ON THE TAMAGAWA NUMBER CONJECTURE FOR HECKE CHARACTERS

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Abstract. In this paper we prove the weak $p$-part of the Tamagawa number conjecture in all non-critical cases for the motives associated to Hecke characters of the form $\varphi^n\overline{\varphi}$ where $\varphi$ is the Hecke character of a CM elliptic curve $E$ defined over an imaginary quadratic field $K$, under certain restrictions which originate mainly from the Iwasawa theory of imaginary quadratic fields.

1. Introduction

The Tamagawa number conjecture for a variety $X$ over a number field of Bloch and Kato [5], or, more precisely, for a motive $M$ of pure weight $w$ over a number field, describes the special values of the $L$-function in terms of cohomological data (see for example Kato [19] or Fontaine and Perrin-Riou [11]) and the $p$-part of the conjecture describes these values up to units in the ring $\mathbb{Z}(p) := \{ \frac{a}{b} \in \mathbb{Q} | a, b \in \mathbb{N}, b \neq 0, (b, p) = 1 \}$.

Recall that the special values of an $L$-function are the leading coefficient of Taylor expansion at integer points. Suppose we have a motive $M$ of weight $w$ such that its $L$-function has meromorphic continuation and satisfies the expected functional equation. We say that an integer $m < \frac{w}{2}$ is non-critical if $L(M, m) = 0$ and that it is critical if $L(M, m) \neq 0$. We extend this definition to the integers $m > \frac{w}{2} + 1$ by saying that $m$ is critical for $M$ if $w - m + 1$ is critical for $\tilde{M}$ the dual motive associated to $M$, and non-critical for $M$ if $w - m + 1$ is non-critical for $\tilde{M}$. The Tamagawa number conjecture can be formulated in terms of period maps (period integrals nor $p$-adic periods appears) and regulator maps ([10], [19]), but in the non-critical situation it can be formulated for almost all non-critical cases (using the hypothetical functional equation and good compatibilities) without the period maps, where by period map we mean a map between de Rham to Betti cohomology and for the $p$-adic period a map between étale cohomology to de Rham cohomology (see [19, §2.3]).

There are few cases proved in the non-critical situation: for the Riemann zeta function ([5, §6]), for Dirichlet motives ([6], [17]), for CM elliptic curves defined over the field of the endomorphism ring ([22]) or defined over $\mathbb{Q}$ ([5, §7], [11]).

The weak $p$-part of the Tamagawa number conjecture for an elliptic curve $E$ with CM by $K$ defined over the field of endomorphisms, proved by Kings [22], is related to the weak $p$-part of the Tamagawa number conjecture for the $L$-function of the Hecke character $\varphi$, associated to $E$, over the imaginary quadratic field $K$ of class number 1. More precisely, Kings proves in [22] the conjecture for the motive $h(\overline{\varphi})(-r)$ with $r \geq 0$ which corresponds to the special value (non-critical) for the $L$ function associated to $\overline{\varphi}$ at $-r$, where $h(\overline{\varphi})$ is the motive associated to $\overline{\varphi}$ over $K$ with $K$-coefficients. As a consequence, he obtains the conjecture for the motive $h^1(E)(-r)$. Using the functional equation for $E$ and good compatibilities one should obtain the conjecture for $h^1(E)(r + 2)$. We generalize the methods of Kings to other Hecke characters over an imaginary quadratic field $K$ in the non-critical situation.

Key words and phrases. Tamagawa number conjecture, Hecke characters, Iwasawa theory of imaginary quadratic fields, regulators.
We consider the motive associated to the Hecke character $\varphi^a\varphi^b$ with $a, b \geq 0$, which has weight $a + b$. It is known that almost all the non-critical values for this motive are the integers lower than $\min(a, b)$. Our work is concentrated in this situation, but we remark that there are results on the Tamagawa number conjecture in the critical situation (Harrison [16], Guo [14], Kimura [23], Han [15] and in greater generality by Tsuji [27]).

The aim of this paper is to prove the $p$-part of the Tamagawa number conjecture (under the formulation in [19, (2.2)]) for all the non-critical values for the $L$-function of the motive associated Hecke characters $\psi\vartheta$ of an imaginary quadratic field $K$ with class number 1, which under a fixed embedding corresponds to $\varphi^a\varphi^b$.

The main results of this paper are Theorems 5.12, 5.13 and Theorems 6.3, 6.4. These results are the weak $p$-part of the Tamagawa number conjecture for the geometric object associated to the Hecke characters over $K$ with $a \neq b$ (modulo $(\#O_K^\times)$) with $K$ or $\mathbb{Q}$-coefficients, under certain restrictions which originate mainly from Iwasawa theory of imaginary quadratic fields, where $O_K^\times$ means the invertible elements of the ring of integers of $K$. See the last section for numerical examples.

To obtain these main results, we study in detail the image with respect to the regulator map of a certain non-trivial submodule of some $K$-theory group. The basic ingredients used in the proof of these results are Deninger’s proof of the Beilinson’s conjecture for Hecke characters in [8], the specialization of the polylogarithm sheaf [22] and the Iwasawa main conjecture for imaginary quadratic fields [24], as in [22]. This paper need to deal with negative twists. This problem does not appear in [22]. For negative twists, we modify Deninger’s elements [8] in order to apply the $p$-adic techniques of [22].

The results of this paper generalize the results in [2, Chapter 3] which restricts to the Hecke characters $\varphi^a\varphi^b$.

2. THE MOTIVE ASSOCIATED TO HECKE CHARACTERS

Let $K$ be an imaginary quadratic field with class number $cl(K)$ equal to 1 and $O_K$ be its ring of integers. Let $D_K$ be the discriminant of $K$. Let $E$ be an elliptic curve over $K$ with $\mathbb{Q}$-coefficients, under certain restrictions which originate mainly from Iwasawa theory of imaginary quadratic fields, where $O_K^\times$ means the invertible elements of the ring of integers of $K$. See the last section for numerical examples.

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Let $\varphi : I_K \rightarrow K^*$ be the CM character associated to the elliptic curve $E$ where $I_K$ is the idèles of $K$. Denote by $T_w := \otimes_w^\varphi K$ with $w$ a positive integer. Observe that $T_w$ is equal to a product of fields $\prod_w T_\vartheta$, where $\vartheta$ runs through the $Aut(\mathbb{C})$-orbits of $J = Hom(T_w, \mathbb{C})$, $\vartheta \subseteq J$. Let $e_\vartheta$ be the idempotent corresponding to $T_\vartheta$ of $T_w$.

Define the CM character $\underline{\psi}_\vartheta : I_K \rightarrow T_\vartheta^*$ by $\underline{\psi}_\vartheta = e_\vartheta \cdot (\otimes_w^\varphi \underline{\varphi})$, and denote by $f_\vartheta$ the conductor of $\underline{\psi}_\vartheta$. Observe that $f_\vartheta$ since $\underline{\psi}_\vartheta$ is a sub-representation of $\otimes_w^\varphi \underline{\varphi}$.

Let us fix once and for all an embedding $K \rightarrow \mathbb{C}$ like in the last paragraph on [8, p.132]. We have a natural embedding $\mathcal{Y}$

$$K \rightarrow \otimes_w K \rightarrow T_w \rightarrow T_\vartheta$$

where the first map corresponds to the diagonal map.

For any $\vartheta \in \vartheta$ which its orbit in $J$ is equal to $\vartheta$, we have a map $\vartheta : T_w \rightarrow \vartheta(T_w) \subseteq \mathbb{C}$ inducing an isomorphism $T_\vartheta = T_w/ker(\vartheta) \rightarrow \vartheta(T_w)$, and $\vartheta(T_w)$ is the field generated by $\lambda_1(K) \cdot \ldots \cdot \lambda_w(K)$ where $\lambda_i \in Hom(K, \mathbb{C})$, which is isomorphic to $K$. 

2.1. THE MOTIVE ASSOCIATED TO HECKE CHARACTERS
Define by $\theta_K$ the subset of $\theta$ which contains the $\lambda \in \theta$ which orbit is equal to $\theta$ such that $T_\theta = T_{w/\ker(\theta)} \to \theta(T_{w/}) \subset \mathbb{C}$ is in $\text{Hom}_K(T_\theta, \mathbb{C})$ for the natural embedding $\Psi$. In our case $\theta_K$ contains only one element. Let $\mathfrak{v} = (\lambda_1, \ldots, \lambda_w) \in \mathbb{Z}$, we set $a_\mathfrak{v} = \#(\lambda_i \in \text{Hom}_K(K, \mathbb{C}))$ and $b_\mathfrak{v} = w - a_\mathfrak{v}$. The infinite type of $\psi_{\mathfrak{v},p}$ is defined by $\mathfrak{v}_{\mathfrak{v},p} := (a_\mathfrak{v}, b_\mathfrak{v})$ where $\mathfrak{v}$ is the element in $\theta_K$. Observe that $\theta$ only contains two elements, the element of $\theta$ different for $\vartheta = (\lambda_1, \ldots, \lambda_w) \in \theta_K$ is $\vartheta = (\lambda_1, \ldots, \lambda_w)$ where $\lambda_i$ denotes the composition of $\lambda$ with the complex conjugation.

Consider the category of Chow motives $M(K)$ over $K$ with morphisms induced by graded correspondences in Chow theory. We have then a natural covariant functor $h$ from the category of smooth and projective varieties over $K$ to $M(K)$. Then the motive $h(E)$ of an elliptic curve $E$ over $K$ has a decomposition with respect to the zero section $h(E) \cong h^0E \oplus h^1E \oplus h^2E$, where $h^0E = h(\text{Spec}(K))$ and $h^2E = h(\text{Spec}(K))(-1)$. We can also consider the category $M_q(K)$ which consist of the same objects but tensoring by $\mathbb{Q}$ the group of morphisms.

The motive $h^1E$ has multiplication by $O_K$. Consider then the motive $\otimes \omega h^1E$, has multiplication by $O_w := \otimes \omega O_K$. Then $\otimes \omega h^1E \mathbb{Q}$ has multiplication by $T_w$. Notice that $\psi_{\omega}$ is not integral in general for $w > 1$, but is contained in $O_K[1/D_K]$. Let's denote by

$$M_\theta := e_\theta(\otimes \omega h^1(E) \mathbb{Q} \otimes O_K[1/D_K]),$$

considered as a motive with coefficients in $O_K[1/D_K]$, and by $M_{\theta \mathbb{Q}}$ its image in $M_{\mathbb{Q}}(K)$. As $\lambda_i \in \{\lambda, \lambda\}$ where $\lambda$ is the fixed embedding of $K$ in $\mathbb{C}$, and $\lambda_i(O_K) = O_K$, we have then that $e_\theta(\otimes \omega h^1(E) \otimes O_K[1/D_K]) \in M(K)$ has multiplication by $O_\theta := e_\theta(O_w \otimes O_K[1/D_K])$, and $M_{\theta \mathbb{Q}}$ has multiplication by $T_\theta \cong K$.

Our objective in this section is to study the $p$-adic and Betti realizations of this motive $M_\theta$, called Hecke motive, and to determine its $L$-function.

There are at least three equivalent notions of a Hecke character, see [12, p.48]. One is the notion of CM-character used above, [12, p.48, definition 2]. For Hecke $L$-functions and the Galois group action on the $p$-adic realization associated to the Hecke motive, we use the notion of a character which is trivial on $K^*$ and with image in some idèle group, $\psi_\theta : I_K/K^* \to I_{\theta}$ [12, p.48, definition 3]. The associated complex Hecke character, in order to define the Hecke $L$-function, is constructed from $\psi_\theta$ by taking the archimedean places of $I_{\theta}$ which correspond to the fixed immersion of $K$ in $\mathbb{C}$ in our situation, which we also call $\psi_\theta$. The character constructed from $\psi_\theta$ by taking the components of the places of $I_{\theta}$ above $p$ is called $\psi_{\theta,p}$, and is related with the Galois action on the $p$-adic realization associated to the motive. The character $\psi_{\theta,p}$ factors through $\text{Gal}(K^{ab}/K)$ and has image in $(T_\theta \otimes \mathbb{Z}_p)^\times$. We will use the term Hecke character when we want to consider this second notion from now on. The third notion [12, definition 1,p.48], corresponds to certain map $\tilde{\psi}_\theta : I_{\theta} \to T_\theta$, where $I_{\theta}$ is the ideal classes of $K$ prime to $\{p\}$. So, if $p$ is a prime ideal of $O_K$ prime to $\{p\}$, we mean for $\psi_\theta(p)$ or $\psi_{\theta,p}(p)$ the value of the Hecke character or CM character at the idèle which has an uniformizer $\pi$ at the place $p$ and 1 in the other places.

We have $\psi_\theta(p) = \psi_{\theta,p}(p) = \tilde{\psi}_\theta(p)$ (see [12, p.49-50]).

The $p$-adic realization of the motive $M_{\theta \mathbb{Q}}(w)$ is, by definition, $H^w_{\text{c}}(M_{\theta \mathbb{Q}} \times_{K} K, \mathbb{Q}_p(w))$ and we denote it by $M_{\theta \mathbb{Q}_p}(w)$.

**Lemma 2.1** Let $p$ be a prime such that $p \nmid D_K$. The integral $p$-adic realization of $M_{\theta}(w)$, $H^w_{\text{c}}(M_{\theta} \times_{K} K, \mathbb{Q}_p(w)) \otimes_{O_K} O_K[1/D_K]$, is isomorphic to

$$e_\theta(\otimes \omega T_p E)$$

as free $e_\theta(\otimes \omega O_K[1/D_K] \otimes \mathbb{Z}_p)$-modules of rank 1, with $\text{Gal}(K/K)$-action on $e_\theta(\otimes \omega T_p E)$ given by the Hecke character $\psi_{\theta,p}$. 


Proof. Observe first that $T_p E$ is isomorphic as Galois modules to $H^2_{et}(h^1(E) \times_K \overline{K}, \mathbb{Z}_p(1)) = \text{Hom}(T_p E, \mathbb{Z}_p(1))$ by the use of Weil pairing but the isomorphism change the action of $O_K$ to its conjugate.

The claim that $e_\theta(\otimes^w T_p E)$ is a free module of rank 1 follows because $T_p E$ is a free $O_K \otimes \mathbb{Z}_p \cong O_K[1/D_K] \otimes \mathbb{Z}_p$-module of rank 1 and then $e_\theta \cdot (T_p E \otimes \ldots \otimes T_p E)$ is a free $e_\theta \cdot (\otimes^w (O_K[1/D_K] \otimes \mathbb{Z}_p))$-module of rank one.

Now, consider the natural action of $G_K := \text{Gal}(\overline{K}/K)$ on $H^2_{et}(h^1(E) \times_K \overline{K}, \mathbb{Z}_p(1))$. Since $G_K$ acts on the Tate module by the Hecke character $\varphi : G_K \to (O_K \otimes \mathbb{Z}_p)^\times$, so it acts on $H^2_{et}(h^1(E) \times_K \overline{K}, \mathbb{Z}_p(1))$ by $\overline{\varphi}_p$. Using

$$H^w((h^1(E) \times_K \overline{K})^w, \mathbb{Z}_p(w)) = H^1(h^1(E) \times_K \overline{K}, \mathbb{Z}_p(1))^\otimes w$$

and taking our idempotent, the action is given by $e_\theta(\otimes^w \overline{\varphi}_p) = \overline{\psi}_p$.

Twisting by $l+1$ we get for $p \nmid D_K$ that the integral $p$-adic realization for $M_\theta(w+l+1)$ is isomorphic to $e_\theta(\otimes^w T_p E)(l+1)$ with $G_K$-action on $e_\theta(\otimes^w T_p E)(l+1)$ given by $\overline{\varphi}_p$, multiplied by the $l+1$-th power of the $p$-adic cyclotomic character.

The Betti realization for the motive $M_{\theta}(w+l)$ is isomorphic to $e_\theta(\otimes^w H^1_{et}(E(C), \mathbb{Q}(w+l)))$ on $e_\theta(\otimes^w H^1_{et}(E(C), \mathbb{Q}(w+l)))$, we remember that we fixed once and for all an immersion for $K \subseteq C$. $E(C)$ is the set of closed points with the analytic topology. We have

$$\otimes^w H^1_{et}(E \times_K \overline{C}, \mathbb{Z}(1)) \otimes_{O_K} O_K[1/D_K]$$

a $O_K$-module of rank 1 and taking the idempotent $e_\theta$ we obtain

$$e_\theta(\otimes^w H^1_{et}(E \times_K \overline{C}, \mathbb{Z}(1)) \otimes_{O_K} O_K[1/D_K](l),$$

a $O_{\theta}$-module of rank 1 which is the submodule of $H^1_{et}(M_{\theta}, \mathbb{Q}(w+l))$ corresponding to $H^1_{et}(M_{\theta}, \mathbb{Q}(w+l)) \otimes_{O_K} O_K[1/D_K]$.

Now, we are going to study the $L$-function that corresponds to the $p$-adic representation $M_{\theta} = H^1_{et}(M_{\theta} \times_K \overline{K}, \mathbb{Q}_p)$ of $M_{\theta}$.

The Tamagawa number conjecture describes conjecturally special values of the $L$-function for the motive and this $L$-function involves the product of all Euler factors (one for every non-archimedean place), but for the part of this conjecture, Kato reformulates the conjecture in terms of the partial $L$-function avoiding a non-vanishing finite set of Euler factors, more concretely the finite set contains the Euler factors coming from the primes above $p$ and the primes where the motive has bad reduction (see [20, Proposition 7.8], or [2, Chapter 1] for an overview, and see remark 2.5 for these no-vanishing in our setting).

Let $S$ be the set of places of $K$ that divide $f_{\theta}$ or $p$. Define as usual

$$L_S(M_{\theta, s}) := \prod_{q \in S} \det_{q_p}(1 - \text{Frob}_q N^{-s} | M_{\theta}^1_{q_p})^{-1}$$

where $\text{Frob}_q$ means the geometric Frobenius, $I_l$ the inertia group at $l$ and $NI$ the norm $N_{K/Q}$.  

Our goal is to compute this determinant and to relate it to the local factors of the $L$-function of the Hecke character $\psi_\theta$ that is defined by

$$L_S(\psi_\theta, s) := \prod_{q \in S} (1 - \psi_\theta(q))^{-1}.$$
Lemma 2.2 (Deninger, prop. 1.3.2.a [8]) Let \( l \) be a finite prime of \( K \) with \( l \nmid \rho_{\text{theta}} \). where \( \rho_{\theta} \) is the conductor of the Hecke character \( \psi_{\theta} \). Then
\[
\det_{\Q_{\theta}}(1 - Fr_{l}N^{-s}|M^{l}_{d\Q_{p}}) = (1 - \psi_{\theta}(l)N^{-s}).
\]

We impose some restrictions for our motive \( M_{\Q}(w+l+1) \) once and for all. We suppose \(-w - 2l \leq -3\). Remember that, with the restriction that \( E \) is defined over \( K \), we have \#\( \theta \) = 2, and, in particular, we have \( T_{\theta} \cong K \) and for \( p \nmid D_{K} \) we have \( \O_{\theta} \otimes \Z_{p} \cong \O_{K} \otimes \Z_{p} \).

The \( L \)-function for \( M_{\Q} \) can be described by using lemma 2.2 and by taking the norm map.

Lemma 2.3 Let \( l \) a prime of \( K \) such that \( l \nmid \rho_{\theta} \) and it is prime to \( p \). We have then the following equality
\[
\det_{\Q_{\theta}}(1 - Fr_{l}N^{-s}|M^{l}_{d\Q_{p}}) = (1 - \psi_{\theta}(l)N^{-s})(1 - \overline{\psi}_{\theta}(l)N^{-s}).
\]

As a corollary we obtain a generalization of a result of Deuring.

Theorem 2.4 Let \( S \) be the set of the primes on \( K \) dividing \( \rho_{\theta} \) and primes dividing \( p \). Then:
\[
L_{S}(M_{\Q}, s) = L_{S}(\psi_{\theta}, s)L_{S}(\overline{\psi}_{\theta}, s).
\]

Remark 2.5 The \( p \)-adic realization \( V_{p} := M_{\Q_{p}}(w+l+1) \) satisfies that the local Euler factors
\[
P_{p}(V_{p}^{*}(1), 0) = P_{p}(\overline{\psi}_{\theta}, -l)
\]
are different from 0 for all \( p \in S \) where \( V_{p}^{*} \) is the dual Galois module of \( V_{p} \). Hence, it satisfies the hypothesis of [19, conjecture 2.2.7].

To show this fact, suppose first that \( p \nmid \rho_{\theta} \). Then the inertia group acts non-trivially on \( V_{p} \), which is a one dimensional \( \O_{\theta} \otimes \Q \)-module, and hence
\[
L_{p}(\overline{\psi}_{\theta}, s) = 1
\]
for all \( p \mid \rho_{\theta} \), and in particular for \( s = -l \).

If \( p \) divides \( \rho_{\theta} \), then the result follows from the fact that any proper smooth variety with good reduction at \( p \) satisfies it for weight reasons, and in particular:
\[
\det_{\Q}(1 - Fr_{p}N^{{l'}}|H_{\et}^{w}(\tilde{E} \times \K, \Q_{l}), 0) \neq 0
\]
with \( l' \neq p \), and therefore, since the different idempotents \( e_{\theta} \) give a direct summand of the cohomology group \( H^{w}(\tilde{E} \times \K, \Q_{l}), 0 \),
\[
L_{p}(\overline{\psi}_{\theta}, -l) \neq 0
\]

The motivic cohomology group \( H^{w+1}_{\et}(M_{\Q}, \Q(w+l+1)) \) is the \( K \)-theory group \( K_{2(w+l)}(M_{\Q})(w+w+1) \otimes \Q \) where the \( K \)-groups are the Quillen \( K \)-groups and the superscript denotes the Adam’s filtration on them.

We suppose recall that \( -2(w+l+1) \leq -3 \). We have a Beilinson regulator map,
\[
r_{\et} : H^{w+1}_{\et}(M_{\Q}, \Q(w+l+1)) \to H^{w}_{\et}(M_{\et}, \Q(w+l+1)) \otimes \R,
\]
and the Soulé regulator map:
\[
r_{p} : H^{w+1}_{\et}(M_{\Q}, \Q(w+l+1)) \to H^{1}_{\et}(\O_{K}[1/S], M_{\et}(w+l+1)).
\]

The \( p \)-part of the Tamagawa number conjecture claims in particular that \( r_{\et} \otimes \Q \R \) and \( r_{p} \otimes \Q \Q_{p} \) are isomorphisms. Deninger in [8] [7] constructs a \( \Q \)-subspace \( H^{w+1}_{\et}(M_{\et}, \Q(w+l+1)) \) such that \( r_{\et} \otimes \R \) restricted to \( H^{w+1}_{\et} \otimes \Q \R \) is an isomorphism. The term weak in the formulation of the \( p \)-part of the Tamagawa number conjecture means that this conjecture is proved using the space \( H^{w+1}_{\constr} \otimes \Q \R \) instead of \( H^{w+1}_{\et}(M_{\et}, \Q(w+l+1)) \).
3. The Beilinson Conjecture for Hecke Characters

In this section we review the work on the Beilinson conjecture for the motive $M_{\Theta_2}(w + l + 1)$ done by Deninger in [8], under the language of the Tamagawa number conjecture.

**Theorem 3.1** (Deninger, Theorem 1.4.1 [8]) Let $w = a_\theta + b_\theta \geq 1$. Consider an integer $l$ such that

$$-l \leq \text{Min}(a_\theta, b_\theta) \quad \text{if } a_\theta \neq b_\theta$$

$$-l < a_\theta = b_\theta = w/2 \enspace \text{otherwise}.$$  

Then the $L$-series $L(\overline{\psi}_\theta, s)$ has a zero of order 1 at $s = -l$, (i.e. $\text{ord}_{s=-l}L(\overline{\psi}_\theta, s) = 1$).

Moreover, there exist an element $\xi_\theta$ in $H^{w+1}_M(M_{\Theta_2}, \mathbb{Q}(w + l + 1))$, such that

$$r_D(\xi_\theta) = \lim_{s \to -l} \frac{L(\overline{\psi}_\theta, s)}{s + l - \eta_\theta \text{ mod } T^*}$$

in the free rank one $T_\theta \otimes \mathbb{R}$-module $H^w_B(M_{\Theta_2}, \mathbb{R}(w + l))$, where $\eta_\theta$ is a $T_\theta$-generator of $H^w_B(M_{\Theta_2}, \mathbb{Q}(w + l))$.

Let's recall the construction of $\xi_\theta$, following the results of Deninger. We suppose once for all that $l \geq 0$.

Fix an algebraic differential form $\omega \in H^0(E, \Omega_{E/K})$. Since we have complex multiplication, we can write the period lattice as $\Gamma = \Omega \mathcal{O}_K$, where $\Omega \in \mathbb{C}^\ast$ is the complex period. Fix an element $\gamma$ in $H_1(E(\mathbb{C}), \mathbb{Z})$ such that it is an $\mathcal{O}_K$-generator, and satisfies

$$\Omega = \int_{\gamma} \omega.$$ 

By Poincaré duality, we have that $\gamma$ corresponds to $\eta_\gamma$, an $\mathcal{O}_K$-generator for $H^1(E(\mathbb{C}), \mathbb{Z}(1))$. Thus $\eta_\gamma \otimes \mathcal{O}_K \mathcal{O}_K[1/D_k]$ is an $\mathcal{O}_K[1/D_k]$-generator for the module $H^1(E(\mathbb{C}), \mathbb{Z}(1)) \otimes \mathcal{O}_K \mathcal{O}_K[1/D_k]$ which by abuse of notation we call also $\eta_\gamma$. Consider now the $\mathcal{O}_K$-generator

$$\eta_\theta := (2\pi i)^l \epsilon_\theta (\otimes^w \eta_\gamma)$$

of $H^w_B(M_{\Theta_2}, \mathbb{Z}(w + l)) \otimes \mathcal{O}_K \mathcal{O}_K[1/D_k]$.

To construct $\xi_\theta$, we will define a divisor on the torsion points of the elliptic curve; its image by the composition of the Eisenstein map $E_M$ (\([7, \S 8]\)) with the Deninger projector map $K_M$ (\([8, (2.8)]\)) will define our $\xi_\theta$.

Remember that $f_\theta$ is the conductor of the Hecke character $\psi_\theta$ associated with $M_\theta$, and denote by $f$ a generator of $f_\theta$ (it exists since $\cl(K) = 1$). We have that

$$\Omega f^{-1} \in f_\theta^{-1} \Gamma$$

and that $(\Omega f^{-1})$ gives a divisor in $\mathbb{Z}[E][\ell_\theta] \setminus 0$ defined over $K(E[\ell_\theta])$. Since $\ell$ is the conductor of $\psi$ and $f_\theta\mid f$, the divisor $(\Omega f^{-1})$ is defined also over $K(E[\ell])$. We will define our divisor as

$$\beta_\theta := N_K(\ell_\theta)^{-1}((\Omega f^{-1})).$$

Denote by $\rho_\theta$ a finite idèle such that $(\rho_\theta) = f_\theta$ and $v_p(f^{-1} - \rho^{-1}_p) \geq 0$ for all $p \mid f_\theta$.

If $a_\theta \neq b_\theta \mod \mathcal{O}_K^\ast$, we obtain that (\([8, p.142,(2.11)]\))

$$r_D(K_ME_M(\beta_\theta)) = (-1)^{l-1} \frac{2^{l-1} - 1}{(2l + w)!N_{K/Q}(f_\theta)^l + w} \Phi(\ell) L'(\overline{\psi}_\theta, -l) \eta_\theta,$$

where $\Phi(m) := \mathcal{O}_K/m^\ast$ for any ideal $m$ of $\mathcal{O}_K$.

This is an analog for $M_\theta(w + l + 1)$ of \([22, \text{thm.1.2.2}]\), which corresponds to the case $h^1(E)(1 + l + 1)$.

**Theorem 3.2** (Deninger, §2 [8]) Suppose that $a_\theta \neq b_\theta \mod \mathcal{O}_K^\ast$ and that $a_\theta, b_\theta, l$ satisfy the hypothesis of the theorem 3.1 with $l \geq 0$. Define, by using the previous notation,

$$\xi_{\theta, l} :=$$
For Observing that The submodule We have the following equality The map The first and the second conditions are clear for the dimensions of the spaces Then

\[ r_D(\xi_\theta, l) = L_S^*(\overline{\psi_\theta}, -l)e_\theta(\otimes^w \eta_\gamma), \]

where \( S \) are the primes of \( K \) that divide \( \mathfrak{p} \mathfrak{p} \). Here \( L_S^*(\overline{\psi_\theta}, -l) = \lim_{s+l \to 0} L_S(\overline{\psi_\theta}, s)/(s+l). \)

**Definition 3.3** For \( a_\theta \not\equiv b_\theta \mod(\#\mathcal{O}_K^\times) \) we define

\[ \mathcal{R}_\theta := \xi_{\theta, l}\mathcal{O}_K. \]

**Remark 3.4** Theorem 1.4.1 [8] is more general because it includes the situation \( a_\theta \equiv b_\theta \mod(\#\mathcal{O}_K^\times) \). But in this situation, Deninger defines a divisor \( \delta_\theta \) instead of \( \beta_\theta \) which is not a norm of a positive divisor and moreover it contains the zero point of \( E \). Thus the techniques for constructing an Euler system of \( \Sigma_5 \) can not be applied in this case (see for example Theorem 5.1).

As a consequence of Theorem 3.2, we have that our submodule \( \mathcal{R}_\theta \) verifies some integral version of the Beilinson conjecture for the motive \( M_\theta(w + l + 1). \)

**Theorem 3.5** The \( \mathcal{O}_K \)-submodule \( \mathcal{R}_\theta \) of \( H_{\text{et}}^{w+1}(M_{\theta\mathbb{Q}}, \mathbb{Q}(w + l + 1)) \) satisfies that

\[ \det_{\mathcal{O}_{K[1/D_\mathcal{K}]}}(r_D(\mathcal{R}_\theta \otimes \mathcal{O}_K \mathcal{O}_K[1/D_\mathcal{K}])) = L_S^*(\overline{\psi_\theta}, -l)\det_{\mathcal{O}_{K[1/D_\mathcal{K}]}}(H^w_B(M_\theta, \mathbb{Z}(w + l)) \otimes \mathcal{O}_K \mathcal{O}_K[1/D_\mathcal{K}]) \]

in \( \det_{\mathcal{O}_{K[1/D_\mathcal{K}]}}(H^w_B(M_{\theta\mathbb{Q}}, \mathbb{Z}(w + l)) \otimes \mathcal{O}_K \mathcal{O}_K[1/D_\mathcal{K}]) \otimes \mathbb{R} \).

**Proof.** Observing that \( \eta_\theta \) is a \( \mathcal{O}_\theta \)-base for the free \( \mathcal{O}_\theta \)-module

\[ H^w_B(M_{\theta\mathbb{Q}} \times \mathbb{K}, \mathbb{Z}(w + l)) \otimes \mathcal{O}_K \mathcal{O}_K[1/D_\mathcal{K}] \]

of rank one, the result follows. \( \square \)

**Corollary 3.6** The submodule \( \mathcal{R}_\theta \) defined above satisfies the Beilinson conjecture inside the \( p \)-part of the Tamagawa number conjecture for \( p \nmid D_K \), that is \( \mathcal{R}_\theta \) satisfies the following conditions:

1. The map \( r_\theta \otimes \mathbb{R} \) is an isomorphism when restricted to \( \mathcal{R}_\theta \otimes \mathbb{R} \).
2. \( \dim_{\mathbb{Q}}(H^w_B(M_{\theta\mathbb{Q}}, \mathbb{Q}(w + l))) = \text{ord}_{s=-l} L_S(M_{\theta\mathbb{Q}}, s) = 2. \)
3. We have the following equality

\[ r_D(\det_{\mathbb{Z}[1/D_\mathcal{K}]})|_{\mathcal{R}_\theta \otimes \mathcal{O}_K \mathcal{O}_K[1/D_\mathcal{K}])} = L_S^*(M_{\theta\mathbb{Q}}, -l)\det_{\mathbb{Z}[1/D_\mathcal{K}]}(H^w_B(M_\theta, \mathbb{Z}(w + l)) \otimes \mathcal{O}_K \mathcal{O}_K[1/D]) \]

where \( L_S^*(M_{\theta\mathbb{Q}}, -l) \) means \( \lim_{s \to -l} L_S^*(M_{\theta\mathbb{Q}}, s)/(s+l)^2 \) (this makes sense by using theorem 2.4 and theorem 3.1).

**Proof.** The first and the second conditions are clear for the dimensions of the spaces involved in the Deligne regulator map, and the theorem 3.5. The third condition comes from the previous theorem using the fact that, if we multiply an \( \mathcal{O}_\theta \)-module with an element \( L_S^*(\overline{\psi_\theta}, -l) \) in \( \mathcal{O}_\theta \otimes \mathbb{R} \), the determinant is multiplied by the norm


Using theorem 2.4, we obtain that this is equal to \( L_S^*(M_{\theta\mathbb{Q}}, -l) \). \( \square \)
4. IWASAWA THEORY

We suppose once and for all that $p \nmid \#O_K$ and $p \nmid N_{K/Q}(l)$, (in particular $p \nmid D_K$).

To simplify the notation, we will denote in the following by

$$M_{\mathbb{Q}_p}(w + l + 1) = \epsilon_\theta \otimes^w (H^1_{et}(E \times_K \bar{K}, \mathbb{Z}_p(1))(l + 1))$$

the $p$-adic lattice for the $p$-adic realization of $M_E(w + l + 1)$.

Let $K_n := K(E[p^{n+1}])$ be the field of definition of the $p^{n+1}$-torsion points of $E$, $O_n$ its ring of integers and let $K_\infty := \lim \ker K_n$ be its direct limit. Denote by $O_n$ the ring of integers of $K_n$ (respectively $O_\infty$). We know that $\Delta := \text{Gal}(K_0/K)$ has order prime to $p$ and $\Gamma := \text{Gal}(K_\infty/K_0)$ is isomorphic to $\mathbb{Z}_p^\times$.

Let $\mathcal{O}$ be the Galois group $\text{Gal}(K_\infty/K)$; then $\mathcal{O} \cong \Delta \times \Gamma$.

We use now the notations on Iwasawa theory for imaginary quadratic fields used in [22, §2.1] but with a different definition of elliptic units.

Let us define the elliptic units $C_{n, f_\theta}$ in $K_n$ which are more convenient for us.

For every ideal $a$ of $K$ prime to $6$ we can define a theta function

$$\theta_a : E \setminus \ker([a]) \longrightarrow \mathbb{C}$$

which has divisor $N(a)(e) - \ker([a])$ (for the precise definition see [22, Theorem 4.2.2]).

The function $\theta_a(z)$ is in fact a $12$-th root of the function defined in [9, II.2.4]. Let $g$ be a fixed ideal of $O_K$ such that $O_K^* \rightarrow (O_K/g)^*$ is injective, and suppose that $g$ divides the conductor $f$ of the elliptic curve $E$. Let’s denote by $t_g$ a generator for the $E[g]$-torsion points as $O_K$-module, and let $a$ be an ideal prime to $6g$.  

**Definition 4.1** Let $C_{n, g}$ be the subgroup of units generated over $\mathbb{Z}[\text{Gal}(K_n/K)]$ by

$$\prod_{\sigma \in \text{Gal}(K(g)/K)} \theta_a(t_g^n + h_n),$$

where $a$ runs through all ideals prime to $6g$, $K(g)$ is the ray class field defined by $g$ and $h_n$ is a primitive $p^{n+1}$-torsion point (i.e. a generator of the $p^{n+1}$-torsion points of $E$ as $O_K$-module). Define the group of elliptic units of $K_n$ as

$$C_{n, g} := \mu_\infty(K_n)C_{n, g},$$

where $\mu_\infty(K_n)$ denotes the roots of unity in $K_n$.

Denote by $\overline{C}_{n, g}$ the closure in the local units $\mathcal{U}_n^p$ where $\mathcal{U}_n^p$ is the group of local units of $K_n \otimes_K K_p$ which are congruent to $1$ modulo the primes above $p$ where $p$ is a prime of $K$ above $p$. Define $\overline{C}_{\infty, g} := \lim C_{n, g}$, and $\overline{U}_n^p := \lim U_n^p$ where the limit is taken with respect to the norm maps. Define also $\mathcal{U}_\infty$ by $\mathcal{U}_\infty^p \times \mathcal{U}_\infty^p$ if $p = pp^*$ splits, and if $p$ inert or ramified by $\mathcal{U}_\infty^p$. Let $\mathcal{Y}_n$ be the $p$-adic completion of $(K_n \otimes \mathbb{Q}_p)^*$ and $\mathcal{Y}_\infty := \lim \mathcal{Y}_n$.

Let us once and for all to specialize the elliptic units to $g = f_\theta$.

Recall that $S$ denotes the set of primes of $K$ which divide $f_\theta$ or $p$, and that $S'$ denotes the set of primes of $K$ which divide $p$ or the conductor $f$ of the elliptic curve $E$. Denote $O_S := O_K[1/S]$ and $O_p := O_K \otimes \mathbb{Z}_p$.

We are going to define a map in the spirit of Soulé:

$$(\text{Soul})_p : \overline{C}_{\infty, f_\theta} \otimes_{\mathbb{Z}_p} M_{\mathbb{Q}_p}(w + l) \rightarrow H^1(O_S, M_{\mathbb{Q}_p}(w + l + 1)),$$

observe that $M_{\mathbb{Q}_p}(w + l)$ is unramified outside $S$, thus $H^1(O_S, M_{\mathbb{Q}_p}(w + l + 1))$ makes sense.

Using the definition of $M_{\mathbb{Q}_p}(w)(l + 1)$, we have that

$$H^1(O_S, M_{\mathbb{Q}_p}(w + l + 1)) = \lim \frac{H^1(O_S, (\epsilon_\theta \otimes^w E[p^{l+1}]))(l + 1)}{.}$$
Define \((\text{Soul})_p\) in the following way. Given \((\theta_v, a)\), a norm compatible system of elliptic units and an element \((t_r)_r\) of \(\lim \left\langle \phi_\theta \left( \bigotimes^w E[p^{r+1}] \right) \right\rangle(l)\), we define
\[
(\text{Soul})_p((\theta_v \otimes t_r)_r) := (N_{K_r/K}(\theta_r \otimes t_r))_r.
\]

It is well defined: \(\theta_v \otimes t_r\) is an element in
\[
\mathcal{O}_{r,S}^{\ast}/(\mathcal{O}_{r,S})^{p+1} \otimes (e_\theta(\bigotimes^w E[p^{r+1}])(l)) \subset H^1(\mathcal{O}_{r,S}, (e_\theta(\bigotimes^w E[p^{r+1}])(l+1))
\]
where \(\mathcal{O}_{r,S}\) is the ring of integers of \(K_r\) inverting the primes above \(p\). We denote the norm map in cohomology by and by Soulé’s Lemma 1.4 [26] one gets an element in \(H^1(\mathcal{O}_S, M_{\text{gZ}_p}(w + l + 1))\). The map \((\text{Soul})_p\) factors thought the coinvariants, denoted by \((\mathcal{C}_{\infty,f_\theta} \otimes M_{\text{gZ}_p}(w + l + 1))\).

**Definition 4.2** The Soulé elliptic elements are the elements in the image of the map
\[
(\text{Soul})_p : (\mathcal{C}_{\infty,f_\theta} \otimes M_{\text{gZ}_p}(w + l)) \to H^1(\mathcal{O}_S, M_{\text{gZ}_p}(w + l + 1))
\]
where \(G = \text{Gal}(K(E[p^{\infty}])/K)\).

We consider in the following the representation \(\chi\) of the group \(\Delta\) given by the action of \(\Delta\) in \(H_{\text{mot}}(\mathcal{O}_S, M_{\text{gZ}_p}(w + l), \mathcal{O}_p)\).

We are only able to apply the techniques on Iwasawa theory of [22] for certain representations that we call good representation.

**Definition 4.3** We say that such representation \(\chi\) of the group \(\Delta\) is a good representation if it satisfies two conditions in Iwasawa theory about isomorphism between some concrete Iwasawa modules: (A) the Iwasawa main conjecture of Rubin [22, Theorem 2.1.3] but replacing the elliptic units module there with the elliptic module units \(\mathcal{C}_{\infty,f_\theta}\) and (B) from the inclusion \(\mathcal{U}_S \subset \mathcal{Y}_S\) we get that \(\mathcal{U}_\infty \cong \mathcal{Y}_\infty\) as Iwasawa modules for the Iwasawa ring \(\mathcal{C}_\infty\).

We observe that the elliptic units \(\mathcal{C}_{\infty,f_\theta}\), which are the ones that appear in [25] and [22], satisfies the theorem of Iwasawa main conjecture of [24] for any \(\Delta\)-representation under the hypothesis of the theorem in [24] (personal communication of Rubin).

When \(S = S'\) we have \(\mathcal{C}_{\infty,f_\theta} = \mathcal{C}_{\infty,f}\) [9, Proposition II.2.5], therefore the Iwasawa main conjecture (condition (A)) is true from Rubin’s theorem [24] [25] for \(p\) splits and for \(p\) inert when \(\chi\) is non trivial on the decomposition group \(\Delta_p\) of \(p\) in \(\Delta\).

Condition (B) is always true if \(p\) splits [22, Lemma 2.1.6]. If \(p\) is inert or ramified, the representation \(\chi\) satisfies condition (B) if \(Z_p[\Delta/\Delta_p]^\chi = 0\) (see [22, lemma 2.1.6]), moreover because \(p\) is a prime over which \(E\) has good reduction we have \(\Delta_p = \Delta\) [22, Lemma 2.2.9], thus condition (B) is true if \(\chi\) is non-trivial.

**Remark 4.4** We guess that the Iwasawa main conjecture [22, Theorem 2.1.3] is also true without the condition \(S = S'\) for our elliptic units and our character \(\chi\), so such a character should be good if it just verify condition (B). In the next section we prove that \(\mathcal{C}_{\infty,f_\theta}\) is a rank 1 Iwasawa module and we construct an Euler system there. Using this and [9, Lemma III.1.10], it should be possible to prove the Iwasawa main conjecture for \(\chi\) using the techniques in [25].

**Lemma 4.5** Suppose that \(\psi_\theta\) has infinite type \((a_\theta, b_\theta)\) with \(a_\theta \neq b_\theta \mod \# \mathcal{O}_K^\ast\). Suppose also that \(p\) splits, and that \((a_\theta - b_\theta, p - 1) = 1\). Then \(S = S'\).

**Proof.** Let \(v\) be a prime of \(K\) dividing \(f\). Let \(v_0\) be a prime of \(K_0\) dividing \(v\). Denote by \(\Delta_{v_0}\) the stabilizer of \(v_0\) in \(K_0\). We have then that \(I_{v_0} \subset \Delta_{v_0} \subset \Delta\) acts non-trivially in the Tate module \(T_v E\) via the Hecke character \(\varphi_p\). Hence, \(I_{v_0}\) acts on \(e_\theta(\bigotimes^w T_v E)\) via \(\psi_{\theta, p} I_{v_0} = \varphi^a - \varphi^b\), as \(I_{v_0}\) lies in the kernel of the \(p\)-adic cyclotomic character \((v\) is prime to \(p\)). Since \(p\) splits, we have that \(#\Delta = (p - 1)^2\), and since \((a_\theta - b_\theta, p - 1) = 1\), \(\varphi^a - \varphi^b\) acts non-trivial on \(I_{v_0}\). \(\square\)
Lemma 4.6 Suppose that \( p \) splits in \( K \) and suppose that \( p - 1 \nmid a_0 + l + 1 \) or \( p - 1 \nmid b_0 + l + 1 \) or \( p - 1 \nmid a_0 - b_0 \). Then \( \chi \), as \( \Delta \)-representation, is not the cyclotomic representation.

Proof. The character \( \chi \) is equal to \((\overline{\psi}_p \kappa^l)^{-1} \) where \( \kappa \) is the cyclotomic character. Since \( p \) is split in \( K \), we have that \( p = pp^* \), with \( p \neq p^* \). Let \( \Delta_p \) be the Galois group \( Gal(K(E[p])/K) \); it is a subgroup of the decomposition group since \( p \) is totally ramified in \( \Delta_p \). Observe that \( M_{\mathbb{Z}_p} \) has multiplication isomorphic to \( \mathcal{O}_K \otimes \mathbb{Z}_p \) and, as \( p \) splits, it decomposes in two idempotents. These idempotents decompose the Hecke character. Since we know that \( \ker \psi \) decomposes, it decomposes in two idempotents. These idempotents decompose the Hecke character. Using the same kind of argument for \( p^* \) instead of \( p \) we obtain a similar divisibility result but with \( a_0 \) instead of \( b_0 \). Thus, we obtain the cyclotomic character only in the case that \( p - 1 \nmid a_0 + l + 1 \) and \( p - 1 \nmid b_0 + l + 1 \).

Similar argument for \( \overline{\psi}_{\Omega_2} \) we obtain the same simultaneous arithmetic conditions, i.e. \( p - 1 \mid b_0 + l + 1 \) and \( p - 1 \mid a_0 + l + 1 \) in order to obtain the cyclotomic character. We refer to [13, p.220, pp.223-234] for more details on the characters \( \psi_{\Omega_1} \) and \( \psi_{\Omega_2} \).

Next theorem is the analog of [22, Theorem 2.2.12] in our situation.

Theorem 4.7 Suppose that \( p \) is an odd prime, prime to \( N_{K/Q} \) and to \#\( \mathcal{O}_K^\times \). Let the \( \Delta \)-representation \( \chi \) on \( \text{Hom}_{\mathbb{O}_p}(M_{\mathbb{Z}_p}(w + l), \mathcal{O}_p) \) be a good representation. Then the map \( (\text{Soul})_p \) induces an isomorphism of \( \mathcal{O}_p \)-modules

\[
\text{det}_{\mathcal{O}_p}(R\Gamma(G, H^0(K_{\infty} \otimes \mathbb{Q}_p, M_{\mathbb{Z}_p}(w + l + 1)))^* \cong \text{det}_{\mathcal{O}_p}(R\Gamma(K_{\infty} \otimes \mathbb{Q}_p, M_{\mathbb{Z}_p}(w + l + 1)))^{-1}
\]

Proof. In order to prove the theorem one can follow the same steps as in the proof of [22, Theorem 2.2.12], but with \( M_{\mathbb{Z}_p}(w) \) instead of \( T_pE \). The only results that need some work are [22, Proposition 2.2.13] and [22, Lemma 2.2.16] (see [2, §3.4] for a detailed proof). We will show next the necessary steps to prove these two theorems.

Let us prove the analog of [22, Proposition 2.2.13]:

\[
\text{det}_{\mathcal{O}_p}(R\Gamma(G, H^0(K_{\infty} \otimes \mathbb{Q}_p, M_{\mathbb{Z}_p}(w + l + 1)))^* \cong \mathcal{O}_p
\]

and

\[
\text{det}_{\mathcal{O}_p}(R\Gamma(G, H^0(M_{\mathbb{Z}_p}(w + l + 1)))^* \cong \mathcal{O}_p,
\]

where \( M' \) denotes \( \text{Hom}_{\mathbb{O}_p}(M, \mathcal{O}_p \otimes \mathbb{Q}_p/\mathbb{Z}_p) \) and \( M^* = \text{Hom}_{\mathbb{O}_p}(M, \mathcal{O}_p \otimes \mathbb{Q}_p/\mathbb{Z}_p) \).

It follows from [12, prop. 2.4.6], that the action of \( G \) on \( M_{\mathbb{Z}_p}(w + l) \) is via the character \( \overline{\psi}_{\theta,p} : G \to (\mathcal{O}_p \otimes \mathbb{Z}_p)^* \), multiplied by the \( l \)-th power of the \( p \)-adic cyclotomic character.

Then it induces a surjection of \( \mathcal{O}_p \)-modules \( \rho : \mathcal{O}_p[\Gamma] \to M_{\mathbb{Z}_p}(w + l) \) by the action described above. Thus \( \ker(\rho) \) is an ideal of height \( 2 \) because \( \Gamma \cong \mathbb{Z}_p^2 \). We know that \( \text{det}_R \) is determined by the ideals of height \( 1 \) for the ring \( R \) (cf. [19, 2.1.4]). We are going to show that this implies that

\[
(1) \quad \text{det}_{\mathcal{O}_p}(M_{\mathbb{Z}_p}(w + l) \otimes_{\mathcal{O}_p[\Gamma]} \mathcal{O}_p) \cong \mathcal{O}_p.
\]

In fact, since \( \Delta \) is finite and \( \mathcal{G} \cong \Gamma \times \Delta \), we have the isomorphism

\[
M_{\mathbb{Z}_p}(w + l) \otimes_{\mathcal{O}_p[\Gamma]} \mathcal{O}_p \cong (M_{\mathbb{Z}_p}(w + l))_\Delta \otimes_{\mathcal{O}_p[\Gamma]} \mathcal{O}_p.
\]

Since we know that \( \ker(\rho) \) has height \( 2 \), we have that \( \text{det}_{\mathcal{O}_p[\Gamma]}((M_{\mathbb{Z}_p}(w + l))_\Delta) \cong \mathcal{O}_p[\Gamma] \) and so \( \text{det}_{\mathcal{O}_p((M_{\mathbb{Z}_p}(w + l))_\Delta \otimes_{\mathcal{O}_p[\Gamma]} \mathcal{O}_p) \cong \mathcal{O}_p \). This shows (1). We conclude by using [22, 2.2.6].

Now, we show the analog of [22, Lemma 2.2.16]: the restriction map induces isomorphisms

\[
\text{det}_{\mathcal{O}_p}(H^0(\Delta, R\Gamma(\mathcal{O}_0, S_p, M_{\mathbb{Z}_p}(w + l + 1))) \cong
\]

\[
\text{det}_{\mathcal{O}_p}(H^0(\Delta, R\Gamma(\mathcal{O}_0, S_p, M_{\mathbb{Z}_p}(w + l + 1))).
\]
Let $\delta$ a projective limits of lisse sheaves [22, §T].

Elliptic curve over a base scheme have the connecting map $\delta$ ker over us consider the map multiplication by $p$.

Using this connecting morphism, we can express the Eisenstein classes explicitly.

Sheaf, which is an important key in his proof of the Tamagawa number conjecture.

To show this result, observe that $\theta$.

To show this consider the exact triangle $R\Gamma(O_{v_0}, M_{\mathbb{Z}_p}(w + l + 1)) \rightarrow R\Gamma(O_{v_0}, M_{\mathbb{Z}_p}(w + l + 1))$

where $O_{v_0}$ is the local ring at $v_0$ and $S_p$ is the set of places that divide $p$. Since $T_pE$ is unramified at the places of $K_0$ in $S \setminus S_p$, the same is true for $e_\theta(\otimes T_pE)(l + 1)$. By the purity theorems in étale cohomology we have that

$$R\Gamma_{k(v_0)}(O_{v_0}, M_{\mathbb{Z}_p}(w + l + 1)) \cong R\Gamma_{k(v_0)}(O_{v_0}, M_{\mathbb{Z}_p}(w + l))[-2].$$

It remains to prove only that

$$H^0(\Delta, \oplus_{v_0 \in S \setminus S_p} R\Gamma_{k(v_0)}(O_{v_0}, M_{\mathbb{Z}_p}(w + l))) = 0.$$ 

To show this result, observe that

$$H^1(k(v_0), M_{\mathbb{Z}_p}(w + l)) \cong M_{\mathbb{Z}_p}(w + l)_{\text{Gal}(k(v_0)/k(v_0))}$$

and $H^0(k(v_0), M_{\mathbb{Z}_p}(w + l)) = 0$ because $-w - 2l \leq -3$.

Now, let $v_0$ be a prime of $K_0$ dividing $v$ a prime of $K$ with $v|\mathfrak{p}$ and let $\Delta_{v_0}$ be the stabilizer of $v_0$. Since $1_{v_0} \subset \Delta_{v_0}^v$ acts non trivially on the coinvariants $M_{\mathbb{Z}_p}(w + l)_{\text{Gal}(k(v_0)/k(v_0))}$ because $v_0 \mid \mathfrak{p}$, there are no fix elements.

\section{The Comparison Between the Map $r_p$ and $(\text{Soul})_p$ in the Constructible $K$-elements}

Let’s start recalling the result of Kings on the specialization of the elliptic polylogarithm sheaf, which is an important key in his proof of the Tamagawa number conjecture.

Let $E$ be an elliptic curve over a base scheme $T$, and denote by $\pi : E \rightarrow T$ the structural morphism, which is proper and smooth. Consider $U = E \setminus e$, where $e$ is the zero section of $E$. Consider the elliptic polylogarithm sheaf $\text{Pol}_{\mathbb{Q}_p}$ on $U$, which is a lisse pro-sheaf (i.e. a projective limits of lisse sheaves) [22, §3.2].

Let $\mathcal{H}_{\mathbb{Q}_p} := H_{\text{cont}}(R^1\pi_*, \text{Pol}_{\mathbb{Q}_p}, \mathbb{Q}_p)$. Using $\text{Pol}_{\mathbb{Q}_p}$ one defines the $p$-adic Eisenstein classes associated to any integer $k$ and any $M$-torsion point $t$ in $E$ different from $e$ as some elements in $H^1(T, \text{Sym}^k \mathcal{H}_{\mathbb{Q}_p})$. The definition is extended by linearity to any divisor supported on $M$-torsion points ( [22, Def. 3.5.9]). The main part of the result of Kings is the explicit computation of these Eisenstein classes.

We are going to explain this result. Consider $H_n := \ker[p^n]$ as a scheme over $T$. Let us consider the map multiplication by $p^n$, $p_n : E_n \rightarrow E$, where $E_n$ is the elliptic curve $E$ over $T$ considered as a $H_n$-torsor over $E$. Consider the characteristic group $I[H_n] := \ker(p_n_*\mathbb{Z} \rightarrow \mathbb{Z})$, which is the characteristic group of a torus $T_{H_n}$. In this situation we have the connecting map $\delta$ from the Kummer exact sequence:

\begin{equation}
\delta : H^0(H_n, T_{H_n}) \rightarrow H^1(H_n, T_{H_n}[p^n]).
\end{equation}

Using this connecting morphism, we can express the Eisenstein classes explicitly.

\textbf{Theorem 5.1} (Kings, theorem 4.2.9 in [22]) \textit{Let $p$ be a prime number, and let $E$ be an elliptic curve over a base scheme $T$ where $p$ is invertible.}

\textit{Let $\beta$ be any divisor in $E$ of the form $\beta := \sum_{e \in E[M](T)\setminus e} n_e(t)$,}

\textit{$n_\ell$ an integer and consider $[a] : E \rightarrow E$ any isogeny with degree prime to $Mp$.}

\textit{Then, for any $m > 0$, the $p$-adic Eisenstein class $Na(a^\otimes m Na - 1)(\beta^* \text{Pol}_{\mathbb{Q}_p})^m \in H^1(T, \text{Sym}^m \mathcal{H}_{\mathbb{Q}_p}(1))$}
is given by  
\[ \pm \frac{1}{m!} (\delta \sum_{t \in E[M]\{\emptyset\}} n_t \sum_{[p^n] t = \gamma} \theta_e (-t_\gamma) t_\gamma \otimes m) \]  
where \( t_\gamma \) is the projection of \( t_\gamma \) to \( E[p^n] \) and \( \delta \) is the Sym-extension of the boundary map \( H^0(H_n, T_{H_n}) \to H^1(H_n, T_{H_n}[p^n]) \) where \( H_n := \ker[p^n] \) is considered as a scheme over \( T \) and \( T_{H_n} \) is the torus with character group \( I[H_n] := \ker(p_n, \mathbb{Z} \to \mathbb{Z}) \).

The following result relates the image of \( E_M^n(\beta) \) by the Soulé regulator map with the polylogarithmic sheaf, where \( E_M^n \) is the Eisenstein symbol \([22, \text{§1.2.1}].\)

**Theorem 5.2** Under the same hypothesis of Theorem 5.1, let \( \beta \) be as in the previous theorem. Then  
\[ r_p(E_M^n(\beta)) = -M^{2m} (\beta^* Pol_{Q_p})^m \]  
in \( H^1(T, Sym^m \mathcal{H}_{Q_p}(1)) \).

**Proof.** The same proof of \([22, \text{Theorem 1.2.5}]\) with \( m \) instead of \( 2k + 1 \) works. See also \([2, \text{proof Theorem 3.5.2}]\).

We are going to apply these results to the divisor \( \beta_\theta := N_{K(\ell)/K}(1) \), where \( \ell := \Omega f^{-1} \) is a \( f_\theta \)-torsion point. Take \( M = N_{K/\mathbb{Q}} \theta, m = w + 2l, T = \mathcal{O}_S = \mathcal{H}_{Q_p} = T_p E \otimes \mathbb{Q}_p \), using the notations of the theorem 5.1. Let \( \mathfrak{a} \subset \mathcal{O}_K \) be an ideal prime to \( 6p \ell \), and consider the isogeny given by \( \varphi(\mathfrak{a}) \). Let \( \theta_\mathfrak{a} \) be the classical theta function.

To simplify the notation, define for any \( \tilde{t}_r \in E[p^r] \)  
\[ \gamma(\tilde{t}_r)^m := < \tilde{t}_r, \sqrt{d_K t_\gamma} > \otimes m \]  
where <, > denotes the Weil pairing. Our objective is the computation of  
\[ K_M \cdot E_M^{w+2l}(\beta_\theta). \]

Remember that we are under the restriction \( a_\theta \neq b_\theta \text{ mod } (\# \mathcal{O}_K) \).

We consider the following commutative diagram \([8, (2.8)]\)

\[
\begin{array}{ccc}
H^2_{M} (Sym^{2l+w} H^{1}(E, \mathbb{Q}(w+2l+1))) & \xrightarrow{(\Delta \in M)^{\times} \times \text{id}^*} & H^2_{M} (Sym^{2l+w} H^{1}(E^{2l+w}, \mathbb{Q}(2l+w+1))) \\
K_M \cdot (M_{\mathfrak{a}}, \mathbb{Q}(w+l+1)) & \xrightarrow{\psi_{\mathfrak{a}}} & H^2_{M} (Sym^{2l+w} H^{1}(\mathbb{Q}(l+w+1)),
\end{array}
\]

where \( pr \) is the projection in the last \( w \) components and \( \Delta_{CM} : E \to E \times E \) is given by \( e \mapsto (e, \sqrt{d_K e}) \). We obtain a map in Galois cohomology given by  
\[ H^1(\mathcal{O}_S, Sym^{2l+w} \mathcal{H}_{Q_p}(1)) \to H^1(\mathcal{O}_S, (e_\mathfrak{a} Sym^w \mathcal{H}_{Q_p})(l+1)) = H^1(\mathcal{O}_S, M_{\mathfrak{a}} \mathcal{H}_{Q_p}(w+l+1)) \]

such that  
\[ K_M (\varphi_{\mathfrak{a}}(\mathfrak{o}) \otimes Sym^{2l+w} \mathcal{H}_{Q_p}(1)) = e_\mathfrak{a} (\otimes^w \varphi_{\mathfrak{a}}(\mathfrak{a})) N_{\mathfrak{a}} Sym^w \mathcal{H}_{Q_p}(l+1). \]

**Theorem 5.3** Let \( \mathfrak{p} \) be a prime number such that \( \mathfrak{p} \nmid 6N_{K/\mathbb{Q}}(1) \). Let \( \theta_\mathfrak{p} \) be an idempotent with infinity type \((a_\mathfrak{p}, b_\mathfrak{p})\) which \( a_\mathfrak{p} \neq b_\mathfrak{p} \text{ mod } (\# \mathcal{O}_K) \). For a \( \mathfrak{p}^\mathfrak{l} N_{K/\mathbb{Q}}(1) \)-torsion point \( t_\mathfrak{r} \), denote by \( \tilde{t}_r \) its projection to \( E[p^r] \). Then, if \( t = \Omega f^{-1} \), we have the following equality  
\[ N \mathfrak{a} (\psi_{\mathfrak{a}, \mathfrak{p}}(\mathfrak{a}) N_{\mathfrak{a}}^{l+1} - 1) r_p(\xi_{\mathfrak{a}, \mathfrak{r}}) = \]

\[
\frac{(-1)^l L_p(\psi_{\mathfrak{a}, \mathfrak{r}} - b) N_{\mathfrak{a}}^{l+2w} \Phi(\mathfrak{f})}{2^{l-1} \psi_{\mathfrak{a}}(\mathfrak{p}) \Phi(\mathfrak{f})} \delta N_{K(\ell)/K} \sum_{p^l t_\gamma = t} \theta_e (-t_\gamma) \otimes e_\mathfrak{a} (\otimes^w \tilde{t}_r) \otimes \gamma(\tilde{t}_r)^l.\]
Using theorems 3.5, 5.1 and 5.2, we have that

\[
\psi_{\theta} \in \mathcal{K}_{\mathcal{M}}(E_{2l+w}(\beta))
\]

\[
= \frac{(-1)^l(2l + w)!L_p(\psi_{\theta}, -l)}{2l-1N_{T_2/q}\psi_p(\theta_0)\Phi(\bar{f})} \mathcal{K}_{\mathcal{M}}(\beta^* Pol_{Q_p})^{w+2l}.
\]

We have that

\[
\mathcal{K}_{\mathcal{M}}(\tilde{t}_r \otimes 2l + w) = e_\sigma(\otimes^w \tilde{t}_r) \otimes \gamma(\tilde{t}_r)^l.
\]

Finally, applying Kings’ theorem 5.1, we obtain the desired identity.

We want to rewrite the previous formula in terms of the norm map of the extension \(K(f_0)K(E[\mathfrak{p}^n])/K\). We will work with \(f\) instead of \(f_0\) since then we can use that \(K(E[\mathfrak{p}^n]) = K(\mathfrak{p}^n)\), the ray class field, because \(f\) is the conductor of \(E\) and divides the ideal \(\mathfrak{p}^n\) [9, II, Prop.1.6].

Fix a prime \(p\) of \(K\) where \(E\) has good reduction, and take \(\pi = \varphi(p)\). Denote by

\[
H_{p, r,t} := \{t_r \in E[p^r\mathfrak{f}]|\pi^r t_r = t\}.
\]

Let \(\sigma_p\) be the Frobenius at \(p\) in \(Gal(K(f)/K)\), we write \(t_r = (\tilde{t}_r, \pi^{-r}t) \in E[p^r]\] = \(E[p^r] \oplus E[t]\), where \(\pi^{-r}t\) means \(\pi^{-r}t\).

Consider the filtration of \(F^p_{r,s}\) defined by

\[
F^r_{r,t} := \{t_r \in H_{p, r,t} | \pi^{r-i} \tilde{t}_r = 0\}.
\]

**Theorem 5.4** Let \(p\) be as above and \(t_r = (\tilde{t}_r, \pi^{-r}t) \in F^0_{r,s} \setminus F^1_{r,s}\). Suppose that \(O_K^* \to (O_K/f_0)^*\) is injective. Denote the Euler factor of the Hecke character \(\psi_{\theta}\) at \(p\) evaluated at \(-l\) by \(L_p(\psi_{\theta}, -l)\). Then

\[
L_p(\psi_{\theta}, -l)^{-1} \left( N_{K(\mathfrak{p}^r)} K \sum_{s_r \in H_{p, r,t}^r} \theta_a(-s_r) \otimes e_\sigma(\otimes^w \tilde{t}_r) \otimes \gamma(\tilde{t}_r)^l \right)_{r,s} =
\]

\[
(N_{K(\mathfrak{p}^r)} K (\theta_a(-t_r) \otimes e_\sigma(\otimes^w \tilde{t}_r) \otimes \gamma(\tilde{t}_r)^{l}))_{r,s},
\]

in \(H^1(O_S, e_\sigma(T_p E(1))(l) \otimes \mathbb{Q}_p)\) for all \(r\) relatively prime to \(p\).

**Proof.** The identification \(Hom_{O_S}(T_p E, O_p) \cong T_p E(-1)\) is via the conjugate linear \(O_p\)-action on the right side. Hence \(\varphi(p)\) acts trivially on \(L_p(\psi_{\theta}, -l)\). We have the equality

\[
\left( \psi_{\theta}(p)/N_p(p)^{-1}\right) N_{K(\mathfrak{p}^r)/K(\mathfrak{p}^r-1)}(\theta_a(-t_r) \otimes e_\sigma(\otimes^w \tilde{t}_r) \otimes \gamma(\tilde{t}_r)^{l}) =
\]

\[
N_{K(\mathfrak{p}^r)/K(\mathfrak{p}^r-1)}(\theta_a(-t_r) \otimes e_\sigma(\otimes^w \tilde{t}_r) \otimes \gamma(\tilde{t}_r)^{l}) =
\]

\[
N_{K(\mathfrak{p}^r)/K(\mathfrak{p}^r-1)}(\theta_a(-t_r) \otimes e_\sigma(\otimes^w \tilde{t}_r) \otimes \gamma(\tilde{t}_r)^{l}) =
\]

\[
\theta_a(-t_{r-i}, \pi^{-r}t) \otimes e_\sigma(\otimes^w \tilde{t}_{r-i}) \otimes \gamma(\tilde{t}_{r-i})^{l},
\]

where the last equality uses the distribution relation for \(\theta_a\) [9, II 2.5].

The Galois group of \(K(\mathfrak{p}^{-1})/K(\mathfrak{f})\) acts simply transitively on \(F^1_{r,s} \setminus F^1_{r,s+1}\). We get that

\[
\left( \psi_{\theta}(p)/N_p(p)^{-1}\right) N_{K(\mathfrak{p}^r)/K(\mathfrak{p}^r-1)}(\theta_a(-t_r) \otimes e_\sigma(\otimes^w \tilde{t}_r) \otimes \gamma(\tilde{t}_r)^{l}) =
\]

\[
\sum_{t_{r-i} \in F^1_{r,s} \setminus F^1_{r,s+1}} \theta_a(-t_{r-i}, \pi^{-r}t) \otimes e_\sigma(\otimes^w \tilde{t}_{r-i}) \otimes \gamma(\tilde{t}_{r-i})^{l}.
\]

We know by [9, Prop. II.2.4 ii)] that we have the equality \(\theta_a(-t_{r-i}, \pi^{-r}t) = \theta_a(-t_{r-i}, \pi^{-r}t)\) with \(\sigma_p\) is the Frobenius at \(p\) in the Galois group of \(K(\mathfrak{f})/K\). This and the fact that \(N_{K(\mathfrak{f})/K}\) is the sum over all Galois translates, which act trivially on \(t_{r-i}\), gives that

\[
\left( \psi_{\theta}(p)/N_p(p)^{-1}\right) N_{K(\mathfrak{p}^r)/K(\mathfrak{p}^r-1)}(\theta_a(-t_r) \otimes e_\sigma(\otimes^w \tilde{t}_r) \otimes \gamma(\tilde{t}_r)^{l}) =
\]
Suppose that \( \varphi \) obtain that \( u \) on \( F_{r} \) and \( K \). Adding these equalities with respect to \( w \) and with \( O(\Phi(f, p)) = 1 \), hence the result for the type \((w, 0)\). For the type \((0, w)\) the proof is similar but with \( \varphi \) instead of \( \varphi \).

**Lemma 5.5** Suppose that \( \theta \) has infinity type \((w, 0)\) or \((0, w)\) and \((#O_{K}^\ast, w) = 1\). Then \( O_{K}^\ast \to (O_{K}/I_{0})^\ast \) is injective.

Proof. Let \( u \) be an element in \( O_{K}^\ast \), \( u \neq 1 \) and consider the idèle defined by \( u_{\infty} = 1 \) and \( x_{p} = u \) at all finite places \( p \) of \( K \). Then as complex Hecke character \( \varphi_{w}(x) = \varphi_{w}(u^{-1}x) = u_{w} \neq 1 \) if \((w, #O_{K}^\ast) = 1\). So, by definition of the conductor of \( \psi_{0} \), we obtain that \( u \neq 1(mod f_{0}) \), hence the result for \((w, 0)\). For the type \((0, w)\) the proof is similar but with \( \varphi \) instead of \( \varphi \).

**Corollary 5.6** Suppose \( p \nmid 6N_{K/Q}(f) \), \( O_{K}^\ast \to (O_{K}/I_{0})^\ast \) is injective and the hypothesis of theorem 5.3. Then

\[
\text{Proof. If } p \text{ is inert or prime the first equality is deduced from the previous theorem. If } p \text{ split, it decomposes in a } p \text{ part and a } p^\ast \text{ part. Putting together the previous result with } p \text{ and } p^\ast, \text{ we have the first equality.}
\]

To show the second equality, consider

\[
N_{K(E[p^r])}/K_{(f)} \theta_{\alpha}(-t_{r}) = \prod_{\sigma \in \text{Gal}(K(f)/K(E[p^r]))} \theta_{\alpha}(-t_{r})^{\sigma}
\]

because \( K(f) \) is disjoint with \( K(p^r) \) over \( K \) since \( K = K(1) \), and \( K(f) = K(E[f]) \) is disjoint with \( K(E[p^r]) \) over \( K \). Moreover, since \( \theta_{\alpha}(-t_{r}) \in K(f)K(p^r) = K(fp^r) \) and \((f, p) = 1\), we have that the norm is equal to

\[
\prod_{\tau \in \text{Gal}(K(fp^r)/K(fp^r))} \theta_{\alpha}(-t_{r})^{r}.
\]

But \( \theta_{\alpha}(-t_{r}) \in K(fp^r) \) because \( -t_{r} \) is a point of \( fp^r \)-torsion. To obtain the second equality we need to show that \( \Phi_{K(E[p^r])}/K_{(f)} \) is Galois extension. Observe that \( \Phi_{K(E[p^r])/K_{(f)}} = \Phi_{K(E[p^r])/K(f)} \), where \( \Phi_{K(E[p^r])/K(f)} \) is the representation of \( \Delta \) on \( H_{0}(\mathcal{O}_{K}, \mathcal{O}_{K}(w + l)) \), that we suppose a good representation.
We suppose from now on that the natural map $\mathcal{O}_K^* \to (\mathcal{O}_K/f_\theta)^*$ is injective, assumption also needed to define our elliptic units.

**Proposition 5.7** Consider $p \mid 6N_K/\mathbb{Q}(l)$ and $a$ an ideal in $\mathcal{O}_p$, which is prime to $6p$ and such that $\psi_{a,p}(a)Na^{l+1} \not\equiv 1 \pmod{p}$. Then the $\mathcal{O}_p[[\Gamma]]$-module

$$\mathcal{T}_\infty^{\chi} \otimes_{\mathcal{O}_p} M_{\mathbb{Z}_p}(w + l)$$

is generated by $(\theta_a(t_r) \otimes e_\theta(\otimes^w \bar{t}_r) \otimes \gamma(\bar{t}_r)^{l})$, where $t_r$ is a primitive $p^r f_\theta$-division point.

**Remark 5.8** The existence of an ideal $a$ satisfying the conditions of the proposition 5.7 is equivalent to the condition that the $\Delta$-representation $\chi$ is not the cyclotomic representation $\chi_p$.

**Proof.** Observe first that $e_\theta(\bar{t}_r) \otimes \gamma(\bar{t}_r)$ generates $M_{\mathbb{Z}_p}(w + l)$, because $M_{\mathbb{Z}_p}(w)$ is one dimensional and concerning how it generates $\mathbb{Z}_p(l)$ use the same proof in [22, p.623].

Remember that we have an inclusion of $\mathcal{T}_\infty^{\chi}$ in $\mathcal{U}_\infty$ the local units Iwasawa module, which is torsion free [25, Prop.11.4], thus $\mathcal{T}_\infty^{\chi}$ is a torsion free $\mathcal{O}_p[[\Gamma]]$-module. Is enough to show that is one dimensional. Let $b$ be another ideal prime to $6p$. Take $\sigma_a = [a, K_n/K]$ and $\sigma_b = [b, K_n/K]$. Then, by the properties of the theta function, we have that

$$(\sigma_a - \psi_{a,p}(a)Na^{l+1})(\theta_b(t_n) \otimes e_\theta(\otimes^w \bar{t}_n) \otimes \gamma(\bar{t}_n)^{l}) =$$

$$\psi_{a,p}(a)Na^{l}(\theta_b(t_n)^{\sigma_a - Na} \otimes e_\theta(\otimes^w \bar{t}_n) \otimes \gamma(\bar{t}_n)^{l}) =$$

$$\psi_{a,p}(a)N_{\theta,p}(a)Na^{l}(\theta_b(t_n)^{\sigma_a - Nb} \otimes e_\theta(\otimes^w \bar{t}_n) \otimes \gamma(\bar{t}_n)^{l}).$$

Then, it is enough show that $(\sigma_a - \psi_{a,p}(a)Na^{l+1})$ is invertible in $\mathcal{O}_p[[\Gamma]]$. But the element $\sigma_a$ corresponds to $1$ on $\mathcal{O}_p/p$ and thus $\sigma_a - \psi_{a}(a)Na^{l+1}$ is invertible in $\mathcal{O}_p[[\Gamma]]$ because $1 \not\equiv \psi_{a}(a)Na^{l+1} \pmod{p}$. 

**Corollary 5.9** Assume that $p \mid 6N_K/\mathbb{Q}$. Then the image of $\mathcal{R}_\theta$ by $r_p$ in the cohomology group $H^1(\mathcal{O}_S, M_{\mathbb{Z}_p}(w + l + 1)) \otimes \mathbb{Q}_p$ coincides with

$$(\text{Soul})_p((\mathcal{T}_\infty^{\chi, f_\theta} \otimes M_{\mathbb{Z}_p}(w + l))_{\Gamma}).$$

**Proof.** As

$$N_{\theta}^{[l] + 2w/2^{l-1}} \psi_{\rho}(\rho_\theta)$$

is prime to $p$, it follows from the definition of $(\text{Soul})_p$ and Corollary 5.6. 

**Lemma 5.10** The canonical map

$$\mathcal{T}_\infty^{\chi, f_\theta} \otimes M_{\mathbb{Z}_p}(w + l) \otimes \mathcal{O}_p[[\mathcal{G}]] \mathcal{O}_p \to (\mathcal{T}_\infty^{\chi, f_\theta} \otimes M_{\mathbb{Z}_p}(w + l))_{\Gamma} \cong (\mathcal{T}_\infty^{\chi} \otimes M_{\mathbb{Z}_p}(w + l))_{\Gamma} \cong \mathcal{O}_p.$$  

**Proof.** We observe that the proof of proposition 5.7 shows that $\mathcal{T}_\infty^{\chi, f_\theta} \cong \mathcal{O}_p[[\Gamma]]$ is a free $\mathcal{O}_p[[\Gamma]]$-module of rank $1$. This implies, as in [22, lemma 5.2.3], that $(\mathcal{T}_\infty^{\chi} \otimes M_{\mathbb{Z}_p}(w + l))_{\Gamma} \cong \mathcal{O}_p$. The claim follows since the previous module is induced and hence the higher Tor-terms vanish. 

**Corollary 5.11** The map

$$\mathcal{R}_\theta \otimes \mathbb{Z}_p \to R\Gamma(\mathcal{O}_S, M_{\mathbb{Z}_p}(w + l + 1) \otimes \mathbb{Q}_p)[1]$$

induced by $r_p$, gives an isomorphism

$$\text{det}_{\mathcal{O}_p} \mathcal{R}_\theta \cong \text{det}_{\mathcal{O}_p} R\Gamma(\mathcal{O}_S, M_{\mathbb{Z}_p}(w + l + 1))^{-1}.$$
Before stating the next theorem, let us recall all the hypothesis we used during the paper and that we will need:

(⋄ ⋄ ⋄) Let $p$ be a fix prime such that $p \nmid 6N_{K/Q}$ (hence, in particular $p \nmid \# O_K^*$), and $p \nmid D_K$). Consider $l$ a non-negative integer. Let $(a_\theta, b_\theta)$ be the infinite type of $\psi_\theta$, with $a_\theta, b_\theta$ non-negative integers with $w = a_\theta + b_\theta \geq 1$ such that $a_\theta \equiv b_\theta \mod \# O_K^*$ and $-w - 2l \leq -3$. Assume that $O_K^* \to (O_K/p)^*$ is injective. Suppose moreover that the representation $\chi$ of $\Delta$ in $\text{Hom}_{\mathcal{O}_p}(H^2_{\text{et}}(M_{\theta} \times_K K, \mathbb{Z}_p(w + l)), \mathcal{O}_p)$ is a good representation (see the definition in 4.3) which is not equal to $\Delta$-representation to the cyclotomic representation.

**Theorem 5.12** Under the hypothesis $(⋄ ⋄ ⋄)$ above, there is an $O_K$-submodule $\mathcal{R}_\theta \subset H^w_{\text{et}}(M_{\mathbb{Q}}, \mathbb{Q}(w + l + 1))$ of rank 1 such that:

1. $\det_{O_K[1/D_K]}(r_D(\mathcal{R}_\theta \otimes O_K O_K[1/D_K])) \cong \left( \frac{\mathbb{L}(\theta)}{\theta} \right) l = \lim_{s \to -l} L_S(\overline{\theta}, s)$.

2. The map $r_p$ induces an isomorphism
\[
det_{\mathcal{Q}_p} \cong \det_{\mathcal{Q}_p} (R\Gamma(O_K[1/S], M_{\mathbb{Q}}(w + l + 1)))^{-1}.
\]

Here $L_S(\overline{\theta}, s) = \lim_{s \to -l} L_S(\overline{\theta}, s)$, and $S$ is the set of primes of $K$ dividing $p$ and the ones dividing $\mathcal{I}_\theta$.

Moreover, if $r_p$ is injective on $\mathcal{R}_\theta$, the second part can be written as
\[
det_{\mathcal{O}_K \otimes \mathbb{Q}_p} (H^1(O_K[1/S], M_{\mathbb{Q}}(w + l + 1))/r_p(\mathcal{R}_\theta)) \cong \det_{\mathcal{O}_K \otimes \mathbb{Q}_p} H^2(O_K[1/S], M_{\mathbb{Q}}(w + l + 1)).
\]

Proof. It is a direct consequence of the theorem 3.5 and the above corollary 5.11. □

After taking the norm $N_{K/Q}$, we obtain the following result.

**Theorem 5.13** Under the assumption $(⋄ ⋄ ⋄)$ above, there is a $\mathbb{Z}$-submodule $\mathcal{R}_\theta$ in $H^w_{\text{et}}(M_{\mathbb{Q}}, \mathbb{Q}(w + l + 1))$ of rank 2 such that:

1. The map $r_p \otimes \mathbb{R}$ is an isomorphism restricted to $\mathcal{R}_\theta \otimes \mathbb{R}$.
2. $\dim_{\mathbb{Q}_p}(H^w_{\text{et}}(M_{\mathbb{Q}}, \mathbb{Q}(w + l))) = \text{ord}_{s = -l} L_S(M_{\mathbb{Q}}, s) = 2$.
3. We have the equality
\[
r_D(\det_{\mathbb{Z}[1/D_K]}(\mathcal{R}_\theta \otimes O_K O_K[1/D_K])) = \left( \frac{\mathbb{L}(\theta)}{\theta} \right) l = \lim_{s \to -l} L_S(M_{\mathbb{Q}}, s)
\]
where
\[
\text{and } S \text{ is the set of places of } K \text{ that divides } p \text{ and the places dividing the conductor } \mathcal{I}_\theta.
\]
4. We have that
\[
det_{\mathbb{Q}_p}(\mathcal{R}_\theta \otimes \mathbb{Q}_p) = \det_{\mathbb{Q}_p} (R\Gamma(O_K[1/S], M_{\mathbb{Q}}(w + l + 1)))^{-1}.
\]
If $r_p$ is injective on $\mathcal{R}_\theta$, then $r_p(\det_{\mathbb{Q}_p}(\mathcal{R}_\theta \otimes \mathbb{Q}_p))$ is a basis of the $\mathbb{Z}_p$-lattice
\[
det_{\mathbb{Z}_p}(R\Gamma(O_K[1/S], M_{\mathbb{Q}_p}(w + l + 1)))^{-1}
\]
\[
\subset \det_{\mathbb{Q}_p} (R\Gamma(O_K[1/S], M_{\mathbb{Q}_p}(w + l + 1) \otimes \mathbb{Q})(-1)).
\]

**Remark 5.14** Theorems 5.12 and 5.13 imply the weak $p$-part of the Tamagawa number conjecture for Hecke characters [19] for $K$ or $\mathbb{Q}$ coefficients respectively, up to the finiteness of $H^2_{\text{et}} := H^2(O[1/S], M_{\mathbb{Q}_p}(w + l + 1))$ and the bijectivity of the Soulé regulator map $r_p$. Concerning these requirements, we have the following.
(1) If \( p \) is a regular prime for the field \( K(E[p]) \), then \( H^2_p \) is finite [3]. Moreover without any assumption, one obtains that for almost all \( l \) this Galois cohomology group is finite [21, Theorem 12.4] or [4].

(2) About the bijectivity of the Soulé regulator map observe if \( H^2_p \) is finite, similar arguments as in [22, §5.2.2] implies the injectivity for \((Soul)_p\) and therefore \( r_p \) is injective on \( R_\theta \otimes \mathbb{Q}_p \) by corollary 5.9. Therefore \( r_p \) restricted to \( R_\theta \otimes \mathbb{Q}_p \) is an isomorphism [18, cor. 1].

Therefore for regular primes \( p \), we obtain in full generality the weak \( p \)-part of the Tamagawa number conjecture for Hecke characters of imaginary quadratic fields.

6. The remaining Tate twists

6.1. The remaining non-critical twists. The value of the \( L \)-function at zero for \( M_0(w + l + 1) \) with \( -w - 2l < -2 \) is related with the first non-zero coefficient of the Taylor development at \(-l\) of the \( L \)-function associated to \( \psi_\theta \) by the use of the functional equation of \( L \)-functions. The non-critical values associated to the Hecke character \( \psi_\theta \) (we restrict to the situation \( a_\theta \neq b_\theta (\text{mod} O_K^\times) \)) are the integers \( l \) such that \(-l < \min(a_\theta, b_\theta) \) where \( a_\theta, b_\theta \) are associated to the Hecke character \( \psi_\theta \) (see [8, Theorem 1.4.1]).

The general formulation of the Tamagawa number conjecture at the non-critical values following [19] assumes \( w + l + 1 > w \) [19, Conjecture 2.2.7] because then one avoids the poles in the bad Euler factors, and therefore the assumption \( l \geq 0 \). But, for \( M_0(w + l + 1) \), there are no poles in the bad Euler factors, see Remark 2.5. Thus, we can expect to study the Tamagawa number conjecture for \( l < 0 \) by using only the regulators maps.

In this section we construct elements in \( K \)-theory for \( M_0(w + l + 1) \) with \( 0 < -l < \min(a_\theta, b_\theta) \) and we study the image of these elements by the Beilinson regulator map and the Soulé regulator map, obtaining the weak \( p \)-part of the Tamagawa number conjecture.

Deninger [8, pp.142-144] already constructed elements in \( K \)-theory for the motive \( M_0(w + l + 1) \) with \( l < 0 \) non-critical and obtains their image by the Beilinson regulator map, proving the Beilinson conjecture. He constructed these elements in \( K \)-theory by use of a projector map \( K_M \) without using complex multiplication. The problem of his construction is that the Weil pairing appearing in §5 to a \( E[p^r] \)-torsion point \( \tilde{t}_r \), \( \gamma(t_r) = \beta \) is trivial and the arguments through §5 does not generalize in order to construct an Euler system to control the image by the Soulé regulator map. We modify Deninger’s projector map by \( K'_M \) (we use now complex multiplication), and we construct the elements in \( K \)-theory using \( K'_M \) and we reobtain Beilinson’s conjecture. With this modification the arguments in the \( p \)-part of the weak Tamagawa number conjecture, i.e. the image by the Soulé regulator map of these \( K \)-theory elements §4, §5, apply straightforward obtaining the weak \( p \)-part of the Tamagawa number conjecture for \( l < 0 \), Theorems 6.3, 6.4.

6.2. Modification of Deninger’s projector map. Beilinson conjecture revisited. Let us fix \( w \geq 1 \) and \( l < 0 \) such that \(-w - 2l \leq -3 \) with \( 0 < -l \leq \min(a_\theta, b_\theta) \) and let us consider the motive \( M_0(w + l + 1) \). With the fixed embedding we have \( \vartheta = (\lambda_1, \ldots, \lambda_w) \in \theta_K \) and set \( I_1 = \{i | \lambda_i \in Hom_K(K, C)\} \) and \( I_2 = \{i | \lambda_i \notin Hom_K(K, C)\} \) and we have now that \( 0 < |l| \leq \#I_1 = a_\theta \) and \( 0 < |l| \leq \#I_2 = b_\theta \), where \( |l| \) is the absolute value. Denote by \( \Delta = id^1 \times id^2 : E \to E \times E \) the diagonal map and by \( \Delta_{CM} = \underbrace{id^1 \times id^2}_{CM} : E \to E \times E \) given by \( e \mapsto (e, (\sqrt{d_K}e)) \) where we understand \( \sqrt{d_K} \in End(E) \). Let us choose exactly \( |l| \) elements in the sets \( I_1 \) and \( I_2 \), denote their in increasing order \( i_1, \ldots, i_{|l|} \in I_1 \) and \( j_1, \ldots, j_{|l|} \in I_2 \). Let us define the projector map \( pr : E^{w+l} \to E^{w+2l} \) by the projection of the first \( w + 2l \)-components of \( E^{w+l} \) and define \( (id \times \Delta^{|l|}) : E^{w+l} \to E^w \) (which it depends of the choice in the sets \( I_1 \) and \( I_2 \)) by \( (e_1, \ldots, e_{w+2l}, e_{w+2l+1}, \ldots, e_{w+l}) \mapsto (e_{\alpha_1}, \ldots, e_{\alpha_w}) \) where \( e_{\alpha_i} \) is defined as follows:
• if \( \alpha_s \) appears in one component of the set of tuples \( L := \{ (i_1, j_1), \ldots, (i_{\parallel l}, j_{\parallel l}) \} \) then
  \[
eq \begin{cases} 
    \text{id}(e_{w+2l+m}) & \text{if } \alpha_s = i_m \\
    \text{id}(e_{w+2l+m}) & \text{if } \alpha_s = j_m 
  \end{cases}
\]

• in the other case, then it is defined by \( e_{\alpha_s} := e_\theta \) with \( 1 \leq \tilde{n} \leq w + 2l \) such that \( \alpha_s = \tilde{n} + \sum 1 \) where the sum runs the naturals that appear in some component of the elements of \( L \) and which are lower than \( \alpha_s \).

We define the map \((id \times \Delta_{CM})_\triangleright\) similar as \((id \times \Delta_{\triangleright})\) but replacing \( id^l \) by \( id^l\cdot CM \).

The projector map \( K_M' \) is defined by the commutative diagram

\[
\begin{align*}
K_M^l(w+1)(\text{Sym}^{2l}w^1 & E, Q(w+2l+1)) \xrightarrow{\text{pr}^-} H_M^{2l+w+1}(E^{2l+w+1}, Q(2l+w+1)) \\
K_M'(w+1)(M_{\Phi}, Q(w+l+1)) & \xrightarrow{\epsilon_{\Phi}} H_M^{2l+w+1}(h^1(E)^{\otimes w}, Q(l+w+1)).
\end{align*}
\]

Deninger defines a projector map \( K_M' \) with a similar diagram as for \( K_M' \) but replacing the map \((id \times \Delta_{CM})_\triangleright\) by the map \((id \times \Delta_{\triangleright})_\triangleright\).

Let us choose the element in \( H_M^{2l+w+1}(M_{\Phi Q}, Q(w+l+1)) \)

\[
\Gamma : = K_M'\mathcal{E}_M^{2l+w}(N_{K(E[\Gamma])}/K(\Omega f^{-1})),
\]

where \( \mathcal{E}_M^{2l+w} \) is the Eisenstein symbol, \( f \) a generator of \( f_\theta \), \( \Omega \) the period of \( E \) and \( (\Omega f^{-1}) \) means the divisor in \( \mathbb{Z}[E_{\Gamma}] \setminus \{0\} \).

The next result is a modification of Deninger’s result [8, pp.143-145].

**Theorem 6.1** Suppose \( a_\theta \not\equiv b_\theta \mod \#O_K \) with \( a_\theta, b_\theta \geq 0 \), \( l < 0 \), \( w = a_\theta + b_\theta \), with \( -w - 2l \leq -3 \) and \( -l \leq \min(a_\theta, b_\theta) \). Define, up to sign,

\[
\xi_{\theta, l} := \frac{(\sqrt{d_{K}})^{2l}(2l+w)l_{\Phi}((\psi_{\theta}, -l)^{-1}\Phi(|l\rangle)^{1/2})}{2^{-1}N_{K/H}^{1}\psi_{\theta}((\rho_{0})^{\Phi(|l\rangle}^{1/2})} \Gamma_{\theta}
\]

which belongs to \( H^{w+1}(M_{\Phi Q}, Q(w+l+1)) \) where \( L_{\Phi}((\psi_{\theta}, -l) \) means the product of the Euler factors of the primes above \( p \) of \( K \) at \(-l\) (is well defined by Remark 2.5), and \( \rho_{0} \) is the idele of \( K \) such that \( v_{q}(\rho_{0}^{-1}f^{-1}) \geq 0 \) for \( q \) \( |l\rangle \) and \( v_{q}(\rho_{0}) = 0 \) in the other primes \( q \). Then

\[
\tau_{\partial}(\xi_{\theta, l}) = L_{\mathbb{Z}}((\psi_{\theta}, -l))_{\theta},
\]

where \( S \) are the set of primes of \( K \) that divide \( \{a_{\theta}, b_{\theta}\} \) and \( \Omega_{K[1/D_K]} \)-basis for \( H_{\text{B}}(M_{\Phi Q}, \mathbb{Z}(w+l)) \) \( \otimes \mathcal{O}_{K}, \mathcal{O}_{K[1/D_K]} \) and \( L_{\mathbb{Z}}((\psi_{\theta}, -l)) = \lim_{s \to l - 1} L_{\mathbb{Z}}((\psi_{\theta}, s), x) \).

Proof. We will follow closely Deninger’s papers [7] and [8], we follow also in this proof his notation where his \( n \) is our \( w + 2l \). Deninger defines the element \( \xi_{\theta, l} \) from \( K_M'\mathcal{E}_M^{2l+w}(N_{K(E[\Gamma])}/K(\Omega f^{-1}))) \) instead of \( \Gamma_{\theta} \). We modify only the calculation in [8, (2.13) Lemma] for \( K_M' \) instead of \( K_M' \). One obtains (up to sign)

\[
\frac{1}{(2\pi i)^w} \int_{E_{\mathbb{Z}}} K_M' \wedge d_{\mathbb{Z}}(L) =
\]

\[
B_{\mathbb{Z}}\sqrt{d_{K}}^{n+|l|} A(\Gamma)^{n+|l|} C_{n+|l| - |\xi|}
\]

see the calculation at the top of [7, p.63]. To precise the \( sgn \) we should control the chosen order of the factors of the map \((id \times \Delta_{CM})_\triangleright\), but for our interest is unnecessary. Then the argument [8, p.143-144] applies in our situation obtaining,

\[
\tau_{\partial}(\Gamma_{\theta}) = t_{\theta} L^{*}((\psi_{\theta}, -l))_{\theta}
\]

where \( t_{\theta} \) is given by \( 2^{-1}N_{K/\mathbb{Q}}^{1}\psi_{\theta}((\rho_{0})^{\Phi(|l\rangle}^{1/2}) \) (up to sign). By Remark 2.5 we can introduce the Euler factors above \( p \) in the constant factor \( t_{\theta}, \) obtaining the statement.
6.3. The weak Tamagawa number conjecture for $l < 0$. Following §3 we define for $l < 0$ the constructible module by

$$R_{\theta} := \xi_{\theta,l} \mathcal{O}_K,$$

where $\xi_{\theta,l}$ is defined in theorem 6.1. Let us observe that with this notation we can follow straightforward all the results and proofs of §3 and §4. In §5 we need to compute $\mathcal{K}_M^r \circ \mathcal{E}_M^{\oplus 2l}(N_{K(E[l])}/K(\Omega f^{-1}))$. We remember that we suppose once and for all that $p \nmid D_K$.

Denote by $e = (\tilde{t}_r)$, and element of the Tate module $T_p E$ where $\tilde{t}_r \in E[p^r]$ a $p^r$-torsion point for $E$.

**Lemma 6.2** The realization on Galois cohomology of the projector map $\mathcal{K}_M^r$ has the property, $\mathcal{K}_M^r(\tilde{t}_r \otimes \omega^{2l} \theta) = e_\theta(\tilde{t}_r) \otimes \gamma(\tilde{t}_r)^l$ where $\gamma(\tilde{t}_r) = \langle \tilde{t}_r, \sqrt{D_K} \tilde{t}_r \rangle$.

**Proof.** Observe first that the projector map $\mapsto \otimes \mathcal{O}_K$ of rank 1 such that:

• $\text{Hom}_{\mathcal{K}} \Delta$ as

Now take this equality at level $l$ twisting now by $w$ and because the map $\delta_* \mapsto \text{Hom}(\mathcal{K})$ is straightforward all the results and proofs of §3 and §4. In §5 we need to compute $\mathcal{K}_M^r \circ \mathcal{E}_M^{\oplus 2l}(N_{K(E[l])}/K(\Omega f^{-1}))$. We remember that we suppose once and for all that $p \nmid D_K$.

Denote by $e = (\tilde{t}_r)$, and element of the Tate module $T_p E$ where $\tilde{t}_r \in E[p^r]$ a $p^r$-torsion point for $E$.

**Lemma 6.2** The realization on Galois cohomology of the projector map $\mathcal{K}_M^r$ has the property, $\mathcal{K}_M^r(\tilde{t}_r \otimes \omega^{2l} \theta) = e_\theta(\tilde{t}_r) \otimes \gamma(\tilde{t}_r)^l$ where $\gamma(\tilde{t}_r) = \langle \tilde{t}_r, \sqrt{D_K} \tilde{t}_r \rangle$.

**Proof.** Observe first that the projector map $\mathcal{K}_M^r$ is $e_\theta \circ (id \times A_{CM}^*) \circ \text{pr}^*$. Let us take $\delta_* := (id \times A_{CM}^*) \circ \text{pr}^*$ and observe that its transpose $\delta^* = \text{pr}_* \circ (id \times A_{CM}^*)^*$ is part of the definition of $\mathcal{K}_M = e_\theta \circ \delta^*$ with $l := |l| > 0$ given at [22]. We want only to study these projector maps on the Galois cohomology. Denote by $\mathcal{H}_{\mathcal{Q}_p}$ the étale realization of $h^1(E)(1)$ and observe that there is an isomorphism $\mathcal{H}_{\mathcal{Q}_p}(1) \cong \mathcal{Q}_p$, since $(h^1(E)(1))^* = h_1(E)(-1) \cong h^1(E)(1)(-1) = h^1(E)$. The map $\delta^*$ is given by

$$H^1(O_S, Sym^{2l+2}((\mathcal{H}_{\mathcal{Q}_p})(1)) \rightarrow H^1(O_S, Sym^{2l}((\mathcal{H}_{\mathcal{Q}_p})(1 + 1)),$$

and because the map $\delta_*$ is the transpose for the map $\delta^*$, up to Tate twist by $w + l$ it is represented by global Tate duality by,

$$H^1(\Omega\mathcal{S}, Sym^{2l+2}((\mathcal{H}_{\mathcal{Q}_p})(-l - 1)(1)) \rightarrow H^1(\Omega\mathcal{S}, Sym^{2l}((\mathcal{H}_{\mathcal{Q}_p})(-l - 1)(1)).$$

Is known [22] that

$$\delta^*(\lim_{\mathcal{V}}((\tilde{t}_r)^l)) = \lim_{\mathcal{V}}((\tilde{t}_r)^l)$$

write this equality also by $\delta^*((\tilde{t}_r)^l) = (((\tilde{t}_r)^l) \gamma(\tilde{t}_r)^l)$. Take now the dual map by $Hom(\mathcal{O}_p)$ and with the identification $T_{\mathcal{V}} E \cong Hom(T_{\mathcal{V}} E, \mathcal{Q}_p(1))$, we obtain

$$(\tilde{t}_r)^l \gamma(\tilde{t}_r)^l \mapsto (\tilde{t}_r)^l \gamma(\tilde{t}_r)^l$$

twisting now by $w + l$ we arrive to the definition for $\delta_*$. and,

$$\delta_*(\tilde{t}_r) \mapsto (\tilde{t}_r)^l \gamma(\tilde{t}_r)^l.$$

Now take this equality at level $r$, $w = w^l + l$, $l = -l$, and apply the idempotent $e_\theta$ to finish. \qed

After the lemma 6.2 all the results of §5 and the proofs of §5 follow straightforward up to a power of 2 and $D_K$, (the reader could make these modifications which follow from our definition of $R_{\theta}$). Therefore we obtain the weak $p$-part of the Tamagawa number conjecture with $K$-coefficients and $\mathcal{Q}$-coefficients, under standard hypothesis from Iwasawa theory for imaginary quadratic fields:

(***) Let $p$ be a fix prime such that $p \nmid 6N_{K/\mathcal{Q}}$. Suppose that $\psi_p$ has infinity type $(a_0, b_0)$ with $a_0, b_0$ non-negative integers, such that $a_0 \neq b_0 \mod(\mathcal{O}_K^*)$ and $w = a_0 + b_0$ verifies $w - 2l \leq -3$ with $l < 0$ and $-l \leq \min(a_0, b_0)$. Suppose that $\mathcal{O}_K^* \rightarrow (\mathcal{O}_K/\mathcal{Q})^*$ is injective. Suppose moreover that the representation $\chi$ of $\text{Gal}(K(E[p])/K)$ in $Hom_{\mathcal{O}_p}(H^1(M_0 \times_K K, \mathcal{Q}(w + l)), \mathcal{Q})$ is a good representation which is not equal as $\Delta$-representation to the cyclotomic representation.

**Theorem 6.3** Assume hypotheses (***) then, there is an $\mathcal{O}_K$-submodule $R_{\theta} \subset H_M$ of rank 1 such that:
We have the equality
\[ \det_{\mathcal{O}_K[1/D_K]}(r_D(\mathcal{R}_\theta \otimes \mathcal{O}_K \mathcal{O}_K[1/D_K])) \cong \]
\[ L^*_S(\mathcal{R}_\theta, -l) \det_{\mathcal{O}_K[1/D_K]}(H^0_B(M_{\mathcal{O}_K}, \mathbb{Z}(w + l)) \otimes \mathcal{O}_K[1/D_K]) \]
in \[ \det_{\mathcal{O}_K[1/D_K]}(H^0_B(M_{\mathcal{O}_K}, \mathbb{Z}(w + l)) \otimes \mathcal{O}_K[1/D_K] \otimes \mathbb{R}) \).

(2) The map \( r_p \) induces an isomorphism
\[ \det_{\mathcal{O}_K \otimes \mathbb{Z}_p}(\mathcal{R}_\theta) \cong \det_{\mathcal{O}_K \otimes \mathbb{Z}_p}(\Gamma(\mathcal{O}_K[1/S], M_{\mathcal{O}_K}(w + l + 1))^{-1}. \]
Here
\[ L^*_S(\mathcal{R}_\theta, -l) = \lim_{s \to -1} \frac{L^*_S(\mathcal{R}_\theta, s)}{s + l}, \]
and \( S \) is the set of primes of \( K \) dividing \( p \) and the ones dividing \( f_0 \).

Moreover, if \( r_p \) is injective on \( \mathcal{R}_\theta \), the second part can be written as
\[ \det_{\mathcal{O}_K \otimes \mathbb{Z}_p}(H^1(\mathcal{O}_K[1/S], M_{\mathcal{O}_K}(w + l + 1))/r_p(\mathcal{R}_\theta)) \cong \]
\[ \det_{\mathcal{O}_K \otimes \mathbb{Z}_p}(H^2(\mathcal{O}_K[1/S], M_{\mathcal{O}_K}(w + l + 1))). \]

**Theorem 6.4** Suppose hypotheses (****).

Then, there is a \( \mathbb{Z} \)-submodule \( \mathcal{R}_\theta \) in \( H_M \) of rank 2 such that:

1. The map \( r_p \otimes \mathbb{R} \) is an isomorphism restricted to \( \mathcal{R}_\theta \otimes \mathbb{R} \).
2. \( \dim_{\mathbb{Q}}(H^0_B(M_{\mathcal{O}_K}, \mathbb{Q}(w + l))) = \text{ord}_{s=-1} L_S(M_{\mathbb{Q}}, s) = 2. \)
3. We have the equality
\[ r_D(\det_{\mathcal{O}_K[1/D_K]}(\mathcal{R}_\theta \otimes \mathcal{O}_K \mathcal{O}_K[1/D_K])) = \]
\[ L^*_S(M_{\mathcal{Q}}, -l) \det_{\mathcal{O}_K[1/D_K]}(H^0_B(M_{\mathcal{O}_K}, \mathbb{Z}(w + l)) \otimes \mathcal{O}_K \mathcal{O}_K[1/D_K]) \]
where
\[ L^*_S(M_{\mathcal{Q}}, -l) = \lim_{s \to -1} \frac{L^*_S(M_{\mathcal{Q}}, s)}{(s + l)^2} \]
and \( S \) is the set of places of \( K \) that divides \( p \) and the places dividing the conductor \( f_0 \).
4. We have that
\[ \det_{\mathbb{Z}_p}(\mathcal{R}_\theta \otimes \mathbb{Z}_p) = \det_{\mathbb{Z}_p}(\Gamma(\mathcal{O}_K[1/S], M_{\mathbb{Z}_p}(w + l + 1))^{-1}. \]
If \( r_p \) is injective on \( \mathcal{R}_\theta \), then \( r_p(\det_{\mathbb{Z}_p}(\mathcal{R}_\theta \otimes \mathbb{Z}_p)) \) is a basis of the \( \mathbb{Z}_p \)-lattice
\[ \det_{\mathbb{Z}_p}(\Gamma(\mathcal{O}_K[1/S], M_{\mathbb{Z}_p}(w + l + 1))^{-1} \]
\[ \subset \det_{\mathbb{Q}_p}(\Gamma(\mathcal{O}_K[1/S], M_{\mathbb{Q}_p}(w + l + 1) \otimes \mathbb{Q})[1]). \]

**7. Some Explicit Examples**

Observe first if we consider the Hecke character associated to the idempotent \( e_\theta \) with infinite type \( (1, 0) \), then, Theorem 5.12 is exactly [22, Theorem 1.1.5] (hypothesis \( (\circ \circ \circ) \)) for the infinite type \( (1, 0) \) coincides with the ones that appears in [22, Theorem 1.1.5], use [22, Corollary 2.2.11] and Lemmata 4.5, 4.6, 5.5 and comments after Definition 4.3).

Let us give examples of Hecke characters of infinite type different to \( (1, 0) \). In the following, take \( E \) the elliptic curve \( y^2 = 4x^3 - 4x \) defined over \( K = \mathbb{Q}(i) \) and the differences \( e_\theta \) are defined from this fixed \( E \). \( E \) is a CM elliptic curve with CM by \( \mathbb{Z}[i] \), and for \( p \geq 5 \) we have \( p \nmid 6N_{K/\mathbb{Q}}(i) \).

First, consider idempotents \( e_\theta \) with infinite type \( (w, 0) \) or \( (0, w) \) in this case \( l \geq 0 \). Take \( w, p \) such that \( w \equiv 1 \pmod{2} \), \( p - 1 > \max(3, w) \), \( p \) splits in \( \mathbb{Q}(i) \) and \( -w - 2l \leq -3 \). Such \( w, p \) satisfies all hypothesis \( (\circ \circ \circ) \) except the condition (A) for \( \chi \) a good representation (use Lemmata 4.6 and 5.5). We impose \((w, p - 1) = 1\) to ensure that \( w, p \) satisfy the technical hypothesis (A) (use Lemma 4.5). For such \( w, p \) we obtain the conclusion of theorems 5.12 and 5.13. For an explicit example take the infinite type \( (3, 0) \) and \( p = 5 \), then all hypothesis \( (\circ \circ \circ) \) are satisfied and moreover \( p = 5 \) is a regular prime for \( \mathbb{Q}(i) \) (see [28, p.33]) therefore by theorem 5.13 (and remark 5.14) we get the weak 5-part of
the Tamagawa number conjecture for the dual of the motive $M_{(3,0)}(3 + l + 1)$ with $l \geq 0$ twisted by 1 (the special value for the motive $M$ is the special value associated to $L(M,0)$, and in our formulation we get the special value $L(\overline{\psi},-l) = L(M_0(w+l+1)(1),0)$). By use of the functional equation between the motive and its dual twisted by 1 (recall that this functional equation is proved in the case of Hecke characters) and good compatibilities, one should obtain the 5-part of the Tamagawa number conjecture for the motive $M_{(3,0)}(3 + l + 1)$ with $l \geq 0$.

Now we give a numerical example where §6 applies. Let us consider $\epsilon_\theta$ with infinite type $(a, b)$ satisfying $a \not\equiv b (\text{mod } 2)$, $a > b > 0$ and $-(a + b) - 2l \leq -3$. Take $p$ such that it splits in $\mathbb{Q}(i)$ and $p - 1 > \max(3, a - b)$. We have that $\epsilon_\theta$ and $p$ satisfy all the hypothesis ($\circ \circ \circ$) and ($\ast \ast \ast$), with the exception of the condition (A) in Definition 4.3, (use Lemma 4.6 and is easy to prove that $(\mathbb{Z}[i])^* \to (\mathbb{Z}[i]/\mathfrak{p}_0)^*$ is injective if $(\# \mathcal{O}_K^*, a - b) = 1$ with a similar proof done for Lemma 5.5). We impose $(a - b, p - 1) = 1$ to affirm that $\epsilon_\theta$ and $p$ satisfy the condition (A) (see Lemma 4.5). For an explicit example, take $\epsilon_\theta$ with infinity type $(a, b) = (3, 2)$ and $p = 5$, then all hypotheses ($\circ \circ \circ$) and ($\ast \ast \ast$) are satisfied and moreover $p = 5$ is a regular prime (§28, p.33) therefore we get some sort of weak 5-part of the Tamagawa number conjecture for $M_{(3,2)}(3 + 2 + l + 1)$ for $l \geq -1$ by theorems 5.13, 6.4.

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REFERENCES


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