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Infinitely many cubic points for $X_0^+(N)$ over $\mathbb{Q}$

by

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1. Introduction. A non-singular smooth curve C over a number field K of genus $g_C > 1$ always has a finite set $C(K)$ of K-rational points by a celebrated result of Faltings (here we fix once and for all $\overline{K}$, an algebraic closure of K). We denote the set of all points of degree at most d for C by $\Gamma_d(C,K) = \bigcup_{[L:K]\le d} C(L)$ and of exact degree d by $\Gamma'_d(C,K) = \bigcup_{[L:K]=d} C(L)$, where $L \subseteq \overline{K}$ runs over the finite extensions of K. A point $P \in C$ is said to be a point *of degree* d over K if $[K(P):K] = d$.

The set $\Gamma_d(C,M)$ is infinite for a certain finite extension M/K if C admits a degree at most d map, all defined over M, to a projective line or an elliptic curve with positive M-rank. The converse is true for $d=2$ [HaSi], $d=3$ [AbHa] and $d=4$ under certain restrictions [AbHa, DeFa]. If we fix the number field M in the above results (i.e. an arithmetic statement for $\Gamma_d(C,M)$ with M fixed), we need a precise understanding over M of the set $W_d(C) = \{v \in \mathrm{Pic}^d(C) \mid h^0(C,\mathcal{L}_v) > 0\}$ where Pic^d is the usual d-Picard group and $\mathcal{L}_v$ the line bundle of degree d on C associated to v. If $W_d(C)$ contains no translates of abelian subvarieties with positive M-rank of $\mathrm{Pic}^d(C)$ then $\Gamma'_d(C,M)$ is finite (under the assumption that C admits no maps of degree at most d to a projective line over M).

For $d=2$ the arithmetic statement for $\Gamma_d(C,K)$ follows from [AbHa] (for a sketch of the proof and the precise statement see [Ba, Theorem 2.14]).

For $d=3$, Daeyeol Jeon [Jeo21] introduced an arithmetic statement and its proof following [AbHa] and [DeFa]. In particular, if $g_C \ge 3$ and C has no degree 3 or 2 map to a projective line and no degree 2 map to an elliptic curve over $\overline{K}$ then the set of exact cubic points of C over K, $\Gamma'_3(C,K)$, is

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infinite if and only if C admits a degree 3 map to an elliptic curve over K with positive K-rank.

Observe that if $g_C \leq 1$ (with $C(K) \neq \emptyset$ for $g_C = 1$), then C has a degree 3 map over K to the projective line, thus $\Gamma_3'(C,K)$ is always infinite. Thus for curves C with $C(K) \neq \emptyset$ we restrict to $g_C \geq 2$ in order to study the finiteness of $\Gamma_3'(C,K)$.

Let N be an integer greater than 1 and consider the modular curve $X_0(N)$ whose non-cusp points correspond to isomorphism classes of isogenies between elliptic curves $\phi : E \to E'$ of degree N with cyclic kernel. The rational and quadratic points of $X_0(N)$ have been studied by many authors. In particular, Jeon [Jeo21] determined the finite set of modular curves $X_0(N)$ where $\Gamma_3'(X_0(N), \mathbb{Q})$ is infinite.

Next, the Fricke involution w_N on $X_0(N)$ arises from taking the dual isogeny $\hat{\phi} : E' \to E$. We define the modular curve $X_0^+(N)$ to be the quotient of $X_0(N)$ by the group of two elements generated by w_N. There is a model for $X_0^+(N)$ over $\mathbb{Q}$, and the study of $\mathbb{Q}$-rational points and quadratic points on those curves attracted the attention of Momose [Mo] and Galbraith [Ga02] and many others.

In this paper, we deal with determining whether there are infinitely many cubic points on $X_0^+(N)$ for genus ≥ 2. The values of N for which $X_0^+(N)$ has genus 0 and 1 are listed in Theorem 3 and recall that $X_0^+(N)(\mathbb{Q}) \neq \emptyset$, because it has a rational cusp.

The novelty of the paper compared to previous works on degree 2 and 3 maps to an elliptic curve E with positive $\mathbb{Q}$-rank is considering the cover $\mathbb{Q}(X_0(N))/\mathbb{Q}(E)$ by taking into account the action of an Atkin–Lehner involution.

The main result of the article is the following.

THEOREM 1. *Suppose* $g_{X_0^+(N)} \geq 2$. *Then* $\Gamma_3'(X_0^+(N), \mathbb{Q})$ *is infinite if and only if* $g_{X_0^+(N)} = 2$ *or* N *is in the following list:*

$g_{X_0^+(N)}$	N
3	$58, 76, 86, 96, 97, 99, 100, 109, 113, 127, 128, 139, 149, 151, 169, 179, 239$
4	$88, 92, 93, 115, 116, 129, 137, 155, 159, 215$
5	$122, 146, 181, 185, 227$
6	$124, 163, 164, 269$
7	$196, 243$
10	236

All computation sources used in the paper are available at https://github.com/Tarundalalmath/X_0-N-with-infinitely-many-cubic-points except the ones for counting points over finite fields, where we use modified versions for $X_0^+(N)$ of the ones already available at different links in [BaGo].

Having the result of this paper on cubic points for $C = X_0^+(N)$, Theorem 1, or [Jeo21] for $C = X_0(N)$, one can try to determine the whole set of cubic points for such C's when $\Gamma_3'(C, \mathbb{Q})$ is finite. This problem could be attacked if the Chabauty method given by Siksek [Si09] (or [BoGaGo]) could apply.

2. General considerations. Given a complete curve C over K, the *gonality* of C is defined as

$$\mathrm{Gon}(C) := \min\{\deg(\varphi) \mid \varphi : C \to \mathbb{P}^1 \text{ defined over } \overline{K}\}.$$

By [Jeo21, Lemma 1.2] (and arguments there) we have

LEMMA 2. *Suppose* $\mathrm{Gon}(C) \geq 4$, $P \in C(K)$ *and* C *does not have a degree* ≤ 2 *map to an elliptic curve. If the set* $\Gamma_3'(C, K)$ *is infinite then* C *admits a* K*-rational map of degree* 3 *to an elliptic curve with positive* K*-rank.*

The modular curves $X_0^+(N)$ to which Lemma 2 is not applicable are listed in the next result, corresponding to the works [FuHa,Jeo18,HaSh99b] (the list with $g_{X_0^+(N)} \leq 1$ is well-known and follows easily from [BaGo, Appendix]).

THEOREM 3.

(i) *The modular curve* $X_0^+(N)$ *has* $g_{X_0^+(N)} = 0$ *if and only if* N *is one of the following:*

1–21, 23–27, 29, 31, 32, 35, 36, 39, 41, 47, 49, 50, 59, 71.

(ii) $X_0^+(N)$ *is an elliptic curve* (*equivalently* $g_{X_0^+(N)} = 1$) *if and only if* N *is one of the following:*

22, 28, 30, 33, 34, 37, 38, 40, 43, 44, 45, 48, 51, 53–56, 61, 63–65, 75, 79, 81, 83, 89, 95, 101, 119, 131.

(iii) (Furumoto–Hasegawa) $X_0^+(N)$ *is hyperelliptic if and only if* N *is one of the following:*

42, 46, 52, 57, 60, 62, 66–69, 72–74, 77, 80, 85, 87, 91, 92, 94, 98, 103, 104, 107, 111, 121, 125, 143, 167, 191.

(iv) (Jeon) $X_0^+(N)$ *is bielliptic,i.e. has a degree* 2 *map to an elliptic curve, if and only if* N *is one of the following:*

42, 52, 57, 58, 60, 66, 68, 70, 72, 74, 76–78, 80, 82, 84–86, 88, 90, 91, 96, 98–100, 104, 105, 108, 110, 111, 117, 118, 120, 121, 123, 124, 128, 135, 136, 141–145, 155, 159, 171, 176, 188.

(v) (Hasegawa–Shimura) $\mathrm{Gon}(X_0^+(N)) = 3$ *if and only if* N *is one of the following:*

58, 70, 76, 82, 84, 86, 88, 90, 93, 96, 97, 99, 100, 108, 109, 113, 115, 116, 117,122, 127, 128, 129, 135, 137, 139, 146, 147, 149, 151, 155, 159, 161, 162, 164, 169, 173, 179, 181, 199, 215, 227, 239, 251, 311.

We say that a pair (N, E), where N is a natural number and E is an elliptic curve over $\mathbb{Q}$ with positive $\mathbb{Q}$-rank, is *admissible* if there is a degree 3 map over $\mathbb{Q}$ of the form $X_0^+(N) \to E$. The following lemma gives a criterion to rule out the pairs which are not admissible.

LEMMA 4. *If (N, E) is an admissible pair, then:*

(i) *E has conductor M with $M \mid N$ and for any prime $p \nmid N$ we have*
$$|\overline{X}_0^+(N)(\mathbb{F}_{p^n})| \le 3|\overline{E}(\mathbb{F}_{p^n})| \text{ and } |\overline{X}_0(N)(\mathbb{F}_{p^n})| \le 6|\overline{E}(\mathbb{F}_{p^n})|, \ \forall n \in \mathbb{N};$$
(ii) *if the conductor of E is N, then the degree of the strong Weil parametrization of E divides 6;*

(iii) *for any prime $p \nmid N$ we have*
$$\frac{p-1}{12}\psi(N) + 2^{\omega(N)} \le 6(p+1)^2,$$
where $\omega(N)$ is the number of prime divisors of N and $\psi = N\prod_{q|N,\, q\, prime}(1 + 1/q)$ is the ψ-Dedekind function;

(iv) *for any Atkin–Lehner involution w_r of $X_0(N)$ with $r \ne N$ we have*
$$g_{X_0^+(N)} \le 3 + 2 \cdot g_{X_0^+(N)/w_r} + 2.$$

Proof. Let (N, E) be admissible. Then there is a $\mathbb{Q}$-rational degree 3 mapping $f : X_0^+(N) \to E$ and consequently we have a $\mathbb{Q}$-rational degree 6 mapping $g : X_0(N) \to E$. Hence $\mathrm{cond}(E) \mid N$.

(i) Let $p \nmid N$ be a prime. Since $p \nmid N$, the curves $X_0^+(N)$, $X_0(N)$ and E have good reduction at p and the mappings f, g induce the $\mathbb{F}_p$-rational mappings $\overline{f} : \overline{X}_0^+(N) \to \overline{E}$ and $\overline{g} : \overline{X}_0(N) \to \overline{E}$, where $\overline{X}_0^+(N), \overline{X}_0(N)$ and $\overline{E}$ denote the mod p reductions of $X_0^+(N)$, $X_0(N)$ and E respectively. Hence we have $|\overline{X}_0^+(N)(\mathbb{F}_{p^n})| \le 3|\overline{E}(\mathbb{F}_{p^n})|$ and $|\overline{X}_0(N)(\mathbb{F}_{p^n})| \le 6|\overline{E}(\mathbb{F}_{p^n})|$, for all $n \in \mathbb{N}$.

(ii) If $\mathrm{cond}(E) = N$, and E' denotes the strong Weil curve with strong Weil parametrization $\varphi : X_0(N) \to E'$, then there exists an isogeny $\psi : E' \to E$ such that $g = \psi \circ \varphi$, hence the degree of the strong Weil parametrization divides 6.

(iii) For any prime $p \nmid N$ we know that $|\overline{X}_0(N)(\mathbb{F}_{p^2})| \ge \frac{p-1}{12}\psi(N) + 2^{\omega(N)}$ (cf. [HaSh99a, Lemma 3.1]) and $|\overline{E}(\mathbb{F}_{p^2})| \le (p + 1)^2$. Hence we have $\frac{p-1}{12}\psi(N) + 2^{\omega(N)} \le 6(p + 1)^2$.

(iv) We know $f : X_0^+(N) \to E$ is a degree 3 mapping. If w_r is an Atkin–Lehner involution on $X_0(N)$ with $r \ne N$, then we have a degree 2 mapping $X_0^+(N) \to X_0^+(N)/w_r$. The result follows from Castelnuovo's inequality. ∎

As an immediate application of Lemma 4(iii) we obtain the following:

Corollary 5. *For $N > 623$, the pair (N, E) is not admissible.*

Proof. The proof is similar to that of [HaSh99a, Lemma 3.2]. We will show that for $N \geq 623$, there exists a prime $p \nmid N$ such that

$$\psi(N) > \frac{12}{p-1}(6(p+1)^2 - 2^{w(N)}).$$

- If $2 \nmid N$ and $N > 623$, then choosing $p = 2$ we have
$$\psi(N) \geq N + 1 > 624 = 12(6 \cdot (2+1)^2 - 2) \geq 12(6 \cdot (2+1)^2 - 2^{w(n)}).$$
- If $2 \mid N$, $3 \nmid N$ and $N > 376$, then choosing $p = 3$ we have
$$\psi(N) \geq \tfrac{3N}{2} > 564 = \tfrac{12}{2}(6.16 - 2) > \tfrac{12}{2}(6.16 - 2^{w(N)}).$$
- If $2 \cdot 3 \mid N$, $5 \nmid N$ and $N > 321$, then choosing $p = 5$ we have
$$\psi(N) \geq N \cdot \frac{3}{2} \cdot \frac{4}{3} > \frac{12}{4}(6.36 - 2).$$
- If $2 \cdot 3 \cdot 5 \mid N$, $7 \nmid N$ and $N > 319$, choosing $p = 7$ we have
$$\psi(N) \geq N \cdot \tfrac{3}{2} \cdot \tfrac{4}{3} \cdot \tfrac{6}{5} > \tfrac{12}{6}(6.64 - 2).$$
- If $2 \cdot 3 \cdot 5 \cdot 7 \mid N$, choose p to be the smallest prime not dividing N. ∎

After applying Lemma 4 (see Appendix B for a list of N's that we can discard in each item), we are reduced to a finite set of N's. To deal with the remaining admissible pairs, the next two lemmas will be helpful.

Lemma 6. *Let $E/\mathbb{Q}$ be an elliptic curve of conductor N and let $\varphi : X_0(N) \to E$ be the strong Weil parametrization of degree k defined over $\mathbb{Q}$. Suppose that w_N acts as $+1$ on the modular form f_E associated to E. Then φ factors through $X_0^+(N)$ (and k is even).*

Proof. Consider the mapping $\varphi : X_0(N) \to E$. Following [CaEm, p. 424] (or [De, §2]), the fact that $w_N f_E = f_E$ implies $\varphi \circ w_N = \varphi + P$, where P is a torsion point of E given by $P = \varphi(0) - \varphi(\infty)$, where $0, \infty$ are the corresponding cusps on $X_0(N)$ with $\varphi(\infty) = O_E$ (recall that O_E denotes the zero point of E). Because the sign of the functional equation of f_E is -1, the $\mathbb{Q}$-rank of E is odd (cf. [MaSD, §3.1]); this implies that $P = O_E$ (see [CaEm]), so φ factors through the quotient $X_0(N)/\langle w_N \rangle$ and w_N acts as the identity on E. ∎

Lemma 7. *Consider a degree k map $\varphi : X \to E$ defined over $\mathbb{Q}$ where X is a quotient modular curve $X_0(N)/W_N$ with W_N a proper subgroup of $B(N)$ ($B(N)$ is the subgroup of $Aut(X_0(N))$ generated by all Atkin–Lehner involutions). Assume that* $\mathrm{cond}(E) = M$ *($M \mid N$). Let $d \in \mathbb{N}$ with $d \,\|\, M$, $(d, N/d) = 1$ and $w_d \notin W_N$ be such that w_d acts as $+1$ on the modular form f_E associated to E.*

(i) *If E has no non-trivial 2-torsion over $\mathbb{Q}$, then φ factors through $X/\langle w_d\rangle$ and k is even.*
(ii) *If E has non-trivial 2-torsion over $\mathbb{Q}$ and k is odd, then we obtain a degree k map $\varphi' : X/\langle w_d\rangle \to E'$ by taking the w_d-invariant to φ, where E' is an elliptic curve isogenous to E.*

Proof. Let $E(\mathbb{C}) \cong \mathbb{C}/\Lambda$. The mapping φ can be considered as a mapping in the complex field, $\tilde{\varphi} : \Gamma\backslash\mathbb{H} \to \mathbb{C}/\Lambda$, defined by $\tau \mapsto \int_{i\infty}^{\tau} \mathrm{const}\cdot f(\tau')\,d\tau'$, where $\Gamma := \langle \Gamma_0(N), W_N\rangle$ and $f \in \bigoplus_{d|N/M} \mathbb{Q} f_E(q^d) \in S_2(\Gamma_0(N))^{\langle W_N\rangle}$ (cf. [Go]). Since w_d acts on f_E as $+1$, it also acts on f as $+1$. Moreover, $\tilde{\varphi}(w_d\tau) - \tilde{\varphi}(\tau) = P$ is independent of τ. Thus $\varphi \circ w_d = \varphi + P$. Since w_d is an involution, we obtain $2P \in \Lambda$, and P is a 2-torsion point of $E(\mathbb{C})$ (which could be the trivial zero point of E, i.e. belonging to Λ). Therefore we have the following commutative diagram (proj is the usual projection map):

$$\begin{array}{ccc} X(\mathbb{C}) & \xrightarrow{\ \varphi\ } & \mathbb{C}/\Lambda \\ {\scriptstyle \mathrm{proj}}\downarrow & & \downarrow{\scriptstyle \mathrm{proj}} \\ X/\langle w_d\rangle(\mathbb{C}) & \xrightarrow{\ \varphi^{w_d}\ } & \mathbb{C}/\langle \Lambda, P\rangle \end{array}$$

Thus if E has no non-trivial 2-torsion over $\mathbb{Q}$, then P is the trivial zero of E and φ factors through $X/\langle w_d\rangle$.

On the other hand, if E has non-trivial 2-torsion over $\mathbb{Q}$ and k is odd, then from the above commutative diagram we see that P is a non-trivial 2-torsion point of E and φ induces a $\mathbb{Q}$-rational degree k mapping $\varphi' : X/\langle w_d\rangle \to E'$ where $E'(\mathbb{C}) \cong \mathbb{C}/\langle \Lambda, P\rangle$ and E' is isogenous to E. ∎

As an immediate corollary of Lemma 7 we obtain

COROLLARY 8. *Let N be natural number which is not a power of a prime number. Take a pair (N, E) with conductor of E equal to M with $M \mid N$ and $M \neq N$. Let d be a natural number with $d \,||\, M$, $(d, N/d) = 1$ such that w_d acts as $+1$ on the modular form f_E associated to E. Suppose that E has no non-trivial 2-torsion over $\mathbb{Q}$. Then (N, E) is not admissible.*

Proof. If (N, E) is admissible, then we have a degree 3 mapping $\varphi : X_0^+(N) \to E$. Since w_d acts as $+1$ on f_E and E has no non-trivial 2-torsion over $\mathbb{Q}$, by Lemma 7 the map φ factors through $X_0^+(N)/\langle w_d\rangle$. This is a contradiction since φ has degree 3. ∎

3. The curve $X_0^+(N)$ with N not listed in Theorem 3. Here by Lemma 2 it is enough to determine the admissible pairs (N, E). After applying Lemma 4 (see Appendix B for a list of N's that we can discard in each item), we are reduced to the following finite set of candidates for admissible pairs.

(N,E)	AL-action on E	(N,E)	AL-action on E
$(106,53a)$	$w_{53}=+$	$(195,65a)$	$w_5=+, w_{13}=+$
$(114,57a)$	$w_3=+, w_{19}=+$	$(196,196a)$	$w_{196}=+$
$(130,65a)$	$w_5=+, w_{13}=+$	$(202,101a)$	$w_{101}=+$
$(158,79a)$	$w_{79}=+$	$(231,77a)$	$w_7=+, w_{11}=+$
$(163,163a)$	$w_{163}=+$	$(236,118a)$	$w_2=+, w_{59}=+$
$(166,83a)$	$w_{83}=+$	$(236,236a)$	$w_{236}=+$
$(172,43a)$	$w_{43}=+$	$(237,79a)$	$w_{79}=+$
$(174,58a)$	$w_2=+, w_{29}=+$	$(243,243a)$	$w_{243}=+$
$(178,89a)$	$w_{89}=+$	$(249,83a)$	$w_{83}=+$
$(182,91a)$	$w_7=+, w_{13}=+$	$(258,43a)$	$w_{43}=+$
$(182,91b)$	$w_7=-, w_{13}=-$	$(258,129a)$	$w_3=+, w_{43}=+$
$(183,61a)$	$w_{61}=+$	$(267,89a)$	$w_{89}=+$
$(185,37a)$	$w_{37}=+$	$(269,269a)$	$w_{269}=+$
$(185,185c)$	$w_{185}=+$		

When (N,E) is in the table above with $\mathrm{cond}(E)=N$, the strong Weil parametrization $X_0(N)\to E$ has degree 6. Thus we conclude by Lemma 6 that (N,E) is an admissible pairing. More precisely, we have

COROLLARY 9. *For $N=163,185,196,236,243$, 269 the modular curve $X_0^+(N)$ has infinitely many cubic points over $\mathbb{Q}$.*

To deal with the remaining cases we use Lemma 7.

COROLLARY 10. *For $N=106,114,158,166,172,174,178,182$, 183, 202, 231, 237, $249,258$, 267, the set $\Gamma_3'(X_0^+(N),\mathbb{Q})$ is finite.*

Proof. Let N be as in the statement and (N,E) be a pair appearing in the above table. Then $\mathrm{cond}(E)\mid N$, $\mathrm{cond}(E)\neq N$ and E has no non-trivial 2-torsion over $\mathbb{Q}$. By Corollary 8, we conclude that the pair (N,E) is not admissible (for $(182,91b)$ use the Atkin–Lehner operator w_{91}). The result follows. ∎

PROPOSITION 11. *The modular curves $X_0^+(130)$ and $X_0^+(195)$ each have finitely many cubic points over $\mathbb{Q}$.*

Proof. We need to check the pairs $(130,65a)$ and $(195,65a)$. Considering $\varphi: X_0^+(130)\to 65a$ of degree 3, we know that w_5 and w_{13} act as $+1$. Since the degree of $\mathbb{Q}(X_0^+(130))/\mathbb{Q}(X_0^*(130))$ is coprime to 3 (recall that $X_0^*(N):= X_0(N)/B(N)$ where $B(N)$ is the subgroup of $\mathrm{Aut}(X_0(N))$ generated by all Atkin–Lehner involutions), by applying Lemma 7 twice with w_5 and w_{13} we obtain a degree 3 morphism (moreover an isogeny) $X_0^*(130)\to E'$ between elliptic curves, where E' is isogenous to $65a$ (note that $X^*(130)$ has genus 1 and its Cremona level is $65a$). This is a contradiction since the elliptic curve $65a$ has no non-trivial 3-torsion over $\mathbb{Q}$, and also no 3-isogeny over $\mathbb{Q}$ by [Cr].

Thus $(130, 65a)$ is not admissible. A similar argument holds for the pair $(195, 65a)$: recall that $X_0^*(195)$ has genus 1 and its Cremona level is $65a$. ∎

4. The curve $X_0^+(N)$ with N listed in Theorem 3. Recall that $X_0^+(N)(\mathbb{Q}) \neq \emptyset$. Thus $\Gamma_3'(X_0^+(N), \mathbb{Q})$ is an infinite set when $g_{X_0^+(N)} \leq 1$.

We assume, once and for all, $g_{X_0^+(N)} \geq 2$.

4.1. The levels N with $X_0^+(N)$ hyperelliptic. We deal with the following levels N:

$g_{X_0^+(N)}$	N
2	$42, 46, 52, 57, 62, 67, 68, 69, 72, 73, 74, 77, 80, 87, 91, 98,$ 103, 107, 111, 121, 125, 143, 167, 191
3	60, 66, 85, 104
4	92, 94

For such hyperelliptic curves, we pick the model given by Hasegawa [Ha] if $g_{X_0^+(N)} = 2$, and by Furumoto and Hasegawa [FuHa] when $g_{X_0^+(N)} \geq 3$.

Theorem 12 ([JKS04, Lemma 2.1]). *Let X be a curve of genus 2 over a perfect field k. If X has at least three k-rational points, then there exists a map $X \to \mathbb{P}^1$ of degree 3 which is defined over k.*

As an immediate consequence of the last theorem, we have

Proposition 13. *$X_0^+(N)$ has infinitely many cubic points over $\mathbb{Q}$ for*

$$N \in \{42, 46, 52, 57, 67, 68, 69, 72, 73, 74, 77, 80, 91, 103, 107, 111, 121, 125, 143, 167, 191\}.$$

Proof. Using MAGMA it can be easily checked that in this case the genus 2 hyperelliptic curve $X_0^+(N)$ has at least three $\mathbb{Q}$-rational points. ∎

The remaining values of N with $g_{X_0^+(N)} = 2$ are $N = 62, 87, 98$.

Proposition 14. *For $N \in \{62, 87\}$, the set $\Gamma_3'(X_0^+(N), \mathbb{Q})$ is infinite.*

Proof. Consider $N = 62$. An affine model of $X_0^+(62)$ is given by

$$Y : y^2 = x^6 - 8x^5 + 26x^4 - 42x^3 + 29x^2 + 2x - 11.$$

Then Y has two $\mathbb{Q}$-rational points ($(1 : 1 : 0)$ and $(1 : -1 : 0)$) which are the "points at infinity", and the hyperelliptic involution permutes them. Therefore, from [Jeo21, Lemma 2.2] we conclude that there is a $\mathbb{Q}$-rational degree 3 mapping $X_0^+(62) \to \mathbb{P}^1$, and consequently $X_0^+(62)$ has infinitely many cubic points over $\mathbb{Q}$. A similar argument works for $N = 87$ with the model $Y : y^2 = x^6 - 4x^5 + 12x^4 - 22x^3 + 32x^2 - 28x + 17$. ∎

Lemma 15. *The genus 2 curve $X_0^+(98)$ has infinitely many cubic points over $\mathbb{Q}$.*

Proof. An (affine) model of $X_0^+(98)$ is given by
$$y^2 = 4x^5 - 15x^4 + 30x^3 - 35x^2 + 24x - 8.$$
Suppose D is a degree 3 effective $\mathbb{Q}$-rational divisor on a curve of genus 2. By the Riemann–Roch theorem we have $\dim L(D) = 2$.

Observe that with $y = 1$ in the model we get
$$0 = (x^2 - x + 1)\big(x^3 - \tfrac{11}{4}x^2 + \tfrac{15}{4}x - \tfrac{9}{4}\big).$$
Let t_1, t_2, t_3 be the roots of the equation $x^3 - \frac{11}{4}x^2 + \frac{15}{4}x - \frac{9}{4}$. Then $P_i := (t_i, 1) \in X_0^+(98)(K)$ for $1 \le i \le 3$ (where K is a cubic extension of $\mathbb{Q}$ defined by the polynomial $t^3 - \frac{11}{4}t^2 + \frac{15}{4}t - \frac{9}{4}$). Furthermore, the divisor $[P_1 + P_2 + P_3]$ is a $\mathbb{Q}$-rational effective divisor of degree 3. By Riemann–Roch we have $\dim L([P_1 + P_2 + P_3]) = 2$. Therefore, there exists a $\mathbb{Q}$-rational function f with exactly three poles and consequently there is a degree 3 mapping $X_0^+(98) \to \mathbb{P}^1$ defined over $\mathbb{Q}$. The result follows. ∎

Consider $X_0^+(N)$ hyperelliptic with $g_{X_0^+(N)} \ge 3$. By [Jeo21, §2.3], in order for $X_0^+(N)$ to have infinitely many cubic points, $W_3(X_0^+(N))$ must contain an elliptic curve with positive $\mathbb{Q}$-rank.

Thus, by Cremona tables [Cr] we obtain (because there is no elliptic curve with $\mathbb{Q}$-rank ≥ 1 for levels dividing N):

Corollary 16. *For $N \in \{60, 66, 85, 94, 104\}$, the set $\Gamma_3'(X_0^+(N), \mathbb{Q})$ is finite.*

Proposition 17. *$X_0^+(92)$ has infinitely many cubic points over $\mathbb{Q}$.*

Proof. The strong Weil modular parametrization $\phi : X_0(92) \to 92b$ has degree 6 and $92b$ has $\mathbb{Q}$-rank 1, and w_{92} acts as $+1$ on $92b$; therefore, we have a $\mathbb{Q}$-rational degree 3 map $X_0^+(92) \to 92b$ by Lemma 6. ∎

4.2. Trigonal curves $X_0^+(N)$. Suppose that $\mathrm{Gon}(X_0^+(N)) = 3$. The levels N are:

$g_{X_0^+(N)}$	N
3	58, 76, 86, 96, 97, 99, 100, 109, 113, 127, 128, 139, 149, 151, 169, 179, 239
4	70, 82, 84, 88, 90, 93, 108, 115, 116, 117, 129, 135,
	137, 147, 155, 159, 161, 173, 199, 215, 251, 311
5	122, 146, 181, 227
6	164

If $g_{X_0^+(N)} = 3$, then the projection from a $\mathbb{Q}$-rational cusp defines a degree 3 map $X_0^+(N) \to \mathbb{P}^1$ over $\mathbb{Q}$ (cf. [HaSh99a, p. 136]). On the other hand, it is known that every curve C/K of genus ≥ 5 with $\mathrm{Gon}(C) = 3$ has a degree 3 map $X \to \mathbb{P}^1$ over K (cf. [NS, Theorem 2.1], [HaSh99a, Corollary 1.7]).

Thus, we restrict to $\mathrm{Gon}(X_0^+(N)) = 3$ and $g_{X_0^+(N)} = 4$.

It is well known that a non-hyperelliptic curve of genus 4 lies either on a quadratic cone or on a ruled surface (cf. [HaSh99a, p. 136]), and by Petri's theorem a model of the curve can be computed in $\mathbb{P}^3$ as the intersection of a degree 2 and a degree 3 homogeneous equations. Following [HaSh99a, pp. 131, 136] it can be checked that for $N = 159$ the curve $X_0^+(N)$ lies on a quadratic cone over $\mathbb{Q}$ and for $N = 88, 93, 115, 116, 129, 137, 155, 215$ the curve $X_0^+(N)$ lies on a ruled surface over $\mathbb{Q}$. On the other hand, for $N = 70, 82, 84, 90, 108, 117, 135, 147, 161, 173, 199, 251, 311$ the curve $X_0^+(N)$ lies on a ruled surface either over a quadratic extension of $\mathbb{Q}$ or over a biquadratic extension of $\mathbb{Q}$. Hence in these last levels the trigonal maps are not defined over $\mathbb{Q}$. For example, consider $X_0^+(70)$; the quadratic surface is given by $xz - y^2 + 8yw - z^2 - 10zw - 9w^2$, which after a suitable coordinate change can be converted into the equation

$$x^2 - y^2 - z^2 + 7w^2 = (x + y)(x - y) - (z + \sqrt{7}\,w)(z - \sqrt{7}\,w),$$

and this surface is isomorphic to the ruled surface $uv - st$ over $\mathbb{Q}(\sqrt{7})$. See details in Appendix A for all $X_0^+(N)$ trigonal with $g_{X_0^+(N)} = 4$.

From the discussion so far we have

THEOREM 18. *Assume that $g_{X_0^+(N)} \geq 3$. Then $X_0^+(N)$ is trigonal over $\mathbb{Q}$ if and only if N is in the following list:*

58, 76, 86, 88, 93, 96, 97, 99, 100, 109, 113, 115, 116, 122, 127, 128, 129, 137, 139, 146, 149, 151, 155, 159, 164, 169, 179, 181, 215, 227, 239.

In particular, for such N, the set $\Gamma'_3(X_0^+(N), \mathbb{Q})$ is infinite.

Assume now that $\mathrm{Gon}(X_0^+(N)) = 3$, but $X_0^+(N)$ does not admit a degree 3 map to the projective line $\mathbb{P}^1$ over $\mathbb{Q}$.

Hence in these cases $X_0^+(N)$ contains infinitely many cubic points over $\mathbb{Q}$ when $W_3(X_0^+(N))$ contains a translation of the elliptic curve E with positive $\mathbb{Q}$-rank [Jeo21, p. 352].

PROPOSITION 19. *For $N = 70, 82, 84, 90, 108, 117, 135, 147, 161, 173, 199, 251, 311$, the curve $X_0^+(N)$ has finitely many cubic points over $\mathbb{Q}$.*

Proof. For $N = 70, 84, 90, 108, 147, 161, 173, 199, 251, 311$ there is no elliptic curve E of positive $\mathbb{Q}$-rank with $\mathrm{cond}(E) \mid N$. Hence in these cases, $X_0^+(N)$ contains finitely many cubic points over $\mathbb{Q}$.

For $N = 82, 117$ and 135, $X_0^+(N)$ is bielliptic and there are elliptic curves of positive $\mathbb{Q}$-rank with $\mathrm{cond}(E) \mid N$. By arguments in [Jeo21, p. 353], if there is no $\mathbb{Q}$-rational degree 3 mapping $X_0^+(N) \to E$ where E is an elliptic curve of positive $\mathbb{Q}$-rank and $\mathrm{cond}(E) \mid N$, then $W_3(X_0^+(N))$ has no translation of an elliptic curve with positive $\mathbb{Q}$-rank.

In these cases only the pairs $(82, 82a)$, $(117, 117a)$ or $(135, 135a)$ could appear. If any of these pairs (N, E) is admissible (i.e. there is a $\mathbb{Q}$-rational degre 6 mapping $X_0(N) \to E$), then the degree of the strong Weil parametrization of E should divide 6. For $82a, 117a$ and $135a$ the degrees of the strong Weil parametrization are 4, 8 and 16 respectively. Thus no such pairs are admissible. The result follows. ∎

4.3. $X_0^+(N)$ bielliptic and not hyperelliptic and not trigonal. Suppose $X_0^+(N)$ is bielliptic but neither hyperelliptic nor trigonal. Following [Jeo21, p. 353], if $\Gamma_3'(X_0^+(N), \mathbb{Q})$ is infinite, then $W_3(X_0^+(N))$ contains a translation of an elliptic curve E with positive $\mathbb{Q}$-rank, equivalently (N, E) is an admissible pair.

The levels that remain to study are

78, 105, 110, 118, 120, 123, 124, 136, 141, 142, 144, 145, 171, 176, 188.

PROPOSITION 20. *Suppose $X_0^+(N)$ is bielliptic and not hyperelliptic and not trigonal. Then the only admissible pair is $(124, 124a)$; in particular, for all such curves, $\Gamma_3'(X_0^+(N), \mathbb{Q})$ is infinite if and only if $N = 124$.*

Proof. For $N = 78, 105, 110, 120, 144, 188$ there is no possible (N, E) because there is no elliptic curve satisfying (iii) in Lemma 4 by Cremona tables [Cr]. For $N = 118, 123, 124, 136$, 141, 142, 145 the only possible admissible pairs (N, E) have $\mathrm{cond}(E) = N$. If they were admissible, we get a degree 6 map from $X_0(N) \to E$ and the degree of the strong Weil parametrization of E (see Cremona tables [Cr] for such degrees) should divide 6, and no such case happens except $(124, 124a)$, for which by Lemma 6 the Weil parametrization of degree 6 factors through $X_0^+(124)$ because w_{124} in $124a$ acts as $+1$. Finally, take $N = 171, 176$; the pairs to study are $(171, 171b)$, $(171, 57a)$ and $(176, 88a)$. The pair $(171, 171b)$ we discard as before, because the strong Weil parametrization for $171b$ is 8. We can apply Corollary 8 with w_{19} and w_{11} respectively to deduce that $(171, 57a)$ and $(176, 88a)$ are not admissible. ∎

Appendix A. A model for trigonal $X_0^+(N)$ with $g_{X_0^+(N)} = 4$. For a detailed discussion on how to construct the models we refer the reader to [Si] and [Ga].

Curve	Petri model
$X_0^+(70)$	$x^2w - 7xw^2 - y^3 + 3y^2z + 2y^2w - 3yz^2 - 16yzw + 28yw^2 + z^3 + 11z^2w - 19zw^2 - 27w^3$, $xz - y^2 + 8yw - z^2 - 10zw - 9w^2$

Curve	Petri model
$X_0^+(82)$	$x^2w - 2xyw - 5xw^2 - yz^2 + 5yzw + yw^2 + 2z^3 - 12z^2w + 23zw^2 - 9w^3$, $xz - 3xw - y^2 + 2yz - 4z^2 + 10zw - 4w^2$
$X_0^+(84)$	$x^2w - 2xyw - 5xw^2 - y^2z - y^2w + 3yz^2 + 6yzw + 5yw^2 - 2z^3 - 6z^2w$ $+ 4zw^2 + 4w^3$, $xz - xw - y^2 + 2yz + yw - 3z^2 + w^2$
$X_0^+(88)$	$x^2z - xy^2 - xyz - 2xz^2 + y^3 + 6y^2z - 9y^2w - 8yz^2 + 33yw^2 + 5z^3$ $+ 6z^2w - 12zw^2 - 30w^3$, $xw - yz + yw + z^2 - zw - 5w^2$
$X_0^+(90)$	$x^2w - 2xyw - 3xw^2 - y^2z - y^2w + 3yz^2 + 6yzw + 3yw^2 - 2z^3$ $- 5z^2w + zw^2$, $xz - xw - y^2 + 2yz + yw - 3z^2$
$X_0^+(93)$	$x^2z - xy^2 - xyz - 2xz^2 + y^3 + 7y^2z - 11y^2w - 10yz^2 + 7yzw$ $+ 29yw^2 + 6z^3 + 2z^2w - 16zw^2 - 21w^3$, $xw - yz + yw + z^2 - 2zw - 3w^2$
$X_0^+(108)$	$x^2w - 3xw^2 - y^3 + 2y^2z - 8yzw + 12yw^2 - 2z^3 + 12z^2w - 22zw^2 + 5w^3$, $xz - y^2 + 4yw - 6zw - w^2$
$X_0^+(115)$	$x^2z - xy^2 - xyz - 2xz^2 + y^3 + 5y^2z - 9y^2w - 4yz^2 - 6yzw + 29yw^2$ $+ 2z^3 + 5z^2w - 22w^3$, $xw - yz + yw + z^2 - 4w^2$
$X_0^+(116)$	$x^2z - xy^2 - 2xz^2 + 4y^2z + 2y^2w - 6yz^2 - 8yzw + 3yw^2 + 4z^3 + 9z^2w$ $- 4zw^2 - 4w^3$, $xw - yz + z^2 - 3w^2$
$X_0^+(117)$	$x^2w - xyw - 5xw^2 - y^2z + y^2w + yz^2 + yzw + yw^2 - z^3 + 2zw^2 + 4w^3$, $xz - y^2 + yz + yw - 3z^2 + 2zw - 4w^2$
$X_0^+(129)$	$x^2z - xy^2 - 2xz^2 + 5y^2z - 7yz^2 - 3yzw + 3yw^2 + 4z^3 + 3z^2w$ $- 3zw^2 - w^3$, $xw - yz + z^2 - zw - w^2$
$X_0^+(135)$	$x^2w - 2xyw - 3xw^2 - y^3 + 3y^2z + 2y^2w - 3yz^2 + 2yw^2 + z^3 + w^3$, $xz - 2xw - y^2 + 2yz + 3yw - 2z^2 - zw$
$X_0^+(137)$	$x^2z - xy^2 - xz^2 + 3y^2z + 2y^2w - 6yz^2 - yzw - 3yw^2 + 3z^3 + 2z^2w$ $- zw^2 + 2w^3$, $xw - yz + z^2 - zw - w^2$
$X_0^+(147)$	$x^2w - xyw - 6xw^2 - y^2z + yz^2 + 2yw^2 - z^3 + z^2w + 3zw^2 + 7w^3$, $xz - xw - y^2 + yz - 2z^2 + zw + w^2$
$X_0^+(155)$	$x^2z - xy^2 - xyz - xz^2 + y^3 + 3y^2z - 5y^2w - 2yz^2 + 2yzw + 7yw^2$ $+ z^3 - 2zw^2 - 3w^3$, $xw - yz + yw - 2w^2$

Curve	Petri model
$X_0^+(159)$	$x^2z - xy^2 + xyz - 3xz^2 + 2y^2z + y^2w - 8yzw + 3yw^2 + 7z^2w - zw^2 - 2w^3$, $xw - yw - z^2 + 2zw - 2w^2$
$X_0^+(161)$	$x^2w - 5xw^2 - y^2z + yz^2 + 2yw^2 - 3z^2w + 9zw^2 - 4w^3$, $xz - xw - y^2 + 3yw - z^2 + zw - 3w^2$
$X_0^+(173)$	$x^2w - xyw + 6xw^2 - 2y^2w - yz^2 + 4yzw + 6yw^2 + 4z^2w - 17zw^2 - 6w^3$, $xz + 2xw - y^2 + yz + 3yw - 6zw - 3w^2$
$X_0^+(199)$	$x^2w + 2xyw + xw^2 - y^3 - y^2z + 2y^2w + yz^2 - 5yzw + 3zw^2 - 5w^3$, $xz + 2xw - y^2 - 2yz + 3yw - 4w^2$
$X_0^+(215)$	$x^2z - xy^2 - xyz - xz^2 + y^3 + 2y^2z - 3y^2w - 2yzw + 5yw^2 + z^3 - z^2w + zw^2 - 2w^3$, $xw - yz + yw + zw - 2w^2$
$X_0^+(251)$	$x^2w - 5xw^2 - y^2z - y^2w + yz^2 + yw^2 + z^2w - zw^2 + 4w^3$, $xz - 2xw - y^2 + yw + w^2$
$X_0^+(311)$	$x^2w - xyw - y^3 + y^2z + 2y^2w - yz^2 - 2yzw - yw^2 + z^2w$, $xz - xw - y^2 + yz + 2yw - z^2 - 2zw$

Curve	Quadratic surface
$X_0^+(70)$	Diagonal form: $x^2 - y^2 - z^2 + 7w^2$, lies on a ruled surface over $\mathbb{Q}(\sqrt{7})$
$X_0^+(82)$	Diagonal form: $3x^2 - 12y^2 - 4z^2 - w^2$, lies on a ruled surface over $\mathbb{Q}(\sqrt{-1})$
$X_0^+(84)$	Diagonal form: $2x^2 - 6y^2 - 3z^2 + w^2$, lies on a ruled surface over $\mathbb{Q}(\sqrt{3})$
$X_0^+(88)$	Diagonal form: $5x^2 + 5y^2 - 5z^2 - 5w^2$, lies on a ruled surface over $\mathbb{Q}$
$X_0^+(90)$	Diagonal form: $2x^2 - 6y^2 - 3z^2 - 3w^2$, lies on a ruled surface over $\mathbb{Q}(\sqrt{3}, \sqrt{-1})$
$X_0^+(93)$	Diagonal form: $4x^2 + 3y^2 - 4z^2 - 3w^2$, lies on a ruled surface over $\mathbb{Q}$
$X_0^+(108)$	Diagonal form: $-x^2 - y^2 + z^2 + 3w^2$, lies on a ruled surface over $\mathbb{Q}(\sqrt{3})$
$X_0^+(115)$	Diagonal form: $3x^2 + 4y^2 - 3z^2 - 4w^2$, lies on a ruled surface over $\mathbb{Q}$
$X_0^+(116)$	Diagonal form: $3x^2 - y^2 + z^2 - 3w^2$, lies on a ruled surface over $\mathbb{Q}$
$X_0^+(117)$	Diagonal form: $11x^2 - 33y^2 - 3z^2 - 15w^2$, lies on a ruled surface over $\mathbb{Q}(\sqrt{3}, \sqrt{-5})$
$X_0^+(129)$	Diagonal form: $x^2 - 5y^2 + 5z^2 - w^2$, lies on a ruled surface over $\mathbb{Q}$
$X_0^+(135)$	Diagonal form: $x^2 - 2y^2 - 2z^2 + 9w^2$, lies on a ruled surface over $\mathbb{Q}(\sqrt{2})$
$X_0^+(137)$	Diagonal form: $x^2 - 5y^2 + 5z^2 - w^2$, lies on a ruled surface over $\mathbb{Q}$
$X_0^+(147)$	Diagonal form: $7x^2 - 14y^2 - 2z^2 + w^2$, lies on a ruled surface over $\mathbb{Q}(\sqrt{2})$
$X_0^+(155)$	Diagonal form: $2x^2 + 2y^2 - 2z^2 - 2w^2$, lies on a ruled surface over $\mathbb{Q}$
$X_0^+(159)$	Diagonal form: $2y^2 - z^2 - 2w^2$, lies on a quadratic cone over $\mathbb{Q}$

Curve	Quadratic surface
$X_0^+(161)$	Diagonal form: $x^2 - y^2 - z^2 - 3w^2$, lies on a ruled surface over $\mathbb{Q}(\sqrt{-3})$
$X_0^+(173)$	Diagonal form: $-x^2 - 3y^2 + 3z^2 + 37w^2$, lies on a ruled surface over $\mathbb{Q}(\sqrt{37})$
$X_0^+(199)$	Diagonal form: $-x^2 - y^2 + z^2 + 33w^2$, lies on a ruled surface over $\mathbb{Q}(\sqrt{33})$
$X_0^+(215)$	Diagonal form: $x^2 + 2y^2 - z^2 - 2w^2$, lies on a ruled surface over $\mathbb{Q}$
$X_0^+(251)$	Diagonal form: $-x^2 - y^2 + z^2 + 5w^2$, lies on a ruled surface over $\mathbb{Q}(\sqrt{5})$
$X_0^+(311)$	Diagonal form: $3x^2 - 3y^2 - z^2 - 3w^2$, lies on a ruled surface over $\mathbb{Q}(\sqrt{-3})$

Appendix B. The sieves to reduce to a finite set of N to consider. Here we consider the levels N that do not appear in Theorem 3. Using Ogg's classical argument as in the proof of [HaSh99b, Lemma 3.2] one finds that if $N \geq 624$, there is no $\mathbb{Q}$-rational degree 6 mapping $X_0(N) \to E$ for any E, and consequently no degree 3 map $X_0^+(N) \to E$ over $\mathbb{Q}$ for $N \geq 624$.

Now by Lemma 4(i) we can discard the existence of such a degree 3 map for the following N:

252, 260, 264, 272, 276, 280, 288, 290, 294, 296, 300, 304, 306, 308, 310, 312, 315, 316, 318, 320, 322, 324, 328, 330, 332, 336, 340, 342, 344, 345, 348, 350, 352, 354, 356, 357, 360, 364, 366, 368, 370, 372, 374–376, 378, 380, 382, 384, 385, 386, 388, 390, 392, 394, 396, 398–400, 402, 404–406, 408, 410, 412, 414, 416, 418, 420, 422-426, 428-430, 432, 434–436, 438, 440–442, 444, 446, 448, 450, 452–456, 458–460, 462, 464–466, 468, 470–472, 474–478, 480, 482–486, 488–490, 492, 494–498, 500–502, 504–508, 510–520, 522, 524–528, 530–540, 542–546, 548–556, 558–562, 564–623.

By Lemma 4(iii) we can discard all pairs (N, E) for the following N:

126, 132, 133, 134, 140, 150, 157, 165, 168, 177, 180, 186, 187, 193, 194, 206, 211, 213, 217, 221, 223, 230, 233, 240, 241, 247, 250, 253, 255, 257, 261, 263, 266, 268, 271, 279, 281, 283, 287, 292, 293, 295, 299, 307, 313, 317, 319, 321, 323, 329, 334, 337, 341, 343, 349, 353, 355, 358, 365, 367, 379, 383, 391, 397, 401, 403, 409, 411, 413, 417, 419, 421, 439, 447, 449, 457, 461, 463, 479, 487, 491, 499, 509, 521, 523, 529, 541, 547.

By the use of (iii) and (v) in Lemma 4 we can discard N in the list:

102, 112, 138, 152, 153, 156, 160, 170, 175, 189, 190, 192, 197, 200, 201, 203, 205, 207, 208, 209, 210, 214, 216, 218, 219, 220, 225, 226, 229, 235, 238, 245, 254, 274, 275, 277, 278, 289, 291, 298, 302, 309, 314, 327, 331, 335, 338, 339, 346, 347, 359, 361, 362, 373, 377, 381, 389, 431, 433, 437, 443, 451, 467, 469, 493, 503, 557, 563.

For N in the table below, using Lemma 4(v) we can eliminate all (N, E) with $\mathrm{cond}(E) = N$; the remaining pairs (N, E) where $\mathrm{cond}(E) \mid N$ and $\mathrm{cond}(E) \neq N$ ($\mathrm{rank}_{\mathbb{Q}}(E) \geq 1$) can be eliminated by Lemma 4(ii), i.e. by computing $\mathbb{F}_{p^r}$-points on $X_0(N)$ with $p \nmid N$ in the first two columns and the last one for $X_0^+(N)$ instead of $X_0(N)$. Thus we can discard all the levels N appearing in the table below.

N	E	p^r	N	E	p^r	N	E	p^r
148	$37a$	3^2	297	$99a$	5^2	244	$61a$	3^2
154	$77a$	3^2	301	$43a$	5^2	244	$122a$	3^2
184	$92b$	3^2	325	$65a$	3^2	248	$124a$	5^2
198	$99a$	5^2	326	$163a$	3^2	273	$91a$	2
204	$102a$	5^2	333	$37a$	5^2	273	$91b$	2^2
212	$53a$	3^2	351	$117a$	2^2	282	$141a$	5^2
212	$106a$	5^2	363	$121a$	5^2	282	$141d$	7^2
224	$112a$	3^2	369	$123a$	2^2	305	$61a$	7
228	$57a$	5^2	369	$123b$	7^2	395	$79a$	2^2
232	$58a$	3^2	371	$53a$	3^2			
234	$117a$	5^2	387	$43a$	2^2			
242	$121b$	5^2	387	$129a$	2^4			
246	$82a$	7^2	393	$131a$	5^2			
246	$123a$	5^2	407	$37a$	3^2			
246	$123b$	7^2	415	$83a$	3^2			
256	$128a$	3^2	427	$61a$	3^2			
259	$37a$	3^2	445	$89a$	2^2			
265	$53a$	3^2	473	$43a$	2^2			
270	$135a$	7^2	481	$37a$	2^2			
285	$57a$	2^2						
286	$143a$	3^2						

By Lemma 4(iv) we can discard $N = 222, 262, 284, 303$.

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Abstract (will appear on the journal's web site only)

We determine all modular curves $X_0^+(N)$ that admit infinitely many cubic points over the rational field $\mathbb{Q}$.