline{C'}$. The converse is true by a similar discussion. Consequently, the field of moduli for $\overline{C}$ and $\overline{C'}$ coincides, being the intersection of all fields of definition. □

Corollary 2.7. *Consider a smooth $\overline{\mathbb{Q}}$-plane curve $\overline{C}$ defined by an equation of the form*

$$\frac{c_{\phi^2}}{p^2}(XYZ)^2 - \frac{6c_{\phi\psi}}{p}(XYZ)(X^3 + \frac{1}{p}Y^3 + \frac{1}{p^2}Z^3) - 18c_{\psi^2}(X^3 + \frac{1}{p}Y^3 + \frac{1}{p^2}Z^3)^2 + \frac{1}{p}X^3Y^3 + \frac{1}{p^3}(YZ)^3 + \frac{1}{p^2}X^3Z^3 = 0,$$

where $p \in \mathbb{Q}$, in particular $\overline{C}$ admits $\mathbb{Q}(\sqrt{u}, \sqrt{v}, \zeta_3)$ as a plane model-field of definition for $\overline{C}$. Then $\mathrm{Aut}(\overline{C})$ is isomorphic to Hess_{18}. *Moreover, the field of moduli $K_{\overline{C}}$ is $K = \mathbb{Q}(\zeta_3)$, but it is not a field of definition.*

Proof. Since $\overline{C}$ is $\mathbb{Q}(\sqrt[3]{p})$-isomorphic to $F_{\sqrt{u},\sqrt{v}}(X,Y,Z) = 0$ through a change of variables of the shape $\phi = \mathrm{diag}(1, 1/\sqrt[3]{p}, 1/\sqrt[3]{p^2})$, therefore they have conjugate automorphism groups. Moreover, fields of definition and the field of moduli of both curves are the same by Lemma 2.6. Consequently, the field of moduli $K_{\overline{C}}$ is $K = \mathbb{Q}(\zeta_3)$, but it is not a field of definition, using Theorem 2.4. □

Theorem 2.8. *Consider the family $\mathcal{C}_p$ of smooth plane curves over the plane model-field of definition $L = \mathbb{Q}(\sqrt{u}, \sqrt{v}, \zeta_3)$ given by an equation of the form*

$$\frac{c_{\phi^2}}{p^2}(XYZ)^2 - \frac{6c_{\phi\psi}}{p}(XYZ)(X^3 + \frac{1}{p}Y^3 + \frac{1}{p^2}Z^3) - 18c_{\psi^2}(X^3 + \frac{1}{p}Y^3 + \frac{1}{p^2}Z^3)^2 + \frac{1}{p}X^3Y^3 + \frac{1}{p^3}(YZ)^3 + \frac{1}{p^2}X^3Z^3 = 0,$$

where p is a prime integer such that $p \equiv 3$ or 5 mod 7. Given a smooth plane curve C over L in $\mathcal{C}_p$, then there exists a twist C' of C over L which does not have L as a plane model-field of definition. Moreover, the field of moduli of C' is $\mathbb{Q}(\zeta_3)$, and is not a field of definition for C'.

Proof. Consider the Galois extension M'/L with $M' = L(cos(2\pi/7), \sqrt[3]{p})$, where all the automorphisms of $\overline{C} := C \times_L \overline{\mathbb{Q}}$ are defined. Let σ be a generator of the cyclic Galois group $\mathrm{Gal}(L(cos(2\pi/7))/L)$. We define a 1-cocycle on $\mathrm{Gal}(M'/L) \cong \mathrm{Gal}(L(cos(2\pi/7))/L) \times \mathrm{Gal}(L(\sqrt[3]{p})/L)$ to $\mathrm{Aut}(\overline{C})$ by mapping $(\sigma, id) \mapsto [Y : Z : pX]$ and $(id, \tau) \mapsto id$. This defines an element of $\mathrm{H}^1(\mathrm{Gal}(M'/L), \mathrm{Aut}(\overline{C}))$, coming from the inflation of an element in $\mathrm{H}^1(\mathrm{Gal}(L(\cos(2\pi/7))/L), \mathrm{Aut}(\overline{C}))^{Gal(M'/L(cos(2\pi/7)))})$.

This 1-cocycle is trivial if and only if p is a norm of an element of $L(\cos(2\pi/7)$ over L. However, this is not the case, since $\mathbb{Q}(\cos(2\pi/7))$ and L are disjoint with $[L : \mathbb{Q}]$ and $[\mathbb{Q}(\cos(2\pi/7)) : \mathbb{Q}]$ coprime, and moreover p is not a norm of an element of $\mathbb{Q}(\cos(2\pi/7))$ over $\mathbb{Q}$ being inert by our assumption. Consequently, the twist C' is not L-isomorphic to a non-singular plane model in $\mathbb{P}^2_L$ by [1, Theorem 4.1]. That is, L is not a plane model-field of definition for C'. The last sentence in the theorem follows by Lemma 2.6 and Corollary 2.7. □

Remark 2.9. *By our work in [1], we know that a non-singular plane model of C' exists over at least a degree degree 3 extension of L.*

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