AUTOMORPHISM GROUPS OF NON-SINGULAR PLANE CURVES OF DEGREE 5

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Abstract. Let $M_g$ be the moduli space of smooth, genus $g$ curves over an algebraically closed field $K$ of zero characteristic. Denote by $M_g(G)$ the subset of $M_g$ of curves $\delta$ such that $G$ (as a finite non-trivial group) is isomorphic to a subgroup of $\text{Aut}(\delta)$ and let $M_g^\natural(G)$ be the subset of curves $\delta$ such that $G \cong \text{Aut}(\delta)$ is the full automorphism group of $\delta$. Now, for an integer $d \geq 4$, let $M_g^{P(\delta)}$ be the subset of $M_g$ representing smooth, genus $g$ plane curves of degree $d$ (in this case, $g = (d-1)(d-2)/2$) and consider the sets $M_g^{P(\delta)} := M_g^{P(\delta)} \cap M_g(G)$ and $M_g^{P(\delta)} := M_g^\natural(G) \cap M_g^{P(\delta)}$.

Henn in [7] and Komiya-Kuribayashi in [10], listed the groups $G$ for which $M_g^{P(\delta)}(G)$ is non-empty. In this paper, we determine the loci $M_g^{P(\delta)}(G)$, corresponding to non-singular degree 5 projective plane curves, which are non-empty. Also, we present the analogy results of Henn for quartic curves concerning non-singular plane model equations associated to these loci. Similar arguments can be applied to deal with higher degrees.

1. Introduction

It is classically known from Hurwitz [9] that, given any non-trivial finite group $G$, one can construct a Riemann surface $X$ such that its automorphism group $\text{Aut}(X)$ is isomorphic to $G$.

A natural question is to list the groups such that the associated Riemann surface will have a non-singular plane model. Harui in [6] determined the list of finite groups $G$ that could appear in such case however, for a complete answer to the problem, it remains to introduce the exact list of such groups which might appear for a fixed degree and conversely, for an arbitrary but fixed group in the list, one need to determine the degrees such where a group occur. Therefore, there are the following two open problems:

(1) Fixing a group $G$, for which degrees $d \geq 4$ we have that $M_g^{P(\delta)}(G)$ is a non-empty set? For example, by the work of Crass in [3, p.28], we know that $M_g^{P(\delta)}(A_6)$ is non-empty exactly for $g = 10$, $g = 66$ and $g = 406$, where $A_6$ is the alternating group of 6 letters.

(2) Once the degree $d$ is fixed, determine the groups $G$ (up to isomorphism) where $M_g^{P(\delta)}(G)$ is non-empty.

This note is concerned with the second question. Henn in [7] and Komiya-Kuribayashi in [10] solved the question for $d = 4$.

Recall that any $\delta \in M_g^{P(\delta)}(G)$ corresponds to a set of non-singular plane models $C_1$ in $\mathbb{P}^2(K)$ such that any two of them are related through a change of variables $P \in PGL_3(K)$ (where $PGL_3(K)$ is the classical projective linear group of $3 \times 3$ invertible matrices over $K$) and their automorphism groups are conjugate. We mean by $C$ a plane non-singular model associated to $\delta$. Observe that $\text{Aut}(C)$ is a subgroup of $PGL_3(K)$ isomorphic to $G$ by an injective representation $\rho : G \hookrightarrow PGL_3(K)$ that is $\text{Aut}(C) \cong \rho(G)$ for some $\rho$.

We denote by $\rho(M_g^{P(\delta)}(G))$ the set of all elements $\delta \in M_g^{P(\delta)}(G)$ such that $G$ acts on a plane model associated to $\delta$ as $P\rho(G)P^{-1}$ for some $P$. This gives us the following disjoint union decomposition:

$$M_g^{P(\delta)}(G) = \cup_{[\rho] \in A}\rho(M_g^{P(\delta)}(G))$$

where $A := \{\rho \mid \rho : G \hookrightarrow PGL_3(K)\}/\sim$ such that $\rho_a \sim \rho_b$ if and only if $\rho_a(G) = P\rho_b(G)P^{-1}$ for some $P \in PGL_3(K)$. A similar decomposition follows for $M_g^{P(\delta)}(G)$.

Henn in [7] determined the $[\rho]'s$ and $G$ such that $\rho(M_g^{P(\delta)}(G))$ is non-trivial and associated to such locus (once $\rho$ and $G$ are fixed) a certain projective plane equation which depends on some parameters together with some

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algebraic restrictions to these parameters. More concretely, he obtained a plane non-singular model of any element of the locus by a certain specialization of the values of the parameters and vice versa.

In this paper, we obtain the analogy of the previous Henn’s results for the loci \( \rho(M_6^p(G)) \).

First, we classified in [2] for an arbitrary but a fixed degree \( d \), the \( \rho \)'s and cyclic groups \( \mathbb{Z}/m\mathbb{Z} \) of order \( m \) such that \( \rho(M^p_5(\mathbb{Z}/m\mathbb{Z})) \) is not empty. In particular, \( m \) should divide one of the integers

\[
d, d-1, d(d-1), (d-1)^2, d(d-2) \text{ or } d^2 - 3d + 3.
\]

Furthermore, we characterized the locus \( \widetilde{M}_6^p(G) \) whenever \( G \) has an element of order \( m \) with \( m \) large enough. By large enough, we mean to be one of the following integers: \( d-1, (d-1)^2, d(d-2), d^2 - 3d + 3, \ell d \) (\( \ell \geq 3 \)) or \( \ell(d-1) \) (\( \ell \geq 2 \)). Lastly, it remains to treat case by case the groups \( G \) that appeared in Harui’s list [6] in order to investigate which of them must leave when the locus \( \widetilde{M}_6^p(G) \) is non-trivial and \( G \) has no elements of (large enough) order \( m \).

2. Cyclic subgroups for degree 5 non-singular plane curve

Consider \( \delta \in M_6^5 \) such that the group \( G \cong \text{Aut}(\delta) \) is non-trivial. Let \( C : F(X; Y; Z) = 0 \) in \( \mathbb{P}^2(K) \) be a non-singular plane model of degree 5 over an algebraic closed field \( K \) of characteristic zero where \( \text{Aut}(C) = \rho(G) \leq \text{PGL}_3(K) \) for some \( \rho : G \hookrightarrow \text{PGL}_3(K) \) (any other model \( C \) of \( \delta \) is given by \( \text{PC} : F(P(X; Y; Z)) = 0 \) with \( \text{Aut}(\text{PC}) = \text{PAut}(C) \text{P}^{-1} \) for some \( P \in \text{PGL}_3(K) \), and we say that \( \text{PC} \) is \( K \)-equivalent or \( K \)-isomorphic to \( C \)). Assume that \( \sigma \in \text{Aut}(C) \) is an element of order \( m \) hence by a change of variables in \( \mathbb{P}^2 \) (in particular, changing the plane model to a \( K \)-equivalent one associated to \( \delta \)), we can consider \( \sigma \) as the automorphism \((x : y : z) \mapsto (\xi^a x : \xi^b y : \xi^c z) \) where \( \xi_m \) is a primitive \( m \)-th root of unity in \( K \) and \( a, b \) are integers such that \( 0 \leq a < b \leq m - 1 \). Moreover, if \( ab \neq 0 \) then \( m \) and \( \gcd(a, b) \) are coprime (we can reduce to \( \gcd(a, b) = 1 \)) and if \( a = 0 \) then \( \gcd(b, m) = 1 \). Also, such an automorphism is identified with type \( m, (a, b) \).

Then, by a change of variables, we may have one of the following situations (see [2] for more details, in which we follow the same line of argument as Dolgachev for degree 4 in [4] but for a general degree \( d \geq 4 \)).

<table>
<thead>
<tr>
<th>Type: ( m, (a, b) )</th>
<th>( F(X; Y; Z) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>20, (4, 5)</td>
<td>( X^5 + Y^5 + XZ^4 )</td>
</tr>
<tr>
<td>16, (1, 12)</td>
<td>( X^5 + Y^4Z + XZ^4 )</td>
</tr>
<tr>
<td>15, (1, 11)</td>
<td>( X^5 + Y^4Z + YZ^4 )</td>
</tr>
<tr>
<td>13, (1, 10)</td>
<td>( X^4Y + Y^4Z + Z^4X )</td>
</tr>
<tr>
<td>10, (2, 5)</td>
<td>( X^5 + Y^5 + \alpha XZ^4 + \beta_{2,0}X^3Z^2 )</td>
</tr>
<tr>
<td>8, (1, 4)</td>
<td>( X^5 + Y^4Z + \alpha XZ^4 + \beta_{2,0}X^3Z^2 )</td>
</tr>
<tr>
<td>5, (1, 2)</td>
<td>( X^5 + Y^5 + Z^5 + \beta_{3,1}X^2YZ^2 + \beta_{4,3}XY^3Z )</td>
</tr>
<tr>
<td>5, (0, 1)</td>
<td>( Z^5 + L_5, Z )</td>
</tr>
<tr>
<td>4, (1, 3)</td>
<td>( X^5 + X(Z^4 + \alpha Y^4 + \beta_{4,2}Y^2Z^2) + \beta_{2,1}X^3YZ )</td>
</tr>
<tr>
<td>4, (1, 2)</td>
<td>( X^5 + X(Z^4 + \alpha Y^4) + \beta_{2,0}X^3Z^2 + \beta_{3,2}X^2Y^2Z + \beta_{5,2}Y^2Z^3 )</td>
</tr>
<tr>
<td>4, (0, 1)</td>
<td>( Z^4L_1, Z + Z_5, Z )</td>
</tr>
<tr>
<td>3, (1, 2)</td>
<td>( X^5 + Y^4Z + \alpha Y^4Z^2 + \beta_{2,1}X^3YZ + X^2(\beta_{3,0}Z^3 + \beta_{3,3}Y^3) + \beta_{4,2}XY^2Z^2 )</td>
</tr>
<tr>
<td>2, (0, 1)</td>
<td>( Z^4L_1, Z + Z^2L_3, Z + L_5, Z )</td>
</tr>
</tbody>
</table>
Here \( L_{i,*} \) means a homogenous polynomial of degree \( i \) in the variables \( \{X, Y, Z\} \) such that the variable * does not appear. Also, \( \alpha, \beta_{i,j} \in K \) and \( \alpha \) is always non-zero and it can be transformed by a diagonal change of variables \( P \) to 1.

**Remark 1.** It is to be noted that the above table lists all the possible situations for which \( \rho(M_0(\mathbb{Z}/m\mathbb{Z})) \) is not empty where \( P \rho(\mathbb{Z}/m\mathbb{Z})P^{-1} = (\text{diag}(1, \xi_m^a, \xi_m^b)) \) for some \( P \in \text{PGL}_3(K) \) and \( \rho(M_0(\mathbb{Z}/m\mathbb{Z})) \) corresponds to Type \( m, (a, b) \).

### 3. General properties of the full automorphism group

Before a detailed study of the automorphism groups for degree 5, we recall the following results concerning \( \text{Aut}(\delta) \) for \( \delta \in M^P_2 \) which will be useful throughout this paper. In some cases we will use the notation of the GAP library for finite small groups to indicate some groups.

Because linear systems \( g^2 \) are unique (up to multiplication by \( P \in \text{PGL}_3(K) \) in \( \mathbb{P}^2(K) \) [8, Lemma 11.28]), we always take \( C \) a plane non-singular model of \( \delta \), which is given by a projective plane equation \( F(X; Y; Z) = 0 \) and \( \text{Aut}(C) \) is a finite subgroup of \( \text{PGL}_3(K) \) that fixes the equation \( F \) and is isomorphic to \( \text{Aut}(\delta) \). Any other plane model of \( \delta \) is given by \( PC : F(P(X; Y; Z)) = 0 \) with \( \text{Aut}(PC) = P\text{Aut}(C)P^{-1} \) for some \( P \in \text{PGL}_3(K) \) and \( PC \) is \( K \)-equivalent or \( K \)-isomorphic to \( C \). By an abuse of notation, we also denote a non-singular projective plane curve of degree \( d \) by \( C \). Therefore, \( \text{Aut}(C) \) satisfies one of the following situations (see Mitchel [11] for more details):

1. fixes a point \( Q \) and a line \( L \) with \( Q \notin L \) in \( \text{PGL}_3(K) \).
2. fixes a triangle (i.e. a set of three non-concurrent lines),
3. \( \text{Aut}(C) \) is conjugate to a representation inside \( \text{PGL}_3(K) \) of one of the finite primitive group namely, the Klein group \( \text{PSL}(2, 7) \), the icosahedral group \( A_5 \), the alternating group \( A_6 \), the Hessian group \( \text{Hess}_{216} \) or to one of its subgroups \( \text{Hess}_{72} \) or \( \text{Hess}_{36} \).

It is classically known that if a subgroup \( H \) of automorphisms of a non-singular plane curve \( C \) fixes a point on \( C \) then \( H \) is cyclic [8, Lemma 11.44], and recently Harui in [6, §2] provided the lacked result in the literature on the type of groups that could appear for non-singular plane curves. Before introducing Harui’s statement, we need to define the terminology of being a descendant of a plane curve. For a non-zero monomial \( cX^iY^jZ^k \) with \( c \in K \setminus \{0\} \) we define its exponent as \( \max\{i, j, k\} \). For a homogenous polynomial \( F \), the core of \( F \) is defined to be the sum of all terms of \( F \) with the greatest exponent. Let \( C_0 \) be a smooth plane curve, a pair \((C, H)\) with \( H \leq \text{Aut}(C) \) is said to be a descendant of \( C_0 \) if \( C \) is defined by a homogenous polynomial whose core is a defining polynomial of \( C_0 \) and \( H \) acts on \( C_0 \) under a suitable change of the coordinate system.

**Theorem 2** (Harui). If \( H \leq \text{Aut}(C) \) where \( C \) is a non-singular plane curve of degree \( d \geq 4 \) then \( H \) satisfies one of the following.

1. \( H \) fixes a point on \( C \) and then cyclic.
2. \( H \) fixes a point not lying on \( C \) and it satisfies a short exact sequence of the form

   \[
   1 \to N \to H \to G' \to 1,
   \]

   where \( N \) a cyclic group of order dividing \( d \) and \( G' \) (which is a subgroup of \( \text{PGL}_2(K) \)) is conjugate to a cyclic group \( \mathbb{Z}/m\mathbb{Z} \) of order \( m \) with \( m \leq d - 1 \), a Dihedral group \( D_{2m} \) of order \( 2m \) where \( |N| = 1 \) or \( m|(d - 2) \), the alternating groups \( A_4 \) or \( A_5 \) or the symmetry group \( S_4 \).
3. \( H \) is conjugate to a subgroup of \( \text{Aut}(F_d) \) where \( F_d \) is the Fermat curve \( X^d + Y^d + Z^d \). In particular, \( |H| \leq 6d^2 \) and \( (C, H) \) is a descendant of \( F_d \).
4. \( H \) is conjugate to a subgroup of \( \text{Aut}(K_d) \) where \( K_d \) is the Klein curve curve \( XY^{d-1} + YZ^{d-1} + ZX^{d-1} \) hence \( |H| \leq 3(d^2 - 3d + 3) \) and \( (C, H) \) is a descendant of \( K_d \).
5. \( H \) is conjugate to a finite primitive subgroup of \( \text{PGL}_3(K) \) which are mentioned above.

We mention also the following statement [6, Theorem 2.3]:

**Theorem 3.** Given \( C \) a non-singular plane curve of degree \( d \neq 4, 6 \), then \( |\text{Aut}(C)| \leq 6d^2 \).
In particular, for $d = 5$ we conclude

**Corollary 4.** Given $C$ non-singular plane curve of degree $5$, then $\text{Aut}(C)$ is not conjugate to the Hessian group $\text{Hess}_{216}$, the Klein group $\text{PSL}(2,7)$ or the alternating group $A_6$.

Moreover, we proved in [2] the following next two results for cyclic subgroups inside $\text{Aut}(C)$ (see [2, Corollary 33] and [2, §4] respectively).

**Proposition 5.** Given $C$ a non-singular plane curve of degree $d$ and let $\sigma \in \text{Aut}(C)$ be of order $m$ then $m$ divides one of the following integers: $d - 1$, $d$, $(d - 1)^2$, $d(d - 2)$, $d(d - 1) - 3d + 3$.

**Theorem 6.** Let $C$ a non-singular plane curve of degree $d$ and $\sigma \in \text{Aut}(C)$. Then,

1. if $\sigma$ has order $d(d - 1) - \text{Aut}(C) = < \sigma >$ and $C$ $K$-isomorphic to $X^d + Y^d + ZX^{d-1} = 0$.
2. if $\sigma$ has order $(d - 1)^2$ then $\text{Aut}(C) = < \sigma >$ and $C$ is $K$-isomorphic to $X^d + Y^{d-1}Z + XZ^{d-1} = 0$.
3. if $\sigma$ has order $d - 2$ then $C$ is $K$-isomorphic to $X^d + Y^{d-1}Z + YZ^{d-1} = 0$ and for $d \neq 4, 6$ we have $\text{Aut}(C) = < \sigma, \tau | \tau^2 = \sigma^{d-2} = 1, \tau \sigma \tau = \sigma^{-(d-1)} >$.
4. if $\sigma$ has order $d^2 - 3d + 3$ then $C$ is $K$-isomorphic to the Klein curve $K_d$ and for $d \geq 5$ we have $\text{Aut}(C) = < \sigma, \tau | \sigma^{d-3d+3} = \tau^3 = 1$ and $\tau \sigma \tau = \tau \sigma^{-(d-1)} >$.
5. if $\sigma$ has order $\ell(d - 1)$ with $\ell \geq 2$ then $\text{Aut}(C)$ is cyclic of order $\ell'(d - 1)$ with $\ell | \ell'$. If $\ell = 1$, the same conclusion holds if $\sigma$ is a homology (where $\sigma$ is called a homology if $P \sigma P^{-1} = \text{diag}(1, \xi_m, \xi_m^j)$ such that exactly one of $a$ and $b$ is zero for some $P \in \text{PGL}_3(K)$).
6. if $\sigma$ has order $\ell d$ with $\ell \geq 3$ then $\text{Aut}(C)$ fixes a line and a point off that line and $\text{Aut}(C)$ is an exterior group as in Theorem 2 (2) with $N$ of order $d$. When $\ell \leq 2$ may be a decendent of the Fermat curve or $\text{Aut}(C)$ is an exterior group as in Theorem 2 (2) where $|N| = d$.

Now, assume as usual that $C$ is a non-singular plane curve of degree $d = 5$ with $\sigma \in \text{Aut}(C)$ of order $m$ that acts on $F(X;Y;Z) = 0$ by $(x, y, z) \mapsto (x, \xi_m y, \xi_m^j z)$ such that $m$ is the maximal order in $\text{Aut}(C)$. Recall also that we can take $a = 1$ by a convenient change of variables $P$. The following result determines the full automorphism groups of a quintic curve $C$ where there are automorphisms inside $\text{Aut}(C)$ of large orders.

**Corollary 7.** For non-singular plane curves of degree $5$ over an algebraic closed field $K$ of zero characteristic we have:

1. The cyclic group $\mathbb{Z}/20\mathbb{Z}$ appears as $\text{Aut}(C)$ inside $\text{PGL}_3(K)$ generated by the transformation $(x, y, z) \mapsto (x, \xi_{20}^4 y, \xi_{20}^8 z)$ up to conjugation by $P \in \text{PGL}_3(K)$, and $C$ is $K$-isomorphic (through $P$) to the plane non-singular curve $X^5 + Y^5 + XZ^4 = 0$. In particular $\rho(M^{P}(\mathbb{Z}/20\mathbb{Z}))$ is an irreducible locus with one element, where $\rho(\mathbb{Z}/20\mathbb{Z}) = < \text{diag}(1, \xi_{20}^4, \xi_{20}^8) >$.
2. The cyclic group $\mathbb{Z}/16\mathbb{Z}$ appears as $\text{Aut}(C)$ inside $\text{PGL}_3(K)$ generated by the transformation $(x, y, z) \mapsto (x, \xi_{16}^4 y, \xi_{16}^{12} z)$ up to conjugation by $P \in \text{PGL}_3(K)$, and $C$ is $K$-isomorphic (through $P$) to the plane non-singular curve $X^5 + Y^4 Z + XZ^4 = 0$. In particular $\rho(M^{P}(\mathbb{Z}/16\mathbb{Z}))$ is an irreducible locus with one element, where $\rho(\mathbb{Z}/16\mathbb{Z}) = < \text{diag}(1, \xi_{16}, \xi_{16}^{12}) >$.
3. The group $\text{SmallGroup}(30,1)$ $\cong < \sigma, \tau | \sigma^2 = \sigma^{15} = \tau^2 = 1 >$ of order 30 appears as $\text{Aut}(C)$ inside $\text{PGL}_3(K)$ where $\sigma$ and $\tau$ are given, up to conjugation by $P \in \text{PGL}_3(K)$, by $\sigma : (x, y, z) \mapsto (x, \xi_{13} y, \xi_{13}^2 z)$ and $\tau : (x, y, z) \mapsto (x, z, y)$. Moreover, $C$ is $K$-isomorphic (through $P$) to the curve $X^5 + Y^4 Z + YZ^4 = 0$. In particular $\rho(M^{P}(\text{SmallGroup}(30,1)))$ is an irreducible locus with one element and $\rho$ is given by $\sigma, \tau$.
4. The group $\text{SmallGroup}(39,1)$ $\cong < \tau, \sigma | \sigma^{13} = \tau^3 = 1, \sigma \tau \sigma^2 >$ of order 39 appears as $\text{Aut}(C)$ inside $\text{PGL}_3(K)$ given by $\sigma : (x, y, z) \mapsto (x, \xi_{13} y, \xi_{13}^2 z)$ and $\tau : (x, y, z) \mapsto (y, z, x)$ up to conjugation by $P \in \text{PGL}_3(K)$, and $C$ is $K$-isomorphic (through $P$) to the curve $K_5 : X^4 Y + Y^4 Z + Z^4 X = 0$. In particular $\rho(M^{P}(\text{SmallGroup}(39,1)))$ is an irreducible locus with one element, where $\rho$ is determined by $\sigma, \tau$.
5. The cyclic group $\mathbb{Z}/8\mathbb{Z}$ appears as $\text{Aut}(C)$ inside $\text{PGL}_3(K)$ generated by the transformation $(x, y, z) \mapsto (x, \xi_8 y, \xi_8^3 z)$ up to conjugation by $P \in \text{PGL}_3(K)$, and $C$ is $K$-isomorphic (through $P$) to the plane
non-singular curve $X^5 + Y^4Z + XZ^4 + \beta_{2,0}X^3Z^2$, with $\beta_{2,0} \neq 0, \pm 2$. The locus $\rho(MP^1(Z/8\mathbb{Z}))$ has dimension one where $\rho(Z/8\mathbb{Z}) = \langle \text{diag}(1, \xi_8, \xi_8^4) \rangle$.

Proof. Except the last statement on the loci, the proof is a direct consequence of Theorem 6 because one could apply the result for $d = 5$ using the table in §2 when the curve $C$ has a cyclic automorphism of order: $d(d-1)$, $(d-1)^2$, $d(d-2)$, $d^2 - 3d + 3$ and $\ell(d-1)$ with $\ell = 2$ respectively. It remains, for the last case, to observe that if $\text{Aut}(C)$ is bigger then it is always cyclic and should be the group of order 16. Therefore, $\beta_{2,0} \neq 0$ is the only restriction to impose so that the curve has automorphism group exactly $\mathbb{Z}/8\mathbb{Z}$ ($\beta_{2,0} \neq \pm 2$ in order to ensure non-singularity), here previously by a diagonal change of variables we converted $\alpha$ to 1 in the equation given at §2. Lastly, we refer to [1] for the proof of the dimension and the irreducibility over $\mathbb{C}$ of the locus $\rho(MP^1(Z/8\mathbb{Z}))$. \hfill $\square$

4. Determination of the automorphism group with small cyclic subgroups

In this section, following the abuse of notation of the previous section concerning models and curves, we study $\text{Aut}(C)$ for non-singular plane curves $C$ of degree $d = 5$ that appear in the table of §2 such that the maximal order for any element inside the automorphism group is $2d$ or $d$.

Also, we denote by $C_n$ the cyclic group $\mathbb{Z}/n\mathbb{Z}$ to emphasize the multiplication notation as a subgroup inside $\text{PGL}_3(K)$.

Proposition 8. Suppose that $C$ is a non-singular plane curve of degree 5 with $\sigma \in \text{Aut}(C)$ of order 10 as an automorphism of maximal order. Then, we reduce after conjugation by certain $P \in \text{PGL}_3(K)$ that $\sigma$ acts on $C : X^5 + Y^5 + \alpha XYZ + \beta_{2,0}X^3Z^2 = 0$ such that $\alpha\beta_{2,0} \neq 0$ as $\sigma : (x, y, z) \mapsto (x, \xi_5y, \xi_5^2z)$ and one of the following situations happens:

1. If $\alpha = 5$ and $\beta_{2,0} = 10$ then $C$ is $K$-equivalent to the Fermat quintic $F_5 : X^5 + Y^5 + Z^5 = 0$ and $\text{Aut}(C)$ is isomorphic to $\text{SmallGroup}(150, 5)$.

2. If $(\alpha, \beta_{2,0}) \neq (5, 10)$ then $\text{Aut}(C)$ is cyclic of order 10. Moreover, if $1 + \alpha + \beta_{2,0} \neq 0$ then $C$ is $K$-isomorphic through some $P \in \text{PGL}_3(K)$ to a descendant of the Fermat curve which is defined by $PC : X^5 + Y^5 + Z^5 + \frac{5 - 3\alpha + \beta_{2,0}}{1 + \alpha + \beta_{2,0}}(X^4Z + XZ^4) + \frac{2(5 + \alpha - \beta_{2,0})}{1 + \alpha + \beta_{2,0}}(X^3Z^2 + X^2Z^3)$.

Proof. Because the maximal order is 10 then, by the results of the previous section, we reduce $C$ to be, up to $K$-isomorphism, of the form $X^5 + Y^5 + \alpha XYZ + \beta_{2,0}X^3Z^2 = 0$ with $\alpha\beta_{2,0} \neq 0$. This curve admits a homology $\sigma^2$ of order 5 $> 3$ therefore, by Theorem 6, $\text{Aut}(C)$ fixes a line and a point off that line or $C$ is a descendant of Fermat curve. Moreover, the center $(0; 1; 0)$ of this homology is an outer Galois point (by Lemma 3.7 [6]) and if $C$ is not $K$-isomorphic to the Fermat curve $F_5 : X^5 + Y^5 + Z^5$ then it is unique (Theorem 4' [12]). Hence it should be fixed by $\text{Aut}(C)$.

Assume first that $\text{Aut}(C)$ fixes a line and a point off that line and $C$ is not the Fermat quintic. Hence, $\text{Aut}(C)$ satisfies a short exact sequence $1 \rightarrow C_5 \rightarrow \text{Aut}(C) \rightarrow G' \rightarrow 1$ where $C_5$ is generated by $\sigma^2 = [X; \xi_5^2Y; Z]$. In particular, $G'$ contains an element of order 2 obtained by the image of $\sigma$ under the restriction of the natural map from $\text{PBD}(2, 1)$ to $\text{PGL}_3(K)$. Consequently $G'$ is conjugate to $C_2, C_4, S_3, A_4, S_4$ or $A_5$. We claim that $G'$ is conjugate to $C_2$ in particular $\text{Aut}(C)$ is cyclic of order 10.

Since there are no groups of order 30 (respectively, 60) which contain elements of order 10 and no higher orders then $G'$ is not conjugate to $S_3$ or $A_4$. Also, if $G'$ is conjugate to $S_4$ then $\text{Aut}(C)$ is conjugate to $\text{SmallGroup}(120, 5)$ or $\text{SmallGroup}(120, 35)$ because these are the only groups of order 120 with elements of order 10 and no higher orders appear. But one can verify that there are no elements $\tau \in \text{Aut}(C)$ of order 3 or 10 that commute with $\sigma^5$ therefore $\text{Aut}(C)$ is not conjugate to any of these two groups. In particular, $G'$ is not conjugate to $S_4$. On the other hand, groups of order 20 that contain elements of order 10 and no higher orders are $\text{SmallGroup}(20, \ell)$ where $\ell = 1, 3, 4$ or 5. Since there is no element $\tau \in \text{Aut}(C)$ of order 4 such that $\sigma^5 \tau \sigma^5 = \tau$ or $\sigma^2 \tau \sigma^2 = \tau \sigma^2$ then $\ell \neq 1, 3$. Furthermore, there is no element $\tau$ of order 2 in $\text{Aut}(C)$ which commutes with $\sigma^5$ thus $\ell \neq 4, 5$. This implies that $G'$ is not conjugate to $C_4$. Lastly, groups of order 300 that contain elements of order 10 and no higher orders are $\text{SmallGroup}(300, \ell)$ where $\ell = 25, 26, 27, 41$ or 43. If
\( \ell = 43 \) or 41 then \( \text{Aut}(C) \) contains exactly 3 elements of order 2 which contradicts the fact that \( \text{Aut}(C) \) should have at least 15 such elements as \( A_5 \) does. Moreover, there are no elements of order 2 in \( \text{Aut}(C) \) such that \( \tau \sigma^5 = \sigma^5 \tau \) hence \( \ell \neq 25, 26 \) or 27. Consequently, \( G' \) is not conjugate to \( A_5 \). This proves the claim in this situation.

Secondly, assume that \( C \) is a descendant of the degree 5 Fermat curve. This should happen through a transformation \( P \in PGL_3(K) \) such that \( P \sigma P^{-1} = \lambda \sigma_1 \) where \( \sigma_1 = [X; \zeta_{10}^5 Y; Y] \) and \( \sigma_1' = [\zeta_{10}^5 Y; X; \zeta_{10}^5 Z] \) and \( \sigma_1'' = [\zeta_{10}^5 Z; \zeta_{10}^5 Y; X] \) with 5 \( \{a + b\} \). In what follows, we treat each case.

If \( P \sigma P^{-1} = \lambda \sigma_1 \) then \( \lambda = \zeta_{10}^5, 5[a + b + 2] \) and \( P = [a_2 Y; b_2 X + b_2 Z; a_{2a} + 2b_2 X + a_{2b} - 2b - 3 + 3] \). This transforms \( C \) into \( PC \), a descendant of the Fermat quintic where \( Y^3 Z \in PC \) and \( Y^3 Z \notin PC \) which is a contradiction to \( \sigma_1' \in \text{Aut}(PC) \). Similarly, if \( P \sigma P^{-1} = \lambda \sigma_1'' \) then \( \lambda = \zeta_{10}^{2-2b}, 5[a - 2b + 2, a - 2b - 3] \) and \( P = [a_3 X + a_3 Z; \zeta_{10}^4 a_1 X + \zeta_{10}^4 a_1 X; \zeta_{10}^4 a_3 Z; a_3 Y] \). In particular, \( Y^3 \notin PC \) and then \( PC \) is not a descendant of the the Fermat curve \( F_5 \). Finally, if \( P \sigma P^{-1} = \lambda \sigma_1 \) then \( \lambda = \zeta_{10}^{2-2a}, 5[b - 2a + 2, b - 2a - 3] \) and \( P = [a_3 X + a_3 Z; \zeta_{10}^4 a_1 X + \zeta_{10}^4 a_3 Z] \). Since \( (a, b) \in \{(0, 3), (1, 0), (2, 2), (3, 4)\} \) and

\[
[X; \zeta_{10}^{2a} Y; \zeta_{10}^{2a} Z; \zeta_{10}^{2a} Y; X][X; \zeta_{10}^{2a} Y; \zeta_{10}^{2a} Y; X][X; \zeta_{10}^{2a} Y; \zeta_{10}^{2a} Y; X] = [\zeta_{10}^{2a} Y; \zeta_{10}^{2a} Y; X].
\]

Then all possible \( \sigma_1'' \) are conjugate inside \( \text{Aut}(F_5) \) by substituting \( b' = 1, 4 \) and 3. So it suffices to consider the case \( a = 1 \) and \( b = 0 \) and hence \( C \) is transformed through \( P \) to \( PC \), a descendant of \( F_5 \) only if \( 1 + a + \beta_{2, 0} \neq 0 \). Moreover, from the coefficients of \( X^5, Z^5 \) in \( PC \) we should have \( a_3 = a_1 \) and \( a_3^2 (a + \beta_{2, 0} + 1) = 1 \) where \( c \) is 5-th root of unity. Comparing the coefficients of \( X^4 Z, XZ^4 \) and \( Z^2 X^2, Z^2 X^2 \) we obtain \( (a, \beta_{2, 0}) = (5, 10) \) or \( c = 1 \). In other words, \( C \) is transformed to

\[
PC : X^5 + Y^5 + Z^5 + 5 - 3a + \beta_{2, 0} \left( \frac{1 + a + \beta_{2, 0}}{1 + a + \beta_{2, 0}} \right) X^4 Z + XZ^4 + \frac{2(5 + a - \beta_{2, 0})}{1 + a + \beta_{2, 0}} X^2 Z^2 + X^2 Z^3
\]

Now, if \( (a, \beta_{2, 0}) \neq (5, 10) \) then \( [X; \zeta_{10}^5 Y; \zeta_{10}^5 Y; Y], [\zeta_{10}^5 Y; X; \zeta_{10}^5 Z], [\zeta_{10}^5 Y; Z; \zeta_{10}^5 Y; X], [\zeta_{10}^5 Z; \zeta_{10}^5 Y; X] \notin \text{Aut}(PC) \). Furthermore, \( [\zeta_{10}^5 Y; \zeta_{10}^5 Y; X] \) or \( [X; \zeta_{10}^5 Y; \zeta_{10}^5 Y; Z] \) or \( [X; \zeta_{10}^5 Y; \zeta_{10}^5 Z; X] \in \text{Aut}(PC) \) only if \( b' = 0 \). That is \( \text{Aut}(PC) \) is cyclic of order 10. Finally, if \( (a, \beta_{2, 0}) = (5, 10) \) then \( PC \) is the Fermat curve itself. This completes the proof by assuming the next remark.

**Remark 9.** Recall that \( \text{Aut}(F_5) \) it is generated by \( \eta_1 := [X; Z; Y], \eta_2 := [Y; Z; X], \eta_3 := [X; \xi_5 Y; Z] \) and \( \eta_4 := [X; \xi_5 Y; Z] \) of orders 2, 3, 5, 5 respectively such that

\[
(\eta_1 \eta_2)^3 = (\eta_1 \eta_3)(\eta_1 \eta_4)^{-1} = (\eta_2 \eta_3)(\eta_1 \eta_4)^{-1} = \eta_1 \eta_2 \eta_1 \eta_2 \eta_1 \eta_2 \eta_1 \eta_2^{-1} \eta_1 \eta_2^{-1} = 1.
\]

The following lemma is very useful to discard all the groups with a subgroup isomorphic to \( C_2 \times C_2 \) for non-singular plane curves of degree 5.

**Lemma 10.** There is no non-singular plane curve \( C \) of degree 5 with \( C_2 \times C_2 \leq \text{Aut}(C) \). In particular, the full automorphism group \( \text{Aut}(C) \) is not isomorphic to any of the groups: \( C_2 \times C_2, A_4, S_4 \) or \( A_5 \).

**Proof.** By Mitchel [11] and Harui [6], the group \( C_2 \times C_2 \) inside \( PGL_3(K) \) which gives invariant a non-singular plane curve \( C \) of degree 5 fix a point not lying on \( C \) or \( C \) is a descendant of the Fermat or the Klein curve. For \( d = 5 \), it could not be a descendant of the Fermat or the Klein curve because 4 does not divide \( |\text{Aut}(F_3)| = 150 \) or \( |\text{Aut}(K_3)| = 39 \). Therefore, the automorphism subgroup \( C_2 \times C_2 \) fixes a point not in \( C \). Moreover, because \( 2 \) does not divide the degree \( d = 5 \), then by Harui’s main theorem [6], we can think about the elements of \( C_2 \times C_2 \) in a short exact sequence: \( 1 \to N = 1 \to H \to H \to 1 \) where \( H \) is conjugate to \( C_2 \times C_2 \) inside \( PGL_3(K) \). That \( H \) acts only on the variables \( Y, Z \) because \( N \) is the subgroup of \( \text{Aut}(C) \) that acts on \( X \). Now, let \( \sigma, \tau \in H \subseteq PGL_2(K) \) be of order two such that \( \sigma \tau = \tau \sigma \) then we can suppose, up to a coordinate change inside \( \mathbb{P}^2 \), that \( \sigma = \text{diag}(1, -1) \) and \( \tau = [a Y + b Z, c Y - a Z] \neq \sigma \). Consequently, the curve \( C \) has a model of type \( 2, (0, 1) \). But all possible \( \tau \) does not retain invariant the equation of the type \( 2, (0, 1) \) for any choice of the free parameters and hence the result follows. Indeed, because \( \tau \) commutes with \( \sigma \) then \( \tau = \text{diag}(-1, 1) \) or \( [X, b Z, c Y] \) with \( bc \neq 0 \) and therefore \( C \) has the forms: \( Z^3 L_{1, Z} + Z^2 L_{3, Z} + L_{5, Z} \) and \( Y^4 L_{1, Y} + Y^2 L_{3, Y} + L_{5, Z} \) simultaneously which is impossible. □
Now, we deal with quintic curves with a cyclic element of order 5 (respectively, 4) as an automorphism of maximal order.

**Proposition 11.** If $C$ is a degree 5, non-singular plane curve with an automorphism $\sigma$ of maximal order 5, then we reduce, up to projective equivalence, to one of the following situations: $\text{Aut}(C)$ is cyclic of order 5 and $C$ is $K$-equivalent to the type 5, $(0,1)$ of the form $Z^5 + L_5$ or $\text{Aut}(C)$ is isomorphic to the Dihedral group $D_{10}$ of order 10 where $\text{Aut}(C) = \langle \sigma, \tau \rangle$ with $\sigma(x, y, z) = (x, \xi y, \xi^2 z)$ and $\tau(x, y, z) = (z, y, x)$ and the curve $C$ has the form $X^5 + Y^5 + Z^5 + \beta_{3,1} X^2 Y Z^2 + \beta_{4,3} X Y^3 Z = 0$ such that $(\beta_{3,1}, \beta_{4,3}) \neq (0, 0)$.

**Proof.** We consider the situations in §1 concerning types 5, $(a, b)$.

1. Type 5, $(1,2)$: $\text{Aut}(C)$ is not conjugate to any of the subgroups of $\text{Aut}(K)$ since there are no elements of order 5. On the other hand, always $C$ admits a bigger automorphism group $G$ of order 5 in particular, $\text{Aut}(C)$ is not cyclic. Moreover, by the previous Lemma 10, $\text{Aut}(C)$ is not conjugate to $A_5$ (as a finite primitive subgroup of $PGL_3(K)$). Consequently, $C$ is a descendant of the Fermat quintic or $\text{Aut}(C)$ fixes a line and a point off that line.

   - If $\text{Aut}(C)$ fixes a line and a point off that line then the coefficient of $X$ is $\sigma(y, z) = (y, \xi z, \xi^2 y)$ and $\tau(y, z) = (z, y, x)$ in particular, elements of $\text{Aut}(C)$ are of the form $[\alpha_1 X + \alpha_3 Z; \gamma_1 X + \gamma_3 Z]$. Hence, from the coefficients of $Y^2 Z^2$ and $Y^3 X^2$ (respectively, $X^2 Y$ and $Y^4 Z^4$), we should have $\alpha_1 = 0 = \gamma_3$ or $\alpha_3 = 0 = \gamma_1$ and $(\beta_{3,1}, \beta_{4,3}) \neq (0, 0)$ then $a^2_{3,1} = a^2_{4,3} = 1$ and $a_3, 1\gamma_{1,3} = 1$ or $(a_3, 1\gamma_{1,3})^2 = 1$ from which we obtain that $\text{Aut}(C)$ has order 10.

2. Type 5, $(0,1)$: This curve has a homology $\sigma$ of order $d$ with center $(0; 0; 1)$ and axis $Z = 0$ then (by Lemma 3.7 in [6]) this point is an outer Galois point of $C$. Moreover, $C$ is not $K$-isomorphic to the Fermat curve because automorphisms of $C$ has orders $\leq 5$ then (by Yoshihara [12]) this Galois point is unique and hence should be fixed by $\text{Aut}(C)$.

   In particular, $\text{Aut}(C)$ fixes a line $(Z = 0)$ and a point off that line $(0; 0; 1)$ that is elements of $\text{Aut}(C)$ have the form $[\alpha_1 X + \alpha_2 Y; \beta_1 X + \beta_2 Y; Z]$. Furthermore, $\text{Aut}(C)$ satisfies a short exact sequence $1 \rightarrow N \rightarrow \text{Aut}(C) \rightarrow G' \rightarrow 1$ with $N$ cyclic of order dividing 5 and $G'$ is conjugate to $G_m, D_{2m}, A_4, S_4$ or $A_5$ if $m \leq 4$ and for the case $G' = D_{2m}$ we have $m \mid 3$ or $N$ is trivial.

   If $N$ is trivial then $G'$ should be conjugate to $A_5$ (because non of the other groups contains elements of order 5) then $C_2 \times C_2$ is a subgroup of $\text{Aut}(C)$ which is not possible by Lemma 10. Hence, $N$ can not be the trivial group.

   If $N$ has order 5 then for any value of $G'$ (except possibly the trivial group, $C_2, C_4$ or $A_4$ such that $\text{Aut}(C)$ is conjugate to $D_{10}$, SmallGroup$(20,3)$ or $A_5$) there are elements of order $> 5$ in $\text{Aut}(C)$ a contradiction. Again, by Lemma 10, we conclude that $G'$ can not be $A_4$. On the other hand, there exists no elements $\tau \in \text{Aut}(C)$ of order 2 such that $\tau \sigma = \sigma^{-1}$ hence $G'$ is not $C_2$. Moreover, there are no elements $\tau \in \text{Aut}(C)$ of order 4 such that $(\tau \sigma)^2 = 1$ and $\tau \sigma^{-1} = \tau \sigma$ thus $G'$ is not conjugate to $C_4$. Consequently, $\text{Aut}(C)$ is cyclic of order 5.

**Proposition 12.** Suppose that $C$ is a non-singular plane curve of degree 5 with $\sigma \in \text{Aut}(C)$ of order 4 as an element of maximal order, then we reduce, up to $K$-isomorphism, to one of the following two situations: $\text{Aut}(C)$ is cyclic of order 4 generated by $\sigma(x, y, z) = (x, y, \xi z)$ and $C$ is given by $Z^4 Y + L_5 Z(X, Y) = 0$ such that $L_5 Z(X, \xi^n Y) \neq \xi^n L_5 Z(X, Y)$ where $(n, r) \in \{(6, 1), (16, 1), (20, 4)\}$ or $\sigma(x, y, z) = (x, \xi y, \xi^2 z)$ and $C$ is defined by $X^5 + X(Z^4 + \alpha Y^4) + \beta_{2,0}X^3 Z^2 + \beta_{3,2}X^2 Y^2 Z + \beta_{5,2}Y^2 Z^3 = 0$ such that $\alpha \beta_{5,2} \neq 0$. 


Proof. We consider the situations in §1 concerning types 4, (a, b).

First, we observe that C can not be a descendant of the Fermat curve $F_5$ or the Klein curve $K_5$ because $|\text{Aut}(F_5)| = 150$ and $|\text{Aut}(K_5)| = 39$ and $4 \nmid |\text{Aut}(F_5)|$ or $|\text{Aut}(K_5)|$ and $\text{Aut}(C)$ is not conjugate to $A_4$ since there are no elements of order 4. Consequently, $\text{Aut}(C)$ is conjugate to $\text{Hess}_36, \text{Hess}_72$ or it should fix a line and a point off that line by the result of Harui. Moreover, for the last case, we need to consider the situation of a short exact sequence of the form $1 \to N = 1 \to \text{Aut}(C) \to G' \to 1$, where $G'$ should contain an element of order 4. That is, $G'$ is conjugate to a cyclic group $C_4$ or a Dihedral group $D_8$ (by use of Lemma 10).

(1) Type 4, (1, 3); here the automorphism group is bigger and also has a subgroup of order 32 when we impose the coefficient of $X^3YZ$ to be trivial which would give a contradiction on Harui result. But for this type the equation decomposes into $X \cdot G(X, Y, Z)$ and therefore is singular. Thus this situation is out of the scope of this work.

(2) Type 4, (0, 1); This curve admits a homology of order $d = 1$ with center $(0;0;1)$ then it follows by Harui [6] that this point is an inner Galois point of $C$ and moreover it is unique by Yoshihara [12]. Therefore, this point should be fixed by $\text{Aut}(C)$ consequently, $\text{Aut}(C)$ is cyclic. It follows by the assumption that $C$ is not conjugate to any of the above that $\text{Aut}(C)$ is cyclic of order 4. More precisely, we can rewrite type 4, (0, 1) as $Z^2Y' + L'_{bZ}(X, Y') = 0$ and it is necessary to impose the condition that $L'_{bZ}(X, \zeta_3Y') \neq \zeta_3L'_{bZ}(X, Y')$ where $(m, r) \in \{(8, 1), (16, 1), (20, 4)\}$ (otherwise; we get a bigger automorphism group conjugate to those for types 8, (1, 4), 16, (1, 12) or 20, (4, 5)).

(3) Type 4, (1, 2); First, by the same reason as type 4, (1, 3), we need to assume that $\beta_5 \neq 0$. Secondly, we’ll show that $\text{Aut}(C)$ is not conjugate to any of the Hessian subgroups $\text{Hess}_36$ or $\text{Hess}_72$ as follows. Both groups contains reflections but no four groups hence all reflections in the group will be conjugate (see Theorem 11 in [11]). Therefore, it suffices to consider the case $Pa^2P^{-1} = X[Z; Y; X]$. But non of the solutions transform $C$ to PC with $[X; Z; Y], [Y; X; Z], [Z; Y; X] \subseteq \text{Aut}(PC)$. Indeed, $P$ is of the form $[a_1X + a_2Y + a_3Z; b_1X + b_2Z; a_1X - a_2Y + a_3Z] \text{ or } [a_1X + a_2Y + a_3Z; b_2Y - a_1X + a_2Y - a_3Z]$. For both cases, we must have $a_1 = a_3$ (coefficients of $XY^4$ and $Y^4Z$) in particular, the second case does not occur. Moreover, from the coefficients of $X^3Y^2$ and $Y^2Z^3$, we get $\gamma_1 = \gamma_2$ a contradiction. Consequently, the claim follows and $\text{Aut}(C)$ should fix a line and a point off that line.

Now, if $C$ admits a bigger non-cyclic automorphism group then it should be non-commutative by Harui and contain an element of order 2 with does not commute $\sigma$. We can reduce to a subgroup which is conjugate to the dihedral group $D_8$ and moreover $\text{Aut}(C)$ fixes the point $(1;0;0)$ and the line $X = 0$. That is, automorphisms of $C$ are of the form $[X; vY + wZ; sY + tZ]$. Since there is no element $\tau \in \text{Aut}(C)$ of order 2 such that $\tau \sigma \tau = \sigma^{-1}$ then $\text{Aut}(C)$ is not conjugate to $D_8$ or $S_4$. In particular, it is cyclic of order 4. To be more accurate, if $\text{Aut}(C)$ is cyclic of order $4 k > 4$ then $k = 2$, 4 or 5. If $k = 5$ then $C$ is $K$-isomorphic to type 20, (4, 5) and hence $\sigma$ is conjugate to a homology of order 4 a contradiction. Similarly, if $k = 2$ or 4.

Remains yet the study of curves $C$ where their automorphisms have orders at most 3. In particular, $\text{Aut}(C)$ is not conjugate to $A_5, \text{Hess}_36$ or $\text{Hess}_72$ because each of these groups contains elements of order $> 3$. Therefore, $\text{Aut}(C)$ should fix a line and a point off that line or it is conjugate to a subgroup of $\text{Aut}(F_5)$ or $\text{Aut}(K_5)$.

Proposition 13. Let $C$ be a non-singular plane curve of type 3, (1, 2) such that elements inside $\text{Aut}(C)$ have orders $\leq 3$. Then, $C$ is $K$-isomorphic to $X^3 + Y^4Z + YZ^4 + b_{13, 3}X^3Y^2 + X^2(\beta_{13, 0}Z^3 + \beta_{13, 3}Y^3) + \beta_{14, 2}XY^2Z^2 = 0$. Moreover, if $\beta_{13, 0} \neq \beta_{13, 3}$ then $\text{Aut}(C)$ is cyclic of order 3 and is conjugate to $S_3$ otherwise.

Proof. If $C$ is a descendant of the Klein curve then $\text{Aut}(C)$ is conjugate to a subgroup of $\text{Aut}(K_5)$ and hence can not be of order $> 3$. Indeed, we have $|\text{Aut}(K_5)| = 3^2 \cdot 7$ then any subgroup of order $> 3$ inside $\text{Aut}(K_5)$ which does not contain elements of order $> 3$ is isomorphic to $C_3 \times C_3$. Therefore, there exist $\tau \in \text{Aut}(C)$ of order 3 that commutes with $\sigma := [X; \xi_3Y, I^3Z]$. In particular, $\tau$ has the forms $[X; \alpha Y; \mu Z], [Y; \alpha Z; \mu X]$ or $[Z; \alpha X; \mu Y]$. But permuting $X$ with $Y$ or $Z$ does not preserve $C$ hence $\tau = [X; \alpha Y; \mu Z]$ with $\alpha = \omega$ (resp. $\omega^2$) and $\mu = \omega^\sigma$ (resp. $\omega$). That is, $\tau \in \sigma > \sigma$ a contradiction.
If \( C \) is a descendant of the Fermat curve then \( \text{Aut}(C) \) is cyclic of order 3 or conjugate to \( S_3 \) inside \( \text{Aut}(F_3) \).
Indeed, \(|\text{Aut}(F_3)| = 2 \cdot 3 \cdot 5^2\) hence any subgroup of order \( > 3 \) is conjugate to \( S_3 \) (note that \( \text{Aut}(F_3) \) contains no elements of order 6) or it contains elements of order 5 \( > 3 \). Now, if \( \text{Aut}(C) \) is conjugate to \( S_3 \) then there exists \( \tau \in \text{Aut}(C) \) of order 2 such that \( \tau \sigma \tau = \sigma^{-1} \) which reduces \( \tau \) to be of the form \([X;\beta Z;\beta^{-1} Y]\). But, \([X;\beta Z;\beta^{-1} Y] \in \text{Aut}(C) \) iff \( \beta^3 = 1 \) and \( \beta_{3,0} = \beta_{3,3} \).

If \( \text{Aut}(C) \) fixes a point then should be one of the reference points \( P_1 := (1; 0; 0) \), \( P_2 := (0; 1; 0) \) or \( P_3 := (0; 0; 1) \) since these are the only points which are fixed by \( \sigma \). If the fixed points is \( P_2 \) or \( P_3 \) then \( \text{Aut}(C) \) is cyclic of order 3 because both points lie on \( C \). If the fixed point is \( P_1 \) then the line that is left invariant should be \( X = 0 \) hence automorphisms of \( C \) have the form \([X;\beta_2 Z + \gamma_2 Y + \gamma_3 Z]\). Moreover, it follows by Theorem 2(2), Lemma 10 and the assumption that there are no elements in \( \text{Aut}(C) \) of order \( > 3 \) that \( \text{Aut}(C) \) satisfies a short exact sequence of the form \( 1 \rightarrow N = 1 \rightarrow \text{Aut}(C) \rightarrow G' \rightarrow 1 \), where \( G' \) is conjugate to \( C_3, S_3 \). This completes the proof.

\[ \square \]

**Proposition 14.** Let \( C \) be a non-singular plane curve of type \( 2, (0, 1) \) such that elements inside \( \text{Aut}(C) \) have orders \( \leq 2 \). Then, \( C \) is \( K \)-isomorphic to \( C : Z^4 L_{1,2} + Z^2 L_{3,2} + L_{5,2} = 0 \) and \( \text{Aut}(C) \) is cyclic of order 2.

**Proof.** \( C \) is not a descendant of the Klein curve because \( 2 \mid |\text{Aut}(K_2)| (= 63) \). Also, if \( C \) is a descendant of the Fermat curve then \( \text{Aut}(C) \) can not be conjugate to a bigger subgroup of \( \text{Aut}(F_3) \) because \(|\text{Aut}(F_3)| = 2 \cdot 3 \cdot 5^2\) thus subgroups of order \( > 2 \) should contain an element of order 3 or 5 a contradiction. Finally, if \( \text{Aut}(C) \) fixes a line and a point off that line then by [6] and our assumption that there are no automorphisms of order \( > 2 \) we get that \( \text{Aut}(C) \) satisfies a short exact sequence of the form \( 1 \rightarrow N = 1 \rightarrow \text{Aut}(C) \rightarrow G' \rightarrow 1 \), where \( G' \) contain an element of order 2 and no higher orders thus \( \text{Aut}(C) \) should be conjugate to \( C_2 \) or \( C_2 \times C_2 \). Consequently, the result follows by Lemma 10.

\[ \square \]

Lastly, we need to ensure the existence of a non-singular plane curve \( C \), via certain specializations of the parameters, for which the maximal order of the elements in its full automorphism group is exactly \( m \) where \( m \leq 5 \).

This is a tedious computation because we do not know a priori the dimension of the locus \( \rho(M_6(G)) \), see for example the situations with \( m = 4 \) in [1]. To treat the case \( m \neq 4 \), we can apply similar arguments as the situation \( m = 4 \), which we will not reproduce here (nevertheless, we know all the possible groups and the representations that could appear such that \( m \) divides its order. This in turns simplifies the computations), in order to conclude,

**Lemma 15.** Take \( F(X; Y; Z) \) the equation of degree 5 associated with Type \( m, (a, b) \) where \( m \leq 5 \) in §2, table 1 with \( (a, b) \neq 4, (1, 3) \). Then, there exists a non-singular plane curve \( C \) obtained by a concrete specialization of the parameters of the equation \( F(X; Y; Z) \), such that all the elements of \( \text{Aut}(C) \) are of order \( \leq m \). Moreover, for type 3, \((1, 2)\), we have curves with this property on the elements of \( \text{Aut}(C) \) of order \( \leq 3 \) satisfying \( \beta_{3, 0} \neq \beta_{3, 3} \) and also satisfying \( \beta_{3, 0} = \beta_{3, 3} \).

Lastly, we remark the following.

In the following table, we list the exact groups \( G \) (some of them are given in the GAP notations) that appears as an \( \text{Aut}(\delta) \) of a non-singular plane curve \( \delta \) of genus 6. The second column corresponds to the injective representation \( \rho \) of the group \( G \) inside \( PGL_3(K) \). The third column corresponds to a homogenous equation of degree 5 associated with certain parameters such that for any \( \delta \in \rho(M_6^{P_1}(G)) \), a plane non-singular model associated to \( \delta \) can be obtained by a specialization of the parameters and vice versa.

It remains to assign to each equation the parameters’ restrictions to ensure that the equation is geometrically irreducible, non-singular and it does not have a bigger automorphism group (so that, any \( \delta \in \rho(M_6^{P_1}(G)) \) corresponds to some specialization of the parameters with respect to the restrictions, and conversely, any specialization of the parameters which respect the restrictions gives a non-singular plane model of some element in \( \rho(M_6^{P_1}(G)) \)). We recall that always \( \alpha \neq 0 \) can be transformed by a diagonal change of variables to 1.
Table 2. Full Automorphism of quintics

<table>
<thead>
<tr>
<th>$G$</th>
<th>$\rho(G)$</th>
<th>$F_{\rho(G)}(X;Y;Z)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(150,5)$</td>
<td>$[\xi_5 X; Y; Z], [X; \xi_5 Y; Z]$</td>
<td>$X^5 + Y^5 + Z^5$</td>
</tr>
<tr>
<td>$(39,1)$</td>
<td>$[X; \xi_1 Y; \xi_{10}^5 Z], [Y; Z; X]$</td>
<td>$X^4 Y + Y^4 Z + Z^4 X$</td>
</tr>
<tr>
<td>$(30,1)$</td>
<td>$[X; \xi_1 Y; \xi_{11}^5 Z], [X; Z; Y]$</td>
<td>$X^5 + Y^5 Z + YZ^4$</td>
</tr>
<tr>
<td>$Z/20Z$</td>
<td>$[X; \xi_2^5 Y; \xi_{20}^2 Z]$</td>
<td>$X^5 + Y^5 + XZ^4$</td>
</tr>
</tbody>
</table>
| $Z/16Z$        | $[X; \xi_4 Y; \xi_2 Z]$ | $X^5 + Y^5 + \alpha XZ^4 + \beta_{2,0} X^3 Z^2$  
|                |           | $\alpha \beta_{2,0} \neq 0$ and ($\alpha, \beta_{2,0} \neq (5, 10)$) |
| $D_{10}$       | $[X; \xi_3 Y; \xi_3^2 Z], [Z, Y, X]$ | $X^5 + Y^5 + Z^5 + \beta_{3,1} X^2 Y Z^2 + \beta_{4,3} X Y^3 Z$
|                |           | $(\beta_{3,1}, \beta_{4,3}) \neq (0, 0)$ |
| $Z/8Z$         | $[X; \xi_8 Y; \xi_8^6 Z]$ | $X^5 + Y^4 Z + XZ^4 + \beta_{2,0} X^3 Z^2$ $(\beta_{2,0} \neq 0 \pm \overline{2})$ |
| $S_3$          | $[X; \xi_3 Y; \xi_3^3 Z]$ | $X^5 + Y^4 Z + YZ^4 + \beta_{2,1} X^3 Y Z + X^2 (Z^3 + Y^3)+$
|                |           | $+\beta_{4,2} X Y^2 Z^2$ (not above) |
| $Z/5Z$         | $[X; Y; \xi_5 Z]$ | $Z^5 + L_{5, Z}$ (not above) |
| $Z/4Z$         | $[X; \xi_4 Y; \xi_3^3 Z]$ | $X^5 + X (Z^4 + \alpha Y^4) + \beta_{2,0} X^3 Z^2 + \beta_{3,2} X^2 Y^2 Z + \beta_{5,2} Y^2 Z^3$
|                |           | $(\beta_{5,2} \neq 0)$ (not above) |
| $Z/4Z$         | $[X; Y; \xi_4 Z]$ | $Z^4 L_{1, Z} + L_{5, Z}$ (not above) |
| $Z/3Z$         | $[X; \xi_3 Y; \xi_3^3 Z]$ | $X^5 + Y^4 Z + \alpha Y^4 + \beta_{2,1} X^3 Y Z +$
|                |           | $+X^2 (\beta_{3,0} Z^3 + \beta_{3,3} Y^3) + \beta_{4,2} X Y^2 Z^2$ (not above) |
| $Z/2Z$         | $[X; Y; \xi_2 Z]$ | $Z^4 L_{1, Z} + Z^2 L_{3, Z} + L_{5, Z}$ (not above) |

Remark 16. It should be noted that here appears a new phenomena that did not appear for degree $d = 4$. Henn in [7] observed that for each finite group $G$ that appeared as an automorphism group of a non-singular plane curves $\delta \in M_4^G$ there is a unique equation $F_G(X, Y, Z)$ (endowed with a set of restrictions on the parameters), up to change of variables, where the specializations of such equation is a plane non-singular model for the elements of $M_4^G$ and vice versa. This is not the case any more for degree 5, since from the above table with the group $Z/4Z$ we obtain two $\rho$'s where their equations $F_{\rho}(X; Y; Z)$ are not $K$-isomorphic, and corresponds to the disjoint decomposition of $M_5^G(Z/4Z)$ in terms of non-empty $\rho(M_5^G(Z/4Z))$. Similar situations happen for higher degrees, for more details, we refer to [1].

References


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