The local Tamagawa number conjecture on Hecke characters

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Abstract

In this paper we prove the local Tamagawa number conjecture in the non-critical case for the motives associated to Hecke characters $\psi_\theta : \mathbb{A}_K \to K^*$, where K is an imaginary quadratic field with $cl(K) = 1$, under certain restrictions which originate from the Iwasawa theory of imaginary quadratic fields.

Introduction

The local Tamagawa number conjecture for a variety X over a number field of Bloch and Kato [3], or, more precisely, for a pure Chow motive M of weight w over a number field, describes the special values of the L-function in terms of cohomological data (see for example Kato [17] or Fontaine and Perrin-Riou [9]).

Recall that the special values of an L-function are the leading Fourier coefficients at integer points. Suppose we have a weight w motive M such that it's L-function has meromorphic continuation and satisfies the expected functional equation. Then, we say that an integer $m < \frac{w}{2}$ is non-critical if $L(M, m) = 0$ and it is critical if $L(M, m) \neq 0$. We extend this definition to the integers $m > \frac{w}{2} + 1$ by saying that m is critical for M if $w - m + 1$ is critical, and is non-critical for M if $w-m+1$ is non-critical. The Tamagawa number conjecture can be formulated in terms of period maps and regulator maps ([8],[17]), but in the non-critical situation it can be formulated without the period maps (using the hypothetical functional equation and good compatibilities).

There are few cases proved in the non-critical situation: for the Riemann zeta function ([3, §6]), for Dirichlet motives ([4], [15]), for CM elliptic curves defined over the field of the endomorphism ring ([19]) or defined over $\mathbb{Q}$ ([3, §7], [1]).

The weak local Tamagawa number conjecture for an elliptic curve E with CM defined over the field of the endomorphism, proved by Kings [19], is related to the weak local Tamagawa number conjecture for the L-function the Hecke character ψ, associated to E, over the imaginary quadratic field K. More precisely, Kings proves in [19] the conjecture for the motive $h(\overline{\psi})(-r)$ with $r \geq 0$ which corresponds to the special value (non-critical) for the L function associated to $\overline{\psi}$ at $-r$, where $h(\overline{\psi})$ is the motive associated to $\overline{\psi}$ over K with K-coefficients. Then, he obtains the conjecture for the Chow motive $h^1(E)(-r)$ when one rewrites the above result with $\mathbb{Q}$-coefficients. Using the functional equation for E and good compatibilities one should obtain the conjecture for $h^1(E)(r+2)$. We generalize the methods of Kings to Hecke characters over an imaginary quadratic field K in the non-critical situation.

Consider ψ a Hecke character over an imaginary quadratic field associated to an elliptic curve with CM defined over the endomorphism ring and let us

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take the motive associated to $\psi^a\overline{\psi}^b$ with $a, b \geq 0$, which has weight $a+b$. It is known that the non-critical values for this motive are the integers lower than $min(a, b)$ (see §3 for the precise statement). Our work is concentrated in this situation, but note that there are results on the Tamagawa number conjecture in the critical situation (Harrison [14], Guo [12], Kimura [20], Han [13] and in big generality by Tsuji [24]).

The aim of this paper is to prove the local Tamagawa number conjecture (under the formulation in [17]) for all the non-critical values for the L-function of the motive associated to any Hecke character $\psi_\theta : \mathbb{A}_K \to K^*$ where K is an imaginary quadratic field which $cl(K) = 1$, (see [2, Chapter 3] for a more concrete result only for powers of the Grössencharacter associated to the elliptic curve E with CM).

The basic ingredients used in the proof of this result are the specialization of the polylogarithm sheaf and the main conjecture for the Iwasawa theory for imaginary quadratic fields, as in [19]. The main tool in the proof is a precise description of the image of some concrete elements in K-theory under the Soulé and Beilinson regulator maps. Using Deninger proof of the Beilinson's conjecture for Hecke characters in [6], we only need to check the p-adic part of the Tamagawa number conjecture.

Let us finally give a rough sketch of the contents of this paper. In the first section §1 we recall the formulation of the local Tamagawa number conjecture for pure motives using Kato's formulation [17] for some non-critical situation. In section §2 we define the motives associated to Hecke characters coming from elliptic curves defined over an imaginary quadratic field K. We prove a generalization of a result of Deuring for these motives. In §3 we define the constructible elements of Deninger [6] for these motives and we reformulate Deninger's theorem in the notation of the local Tamagawa number conjecture. In the next sections we study the image of these constructible elements for our motives via the Soulé regulator map. In §4 we define a map from some Iwasawa modules to the étale cohomology groups that appear in the conjecture. Here the Rubin's theorem of the main conjecture of the Iwasawa theory for imaginary quadratic fields plays an important role. This section follows basically the arguments of [19, §2] with a definition of elliptic units more useful for us. In section §5 we use the specialization of the elliptic polylogarithm to compare the image of the map defined in §4 with the Soulé regulator map. With these ingredients we obtain the main results of the paper, theorems 5.12 (for K-coefficients) and 5.13 (for $\mathbb{Q}$-coefficients). In the appendix we study the remaining non-critical values, the ones in the band of possible poles of bad Euler factors. We modify Deninger's projector map to define more convenient elements in K-theory and we reprove the Beilinson's conjecture for them. With this modification we can apply all the techniques of §4 and §5, obtaining the main theorems of §5.

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1 The Tamagawa number conjecture

We formulate the conjecture for the category of Chow motives over a number field F with coeficients in $\mathbb{Q}$ denoted by $\mathcal{M}(F)_{\mathbb{Q}}$ (for a more detailed explanation we refer to [2, §1.2.3] and [17] [18]). For every $M \in \mathcal{M}(F)_{\mathbb{Q}}$ one expects a decomposition into direct sum of pure Chow motives in this category.

Using Kato's notation as in [18, §2], a pure motive of weight $w \in \mathbb{Z}$ over a number field F is a finite family of 4-tuples $\{(X_j, m_j, r_j, \epsilon_j)\}$ where X_j is a smooth proper scheme of pure dimension d_j over F and r_j are integers with $w = m_j - 2r_j$, and ϵ_j is an idempotent in the ring of algebraic cycles on $X_j \times_F X_j$ with $\mathbb{Q}$-coefficients modulo rational equivalence.

We denote by $h^m(X)(r)$ the pure motive determined by the tuple (X, m, r, Δ_X), where Δ_X denotes the diagonal, regarded as the identity correspondence. We interpret $\{(X, m, r, \epsilon)\}$ as the direct sumand of $h^m(X)(r)$ corresponding to ϵ. For simplicity we restrict from now on our analysis to pure motives M of the form (X, m, r, ϵ) which we denote by $M_\epsilon(r)$.

Fix integers $m \geq 0$ and r such that $m-2r = w \leq -3$ and $r > \inf(m, dim(X))$. Let also fix a prime p of $\mathbb{Q}$, $p \neq 2$. For every $M_\epsilon(r)$ let associate the motivic cohomology

$$H^{m+1}_{\mathcal{M}}(M_\epsilon, r) := (K_{2r-m-1}(M_\epsilon) \otimes \mathbb{Q})^{(r)}$$

which it is the r-th Adams eigenspace of the $2r - m - 1$-th Quillen K-theory of M_ϵ. We consider some realizations for the motive which are related via regulator maps with the above motivic cohomology. Let define $H_{h,\mathbb{Q}}$ by

$$\epsilon^* H^m_{sing}(X \times_{\mathbb{Q}} \mathbb{C}, (2\pi i)^{r-1}\mathbb{Q})^+,$$

a Betti realization, where the upper-script + means the fixed part by the $Gal(\mathbb{C}/\mathbb{R})$-action acting simultaneously on $\mathbb{C}$ and on $\mathbb{Q}(r-1)$. The p-adic realization for our motive corresponds to

$$V_p := \epsilon^* H^m_{\acute{e}t}(X \times_F \overline{F}, \mathbb{Q}_p(r)) = H^m_{\acute{e}t}(M_\epsilon \times_F \overline{F}, \mathbb{Q}_p(r))$$

which is a $Gal(\overline{F}/F)$-module, unramified outside a finite set of places of F. Consider

$$H_{h,\mathbb{Z}} := \epsilon^* H^m_{sing}(X \times_{\mathbb{Q}} \mathbb{C}, (2\pi i)^{r-1}\mathbb{Z})^+$$

which satisfy $H_{h,\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q} = H_{h,\mathbb{Q}}$, and

$$T_p := \epsilon^* H^m_{\acute{e}t}(X \times_F \overline{F}, \mathbb{Z}_p(r))$$

a $Gal(\overline{F}/F)$-module such that $T_p \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = V_p$. Let S be a set of finite primes of F containing the primes lying over p and the ones where T_p is ramified. Let $\mathcal{O}_F$ be the ring of integers of F and write $\mathcal{O}_S := \mathcal{O}_F[1/S]$. Consider $j : Spec(F) \to Spec(\mathcal{O}_S)$ the natural map and define,

$$H^i_p := H^i_{\acute{e}t}(\mathcal{O}_S, j_* T_p).$$

We omit j_* if no confusion is likely. Then, there are regulator maps due to Beilinson and Soulé,

$$r_{\mathcal{D}} : H^{m+1}_{\mathcal{M}}(M_\epsilon, r) \otimes_{\mathbb{Q}} \mathbb{R} \to H_{h,\mathbb{Z}} \otimes_Z \mathbb{R}$$

$$r_p : H^{m+1}_{\mathcal{M}}(M_\epsilon, r) \otimes_{\mathbb{Q}} \mathbb{Q}_p \to H^1_p \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

For any prime $\mathfrak{p} \nmid p$ in $\mathcal{O}_F$, we have local Euler factors which are defined by

$$P_{\mathfrak{p}}(V_p, s) := det_{\mathbb{Q}_p}(1 - Fr_{\mathfrak{p}} N\mathfrak{p}^{-s} | V_p^{I_{\mathfrak{p}}}),$$

where $Fr_{\mathfrak{p}}$ is the geometric Frobenius at $\mathfrak{p}$ and $I_{\mathfrak{p}}$ is the inertia group at $\mathfrak{p}$. For $\mathfrak{p} \mid p$, they are defined by

$$P_{\mathfrak{p}}(V_p, s) := det_{F_{0,\mathfrak{p}}}(1 - \psi_{\mathfrak{p}}^{-1} N\mathfrak{p}^{-s} | D_{cris}(V_p)),$$

where $\psi_{\mathfrak{p}}$ is the arithmetic Frobenius, $D_{cris}(V_p) = (B_{cris,\mathfrak{p}} \otimes V_p)^{Gal(\overline{F}_{\mathfrak{p}}/F_{\mathfrak{p}})}$ and $F_{0,\mathfrak{p}}$ means the maximal unramified extension inside $F_{\mathfrak{p}}$. Recall that $P_{\mathfrak{p}}(V_p, s)$ is conjecturally independent of the choice of the finite prime p.

The L-function of X is defined by

$$L_S(V_p, s) := \prod_{\mathfrak{p} \notin S} P_{\mathfrak{p}}(V_p, s)^{-1}.$$

Conjecture 1.1. *(cf. §4[18], conj. 2.2.7 in [17]) Let $V_p^* = Hom(V_p, \mathbb{Q}_p)$ be the dual Galois module. Let $p \neq 2$, r, m and S be as above. Assume that*

$$P_{\mathfrak{p}}(V_p^*(1), 0) \neq 0$$

for all $\mathfrak{p} \in S$ and that $L_S(V_p^*(1), s)$ has an analytic continuation to all $\mathbb{C}$, then:

1. *The maps $r_{\mathcal{D}}$ and r_p are isomorphisms and H_p^2 is finite.*
2. *$dim_{\mathbb{Q}}(H_{h,\mathbb{Z}}) = ord_{s=0} L_S(V_p^*(1), s)$; write this number l.*
3. *Let $\eta \in det_{\mathbb{Z}}(H_{h,\mathbb{Z}})$ be a $\mathbb{Z}$-basis. There is an element $\xi \in det_{\mathbb{Q}}(H_{\mathcal{M}})$ such that*
$$r_{\mathcal{D}}(\xi) = (\lim_{s \to 0} s^{-l} L_S(V_p^*(1), s))\eta.$$
This is the "Beilinson conjecture".
4. *Consider $r_p(\xi) \in det_{\mathbb{Q}_p}(H_p^1 \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$. Then $r_p(\xi)$ is a basis of the $\mathbb{Z}_p$-lattice*
$$det_{\mathbb{Z}_p}(R\Gamma(\mathcal{O}_S, T_p))^{-1} \subset det_{\mathbb{Q}_p}(R\Gamma(\mathcal{O}_S, V_p)[-1]).$$
i.e.
$$[det_{\mathbb{Z}_p}(H_p^1) : r_p(\xi)\mathbb{Z}_p] = \#(H_p^2) = det_{\mathbb{Z}_p}(H_p^2).$$

As our knowledge of K-theory is limited, we take a weak version of the conjecture.

Conjecture 1.2. *(cf. §1.1.1 [19]) With the same hypothesis of the above conjecture 1.1, then there is a subspace $H_{\mathcal{M}}^{constr}$ in $H_{\mathcal{M}}$ such that:*

1. *$r_{\mathcal{D}}$ and r_p restricted to $H_{\mathcal{M}}^{constr}$ are isomorphisms and H_p^2 is finite.*
2. *$dim_{\mathbb{Q}}(H_{h,\mathbb{Z}}) = ord_{s=0} L_S(V_p^*(1), s)$; write this number l.*
3. *There is an element $\xi \in det_{\mathbb{Q}}(H_{\mathcal{M}}^{constr})$ such that*
$$r_{\mathcal{D}}(\xi) = (\lim_{s \to 0} s^{-l} L_S(V_p^*(1), s))\eta.$$
4. *The element $r_p(\xi)$ is a basis of the $\mathbb{Z}_p$-lattice*
$$det_{\mathbb{Z}_p}(R\Gamma(\mathcal{O}_S, T_p))^{-1} \subset det_{\mathbb{Q}_p}(R\Gamma(\mathcal{O}_S, V_p)[-1]).$$

2 The motive associated to Hecke characters

Let K be an imaginary quadratic field and $\mathcal{O}_K$ be its ring of integers. Denote by d_K the discriminant of K. Let E be an elliptic curve over K with CM by $\mathcal{O}_K$. In this section we describe some pure motives coming from a self product of the motive $h^1(E)$ and their realizations, and we prove that the L-functions associated to these motives correspond to Hecke characters. We obtain finally an analog for these motives of a result of Deuring for CM elliptic curves.

Let p be an odd prime, fixed once and for all, such that E has good reduction for all primes over p. Let $S^{'}$ be the set of places that divide the conductor of the elliptic curve $\mathfrak{f}$ (that are the same places where E has bad reduction) and the places that divide p.

Consider the category of Chow motives $\mathcal{M}(K)$ over K with morphisms induced by graded correspondences in Chow theory. We have then a natural covariant functor h from the category of smooth and projective varieties over K to $\mathcal{M}(K)$. Then, the motive of an elliptic curve E over K has a canonical decomposition $h(E) = h^0E \oplus h^1E \oplus h^2E$ with respect to the zero section, where $h^0E = h(Spec(K))$ and $h^2E = h(Spec(K))(-1)$. We can also consider the category $\mathcal{M}_{\mathbb{Q}}(K)$ which consist with the same object but enlarging the morphisms tensoring by $\mathbb{Q}$ of the category $\mathcal{M}(K)$. For an elliptic curve we have also the above decomposition for $h(E)_{\mathbb{Q}}$ in $\mathcal{M}_{\mathbb{Q}}(K)$.

If E is an CM elliptic curve as before, the motive h^1E has multiplication by $\mathcal{O}_K$. Consider then the motive $\otimes^w h^1E$, for w a positive integer, which has multiplication by $\mathcal{O}_w := \otimes^w_{\mathbb{Z}}\mathcal{O}_K$. Then, $\otimes^w h^1E_{\mathbb{Q}}$ has multiplication by $T_w := \otimes^w_{\mathbb{Q}}K$. If we denote by $\Upsilon = Hom(K,\mathbb{C})$, observe that T_w is equal to $\prod_\theta T_\theta$, where θ runs through the $Aut(\mathbb{C})$-orbits of $\Upsilon^w = Hom(T_w,\mathbb{C})$ and T_θ are fields. Let e_θ be the idempotent corresponding to the component T_θ of T_w, and consider the motive

$$e_\theta(\otimes^w h^1(E)_{\mathbb{Q}}).$$

Then, on the integral motive $\otimes^w h^1E$, considering the idempotent e_θ as before, we get that the integral motive $e_\theta(\otimes^w h^1(E))$ which has multiplication by $\mathcal{O}_\theta := e_\theta\mathcal{O}_w$, a ring in T_θ.

Remark 2.1. *These motives were introduced by Deninger in [6] in a more general setting. He constructs a motive for every Hecke character on $\mathbb{A}_K$, coming from the self product of $h^1(E)$ for a particular CM elliptic curve related with a weight 1 Hecke character. Under the restriction $cl(K) = 1$, the Hecke characters on $\mathbb{A}_K$, are the ones which maps to K^*, with any Dirichlet character ([6, Prop.1.3.1]). Then our presentation is the general situation for Hecke characters over an imaginary quadratic field with $cl(K) = 1$.*

Let $\psi : I_K \to K^*$ be the grössencharacter associated to the elliptic curve E. Define the CM character

$$\psi_\theta : I_K \to T_\theta^*$$

by $\psi_\theta = e_\theta \cdot (\otimes^w\psi)$, and denote by $\mathfrak{f}_\theta$ the conductor of ψ_θ. Observe that $\mathfrak{f}_\theta|\mathfrak{f}$ since ψ_θ is a sub-character of $\otimes^w\psi$.

The infinity type of this character is obtained as follows: Let fix once and for all an embedding $K \to \overline{\mathbb{Q}}$ like in the last paragraph on [6, p.132]. We have then a natural embedding

$$K \to \otimes^w K \to T_w \to T_\theta$$

where the first map corresponds to the diagonal map. For any $\vartheta \in \theta_K := \theta \cap Hom_K(T_\theta, \mathbb{C})$, ($\theta$ here means the elements of Υ^w which includes T_θ in $\mathbb{C}$), $\vartheta = (\lambda_1, \ldots, \lambda_w) \in \Upsilon^w$, we set $a_\vartheta = \#\{i | \lambda_i \in Hom_K(K, \mathbb{C})\}$ and $b_\vartheta = w - a_\vartheta$. These numbers do not depend on the element ϑ in θ_K, and they determine the infinity type for the Hecke character ψ_θ (cf. last paragraph 1.3 [6]). We will denote the type of ψ_θ as $(a_\theta, b_\theta) := (a_\vartheta, b_\vartheta)$ where ϑ is any element in θ_K. Moreover, in our situation, the CM field T_θ is the field generated by $(\lambda_1(K) \cdot \ldots \cdot \lambda_w(K))$ which is K, where $\vartheta = (\lambda_1, \ldots, \lambda_w) \in \theta$, then $e_\theta(\otimes^w h^1(E)_{\mathbb{Q}}) \in \mathcal{M}_{\mathbb{Q}}(K)$ has multiplication by K. As $\lambda_i \in \{\lambda, \overline{\lambda}\}$ where λ is the fixed embedding of K in $\mathbb{C}$, and $\lambda_i(\mathcal{O}_K) = \mathcal{O}_K$, we have then that $e_\theta(\otimes h^1(E)) \in \mathcal{M}(K)$ has multiplication by $\mathcal{O}_K$ and we can consider $\mathcal{O}_\theta = \mathcal{O}_K$. We mention also that θ only contains two ϑ, because in Υ^w two embeddings of the same field differ by an automorphism of $\mathbb{C}$, but the field T_θ is K which has only two automorphism then we have that the other element of θ diferent for ϑ is $(\overline{\lambda_1}, \ldots, \overline{\lambda_w})$.

Let's denote by

$$M_\theta := e_\theta(\otimes^w h^1(E)),$$

considered as an integral Chow motive ([11, p.57]), that is and element in the category of $\mathcal{M}(K)$, and $M_{\theta\mathbb{Q}}$ its image in $\mathcal{M}_{\mathbb{Q}}(K)$. Our objective in this section is to study the p-adic and Betti realizations of this motive twisted by w, and to determine its L function in terms of the Hecke character ψ_θ.

The p-adic realization of the motive $M_{\theta\mathbb{Q}}(w)$ is, by definition, $H^w_{et}(M_{\theta\mathbb{Q}} \times_K \overline{K}, \mathbb{Q}_p(w))$. We need to choose a lattice on it.

Lemma 2.2. *The integral p-adic realization of $M_\theta(w)$, $H^w_{et}(M_\theta \times_K \overline{K}, \mathbb{Z}_p(w))$, is isomorphic to*

$$e_\theta(\otimes^w T_p E)$$

as free $e_\theta(\otimes^w \mathcal{O}_K)$-modules of rank 1 with G_K-action, with the $\mathcal{O}_\theta$-action on $e_\theta(\otimes^w T_p E)$ given by $\overline{\psi}_\theta \otimes \mathbb{Z}_p$.

Proof. Observe first that T_pE is isomorphic to $H^1_{et}(h^1(E) \times_K \overline{K}, \mathbb{Z}_p(1))$ by Kummer theory, but with G_K-action given by the complex conjugation by Artin reciprocity.

The claim that $e_\theta(\otimes^w T_p E)$ is a free module of rank 1 follows because T_pE is a free $\mathcal{O}_K$-module of rank 1 and then $e_\theta \cdot (T_pE \otimes \ldots \otimes T_pE)$ is a free $e_\theta \cdot (\mathcal{O}_K \otimes \ldots \otimes \mathcal{O}_K)$-module of rank one.

Now, consider the natural action of G_K on $H^1_{et}(h^1(E) \times_K \overline{K}, \mathbb{Z}_p(1)) = Hom(T_pE, \mathbb{Z}_p(1))$. Since G_K acts on the Tate module by $\psi : G_K \to (\mathcal{O}_K \otimes \mathbb{Z}_p)^*$, then it acts on our G_K-module via the complex conjugate. Using that

$$H^w((h^1(E) \times_K \overline{K})^w, \mathbb{Z}_p(w)) = H^1(h^1(E) \times_K \overline{K}, \mathbb{Z}_p(1))^{\otimes w}$$

by [10, Prop.2.4.3 a)], and taking our idempotent, we obtain the result. □

We can apply this result to the motive $M_\theta(w+l+1)$ for any integer $l+1$. We get therefore a $\mathbb{Z}_p$-lattice in the p-adic realization of the motive $M_{\theta,\mathbb{Q}}(w+l+1)$.

We are going now to define a submodule in $H^w_{sing}(E^w \times_{\mathbb{Q}} \mathbb{C}, \mathbb{Z}(2\pi i)^{w+l})^+$ such that their elements are a $\mathbb{Z}$-lattice in the Betti realization with $\mathbb{Q}$-coefficients for $M_\theta(w+l+1)$.

Observe first that $H^1_B(E(\mathbb{C}), \mathbb{Z}(1)) \cong H^1_B(E \times_{\mathbb{Q}} \mathbb{C}, \mathbb{Z}(1))^+$ is an $\mathcal{O}_K$-module of rank 1, and hence

$$\otimes^w_{\mathbb{Z}} H^1_B(E(\mathbb{C}), \mathbb{Z}(1))$$

is a $\otimes^w \mathcal{O}_K$-module of rank 1. By considering now the idempotent e_θ and taking the corresponding submodule

$$e_\theta(\otimes^w H^1_B(E(\mathbb{C}),\mathbb{Z}(1)))(l),$$

we get a $\mathcal{O}_\theta := e_\theta(\otimes^w \mathcal{O}_K)$-module of rank 1. This module corresponds to the $\mathbb{Z}$-module $H^w_B(M_{\theta\mathbb{C}},\mathbb{Z}(w+l))$, the searched $\mathbb{Z}$-lattice of $H^w(M_{\theta\mathbb{C}},\mathbb{Q}(w+l))$ for the Betti realization of our motive in the formulation of the conjecture.

Now, we are going to study the L-function that corresponds to the p-adic representation $H^w_{et}(M_{\theta\mathbb{Q}}\times_K \overline{K},\mathbb{Q}_p)$ of $M_{\theta\mathbb{Q}}$. Let be S the places of K that divide $\mathfrak{f}_\theta$ and the places that divide p. Define as usual

$$L_S(M_\theta,s) := \prod_{\mathfrak{l}\notin S} det_{\mathbb{Q}_p}(1 - Frob_{\mathfrak{l}} N\mathfrak{l}^{-s}|(H^w_{et}(M_{\theta\mathbb{Q}}\times_K \overline{K},\mathbb{Q}_p))^{I_{\mathfrak{l}}})^{-1}$$

where $Frob_{\mathfrak{l}}$ means the geometric Frobenius.

Our goal is to compute this determinant and to relate it with the local factors of the L-function of the Hecke character ψ_θ that is defined by,

$$L_S(\psi_\theta,s) := \prod_{\mathfrak{l}\notin S}(1 - \frac{\psi_\theta(\mathfrak{l})}{N\mathfrak{l}^s})^{-1}.$$

Recall that the operation of the decomposition group $D_{\mathfrak{p}}$ for $\mathfrak{p}\nmid p$ on $H^1_{et}(h^1(E)\times_K\overline{K},\mathbb{Q}_p)$ is given by the operation of $\psi^{-1}|_{K^*_{\mathfrak{p}}}$, and hence $D_{\mathfrak{p}}$ operates on $H^w_{et}(M_{\theta\overline{K}},\mathbb{Q}_p)$ via ψ_θ^{-1}. On one hand, the inertia group $I_{\mathfrak{p}}$ acts non-trivially if and only if $\mathfrak{p}$ divides the conductor $\mathfrak{f}_\theta$. On the other hand, for $\mathfrak{p}\nmid\mathfrak{f}_\theta$, the geometric Frobenius $Fr_{\mathfrak{p}}$ at $\mathfrak{p}$ acts via $\psi_\theta(\mathfrak{p})$. We obtain hence the following result.

Lemma 2.3 (Deninger,§1.3.1[6]). *Let $\mathfrak{l}$ a finite prime of K not over p, then*

$$det_{T_\theta\otimes\mathbb{Q}_p}(1 - Fr_{\mathfrak{l}}N\mathfrak{l}^{-s}|H^w_{et}(M_\theta,\mathbb{Q}_p)^{I_{\mathfrak{l}}}) = (1-\psi_\theta(\mathfrak{l})N\mathfrak{l}^{-s})$$

if $\mathfrak{l}\nmid\mathfrak{f}_\theta$, where $\mathfrak{f}_\theta$ is the conductor of the Hecke character ψ_θ.

We fix some restrictions for our motive $M_\theta(w+l+1)$ once and for all. We suppose $-w-2l\leq -3$. Remember that, with our restriction that E is defined over K, we have $\#|\theta| = 2$, and in particular we have $T_\theta\cong K$ and $\mathcal{O}_\theta\cong\mathcal{O}_K$.

The L-function for M_θ can be described by using lemma 2.3 and by taking the norm map.

Lemma 2.4. *Let $\mathfrak{l}$ a prime of K such that $\mathfrak{l}\nmid\mathfrak{f}_\theta$ and it is prime to p. We have then the following equality*

$$det_{\mathbb{Q}_p}(1 - Fr_{\mathfrak{l}}N\mathfrak{l}^{-s}|(H^w(M_\theta,\mathbb{Q}_p))^{I_{\mathfrak{l}}}) = (1-\psi_\theta(\mathfrak{l})N\mathfrak{l}^{-s})(1-\overline{\psi_\theta}(\mathfrak{l})N\mathfrak{l}^{-s}).$$

As a corollary we obtain a generalization of a result of Deuring.

Theorem 2.5. *Let S be the set of the primes on K dividing $\mathfrak{f}_\theta$ and primes dividing p. Then:*

$$L_S(H^w(M_\theta,\mathbb{Q}_p),s) = L_S(\psi_\theta,s)L_S(\overline{\psi}_\theta,s).$$

Remark 2.6. *The p-adic realization* $V_{p,w,w+l+1} = H^w_{et}(M_{\theta\mathbb{Q}} \times_K \overline{K}, \mathbb{Q}_p(w+l+1))$ *of our motive* M_θ *satisfies that the local Euler factors*

$$P_{\mathfrak{p}}(V_p^*(1), 1) = P_{\mathfrak{p}}(\overline{\psi}_\theta, -l)$$

are different from 0 for all $\mathfrak{p} \in S$. *Hence, it satisfies the hypothesis in the conjecture 1.2.*

To show this fact, suppose first that $\mathfrak{p}|\mathfrak{f}_\theta$. *Then, the inertia group acts non-trivially on* $V_{p,w,w+l+1}$, *which is a one dimensional* $\mathcal{O}_\theta \otimes \mathbb{Q}$*-module, and hence*

$$L_{\mathfrak{p}}(\overline{\psi}_\theta, s) = 1$$

for all $\mathfrak{p}|\mathfrak{f}_\theta$, *and in particular for* $s = -l$.

If $\mathfrak{p}$ *divides* p, *then the result follows from the fact that the abelian varieties with CM satisfy this condition. i.e.*

$$det_{\mathbb{Q}_p}(1 - Fr_{\mathfrak{p}} N\mathfrak{p}^l | H^w_{et}(\overline{E}^w, \mathbb{Q}_p)) \neq 0,$$

and therefore, since the different idempotents e_θ *give a direct summand of the cohomology group* $H^w(\overline{E}^w, \mathbb{Q}_p)$,

$$L_{\mathfrak{p}}(\overline{\psi}_\theta, -l) \neq 0.$$

3 The Beilinson conjecture for Hecke characters

In this section we will review briefly the study of the Beilinson conjecture for the motive $M_{\theta\mathbb{Q}}(w+l+1)$ done by Deninger in [6]. First of all, recall the main theorem in [6].

Theorem 3.1 (Deninger, 1.4.1 [6]). *Let* $w = a_\theta + b_\theta \geq 1$. *Consider an integer* l *such that*

$$-l \leq Min(a_\theta, b_\theta) \quad if\ a_\theta \neq b_\theta$$
$$-l < a_\theta = b_\theta = w/2 \quad otherwise.$$

Then the L-series $L(\overline{\psi}_\theta, s)$ *has a zero of order 1 in* $s = -l$.

Moreover, there exist an element ξ_θ *in* $H^{w+1}_{\mathcal{M}}(M_\theta, \mathbb{Q}(w+l+1))$ *such that, under the Deligne regulator map*

$$r_{\mathcal{D}} : H^{w+1}_{\mathcal{M}}(M_\theta, \mathbb{Q}(w+l+1)) \to H^w_B(M_\theta, \mathbb{R}(w+l)),$$

has image

$$r_{\mathcal{D}}(\xi_\theta) = lim_{s \to -l} \frac{L(\overline{\psi_\theta}, s)}{s+l} \eta_\theta \ mod\ T^*_\theta$$

in the free rank one $T_\theta \otimes \mathbb{R}$*-module* $H^{w+1}_{\mathcal{D}}(M_\theta, \mathbb{R}(w+l+1) = H^w_B(M_{\theta\mathbb{C}}, \mathbb{R}(w+l))$, *where* η_θ *is a* T_θ*-generator of* $H^w_B(M_{\theta\mathbb{C}}, \mathbb{Q}(w+l))$.

Observe that the L-series $L(\overline{\psi}_\theta, s)$ is equal to the L-function for the dual Beilinson motive of M_θ, i.e. the same motive but with action by T_θ given by its complex conjugation.

Let's recall the construction of ξ_θ, following the results of Deninger. We suppose once for all that $l \geq 0$. Remember that S is the set of finite primes $\{\mathfrak{p} \in Spec(\mathcal{O}_K) | \mathfrak{p}|\mathfrak{f}_\theta \ or\ \mathfrak{p}|p\}$.

Fix an algebraic differential form $\omega \in H^0(E, \Omega_{E/K})$. Since we have complex multiplication, we can write the period lattice as $\Gamma = \Omega \mathcal{O}_K$, where $\Omega \in \mathbb{C}^*$ is the complex period. Fix an element γ in $H_1(E(\mathbb{C}), \mathbb{Z})$ such that it is an $\mathcal{O}_K$-generator, and satisfies

$$\Omega = \int_\gamma \omega.$$

By Poincaré duality, we have that to γ it corresponds η_γ, an $\mathcal{O}_K$-generator for $H^1(E(\mathbb{C}), \mathbb{Z}(1))$. Consider now the $\mathcal{O}_\theta$-generator

$$\eta_\theta := (2\pi i)^l e_\theta(\eta_\gamma \otimes \ldots \otimes \eta_\gamma)$$

of

$$H_B^w(M_{\theta\mathbb{C}}, \mathbb{Z}(w+l)) = e_\theta\left((\otimes H_B^1(E(\mathbb{C}), \mathbb{Z}(1)))(l)\right).$$

To construct ξ_θ, we will define a divisor on the torsion points of the elliptic curve; its image by the composition of the Eisenstein map $\mathcal{E}_\mathcal{M}$ ([5, §8]) with the Deninger projector map $\mathcal{K}_\mathcal{M}$ ([6, (2.8)]) will define our ξ_θ.

Remember that $\mathfrak{f}_\theta$ is the conductor of the Hecke character ψ_θ associated with M_θ, and denote by f a generator of $\mathfrak{f}_\theta$ (it exists since the ideal class group of K is equal to 1). We have that

$$\Omega f^{-1} \in \mathfrak{f}_\theta^{-1}\Gamma$$

and that (Ωf^{-1}) gives a divisor in $\mathbb{Z}[E[\mathfrak{f}_\theta] \setminus 0]$ defined over $K(E[\mathfrak{f}_\theta])$. Since $\mathfrak{f}$ is the conductor of ψ and $\mathfrak{f}_\theta | \mathfrak{f}$, the divisor (Ωf^{-1}) is defined also over $K(E[\mathfrak{f}])$. We will define our divisor as

$$\beta_\theta := N_{K(E[\mathfrak{f}])/K}((\Omega f^{-1})).$$

If $a_\theta \not\equiv b_\theta \mod |\mathcal{O}_K^*|$, we obtain that ([6, p.142,(2.11)]),

$$r_\mathcal{D}(\mathcal{K}_\mathcal{M}\mathcal{E}_\mathcal{M}(\beta_\theta)) = (-1)^{l-1} \frac{2^{l-1} N_{K/\mathbb{Q}}\mathfrak{f}_\theta^{w+2l}\psi_\theta(f)}{(2l+w)! N_{K/\mathbb{Q}}(\mathfrak{f}_\theta)^{l+w}} \frac{\Phi(\mathfrak{f})}{\Phi(\mathfrak{f}_\theta)} L'(\overline{\psi_\theta}, -l)\eta_\theta,$$

where $\Phi(\mathfrak{m}) := |(\mathcal{O}_K/\mathfrak{m})^*|$ for any ideal $\mathfrak{m}$ of $\mathcal{O}_K$.

This is an analog for $M_\theta(w+l+1)$ of [19, thm.1.2.2] which corresponds to $h^1(E)(1+l+1)$. Therefore, it suggests the element that we have to take to prove an analog for the p-Tamagawa number conjecture for $M_\theta(w+l+1)$, that is, to control the coimage of the p-Soulé regulator by the number of elements of a second cohomology group.

Theorem 3.2 (Deninger, §2[6]). *Suppose that $a_\theta \not\equiv b_\theta$ mod $|\mathcal{O}_K^*|$ and that a_θ, b_θ, l satisfy the hypothesis of the theorem 3.1 with $l \geq 0$. Define, by using the previous notation,*

$$\xi_{\theta,l} :=$$

$$(-1)^{l-1} \frac{(2l+w)! L_p(\overline{\psi_\theta}, -l)^{-1}\Phi(\mathfrak{f}_\theta)}{2^{l-1} N_{K/\mathbb{Q}}\mathfrak{f}_\theta^l \psi_\theta(f)\Phi(\mathfrak{f})} \mathcal{K}_\mathcal{M} \circ \mathcal{E}_\mathcal{M}^{2l+w}(\beta_\theta) \in H_\mathcal{M}^{w+1}(M_\theta, \mathbb{Q}(w+l+1)),$$

where $L_p(\overline{\psi_\theta}, s)$ is the product of the Euler factors for the primes of K above p.

Then

$$r_\mathcal{D}(\xi_{\theta,l}) = L_S^*(\overline{\psi_\theta}, -l)\eta_\theta,$$

where S are the primes of K that divide $\mathfrak{f}_\theta p$.

Definition 3.3. *For $a_\theta \not\equiv b_\theta$ mod $|\mathcal{O}_K^*|$ we define our constructible space ($H^{cons}_{M,\omega,r}$ in conjecture 1.2) by*

$$\mathcal{R}_\theta := \xi_{\theta,l}\mathcal{O}_K$$

Remark 3.4. *In the situation $a_\theta \equiv b_\theta$ mod$|\mathcal{O}_K^*|$, Deninger defines also a divisor whose image with respect to the composition of the Eisenstein map with the projector map satisfies the above Theorem 3.2 ([6, §5]).*

As a consequence of Theorem 3.2, we have that our submodule $\mathcal{R}_\theta$ verifies the Beilinson conjecture for the motive $M_\theta(w+l+1)$.

Theorem 3.5. *The $\mathcal{O}_K$-submodule $\mathcal{R}_\theta$ of $H^{w+1}_{\mathcal{M}}(M_\theta, \mathbb{Q}(w+l+1))$ satisfies that*

$$det_{\mathcal{O}_K}(r_{\mathcal{D}}(\mathcal{R}_\theta)) = L_S^*(\overline{\psi_\theta}, -l) det_{\mathcal{O}_K}(H_B^w(M_\theta, \mathbb{Z}(w+l))$$

in $det_{\mathcal{O}_K\otimes\mathbb{R}}(H_B^w(M_{\theta\mathbb{C}}, \mathbb{Z}(w+l))\otimes\mathbb{R})$. Here

$$L_S^*(\overline{\psi_\theta}, -l) = \lim_{s+l\to 0} L_S(\overline{\psi_\theta}, s)/(s+l),$$

and S is the set of primes dividing $\mathfrak{f}_\theta$ and primes dividing p.

Proof. Observing that η_θ is a $\mathcal{O}_K = \mathcal{O}_\theta$-base for the free $\mathcal{O}_\theta$-module

$$H_B^w(M_\theta \otimes_K \mathbb{C}, \mathbb{Z}(w+l))$$

of rank one, the result follows. □

Corollary 3.6. *The submodule $\mathcal{R}_\theta$ defined above satisfies the Beilinson conjecture inside the p-local Tamagawa number conjecture, that is $\mathcal{R}_\theta$ satisfies the following conditions:*

1. *The map $r_{\mathcal{D}}\otimes\mathbb{R}$ is a isomorphism when restricted to $\mathcal{R}_\theta\otimes\mathbb{R}$.*
2. *$dim_{\mathbb{Q}}(H_B^w(M_\theta, \mathbb{Z}(w+l))\otimes\mathbb{Q}) = ord_{s=-l}L_S(H^w(M_\theta, \mathbb{Q}_p), s) = 2$.*
3. *We have the following equality*

$$r_{\mathcal{D}}(det_{\mathbb{Z}}(\mathcal{R}_\theta)) = L_S^*(H_{et}^w(M_\theta, \mathbb{Q}_p), -l) det_{\mathbb{Z}}(H_B^w(M_\theta, \mathbb{Z}(w+l)))$$

where $L_S^*(H_{et}^w(M_\theta, \mathbb{Q}_p), -l)$ means $\lim_{s\to -l} L_S^*(H_{et}^w(M_\theta, \mathbb{Q}_p), s)/(s+l)^2$ (this makes sense by using theorem 3.1 and theorem 2.5).

Proof. The first and the second conditions are clear for the dimensions of the spaces involved in the Deligne regulator map, and the theorem 3.5. The third condition comes from the previous theorem using the fact that, if we multiply an $\mathcal{O}_\theta = \mathcal{O}_K$-module with an element $L_S^*(\overline{\psi_\theta}, -l)$ in $\mathcal{O}_\theta\otimes\mathbb{R}$, the determinant is multiplied by the norm

$$N_{\mathcal{O}_\theta\otimes\mathbb{R}/\mathbb{R}}(L_S^*(\overline{\psi_\theta}, -l)) = L_S^*(\overline{\psi_\theta}, -l)\overline{L_S^*(\overline{\psi_\theta}, -l)} = L_S^*(\overline{\psi_\theta}, -l) L_S^*(\psi_\theta, -l).$$

Using theorem 2.5, we obtain that this is equal to $L_S^*(H_{et}^w(M_\theta, \mathbb{Q}_p), -l)$. □

4 Iwasawa theory

In this section we will define a map that relates the cohomology groups $H^i_{p,\otimes^w}$ for the p-adic lattice in the p-adic realization of $M_\theta(w+l+1)$ defined in §2, with some $\mathbb{Z}_p[[X,Y]]$-modules in Iwasawa theory.

To simplify the notation, we will denote in this section by

$$M_{\theta\mathbb{Z}_p}(m) = (e_\theta \otimes^w H^1_{et}(E \times_K \overline{K}, \mathbb{Z}_p))(m)$$

the p-adic lattice for the p-adic realization of $M_\theta(m)$.

Suppose in the following and once and for all that $p \nmid \#|\mathcal{O}_K^*|$ (if $p > 3$ this condition is satisfied). Let $K_n := K(E[p^{n+1}])$ be the field of definition of the p^{n+1}-torsion points of E, and let $K_\infty := \varprojlim K_n$ its inverse limit. Denote by $\mathcal{O}_n$ the ring of integers of these fields (respectively $\mathcal{O}_\infty$). Then $\Delta := Gal(K_0/K)$ has order prime to p and $\Gamma := Gal(K_\infty/K_0)$ is isomorphic to $\mathbb{Z}_p^2$.

Let $\mathcal{G}$ be the Galois group $Gal(K_\infty/K)$; then $\mathcal{G} \cong \Delta \times \Gamma$. Let $\mathcal{A}_n$ be the p-part of the ideal class group of K_n, and let $\mathcal{E}_n$ be the group of global units $\mathcal{O}_n^*$ of K_n. Let $\mathcal{U}_n^{\mathfrak{p}}$ be the group of local units of $K_n \otimes_K K_{\mathfrak{p}}$ which are congruent to 1 module the primes above $\mathfrak{p}$, where $\mathfrak{p}$ is a prime of $\mathcal{O}_K$ lying over p.

For every prime v of K_n above $\mathfrak{p}$, there is then an exact sequence

$$1 \to \mathcal{U}_{n,v} \to K^*_{n,v} \to \mathbb{Z} \times \kappa_n^* \to 1$$

and $\mathcal{U}_n^{\mathfrak{p}} = \oplus_{v|\mathfrak{p}} \mathcal{U}_{n,v}$. Here $\mathcal{U}_{n,v}$ are the local units congruent to 1 modulo v and κ_n is the residue class field of $K_{n,v}$.

First of all, let's recall briefly the definition of the elliptic units $\mathcal{C}_n$ in K_n. For every ideal $\mathfrak{a}$ of K prime to 6 we can define a theta function

$$\theta_{\mathfrak{a}} : E \setminus ker([\mathfrak{a}]) \longrightarrow \mathbb{C}$$

which has divisor $N(\mathfrak{a})(e) - ker([\mathfrak{a}])$ (for the precise definition see [19, (4.2.2)]). The function $\theta_{\mathfrak{a}}(z)$ is in fact a 12-th root of the function defined in [7, II.2.4]. Let $\mathfrak{f}_\theta$ be a fixed ideal of $\mathcal{O}_K$ such that $\mathcal{O}_K^* \to \mathcal{O}_K/\mathfrak{f}_\theta$ is injective, and suppose that $\mathfrak{f}_\theta$ divides the conductor $\mathfrak{f}$ of the elliptic curve E. Let's denote by $t_{\mathfrak{f}_\theta}$ a generator for $E[\mathfrak{f}_\theta]$-torsion points as $\mathcal{O}_K$-module, and let $\mathfrak{a}$ be an ideal prime to $6\mathfrak{f}_\theta$.

Definition 4.1. *Let C_n be the subgroup of units generated over $\mathbb{Z}[Gal(K_n/K)]$ by*

$$\prod_{\sigma \in Gal(K(\mathfrak{f}_\theta)/K)} \theta_{\mathfrak{a}}(t^\sigma_{\mathfrak{f}_\theta} + h_n),$$

where $\mathfrak{a}$ runs through all ideals prime to $6p\mathfrak{f}$, $K(\mathfrak{f}_\theta)$ is the ray class field defined by $\mathfrak{f}_\theta$ and h_n is a primitive p^n-torsion point (i.e. a generator of the p^n-torsion points of E as $\mathcal{O}_K$-module). Define the group of elliptic units of K_n as

$$\mathcal{C}_n := \mu_\infty(K_n)C_n,$$

where $\mu_\infty(K_n)$ denotes the roots of unity in K_n.

Denote by $\overline{\mathcal{E}}_n$ and $\overline{\mathcal{C}}_n$ the closures of $\mathcal{E}_n \cap \mathcal{U}_n^{\mathfrak{p}}$ respectively $\mathcal{C}_n \cap \mathcal{U}_n^{\mathfrak{p}}$ in $\mathcal{U}_n^{\mathfrak{p}}$. Define

$$\mathcal{A}_\infty := \varprojlim \mathcal{A}_n,\ \overline{\mathcal{E}}_\infty := \varprojlim \overline{\mathcal{E}}_n,\ \overline{\mathcal{C}}_\infty := \varprojlim \overline{\mathcal{C}}_n,\ \mathcal{U}_\infty^{\mathfrak{p}} := \varprojlim \mathcal{U}_n^{\mathfrak{p}}$$

where the limits are taken with respect to the norm maps.

Denote by $M_\infty^{\mathfrak{p}}$ the maximal abelian p-extension of K_∞ which is unramified outside of the primes above $\mathfrak{p}$, and write $\mathcal{X}_\infty^{\mathfrak{p}} := Gal(M_\infty^{\mathfrak{p}}/K_\infty)$. Define the Iwasawa algebra

$$\mathbb{Z}_p[[\mathcal{G}]] := \varprojlim \mathbb{Z}_p[Gal(K_n/K)]$$

which has a natural action of $\mathbb{Z}_p[\Delta]$.

For any irreducible $\mathbb{Z}_p$-representation χ of Δ, consider

$$e_\chi := \frac{1}{\#\Delta} \sum_{\tau \in \Delta} Tr(\chi(\tau))\tau^{-1} \in \mathbb{Z}_p[\Delta]$$

and for every $\mathbb{Z}_p[\Delta]$-module Z denote by $Z^\chi := e_\chi Z$.

We define

$$\Lambda^\chi := \mathbb{Z}_p[[\mathcal{G}]]^\chi = R_\chi[[\Gamma]]$$

where R_χ is the ring of integers in the unramified extension of $\mathbb{Z}_p$ of degree $\dim(\chi)$.

We denote by $\Lambda := \mathcal{O}_p[[\Gamma]]$ where $\mathcal{O}_p = \mathcal{O}_K \otimes \mathbb{Z}_p$. The modules $\mathcal{A}_\infty^\chi$ and $\overline{\mathcal{E}}_\infty^\chi/\overline{\mathcal{C}}_\infty^\chi$ are torsion Λ^χ-modules. The classical theorem for the main conjecture in Iwasawa theory, using the determinant notation instead of the characteristic ideal (see [18, Proposition 6.1]), states:

Theorem 4.2 (Rubin, theorem 4.1 in [21]). *Let $p \nmid \#\mathcal{O}_K^*$.*
i) Suppose that p splits in K. Then

$$det_{\Lambda^\chi}(\mathcal{A}_\infty^\chi) = det_{\Lambda^\chi}(\overline{\mathcal{E}}_\infty^\chi/\overline{\mathcal{C}}_\infty^\chi).$$

ii) Suppose that p remains prime or ramifies in K and that χ is nontrivial on the decomposition group of $\mathfrak{p}$ in Δ. Then

$$det_{\Lambda^\chi}(\mathcal{A}_\infty^\chi) = det_{\Lambda^\chi}(\overline{\mathcal{E}}_\infty^\chi/\overline{\mathcal{C}}_\infty^\chi).$$

Remark 4.3. Rubin uses for the above theorem 4.2 in [21], another definition for the elliptic units as the ones defined above, but the result remains true with our definition of elliptic units. Let me briefly explain the reasons for this claim. The original definition [21] defines $C_n(F)$ with F an abelian extension of K by

$$\{N_{FK(\mathfrak{g})/F}\theta_{\mathfrak{a}}(\tau)^{\sigma-1} | \tau, \mathfrak{g}, \mathfrak{a} \textit{ as follows, and } \sigma \in Gal(F/K)\},$$

where $\mathfrak{g}$ runs any ideal of $\mathcal{O}_K$ such that $\mathcal{O}_K^*$ injects into $\mathcal{O}_K/\mathfrak{g}$, τ is an exact $\mathfrak{g}$ division point and $\mathfrak{a}$ be an ideal prime to $6\mathfrak{g}$. Denote by I the augmentation ideal for $Gal(K_\infty/K)$ and by $\mathcal{S}$ the annihilator in $\mathbb{Z}_p[[X,Y]]$ of $\mu_{p^\infty}(K_\infty)$, the group of p-power roots of unity in K_∞. Then the elliptic units in [21] are isomorphic as $\mathbb{Z}_p[[X,Y]]$-modules to $I\mathcal{S}$ (cf. [21, thm 7.7]) . We observe, taking invariants by χ, if χ is not trivial, that I becomes invertible; this implies that we can change the Rubin's definition of elliptic units for another one without multiply our theta function by $\sigma - 1$. Our elliptic units are isomorphic to $\mathcal{S}$, then using the observation by the ideal augmentation we can modify them multiplying by $\sigma - 1$. After this modification, our elliptic units are included in the ones of Rubin in [21], and the Λ^χ-determinant remains equal with this modification. If $\mathfrak{f}_\theta = \mathfrak{f}$ the conductor of the elliptic curve, is known by the specialist theorem

4.2 is true with the definition 4.1. If $\mathfrak{f}_\theta$ divides $\mathfrak{f}$, we can relate our definition of elliptic units (4.1) for the fix ideal $\mathfrak{f}_\theta$ with the definition of elliptic units (4.1) with $\mathfrak{f}$ as a fix ideal. We obtain that the one defined by $\mathfrak{f}$ is exactly multiply by $\prod_{\mathfrak{l}}(\sigma_{\mathfrak{l}}^{-1}-1)$ the one with $\mathfrak{f}_\theta$, where $\sigma_{\mathfrak{l}}$ is the Frobenius at $\mathfrak{l}$, where $\mathfrak{l}$ runs in the prime ideals of K which divide $\mathfrak{f}$ but do not divide $\mathfrak{f}_\theta$. Then we relate both definitions multiplying with an element of the augmentation ideal. Then we have an inclusion:

$$\mathcal{C}_{\infty,\mathfrak{f}} \subset \mathcal{C}_{\infty,\mathfrak{f}_\theta} \subset \mathcal{C}_\infty,$$

because we modify our definition by the augmentation ideal and the element that multiply belongs to this ideal. Now, we take χ-invariant and Λ^χ-determinants. We know that the first determinant coincides with the third one, because the theorem 4.1 is satisfied in the case $\mathfrak{f}_\theta = \mathfrak{f}$ for our definition of elliptic units, then as consequence the determinant of the middle coincides also with the third one, obtaining then our claim.

Let $\mathcal{X}_\infty$ be the Galois group for the maximal abelian p-extension M_∞^p of K_∞ over K_∞ which is unramified outside of the primes above p. Define also

$$\mathcal{U}_\infty := \mathcal{U}_\infty^{\mathfrak{p}} \times \mathcal{U}_\infty^{\mathfrak{p}^*}$$

if $p = \mathfrak{p}\mathfrak{p}^*$ is split, and

$$\mathcal{U}_\infty := \mathcal{U}_\infty^{\mathfrak{p}}$$

if p is inert or ramified. Similarly, let $\mathcal{Y}_n$ be the p-adic completion of $(K_n \otimes \mathbb{Q}_p)^*$ and $\mathcal{Y}_\infty := \varprojlim \mathcal{Y}_n$. We have an inclusion $\mathcal{U}_\infty \subset \mathcal{Y}_\infty$. Using Class field theory we have an exact sequence

$$0 \to \overline{\mathcal{E}}_\infty/\overline{\mathcal{C}}_\infty \to \mathcal{U}_\infty/\overline{\mathcal{C}}_\infty \to \mathcal{X}_\infty \to \mathcal{A}_\infty \to 0, \tag{1}$$

where $\overline{\mathcal{C}}_\infty$ is diagonally embedded into $\mathcal{U}_\infty^{\mathfrak{p}} \times \mathcal{U}_\infty^{\mathfrak{p}^*}$ if p is split.

We can consider moreover the representations $\chi : \Delta \to \mathcal{O}_K^*$. For any topological group B, we denote by $\hat{B}$ the completion by the ideal p. Denote then by

$$B^\chi := \{b \in \hat{B} \otimes \mathcal{O}_K | \sigma b = \chi(\sigma) b \;\; \forall \sigma \in \Delta\}.$$

Using this notation, we have also an idempotent

$$e_\chi := \frac{1}{\#\Delta} \sum_{\tau \in \Delta} Tr(\chi(\tau))\tau^{-1} \in \mathcal{O}_K \otimes \mathbb{Z}_p[\Delta].$$

Using the sequence (1) one gets easily [19, cor. 2.1.5]:

Corollary 4.4. *Under the same hypothesis of theorem 4.2, and if $\mathcal{X}_\infty^\chi$ denotes $e_\chi \mathcal{X}_\infty$, we have that*

$$det_{\Lambda^\chi}(\mathcal{X}_\infty^\chi) = det_{\Lambda^\chi}(\mathcal{U}_\infty^\chi/\overline{\mathcal{C}}_\infty^\chi).$$

To control the coimage of the inclusion of $\mathcal{U}_\infty \subset \mathcal{Y}_\infty$ we will use the following result, [19, lemma 2.1.6].

Lemma 4.5. *Let p be a prime such that $p \nmid N_{K/\mathbb{Q}}\mathfrak{f}$. If p splits in K, the inclusion $\mathcal{U}_\infty \to \mathcal{Y}_\infty$ is an isomorphism and if p is inert or ramified in K, there is an exact sequence*

$$0 \to \mathcal{U}_\infty \to \mathcal{Y}_\infty \to \mathbb{Z}_p[\Delta/\Delta_p] \to 0$$

where Δ_p is the decomposition group of p in Δ.

Recall that $\mathcal{O}_S = \mathcal{O}_K[1/S]$. Here and in the following S denotes the set of primes of K which divide $\mathfrak{f}_\theta$ plus the primes of K above p, and S' denotes the set of primes of K above p and the primes which divide the conductor $\mathfrak{f}$ of the elliptic curve E. Denote by S_p the set of finite primes over p, and by $\mathcal{O}_{n,S_p}$ the ring of integers of K_n where all the primes above p are inverted. We define then $\mathcal{O}_{\infty,S_p} := \varinjlim \mathcal{O}_{n,S_p}$. Similarly we define $\mathcal{O}_{n,S}$ and $\mathcal{O}_{\infty,S}$, where the primes above S are inverted in K_n or K_∞ respectively.

After introducing all the above notations, we are going to define a map in the spirit of Soulé:

$$e_p : \overline{\mathcal{C}}_\infty \otimes_{\mathbb{Z}_p} (e_\theta \otimes^w T_pE)(l) \to H^1(\mathcal{O}_S, (e_\theta \otimes^w T_pE)(l+1)).$$

All makes sense because $M_{\theta\mathbb{Z}_p}(w+l)$ is unramified outside S and it is consider as $\mathcal{O}_S$-sheaf. Denote in the following by $\mathcal{O}_p := \mathcal{O}_K \otimes \mathbb{Z}_p$ and by $\mathcal{O}_{\theta,p} := \mathcal{O}_\theta \otimes \mathbb{Z}_p$.

Using the definition of $M_{\theta\mathbb{Z}_p}(w)(l+1)$, we have that

$$H^1(\mathcal{O}_S, (e_\theta \otimes^w T_pE)(l+1)) = \varprojlim H^1(\mathcal{O}_S, (e_\theta \otimes^w E[p^{r+1}])(l+1)).$$

Define e_p in the following way. Consider $(\theta_r)_r$ a norm compatible system of elliptic units and an element $(t_r)_r$ of $\varprojlim(e_\theta(\otimes^w E[p^{r+1}]))(l)$, then we define

$$e_p((\theta_r \otimes t_r)_r) := (Norm_{K_r/K}(\theta_r \otimes t_r))_r.$$

It is well defined because $\theta_r \otimes t_r$ is an element in

$$\mathcal{O}^*_{r,S}/(\mathcal{O}_{r,S})^{p^{r+1}} \otimes (e_\theta(\otimes^w E[p^{r+1}]))(l) \subset H^1(\mathcal{O}_{r,S}, (e_\theta(\otimes^w E[p^{r+1}]))(l+1))$$

Definition 4.6. *The Soulé elliptic elements are the elements in the image of the map*

$$e_p : (\overline{\mathcal{C}}_\infty \otimes M_{\theta\mathbb{Z}_p}(w+l))_{\mathcal{G}} \to H^1(\mathcal{O}_S, M_{\theta\mathbb{Z}_p}(w+l+1))$$

where $\mathcal{G} = Gal(K(E[p^\infty])/K)$.

We need to find a finite field extension field of K such that $M_{\theta\mathbb{Z}_p}(w+m)$ is unramified in all places outside p. We know that the elliptic curve has good reduction over K_0 at all the places not dividing p. And hence, using Serre-Tate theorem, the Tate module of the elliptic curve is unramified in those places. In particular, our tensor product of Tate modules twisted by m, corresponding to $M_{\theta\mathbb{Z}_p}(w+m)$, is unramified in all the places not dividing p in K_0. Therefore, K_0 verifies the conditions.

For a G_K-module M, we define

$$H^1(K_\infty \otimes \mathbb{Q}_p, M) := \varprojlim H^1(K_n \otimes \mathbb{Q}_p, M) = \varprojlim \oplus_{v|p} H^1(K_{n,v}, M).$$

Suppose that M is a $\mathcal{O}_p := \mathcal{O}_K \otimes \mathbb{Z}_p$-module. We will use the following notation: If M has and action of absolute Galois group G_K of K,

$$M' := Hom_{\mathcal{O}_p}(M, \mathbb{Q}_p/\mathbb{Z}_p(1) \otimes_{\mathbb{Z}_p} \mathcal{O}_p)$$

with the natural action of G_K; and if M has an action of a group G

$$M^* := Hom_{\mathcal{O}_p}(M, \mathbb{Q}_p/\mathbb{Z}_p \otimes_{\mathbb{Z}_p} \mathcal{O}_p)$$

with the natural action of G.

We follow closely the argument in [19, §2], but now the p-adic integer realization is $M_{\theta\mathbb{Z}_p}(w)$ instead of the Tate module of the elliptic curve E.

Proposition 4.7. *There are isomorphisms of $\mathcal{O}_p[[\mathcal{G}]]$-modules*

$$\mathcal{X}_\infty \otimes_{\mathbb{Z}_p} M_{\theta\mathbb{Z}_p}(w+l) \cong H^1(\mathcal{O}_{\infty,S_p}, M_{\theta\mathbb{Z}_p}(w+l+1)')^*$$

$$\mathcal{Y}_\infty \otimes_{\mathbb{Z}_p} M_{\theta\mathbb{Z}_p}(w+l) \cong H^1(K_\infty \otimes \mathbb{Q}_p, M_{\theta\mathbb{Z}_p}(w+l+1)')^*.$$

Proof. See [19, Proof prop. 2.2.3] or [2, Proof prop. 3.4.7]. □

Proposition 4.8. *The groups*

$$H^2(K_\infty \otimes \mathbb{Q}_p, M_{\theta\mathbb{Z}_p}(w+l+1)') \text{ and } H^2(\mathcal{O}_{\infty,S_p}, M_{\theta\mathbb{Z}_p}(w+l+1)')$$

are zero.

Proof. See [19, Proof prop. 2.2.4] or [2, Proof prop. 3.4.8]. □

Let's now recall [19, lemma 2.2.6].

Lemma 4.9. *Let M be a perfect complex of $\Lambda = \mathcal{O}_p[[\Gamma]]$-modules. Then here are canonical isomorphism*

$$M^* \otimes^{\mathbb{L}}_\Lambda \mathcal{O}_p \cong R\Gamma(\Gamma, M)^*$$

where the right hand side is the (continuous) group cohomology of Γ and $M^* = Hom(M, \mathbb{Q}_p/\mathbb{Z}_p \otimes \mathcal{O}_p)$.

Using this lemma, we obtain the following result.

Corollary 4.10. *There are exact triangles*

$$(\mathcal{Y}_\infty \otimes M_{\theta\mathbb{Z}_p}(w+l)) \otimes^{\mathbb{L}}_\Lambda \mathcal{O}_p \to R\Gamma\left(K_0 \otimes \mathbb{Q}_p, M_{\theta\mathbb{Z}_p}(w+l+1)'\right)^*[-1]$$

$$\to R\Gamma\left(\Gamma, H^0(K_\infty \otimes \mathbb{Q}_p, M_{\theta\mathbb{Z}_p}(w+l+1)')\right)^*[-1]$$

and

$$(\mathcal{X}_\infty \otimes M_{\theta\mathbb{Z}_p}(w+l)) \otimes^{\mathbb{L}}_\Lambda \mathcal{O}_p \to R\Gamma(\mathcal{O}_{0,S_p} \otimes \mathbb{Q}_p, M_{\theta\mathbb{Z}_p}(w+l+1)')^*[-1]$$

$$\to R\Gamma\left(\Gamma, H^0(\mathcal{O}_{\infty,S_p}, M_{\theta\mathbb{Z}_p}(w+l+1)')\right)^*[-1].$$

If in the definition of e_p only take the norm maps to K_0, we get a map of complexes

$$e_p : \overline{\mathcal{C}}_\infty \otimes M_{\theta\mathbb{Z}_p}(w+l) \to H^1(\mathcal{O}_{0,S_p}, M_{\theta\mathbb{Z}_p}(w+l+1))$$

Recall that, for weight reasons, we have that $H^0(\mathcal{O}_{0,S_p}, M_{\theta\mathbb{Z}_p}(w+l+1)) = 0$. So, we obtain a map of complexes

$$e_p : (\overline{\mathcal{C}}_\infty \otimes M_{\theta\mathbb{Z}_p}(w+l)) \otimes^{\mathbb{L}}_\Lambda \mathcal{O}_p \to R\Gamma(\mathcal{O}_{0,S_p}, M_{\theta\mathbb{Z}_p}(w+l+1))[1]$$

Lemma 4.11. *-The following diagram is commutative*

$$\begin{array}{ccc}
(\overline{\mathcal{C}}_\infty \otimes M_{\theta\mathbb{Z}_p}(w+l)) \otimes^{\mathbb{L}}_{\Lambda} \mathcal{O}_p & \xrightarrow{e_p} & R\Gamma(\mathcal{O}_{0,S_p}, M_{\theta\mathbb{Z}_p}(w+l+1))[1] \\
\downarrow & & \downarrow \\
(\mathcal{Y}_\infty \otimes M_{\theta\mathbb{Z}_p}(w+l)) \otimes^{\mathbb{L}}_{\Lambda} \mathcal{O}_p & \longrightarrow & R\Gamma(K_0 \otimes \mathbb{Q}_p, M_{\theta\mathbb{Z}_p}(w+l+1)')^*[-1] \\
\alpha \downarrow & & \downarrow \\
(\mathcal{X}_\infty \otimes M_{\theta\mathbb{Z}_p}(w+l)) \otimes^{\mathbb{L}}_{\Lambda} \mathcal{O}_p & \longrightarrow & R\Gamma(\mathcal{O}_{0,S_p}, M_{\theta\mathbb{Z}_p}(w+l+1)')^*[-1]
\end{array}$$

where the map α is induced by the natural map $\mathcal{Y}_\infty/\overline{\mathcal{C}}_\infty \to \mathcal{X}_\infty$.

Proof. See [19, Proof lemma 2.2.8] or [2, Proof lemma 3.4.11]. □

We consider in the following the representation χ of the group Δ given by the action of Δ in

$$Hom_{\mathcal{O}_p}(M_{\theta\mathbb{Z}_p}(w+l), \mathcal{O}_p).$$

We impose that the above representation is irreducible. If (a_θ, b_θ) denotes the infinity type for ψ_θ, then ψ_θ is irreductible if $p-1 \nmid a_\theta - b_\theta$ and p splits in K or if $p+1 \nmid a_\theta - b_\theta$ and p is inert in K.

Definition 4.12. *A character χ of the group Δ is called a good character if it is irreducible and it satisfies the Iwasawa main conjecture (that is, if χ satisfies that $det_{\Lambda^\chi}(\mathcal{A}^\chi_\infty) \cong det_{\Lambda^\chi}(\overline{\mathcal{E}}^\chi_\infty/\overline{\mathcal{C}}^\chi_\infty)$), and that*

$$\mathcal{U}^\chi_\infty \cong \mathcal{Y}^\chi_\infty.$$

These conditions are always satisfied when the prime p splits in K. If p is inert or ramified, χ is good if χ is non trivial on the decomposition group of $\mathfrak{p}$ in Δ (by theorem 4.2(b)) and $\mathbb{Z}_p[\Delta/\Delta_p]^\chi = 0$ by lemma 4.5. We impose once for all that χ is not the cyclotomic character and that χ is a good character.

Lemma 4.13. *Observe that our character χ is equal to $(\overline{\psi}_\theta \kappa^l)^{-1}$ where κ is the cyclotomic character. Suppose that p splits in K and suppose that $p-1 \nmid a_\theta+l+1$ or $p-1 \nmid b_\theta+l+1$ or $p-1 \nmid a_\theta - b_\theta$. Then χ is not the cyclotomic character.*

Proof. Since p is split in K we have that $p = \mathfrak{p}\mathfrak{p}^*$, where $\mathfrak{p}$ is not equal to $\mathfrak{p}^*$. Let $\Delta_\mathfrak{p}$ be the Galois group $Gal(K(E[\mathfrak{p}])/K)$; it is a subgroup of the decomposition group since $\mathfrak{p}$ is totally ramified in $\Delta_\mathfrak{p}$. Observe that $\overline{\psi}_\theta \otimes \mathbb{Z}_p = \psi_{\Omega_1} \oplus \psi_{\Omega_2}$, since p is split. It is known $\overline{\psi}_{\Omega_1}|_{\Delta_\mathfrak{p}} = \kappa^{b_\theta}$ (see for example [10, §2.5]), so we get that our character is different for κ as long as $\#\Delta_\mathfrak{p} = p-1 \nmid b_\theta+l+1$, since κ is a generator for the character group of $\Delta_\mathfrak{p}$.

Using the same kind of argument for $\mathfrak{p}^*$ applied also to the character ψ_{Ω_1} we obtain the same result but with a_θ instead of b_θ. Thus, we will obtain the cyclotomic character only in the case that $p-1 \mid a_\theta+l+1$ and $p-1 \mid b_\theta+l+1$.

We can also use the same argument for ψ_{Ω_2}, obtaining the same simultaneous arithmetic conditions, i.e. $p-1|l+b_\theta+1$ and $p-1|a_\theta+l+1$. We refer to [11, p.220,pp.223-234] for more details on the above characters. □

The goal of the rest of the section is to show that the map e_p induces an isomorphism of $\mathcal{O}_p$-modules

$$det_{\mathcal{O}_p}((\overline{\mathcal{C}}^\chi_\infty \otimes M_{\theta\mathbb{Z}_p}(w+l)) \otimes^{\mathbb{L}}_{\Lambda} \mathcal{O}_p) \cong det_{\mathcal{O}_p}(R\Gamma(\mathcal{O}_S, M_{\theta\mathbb{Z}_p}(w+l+1)))^{-1}.$$

Proposition 4.14. *With the same hypothesis as above, we have that*

$$det_{\mathcal{O}_p}(R\Gamma(\mathcal{G}, H^0(K_\infty \otimes \mathbb{Q}_p, M_{\theta\mathbb{Z}_p}(w+l+1)'))) \cong \mathcal{O}_p$$

$$det_{\mathcal{O}_p}(R\Gamma(\mathcal{G}, H^0(\mathcal{O}_{\infty,S_p}, M_{\theta\mathbb{Z}_p}(w+l+1)'))) \cong \mathcal{O}_p$$

Proof. It follows from [10, prop. 2.4.6], that the action of $\mathcal{G}$ on

$$M_{\theta\mathbb{Z}_p}(w+l) \cong (e(\mathcal{O}_K \otimes \cdots \mathcal{O}_K) \otimes \mathbb{Q}_p) \cong \mathcal{O}_{\theta,p}$$

is via the character

$$\psi_\theta : \mathcal{G} \to \mathcal{O}_{\theta,p}^*.$$

We have hence a surjection of $\mathcal{O}_p$-modules $\rho : \mathcal{O}_p[[\Gamma]] \to M_{\theta\mathbb{Z}_p}(w+l)$. Thus $ker(\rho)$ is an ideal of height 2 because $\Gamma \cong \mathbb{Z}_p^2$. We know that det_R is determined by the ideals of height 1 for the ring R (cf. [17, 2.1.4]). This implies that

$$det_{\mathcal{O}_p}(M_{\theta\mathbb{Z}_p}(w+l) \otimes^{\mathbb{L}}_{\mathcal{O}_p[[\mathcal{G}]]} \mathcal{O}_p) \cong \mathcal{O}_p. \tag{2}$$

In fact, since Δ is finite and $\mathcal{G} \cong \Gamma \times \Delta$, we have the equality

$$M_{\theta\mathbb{Z}_p}(w+l) \otimes^{\mathbb{L}}_{\mathcal{O}_p[[\mathcal{G}]]} \mathcal{O}_p \cong (M_{\theta\mathbb{Z}_p}(w+l))_\Delta \otimes^{\mathbb{L}}_{\mathcal{O}_p[[\Gamma]]} \mathcal{O}_p.$$

Since we know that $ker(\rho)$ has height 2, we have $det_{\mathcal{O}_p[[\Gamma]]}((M_{\theta\mathbb{Z}_p}(w+l))_\Delta) \cong \mathcal{O}_p[[\Gamma]]$ and so $det_{\mathcal{O}_p}((M_{\theta\mathbb{Z}_p}(w+l))_\Delta \otimes^{\mathbb{L}}_{\mathcal{O}_p[[\Gamma]]} \mathcal{O}_p) \cong \mathcal{O}_p$. This checks (2). We conclude by using lemma 4.9. □

Using the isomorphism

$$\mathcal{Y}_\infty^\chi \cong \mathcal{U}_\infty^\chi$$

coming from the property that χ is a good character, and applying the derived functor $R\Gamma(\Delta,)$ to the sequences in corollary 4.10 it proves:

Corollary 4.15.

$$det_{\mathcal{O}_p}((\mathcal{U}_\infty^\chi \otimes_{\mathcal{O}_p} M_{\theta\mathbb{Z}_p}(w+l)) \otimes^{\mathbb{L}}_\Lambda \mathcal{O}_p) \cong$$

$$det_{\mathcal{O}_p}\left(H^0(\Delta, R\Gamma(K_0 \otimes \mathbb{Q}_p, M_{\theta\mathbb{Z}_p}(w+l+1)')^*[-1])\right)$$

and that

$$det_{\mathcal{O}_p}\left((\mathcal{X}_\infty^\chi \otimes_{\mathcal{O}_p} M_{\theta\mathbb{Z}_p}(w+l)) \otimes^{\mathbb{L}}_\Lambda \mathcal{O}_p\right) \cong$$

$$det_{\mathcal{O}_p}\left(H^0(\Delta, R\Gamma(\mathcal{O}_{0,S_p} \otimes \mathbb{Q}_p, M_{\theta\mathbb{Z}_p}(w+l+1)')^*[-1])\right)$$

So, if we apply the functor $R\Gamma(\Delta, \quad)$ to the triangle

$$R\Gamma(\mathcal{O}_{0,S_p}, M_{\theta\mathbb{Z}_p}(w+l+1)) \to R\Gamma(K_0 \otimes \mathbb{Q}_p, M_{\theta\mathbb{Z}_p}(w+l+1)^{'})^*[-2]$$

$$\to R\Gamma(\mathcal{O}_{0,S_p}, M_{\theta\mathbb{Z}_p}(w+l+1)^{'})^*[-2],$$

we obtain the following result.

Corollary 4.16. *There is an isomorphism of determinants*

$$det_{\mathcal{O}_p}(H^0(\Delta, R\Gamma(\mathcal{O}_{0,S_p}, M_{\theta\mathbb{Z}_p}(w+l+1))))^{-1} \cong$$

$$det_{\mathcal{O}_p}((\mathcal{U}_\infty^\chi \otimes_{\mathcal{O}_p} M_{\theta\mathbb{Z}_p}(w+l)) \otimes^{\mathbb{L}}_\Lambda \mathcal{O}_p) det_{\mathcal{O}_p}((\mathcal{X}_\infty^\chi \otimes_{\mathcal{O}_p} M_{\theta\mathbb{Z}_p}(w+l)) \otimes^{\mathbb{L}}_\Lambda \mathcal{O}_p)^{-1}$$

We need now to find the relation between $\mathcal{O}_{0,S_p}$ and $\mathcal{O}_{0,S}$.

Lemma 4.17. *The restriction map induces an equality of determinants*

$$det_{\mathcal{O}_p}(H^0(\Delta, R\Gamma(\mathcal{O}_{0,S_p}, M_{\theta\mathbb{Z}_p}(w+l+1))) \cong$$

$$det_{\mathcal{O}_p}(H^0(\Delta, R\Gamma(\mathcal{O}_{0,S}, M_{\theta\mathbb{Z}_p}(w+l+1)))) \cong det_{\mathcal{O}_p}(R\Gamma(\mathcal{O}_S, M_{\theta\mathbb{Z}_p}(w+l+1))).$$

Proof. Consider the exact triangle

$$R\Gamma(\mathcal{O}_{0,S_p}, M_{\theta\mathbb{Z}_p}(w+l)) \to R\Gamma(\mathcal{O}_{0,S}, M_{\theta\mathbb{Z}_p}(w+l))$$

$$\to \oplus_{v\in S\setminus S_p} R\Gamma_{k(v)}(\mathcal{O}_v, M_{\theta\mathbb{Z}_p}(w+l))[1]$$

where $\mathcal{O}_v$ is the local ring at v. Since T_pE is unramified at the places of K_0 in $S \setminus S_0$, the same is true for $e_\theta(\otimes T_pE)(l)$. By purity we have that

$$R\Gamma_{k(v)}(\mathcal{O}_v, M_{\theta\mathbb{Z}_p}(w+l)) \cong R\Gamma(k(v), M_{\theta\mathbb{Z}_p}(w+l)).$$

But we have that

$$H^0(\Delta, \oplus_{v\in S\setminus S_p} R\Gamma(k(v), M_{\theta\mathbb{Z}_p}(w+l)) = 0.$$

To show this result, observe that

$$H^1(k(v), M_{\theta\mathbb{Z}_p}(w+l)) \cong M_{\theta\mathbb{Z}_p}(w+l)_{Gal(\overline{k(v)}/k(v))}$$

and $H^0 = 0$ since we are under the hypothesis that $-w-2l \leq -3$. Now, let v_o be a prime dividing v in K_0 and let Δ_{v_0} be the stabilizer of v_o. Since $I_{v_0} \subset \Delta_{v_0}$ acts non trivially in the coinvariants

$$M_{\theta\mathbb{Z}_p}(w+l)_{Gal(\overline{k(v)}/k(v))}$$

because $v_0|\mathfrak{f}_\theta$, we obtain the result. □

As a consequence we have the main result of this section.

Theorem 4.18. *Suppose that p is an odd prime, prime to $N_{K/\mathbb{Q}}\mathfrak{f}_\theta$ and to $\#|\mathcal{O}_K^*|$. Let χ the Δ-representation on $Hom_{\mathcal{O}_p}(M_\theta(w+l), \mathcal{O}_p)$ be a good representation. Then, the map e_p induces an isomorphism of $\mathcal{O}_p$-modules*

$$det_{\mathcal{O}_p}((\overline{\mathcal{C}}^\chi_\infty \otimes_{\mathcal{O}_p} M_{\theta\mathbb{Z}_p}(w+l)) \otimes^{\mathbb{L}}_\Lambda \mathcal{O}_p) \cong det_{\mathcal{O}_p}(R\Gamma(\mathcal{O}_S, M_{\theta\mathbb{Z}_p}(w+l+1)))^{-1}.$$

Proof. See [19, Proof thm.2.2.12] or [2, Proof thm.3.4.19]. □

5 The comparison between the map r_p and e_p in the constructible K-elements

Let's start recalling the result of Kings on the specialization of the elliptic polylogarithm sheaf, which is a new key in the proof of the Tamagawa number conjecture.

Let E be an elliptic curve over a base scheme T, and denote by $\overline{\pi}$ the structural morphism, i.e. $\overline{\pi} : E \to T$ which is proper and smooth. Consider $U = E \setminus e$, where e is the zero section of E. There exist on U a lisse pro-sheaf (i.e. a projective limits of lisse sheaves), which is called the elliptic polylogarithm sheaf and denoted by $\mathcal{P}ol_{\mathbb{Q}_p}$ ([19, §3.2]). Let us consider t a N-torsion point in E diferent of e. Then are defined too some projections pr_t and σ ([19, §3.5.1], [19, §3.5.3] respectively), which define the p-adic k-Eisenstein class associated to a torsion point t by

$$(\sigma^k pr_t t^* \mathcal{P}ol_{\mathbb{Q}_p}),$$

for any integer k. This element is defined inside the cohomology group

$$H^1(T, Sym^k \mathcal{H}_{\mathbb{Q}_p}),$$

where $\mathcal{H}_{\mathbb{Q}_p} := \underline{Hom}_T(R^1\overline{\pi}_*\mathbb{Q}_p, \mathbb{Q}_p)$. The definition of the p-adic Eisenstein classes is extended by linearity to any divisor formed by N-torsion points ([19, Def. 3.5.9]). The main part of the result of Kings is the explicit computation of these Eisenstein classes. He compares this classes with the elements coming from a connecting map of the cohomology of some tori which appears when we consider torsors in the elliptic curve. Let us be a little more precise. Consider $H_n := ker[p^n]$ as a scheme over T. Let us consider the map multiplication by p^n, $p_n : E_n \to E$, where E_n is the elliptic curve E but which has p^n-torsors. Consider the characteristic group $I[H_n] := ker(p_{n,*}\mathbb{Z} \to \mathbb{Z})$ which is the characteristic group to a torus T_{H_n}. In this situation we have a connecting map δ as follows ([19, (10),§4.1]):

$$\delta : H^0(H_n, T_{H_n}) \to H^1(H_n, T_{H_n}[p^r]). \tag{3}$$

Using this connecting morphism, we can express the Eisenstein classes explicitly.

Theorem 5.1 (Kings, theorem 4.2.9 in [19]). *Let p be a prime number, and let E be an elliptic curve over a base scheme T where p is invertible.*

Let β be any divisor in E of the form

$$\beta := \sum_{t \in E[N](T) \setminus e} n_t(t)$$

and consider $[\mathfrak{a}] : E \to E$ any isogeny relatively prime to Np.

Then, for any $m > 0$, the p-adic Eisenstein class

$$N\mathfrak{a}(\mathfrak{a}^{\otimes m} N\mathfrak{a} - 1)(\beta^* Pol_{\mathbb{Q}_p})^m \in H^1(T, Sym^m \mathcal{H}_{\mathbb{Q}_p}(1))$$

is given by

$$\pm \frac{1}{m!}(\delta \sum_{t \in E[N](T) \setminus e} n_t \sum_{[p^n]t_n = t} \theta_{\mathfrak{a}}(-t_n) \tilde{t_n}^{\otimes m})_n$$

where $\tilde{t_n}$ is the projection of t_n to $E[p^n]$ and δ is the Sym-extension of the boundary map $H^0(H_n, T_{H_n}) \to H^1(H_n, T_{H_n}[p^r])$ where $H_n := ker[p^n]$ is consider as a scheme over T and T_{H_n} is the torus with character group $I[H_n] := ker(p_n\mathbb{Z} \to \mathbb{Z})$ (3).

The following result relates the image of $\mathcal{E}^m_{\mathcal{M}}(\beta)$ by the Soulé regulator with the polylogarithmic sheaf.

Theorem 5.2. *Let β be as in the previous theorem. Then we have that*

$$r_p(\mathcal{E}^m_{\mathcal{M}}(\beta)) = -N^{2m}(\beta^* Pol_{\mathbb{Q}_p})^m$$

in $H^1(T, Sym^m \mathcal{H}_{\mathbb{Q}_p}(1))$.

Proof. The same proof of [19, Theorem 1.2.5] with m instead of $2k+1$ works. See also [2, proof Theorem 3.5.2]. ☐

We are going to apply these results above to the divisor $\beta_\theta = N_{K(\mathfrak{f})/K}((t))$, where $t := \Omega f^{-1}$ is a $\mathfrak{f}_\theta$-torsion point. Take $N = N_{K/\mathbb{Q}}\mathfrak{f}_\theta$, $m = w+2l$, $T = \mathcal{O}_S$ and $\mathcal{H}_{\mathbb{Q}_p} = T_pE \otimes \mathbb{Q}_p$, using the notations of the previous theorem 5.1. Let $\mathfrak{a} \subset \mathcal{O}_K$ be an ideal prime to $6p\mathfrak{f}$, and consider the isogeny given by $\psi(\mathfrak{a})$. Finally, $\theta_\mathfrak{a}$ means the classical theta function.

To simplify the notation, define for any $\tilde{t_r} \in E[p^r]$

$$\gamma(\tilde{t_r})^m := < \tilde{t_r}, \sqrt{d_K}\tilde{t_r} >^{\otimes m}$$

where $<,>$ means the Weil pairing. Our objective is the computation of

$$\mathcal{K}_\mathcal{M} \circ \mathcal{E}_\mathcal{M}^{w+2l}(\beta_\theta)$$

Remember that we are under the restriction $a_\theta \not\equiv b_\theta \; mod|\mathcal{O}_K^*|$.

Considering the diagram [6, (2.8)]

$$\begin{array}{ccc} H_\mathcal{M}^{2l+w+1}(Sym^{2l+w}h^1E, \mathbb{Q}(w+2l+1)) & \overset{(\Delta_{CM})^l \times id)^*}{\longrightarrow} & H_\mathcal{M}^{2l+w+1}(E^{l+w}, \mathbb{Q}(2l+w+1)) \\ \mathcal{K}_\mathcal{M} \downarrow & & \downarrow pr_* \\ H_\mathcal{M}^{w+1}(M_{\theta\mathbb{Q}}, \mathbb{Q}(w+l+1)) & \overset{e_\theta}{\longleftarrow} & H_\mathcal{M}^{w+1}(h^1(E)^{\otimes w}, \mathbb{Q}(l+w+1)), \end{array}$$

with pr the projection in the last w components and $\Delta_{CM} : E \to E \times E$ given by $e \mapsto (e, \sqrt{d_K}e)$. We obtain a map in Galois cohomology given by

$$H^1(\mathcal{O}_S, Sym^{2l+w}\mathcal{H}_{\mathbb{Q}_p}(1)) \to$$

$$H^1(\mathcal{O}_S, (e_\theta \mathcal{S}ym^w H_{\mathbb{Q}_p})(l+1)) = H^1(\mathcal{O}_S, H^w(M_\theta \times_K \overline{K}, \mathbb{Q}_p(w+l+1)))$$

such that

$$\mathcal{K}_\mathcal{M}(\psi(\mathfrak{a})^{\otimes 2l+w} Sym^{2l+w}\mathcal{H}_{\mathbb{Q}_p}(1)) = e_\theta(\otimes^w \psi(\mathfrak{a}))N\mathfrak{a}^l Sym^w \mathcal{H}_{\mathbb{Q}_p}(l+1).$$

Theorem 5.3. *Let p be a prime number such that $p \nmid 6N\mathfrak{f}_\theta$. Let θ be the idempotent satisfying $\#|\theta| = 2$ and such that the infinity type (a_θ, b_θ) satisfies $a_\theta \not\equiv b_\theta \bmod |\mathcal{O}_K^*|$. For a $p^rN\mathfrak{f}_\theta$-torsion point t_r, denote by $\tilde{t_r}$ its projection to $E[p^r]$. Then, if $t = \Omega f^{-1}$, we have the following equality*

$$N\mathfrak{a}\left(\psi_\theta(\mathfrak{a})N\mathfrak{a}^{l+1} - 1\right) r_p(\xi_{\theta,l}) =$$

$$\frac{(-1)^l L_p(\overline{\psi}_\theta, -l)^{-1} N_{T_\theta/\mathbb{Q}}\mathfrak{f}_\theta^{3l+2w}\Phi(\mathfrak{f}_\theta)}{2^{l-1}\psi_\theta(f)\Phi(\mathfrak{f})} \left(\delta N_{K(\mathfrak{f})/K} \sum_{p^r t_r = t} \theta_\mathfrak{a}(-t_r) \otimes e_\theta(\otimes^w \tilde{t_r}) \otimes \gamma(\tilde{t_r})^l\right)_r$$

Proof. Using theorem 3.5 and the theorems above, we have that

$$r_p(\xi_{\theta,l}) = \frac{(-1)^{l-1}(2l+w)! L_p(\overline{\psi}_\theta, -l)^{-1}\Phi(\mathfrak{f}_\theta)}{2^{l-1}N_{T_\theta/\mathbb{Q}}\mathfrak{f}_\theta^l \psi_\theta(f)\Phi(\mathfrak{f})} \mathcal{K}_\mathcal{M}(\mathcal{E}_\mathcal{M}^{2l+w}(\beta))$$

$$= \frac{(-1)^l(2l+w)! L_p(\overline{\psi}_\theta, -l)^{-1} N_{T_\theta/\mathbb{Q}}\mathfrak{f}_\theta^{3l+2w}\Phi(\mathfrak{f}_\theta)}{2^{l-1}\psi_\theta(f)\Phi(\mathfrak{f})} \mathcal{K}_\mathcal{M}(\beta^* Pol_{\mathbb{Q}_p})^{w+2l}.$$

Using the same notation as above, we have

$$\mathcal{K}_\mathcal{M}(\tilde{t_r}^{\otimes 2l+w}) = e_\theta(\otimes^w \tilde{t_r}) \otimes \gamma(\tilde{t_r})^l$$

Finally, applying Kings' theorem 5.1, we obtain the desired identity. ☐

We want to rewrite the previous formula in terms of the norm map of the extension $K(\mathfrak{f}_\theta)K(E[p^n])/K$. For technical reasons, we will work with $\mathfrak{f}$ instead of $\mathfrak{f}_\theta$ since then we can use that $K(E[\mathfrak{p}^n\mathfrak{f}]) = K(\mathfrak{p}^n\mathfrak{f})$, the ray class field, because $\mathfrak{f}$ is the conductor of E and divides the ideal $\mathfrak{f}\mathfrak{p}^n$ ([7, Prop.1.6]).

Fix a prime $\mathfrak{p}$ of K where E has good reduction, and take $\pi = \psi(\mathfrak{p})$. Denote by

$$H^{\mathfrak{p}}_{r,t} := \{t_r \in E[\mathfrak{p}^r\mathfrak{f}] | \pi^r t_r = t\}.$$

We write $t_r = (\tilde{t_r}, \pi^{-r}t) \in E[\mathfrak{p}^r\mathfrak{f}] = E[\mathfrak{p}^r] \oplus E[\mathfrak{f}]$. Define a filtration of $H^{\mathfrak{p}}_{r,s}$ as

$$F^i_{r,t} := \{t_r \in H^{\mathfrak{p}}_{r,s} | \pi^{r-i}\tilde{t_r} = 0\}.$$

Theorem 5.4. *Let $\mathfrak{p}$ be as above and $t_r = (\tilde{t_r}, \pi^{-r}t) \in F^0_{r,s} \setminus F^1_{s,t}$. Suppose $\mathcal{O}^* \to (\mathcal{O}/\mathfrak{f}_\theta)^*$ is injective. Denote the Euler factor of ψ_θ at $\mathfrak{p}$ evaluated at $-l$ by $L_{\mathfrak{p}}(\overline{\psi_\theta}, -l)$. Then*

$$L_{\mathfrak{p}}(\overline{\psi}_\theta, -l)^{-1} \left(N_{K(\mathfrak{f})/K} \sum_{s_r \in H^{\mathfrak{p}}_{r,t}} \theta_{\mathfrak{a}}(-s_r) \otimes e_\theta(\otimes^w \tilde{s_r}) \otimes \gamma(\tilde{s_r})^l \right)_r =$$

$$(N_{K(\mathfrak{p}^r\mathfrak{f})/K} \left(\theta_{\mathfrak{a}}(-t_r) \otimes e_\theta(\otimes^w \tilde{t_r}) \otimes \gamma(\tilde{t_r})^l\right))_r$$

in $H^1(\mathcal{O}_S, e_\theta(T_{\mathfrak{p}}E(1))(l) \otimes \mathbb{Q}_p)$ for all $\mathfrak{a}$ relatively prime to $\mathfrak{p}\mathfrak{f}$.

Proof. The identification $Hom_{\mathcal{O}_p}(T_pE, \mathcal{O}_p) \cong T_pE(-1)$ is via the conjugate linear $\mathcal{O}_p$-action on the right side. Hence $\overline{\psi(\mathfrak{p})} t_r = t_{r-1}$. We have the equality

$$(\overline{\psi_\theta(\mathfrak{p})}/N\mathfrak{p}^{-l})^i N_{K(\mathfrak{p}^r\mathfrak{f})/K(\mathfrak{p}^{r-i}\mathfrak{f})}(\theta_{\mathfrak{a}}(-t_r) \otimes e_\theta(\otimes^w \tilde{t_r}) \otimes \gamma(\tilde{t_r})^l) =$$

$$N_{K(\mathfrak{p}^r\mathfrak{f})/K(\mathfrak{p}^{r-i}\mathfrak{f})}(\theta_{\mathfrak{a}}(-t_r) \otimes e_\theta(\otimes^w \overline{\psi(\mathfrak{p})}^i \tilde{t_r}) \otimes \gamma(\overline{\psi(\mathfrak{p})}^i \tilde{t_r})^l) =$$

$$(N_{K(\mathfrak{p}^r\mathfrak{f})/K(\mathfrak{p}^{r-i}\mathfrak{f})}(\theta_{\mathfrak{a}}(-t_r))) \otimes e_\theta(\otimes^w \tilde{t_{r-i}}) \otimes \gamma(\tilde{t_{r-1}})^l) =$$

$$\theta_{\mathfrak{a}}(-(\tilde{t_{r-i}}, \pi^{i-r}t)) \otimes e_\theta(\otimes^w \tilde{t_{r-i}}) \otimes \gamma(\tilde{t_{r-i}})^l,$$

where the last equality uses the distribution relation for $\theta_{\mathfrak{a}}$ ([7, II 2.5]).

The Galois group of $K(\mathfrak{p}^{r-i}\mathfrak{f})/K(\mathfrak{f})$ acts simply transitively on $F^i_{r,t} \setminus F^{i+1}_{r,t}$. We get hence that

$$(\overline{\psi_\theta(\mathfrak{p})}/N\mathfrak{p}^{-l})^i N_{K(\mathfrak{p}^r\mathfrak{f})/K(\mathfrak{f})}(\theta_{\mathfrak{a}}(-t_r) \otimes e_\theta(\otimes^w \tilde{t_r}) \otimes \gamma(\tilde{t_r})^l) =$$

$$\sum_{t_{r-i} \in F^i_{r,t} \setminus F^{i+1}_{r,t}} \theta_{\mathfrak{a}}(-(\tilde{t_{r-i}}, \pi^{i-r}t)) \otimes e_\theta(\otimes^w \tilde{t_{r-i}}) \otimes \gamma(\tilde{t_{r-i}})^l.$$

We know by [7, Prop. II.2.4.ii)] that we have the equality $\theta_{\mathfrak{a}}(-(\tilde{t_{r-i}}, \pi^{i-r}t)) = \theta_{\mathfrak{a}}(-(\tilde{t_{r-i}}, \pi^{-r}t))^{\sigma^i_{\mathfrak{p}}}$ with $\sigma_{\mathfrak{p}}$ is the Frobenius at $\mathfrak{p}$ in the Galois group of $K(\mathfrak{f})/K$. This and the fact that $N_{K(\mathfrak{f})/K}$ is the sum over all Galois translates, which act trivially on $\tilde{t_{r-i}}$, gives that

$$(\overline{\psi_\theta(\mathfrak{p})}/N\mathfrak{p}^{-l})^i N_{K(\mathfrak{p}^r\mathfrak{f})/K}(\theta_{\mathfrak{a}}(-t_r) \otimes e_\theta(\otimes^w \tilde{t_r}) \otimes \gamma(\tilde{t_r})^l) =$$

$$N_{K(\mathfrak{f})/K} \left(\sum_{t_{r-i} \in F^i_{r,t} \setminus F^{i+1}_{r,t}} \theta_{\mathfrak{a}}(-(\tilde{t_{r-i}}, \pi^{-r}t)) \otimes e_\theta(\otimes^w \tilde{t_{r-i}}) \otimes \gamma(\tilde{t_{r-i}})^l \right),$$

Adding this equalities with respect to i and increasing r if necessary we get the result. ∎

Lemma 5.5. *Suppose that θ has infinity type $(w,0)$ or $(0,w)$ and $(\#|\mathcal{O}_K^*|, w) = 1$. Then*

$$\mathcal{O}_K^* \to (\mathcal{O}_K/\mathfrak{f}_\theta)^*$$

is injective.

Proof. Let u be and element in $\mathcal{O}_K^*$, $u \neq 1$ and consider the idele defined by $x_\infty = 1$ and $x_\mathfrak{p} = u$ for all finite places $\mathfrak{p}$ of K. Then $\psi^w(x) = \psi^w(u^{-1}x) = u^w \neq 1$ if $(w, |\mathcal{O}_K^*|) = 1$. So, by definition of the conductor of ψ_θ, we obtain that $u \not\equiv 1 (mod\ \mathfrak{f}_\theta)$, hence the result desired for the type $(w,0)$. For the type $(0,w)$ the proof is the same but with $\overline{\psi}$ instead of ψ. □

Corollary 5.6. *With the same hypothesis of theorem 5.3 but with $p \nmid 6N\mathfrak{f}$ and if $\mathcal{O}_K^* \to (\mathcal{O}_K/\mathfrak{f}_\theta)^*$ is injective, we have the equalities*

$$N\mathfrak{a}(\psi_\theta(\mathfrak{a})N\mathfrak{a}^{l+1} - 1)r_p(\xi_{\theta,l}) =$$

$$\pm\frac{N\mathfrak{f}_\theta^{3l+2w}\Phi(\mathfrak{f}_\theta)}{2^{l-1}\psi_\theta(f)\Phi(\mathfrak{f})}\delta\left(N_{K(E[p^r])K(\mathfrak{f})/K}\theta_\mathfrak{a}(-t_r) \otimes e_\theta(\otimes^w \tilde{t_r}) \otimes \gamma(\tilde{t_r})^l\right)_r =$$

$$\pm\frac{N\mathfrak{f}_\theta^{3l+2w}\Phi(\mathfrak{f}_\theta)}{2^{l-1}\psi_\theta(f)\Phi(\mathfrak{f})}|Gal(K(\mathfrak{f})/K(\mathfrak{f}_\theta))|\cdot$$

$$\delta\left(N_{K(E[p^r])K(\mathfrak{f}_\theta)/K}\theta_\mathfrak{a}(-t_r) \otimes e_\theta(\otimes^w \tilde{t_r}) \otimes \gamma(\tilde{t_r})^l\right)_r$$

where t_r is a primitive $p^r\mathfrak{f}_\theta$-torsion point with $p^r t_r = t$ and $\mathfrak{a}$ is relative prime to $p\mathfrak{f}$.

Proof. If p is inert or prime the first equality is deduced from the previous theorem. If p split, it decomposes in a $\mathfrak{p}$ and a $\mathfrak{p}^*$ part. Putting together the previous result with $\mathfrak{p}$ and with $\mathfrak{p}^*$, we have the first equality.

For the second equality, we have that

$$N_{K(E[p^r])K(\mathfrak{f})/K(E[p^r])K(\mathfrak{f}_\theta)}\theta_\mathfrak{a}(-t_r) =$$

$$\prod_{\sigma \in Gal(K(\mathfrak{f})K(E[p^r])K(p^r)/K(\mathfrak{f}_\theta)K(E[p^r])K(p^r))} \theta_\mathfrak{a}(-t_r)^\sigma$$

since $K = K(1)$ and hence $K(\mathfrak{f})$ is disjoint with $K(p^r)$ over K, and we also know that $K(\mathfrak{f}) = K(E[\mathfrak{f}])$ is disjoint with $K(E[p^r])$ over K. Moreover, since $\theta_\mathfrak{a}(-t_r) \in K(\mathfrak{f})K(p^r) = K(\mathfrak{f}p^r)$ and $(\mathfrak{f}, p) = 1$, we have that the norm is equal to

$$\prod_{\tau \in Gal(K(\mathfrak{f}p^r)/K(\mathfrak{f}_\theta p^r))} \theta_\mathfrak{a}(-t_r)^\tau.$$

But $\theta_\mathfrak{a}(-t_r) \in K(\mathfrak{f}_\theta p^r)$ because $-t_r$ is a point of $\mathfrak{f}_\theta p^r$-torsion. Hence we get the second equality. □

Now we want to show that the elements

$$(N_{K(E[p^r])K(\mathfrak{f}_\theta)/K}\theta_\mathfrak{a}(t_r) \otimes e_\theta(\otimes^w \tilde{t_r}) \otimes \gamma(\tilde{t_r})^l)_r$$

generate $(\overline{\mathcal{C}}_\infty^\chi \otimes M_{\theta\mathbb{Z}_p}(w+l))_\Gamma$, where $\mathfrak{a}$ is prime to $6p\mathfrak{f}$ and where χ is the representation of Δ on $Hom_{\mathcal{O}_p}(M_{\theta\mathbb{Z}_p}(w+l), \mathcal{O}_p)$, that we suppose it is a good representation. Here we use $M_{\theta\mathbb{Z}_p}$ with the same notation as in section 4.

We suppose from now on that the natural map $\mathcal{O}_K^* \to (\mathcal{O}_K/\mathfrak{f}_\theta)^*$ is injective.

Proposition 5.7. *Consider $p \nmid 6N\mathfrak{f}_\theta$ and $\mathfrak{a}$ an ideal in $\mathcal{O}_p$, which is prime to $6p\mathfrak{f}$ and such that $\psi_\theta(\mathfrak{a})N\mathfrak{a}^{l+1} \not\equiv 1(mod\ p)$. Then the $\mathcal{O}_p[[\Gamma]]$-module*

$$\overline{\mathcal{C}}_\infty^\chi \otimes_{\mathcal{O}_p} (e_\theta(\otimes^w T_p E)(l))$$

is generated by $(\theta_\mathfrak{a}(t_r)\otimes e_\theta(\otimes^w \tilde{t_r})\otimes\gamma(\tilde{t_r})^l)_r$, where t_r is a primitive $p^r\mathfrak{f}_\theta$-division point.

Remark 5.8. *Before we begin with the proof, let's remark that the existence of an ideal $\mathfrak{a}$ satisfying the conditions of the proposition 5.7 is equivalent to the condition that the Δ-representation χ is not the cyclotomic character (which is one of the hypothesis that we imposed before for χ). To show this fact, just notice that the $\mathcal{O}_p$-action on $\mathcal{O}_p[[\Gamma]]$ is given by complex conjugation because the $\mathcal{O}_p$-action on $e_\theta(\otimes^w T_p E)$ is given by complex conjugation on $\mathcal{O}_p$.*

Proof. Let $\mathfrak{b}$ be another ideal prime to $6p\mathfrak{f}$. Take $\sigma_\mathfrak{a} = [\mathfrak{a}, K_n/K]$ and $\sigma_\mathfrak{b} = [\mathfrak{b}, K_n/K]$. Then, by the properties of the theta function, we have that

$$(\sigma_\mathfrak{a} - \psi_\theta(\mathfrak{a})N\mathfrak{a}^{l+1})(\theta_\mathfrak{b}(t_n)\otimes e_\theta(\otimes^w\tilde{t_n})\otimes\gamma(\tilde{t_n})^l) =$$
$$\psi_\theta(\mathfrak{a})N\mathfrak{a}^l(\theta_\mathfrak{b}(t_n)^{\sigma_\mathfrak{a}-N\mathfrak{a}}\otimes e_\theta(\otimes^w\tilde{t_n})\otimes\gamma(\tilde{t_n})^l) =$$
$$\psi_\theta(\mathfrak{a})N\mathfrak{a}^l(\theta_\mathfrak{a}(t_n)^{\sigma_\mathfrak{b}-N\mathfrak{b}}\otimes e_\theta(\otimes^w\tilde{t_n})\otimes\gamma(\tilde{t_n})^l).$$

Now, it is enough show that $(\sigma_\mathfrak{a} - \psi_\theta(\mathfrak{a})N\mathfrak{a}^{l+1})$ is invertible in $\mathcal{O}_p[[\Gamma]] = \Lambda$, because $\overline{\mathcal{C}}_\infty^\chi$ is a torsion free Λ-module since $\mathcal{U}_\infty$ is torsion free ([22, Prop.11.4]), and we have an inclusion of $\mathcal{C}_\infty$ in $\mathcal{U}_\infty$.

But the element $\sigma_\mathfrak{a}$ corresponds to 1 on $\mathcal{O}_p/p$ and thus $\sigma_\mathfrak{a} - \psi_\theta(\mathfrak{a})N\mathfrak{a}^{l+1}$ is invertible in Λ because $1 \not\equiv \psi_\theta(\mathfrak{a})N\mathfrak{a}^{l+1}$ *mod p*.

It remains prove that $e_\theta(\tilde{t_r})\otimes\gamma(\tilde{t_r})$ generates $M_{\theta\mathbb{Z}_p}(w+l)$. This follows since $M_{\theta\mathbb{Z}_p}(w)$ is one dimensional and hence to prove that it generates $\mathbb{Z}_p(l)$ one can use the same proof as in [19, p.623]. □

Corollary 5.9. *Assume that $p > N_{K/\mathbb{Q}}\mathfrak{f}$. Then the image of $\mathcal{R}_\theta$ by r_p in $H^1(\mathcal{O}_S, e_\theta(\otimes^w T_p E)(l+1))\otimes\mathbb{Q}_p$ coincides with*

$$e_p((\overline{\mathcal{C}}_\infty^\chi \otimes e_\theta(\otimes^w T_p E)(l))_\Gamma).$$

Proof. As

$$|Gal(K(\mathfrak{f})/K(\mathfrak{f}_\theta))|N\mathfrak{f}_\theta^{3l+2w}\Phi(\mathfrak{f}_\theta)/2^{l-1}\psi_\theta(f)\Phi(\mathfrak{f})$$

is prime to p, it follows from the definition of e_p and Corollary 5.6. □

Let's note the following lemma.

Lemma 5.10. *The canonical map*

$$(\overline{\mathcal{C}}_\infty\otimes M_{\theta\mathbb{Z}_p}(w+l))\otimes^{\mathbb{L}}_{\mathcal{O}_p[[\mathcal{G}]]}\mathcal{O}_p \to (\overline{\mathcal{C}}_\infty\otimes M_{\theta\mathbb{Z}_p}(w+l))_\mathcal{G} \cong (\overline{\mathcal{C}}_\infty^\chi\otimes M_{\theta\mathbb{Z}_p}(w+l))_\Gamma$$

is an isomorphism and moreover $(\overline{\mathcal{C}}_\infty^\chi\otimes M_{\theta\mathbb{Z}_p}(w+l))_\Gamma \cong \mathcal{O}_p$.

Proof. We observe that the proof of proposition 5.7 shows that $\overline{\mathcal{C}}_\infty^\chi \cong \Lambda^\chi = \mathcal{O}_p[[\Gamma]]$ is a free Λ^χ-module of rank 1. This implies, as in [19, lemma 5.2.3], that $(\overline{\mathcal{C}}_\infty^\chi\otimes M_{\theta\mathbb{Z}_p}(w+l))_\Gamma \cong \mathcal{O}_p$. The claim follows since the previous module is induced and hence the higher Tor-terms vanish. □

As a consequence, we get the part 4 of conjecture 1.2 for the constructible subspace $\mathcal{R}_\theta$.

Corollary 5.11. *The map*

$$\mathcal{R}_\theta \otimes \mathbb{Z}_p \to R\Gamma(\mathcal{O}_S, M_{\theta\mathbb{Z}_p}(w+l+1) \otimes \mathbb{Q}_p)[1]$$

induced by r_p, gives an isomorphism

$$det_{\mathcal{O}_p}\mathcal{R}_\theta \cong det_{\mathcal{O}_p} R\Gamma(\mathcal{O}_S, M_{\theta\mathbb{Z}_p}(w+l+1))^{-1}$$

This proves, taking $Norm_{K/\mathbb{Q}}$, the conjecture 1.2 under the hypothesis that the Soulé regulator is not zero for the elements in K-theory constructed for the motive $M_\theta(w+l+1)$. Let us first write the same result but over $\mathcal{O}_K$.

Theorem 5.12. *Let p be a prime different from 2 and 3 (hence, in particular, $p \nmid \#|\mathcal{O}_K^*|$), and $p > N_{K/\mathbb{Q}}\mathfrak{f}$.Consider l a non-negative integer. Suppose that ψ_θ has infinity type (a_θ, b_θ) with a_θ, b_θ non-negative integers, such that $a_\theta \not\equiv b_\theta \mod |\mathcal{O}_K^*|$ and $w = a_\theta + b_\theta \geq 1$ verifies $-w-2l \leq -3$. Suppose that $\mathcal{O}_K^* \to (\mathcal{O}_K/\mathfrak{f}_\theta)^*$ is injective.*

Suppose moreover that the representation χ of Δ in $Hom_{\mathcal{O}_p}(H^w(M_\theta \times_K \overline{K}, \mathbb{Z}_p(w+l)), \mathcal{O}_p)$ is a good representation (see the definition in 4.12) which is not the cyclotomic character.

If we denote by $M_{\theta\mathbb{Z}_p}(w+m) = H^w(M_\theta \times_K \overline{K}, \mathbb{Z}_p(w+m))$, then, there is an $\mathcal{O}_K$-submodule $\mathcal{R}_\theta \subset H^{w+1}_{\mathcal{M}}(M_\theta, \mathbb{Q}(w+l+1))$ of rank 1 such that:

1. *$det_{\mathcal{O}_K}(r_{\mathcal{D}}(\mathcal{R}_\theta)) \cong L_S^*(\overline{\psi}_\theta, -l) det_{\mathcal{O}_K}(H^w_B(M_{\theta\mathbb{C}}, \mathbb{Z}(w+l)))$ in $det_{\mathcal{O}_K\otimes\mathbb{R}}(H^w_B(M_{\theta\mathbb{C}}, \mathbb{Z}(w+l)) \otimes \mathbb{R})$.*
2. *The map r_p induces an isomorphism*

$$det_{\mathcal{O}_K\otimes\mathbb{Z}_p}(\mathcal{R}_\theta) \cong det_{\mathcal{O}_K\otimes\mathbb{Z}_p}(R\Gamma(\mathcal{O}_K[1/S], M_{\theta\mathbb{Z}_p}(w+l+1))^{-1}.$$

Here

$$L_S^*(\overline{\psi}_\theta, -l) = \lim_{s\to -l} \frac{L_S(\overline{\psi}_\theta, s)}{s+l},$$

and S is the set of primes of K dividing p and the ones dividing $\mathfrak{f}_\theta$.

Moreover, if r_p is injective on $\mathcal{R}_\theta$, the second part can be written as

$$det_{\mathcal{O}_K\otimes\mathbb{Z}_p}(H^1(\mathcal{O}_K[1/S], M_{\theta\mathbb{Z}_p}(w+l+1))/r_p(\mathcal{R}_\theta)) \cong$$
$$det_{\mathcal{O}_K\otimes\mathbb{Z}_p} H^2(\mathcal{O}_K[1/S], M_{\theta\mathbb{Z}_p}(w+l+1)).$$

Proof. It is a direct consequence of the theorem 3.5 and the above corollary 5.11. □

As a consequence we obtain part of the local Tamagawa conjecture 1.2 for the motive $M_\theta(w+l+1)$.

Theorem 5.13. *Let p be a prime different from 2 and 3 (hence, in particular, $p \nmid \#|\mathcal{O}_K^*|$), and $p > N_{K/\mathbb{Q}}\mathfrak{f}$. Consider l a non-negative integer. Suppose that ψ_θ has infinity type (a_θ, b_θ) with a_θ, b_θ non-negative integers, such that $a_\theta \not\equiv b_\theta \mod |\mathcal{O}_K^*|$ and $w = a_\theta + b_\theta \geq 1$ verifies $-w-2l \leq -3$. Suppose $\mathcal{O}_K^* \to (\mathcal{O}_K/\mathfrak{f}_\theta)^*$ is injective.*

Moreover, suppose that χ, the representation of Δ in $Hom_{\mathcal{O}_p}(M_{\theta\mathbb{Z}_p}(w+l), \mathcal{O}_p)$, is a good representation which is not the cyclotomic character.

Then, there is a submodule $\mathcal{R}_\theta$ in $H^{w+1}_{\mathcal{M}}(M_\theta, \mathbb{Q}(w+l+1))$ such that:

1. *The map* $r_{\mathcal{D}} \otimes \mathbb{R}$ *is an isomorphism restricted to* $\mathcal{R}_\theta \otimes \mathbb{R}$.

2. $dim_{\mathbb{Q}}(H^w_B(M_{\theta\mathbb{C}}, \mathbb{Z}(w+l)) \otimes \mathbb{Q}) = ord_{s=-l} L_S(H^w(M_\theta, \mathbb{Q}_p), s) = 2$.

3. *We have the equality*

$$r_{\mathcal{D}}(det_{\mathbb{Z}}(\mathcal{R}_\theta)) = L^*_S(H^w_{et}(M_\theta, \mathbb{Q}_p), -l) det_{\mathbb{Z}}(H^w_B(M_\theta, \mathbb{Z}(w+l)))$$

where

$$L^*_S(H^w_{et}(M_\theta, \mathbb{Q}_p), -l) = \lim_{s \to -l} \frac{L_S(H^w_{et}(M_\theta, \mathbb{Q}_p), s)}{(s+l)^2}$$

and S *is the set of places of* K *that divides* p *and the places dividing the conductor* $\mathfrak{f}_\theta$.

4. *We have that*

$$det_{\mathbb{Z}_p}(\mathcal{R}_\theta) = det_{\mathbb{Z}_p}(R\Gamma(\mathcal{O}_K[1/S], M_{\theta\mathbb{Z}_p}(w+l+1)))^{-1}.$$

If r_p *is injective on* $\mathcal{R}_\theta$, *then* $r_p(det_{\mathbb{Z}}(\mathcal{R}_\theta))$ *is a basis of the* $\mathbb{Z}_p$*-lattice*

$$det_{\mathbb{Z}_p}(R\Gamma(\mathcal{O}_K[1/S], M_{\theta\mathbb{Z}_p}(w+l+1)))^{-1}$$

$$\subset det_{\mathbb{Q}_p}(R\Gamma(\mathcal{O}_K[1/S], M_{\theta\mathbb{Z}_p}(w+l+1) \otimes \mathbb{Q})[-1]).$$

Remark 5.14. *Theorem 5.13 is part of the local Tamagawa number conjecture 1.2 for Hecke characters. To prove the general conjecture it remains to prove the finiteness of the second cohomology group and the bijectivity of the p-Soulé regulator map.*

1. *The finiteness of* $H^2_p := H^2(\mathcal{O}[1/S], M_{\theta\mathbb{Z}_p}(w+l+1))$ *follows from a general conjecture of Jannsen [16].*

   Following the argument of Wingberg in [25], is easy to prove that if p *is a regular prime for the field* $K(E[p])$, *then this Galois cohomology group is finite. See [2, Appendix B] or an incoming preprint of the author.*

2. *Moreover, on the bijectivity of the Soulé regulator we make the following remarks.*

   (a) *Deninger proves the Beilinson conjecture for the above Hecke characters. He computes the dimension of the space* $H^w_B(M_\theta, \mathbb{Q}(w+l))$ *which it is mapped the Beilinson regulator map. He constructs elements on the* K*-theory group* $H_{\mathcal{M}}$, *generating a subspace which has the same dimension of* $H^w_B(M_\theta, \mathbb{Q}(w+l))$. *These subspace of constructed elements are denoted in the paper by* $H^{constr}_{\mathcal{M}}$. *It remains to prove on the Beilinson conjecture on Hecke characters of imaginary quadratic fields that the* K*-theory group* $H_{\mathcal{M}}$ *coincides with* $H^{constr}_{\mathcal{M}}$. *On the Soulé regulator map, the image of* $H^{constr}_{\mathcal{M}}$ *tensor by* $\mathbb{Q}$ *should map (if the Soulé regulator map is an isomorphism) to a free group* $H^1(\mathcal{O}_K[1/S], M_{\theta\mathbb{Z}_p}(w+l+1)) \otimes \mathbb{Q}$ *of the same rank as* $H^{constr}_{\mathcal{M}}$. *We note that the group* $H^1(\mathcal{O}[1/S], M_{\theta\mathbb{Z}_p}(w+l+1)) \otimes \mathbb{Q}$ *has exactly this rank when* H^2_p *is finite, and bigger if not, [16, cor. 1].*

(b) *We observe that, if e_p is injective, then r_p is injective on $\mathcal{R}_\theta$, because the image of e_p is an $\mathcal{O}_\theta \otimes \mathbb{Z}_p$-module of rank 1 and $\mathcal{R}_\theta$ is a $\mathcal{O}_\theta \otimes \mathbb{Z}_p$-module of rank 1. Let us suppose that H^2_p is finite. Then, similar arguments as in [19, §5.2.2] gives the injectivity for e_p like in the situation of a Hecke character of type $(1,0)$ which is made in [19].*

As a conclusion, for regular primes p, we obtain in full generality the conjecture 1.2 for Hecke characters of imaginary quadratic fields.

A The remaining non-critical values

In the main body of the paper, we restricted the study of the Tamagawa number conjecture to the motives $M_\theta(w+l+1)$ with $l \geq 0$, basically because of the formulation in §1 of the Tamagawa number conjecture (more precisely for the condition $r = w+l+1 > inf(m, X) = w$). Deninger proves also the Beilinson's conjecture for $M_\theta(w+l+1)$ without the condition $l \geq 0$. Then, the formulation for the above motives is under this restriction in order to avoid the poles in the bad Euler factors. But, in our case, there are no poles in the bad Euler factors, that correspond to primes in S (see remark 2.6). Therefore we can ask which is the image of some K-theory elements by the regulators maps for values with $l < 0$ and $-w-2l \leq -3$.

The restriction $l \geq 0$ for $M_\theta(w+l+1)$ with $-w-2l \leq -3$ does not include all the non-critical values associated to the Hecke character ψ_θ (we restrict to the situation $a_\theta \not\equiv b_\theta (mod|\mathcal{O}^*_K|)$ for simplicity). It remains to study the conjecture for the motives $M_\theta(w+l+1)$ which are related with the non-critical values associated to ψ_θ for the integers l such that

$$0 < -l \leq min(a_\theta, b_\theta)$$

(see theorem 3.1 for the exact non-critical values up to functional equation).

In this appendix we construct elements in K-theory for $M_\theta(w+l+1)$ with $0 < -l \leq min(a_\theta, b_\theta)$ and computate the Beilinson regulator and the Soulé regulator in them, obtaining the same result as in the main theorems in §5.

Deninger [6, pp.142-144]constructs also elements in K-theory for the motive $M_\theta(w+l+1)$ with $l<0$ and proves the Beilinson conjecture. In this situation he constructs a projector map $\mathcal{K}_\mathcal{M}$, without using complex multiplication. The problem of this construction is that in the image by this projector map of the specialization of the elliptic polylogarithm, the Weil pairing appearing in §5 corresponds to $\gamma(\tilde{t_r}) =< \tilde{t_r}, \tilde{t_r} >$ and the arguments in proposition 5.7 does not generalize. We modify the Deninger's projector map by $\mathcal{K}'_\mathcal{M}$, and with use of the K-theory elements $\mathcal{K}'_\mathcal{M}(\mathcal{E}_\mathcal{M}(\beta_\theta))$, we prove Beilinson's conjecture. Hence, the arguments in §4 an §5 apply straightforward in order to obtain a determinant comparison for the image of the Soulé regulator map of these elements.

We follow the same notation introduced in the paper, and let us follow the arguments through §3 to §5 with $l<0$ but we remain the other hypothesis appearing. Let us fix $w \geq 1$ and $l < 0$ such that $-w-2l \leq -3$ and let us consider the motive $M_\theta(w+l+1)$. With the fixed embedding we have $\vartheta = (\lambda_1, \ldots, \lambda_w) \in \theta$ and set $I_1 = \{i | \lambda_i \in Hom_K(K, \mathbb{C})\}$ and $I_2 = \{i | \lambda_i \notin Hom_K(K, \mathbb{C})\}$ and we have now that $0 < |l| \leq \#I_1 = a_\theta$ and $0 < |l| \leq \#I_2 = b_\theta$. Denote by $\Delta = id^1 \times id^2 : E \to E \times E$ the diagonal map and by $\Delta_{CM} = id^{1,CM} \times id^{2,CM}$:

$E \to E \times E$ given by $e \mapsto (e, (\sqrt{d_K})e)$ where we understand $\sqrt{d_K} \in End(E)$. Let us choose exactly $|l|$ elements in the sets I_1 and I_2 denote it in increasing order $i_1, \ldots, i_{|l|} \in I_1$ and $j_1, \ldots, j_{|l|} \in I_2$. Let us define the projector map $pr : E^{w+l} \to E^{w+2l}$ by the projection of the first $w+2l$-components of E^{w+l} and let us define $(id \times \Delta^{|l|}) : E^{w+l} \to E^w$ (which it depends of the choice in the sets I_1 and I_2) by $(e_1, \ldots, e_{w+2l}, e_{w+2l+1}, \ldots, e_{w+l}) \mapsto (e_{\alpha_1}, \ldots, e_{\alpha_w})$ where e_{α_s} is defined as follows:

- if α_s appears in one component of the set of tuples $L := \{(i_1, j_1), \ldots, (i_{|l|}, j_{|l|})\}$ then
$$e_{\alpha_s} := \begin{cases} id^1(e_{w+2l+m}) \ if \ \alpha_s = i_m \\ id^2(e_{w+2l+m}) \ if \ \alpha_s = j_m \end{cases},$$
- in the other case, then it is defined by $e_{\alpha_s} := e_{ny}$ with $1 \le ny \le w+2l$ such that $\alpha_s = ny + \sum 1$ where the sum runs the naturals that appear in some component of the elements of L and which are lower than α_s.

We define also the map $(id \times \Delta^{|l|}_{CM})$ similarly as the map $(id \times \Delta^{|l|})$ but interchanging id^i by $id^{i,CM}$.

We define the projector map $\mathcal{K}'_{\mathcal{M}}$ by

$$\begin{array}{ccc} H^{2l+w+1}_{\mathcal{M}}(Sym^{2l+w}h^1E, \mathbb{Q}(w+2l+1)) & \xrightarrow{pr^*} & H^{2l+w+1}_{\mathcal{M}}(E^{2l+w+|l|}, \mathbb{Q}(2l+w+1)) \\ \mathcal{K}'_{\mathcal{M}} \downarrow & & \downarrow (id \times \Delta^{|l|}_{CM})_* \\ H^{w+1}_{\mathcal{M}}(M_{\theta\mathbb{Q}}, \mathbb{Q}(w+l+1)) & \xleftarrow{e_\theta} & H^{w+1}_{\mathcal{M}}(h^1(E)^{\otimes w}, \mathbb{Q}(l+w+1)) \end{array}$$

(Deninger defines $\mathcal{K}_{\mathcal{M}}$ with a similar diagram as above but with the map $(id \times \Delta^{|l|})_*$ instead of the map $(id \times \Delta^{|l|}_{CM})_*$). The next result is a modification of Deninger's result [6, pp.143-145].

Theorem A.1. *Suppose that $a_\theta \not\equiv b_\theta \ mod \ |\mathcal{O}_K^*|$ and that a_θ, b_θ, l satisfy the hypothesis of the theorem 3.1 with $l < 0$. Define, up to $\pm$ by using the §3 notation,*

$$\xi_{\theta,l} := \pm \frac{(\sqrt{d_K})^{2l}(2l+w)! L_p(\overline{\psi_\theta}, -l)^{-1}\Phi(\mathfrak{f}_\theta)}{2^{-1} N_{K/\mathbb{Q}}\mathfrak{f}_\theta^l \psi_\theta(f)\Phi(\mathfrak{f})} \mathcal{K}'_{\mathcal{M}} \circ \mathcal{E}^{2l+w}_{\mathcal{M}}(\beta_\theta)$$

which belongs to $H^{w+1}_{\mathcal{M}}(M_\theta, \mathbb{Q}(w+l+1))$. Then

$$r_{\mathcal{D}}(\xi_{\theta,l}) = L_S^*(\overline{\psi_\theta}, -l)\eta_\theta,$$

where S are the primes of K that divide $\mathfrak{f}_\theta$ and p.

Proof. We will follow closely Deninger's papers [5] and [6], we follow also for this proof his notation where his n is our $w+2l$. We need to modify the calculation in [6, (2.13)Lemma] for $\mathcal{K}'_{\mathcal{M}}$ instead of $\mathcal{K}_{\mathcal{M}}$, up to a sign because our order of the factors of the map $(id \times \Delta^{|l|}_{CM})$, one obtains (up to sign)

$$\frac{1}{(2\pi i)^w} \int_{E^w} \tilde{\mathcal{K}'_{\mathcal{D}}}(\tilde{\xi}) \wedge dz^{(\underline{\varepsilon})} =$$

$$B_{\underline{\varepsilon}} \sqrt{d_K}^{|l|} \begin{pmatrix} n \\ n+|l|-|\underline{\varepsilon}| \end{pmatrix}^{-1} A(\Gamma)^{n+|l|} c_{n+|l|-|\underline{\varepsilon}|}$$

(we use the calculation at the top of [5, p.63]). Then we obtain, following the argument [6, p.143-144] and our conditions of §3 with $l < 0$,

$$r_{\mathcal{D}}(\mathcal{K}'_{\mathcal{M}}\mathcal{E}_{\mathcal{M}}(\beta_\theta)) = t_\theta L^*(\overline{\psi}_\theta, l)\eta_\theta$$

with t_θ given by (up to sign)

$$\frac{2^{-1}N_{K/\mathbb{Q}}\mathfrak{f}_\theta^l\psi_\theta(f)\Phi(\mathfrak{f})}{(\sqrt{d_K})^{2l}(2l+w)!\Phi(\mathfrak{f}_\theta)}.$$

By the remark 2.6 we obtain the result. □

Following §3 we define for $l < 0$ the constructible space by

$$\mathcal{R}_\theta := \xi_{\theta,l}\mathcal{O}_K,$$

with $\xi_{\theta,l}$ given by the theorem A.1. Let observe that with this notation we can follow straightforward all the results and proofs of §3 and §4. In §5 we need to compute $\mathcal{K}'_{\mathcal{M}} \circ \mathcal{E}_{\mathcal{M}}^{w+2l}(\beta_\theta)$, and for this we only need to observe from the definition of $\mathcal{K}'_{\mathcal{M}}$, our projector has these two properties,

$$\mathcal{K}'_{\mathcal{M}}(\psi(\mathfrak{a})^{\otimes 2l+w}Sym^{2l+w}\mathcal{H}_{\mathbb{Q}_p}(1)) = e_\theta(\otimes^w\psi(\mathfrak{a}))N\mathfrak{a}^l Sym^w\mathcal{H}_{\mathbb{Q}_p}(l+1),$$

and $\mathcal{K}_{\mathcal{M}}(\tilde{t_r}^{\otimes 2l+w}) = e_\theta(\otimes^w\tilde{t_r}) \otimes \gamma(\tilde{t_r})^l$ with $\gamma(\tilde{t_r}) =< \tilde{t_r}, \sqrt{d_K}\tilde{t_r} >$. After this observation all the results of §5 and the proof follows straightforward up to a power of 2 and d_K. The reader could make these modifications which follow from our definition of $\mathcal{R}_\theta$. Therefore we obtain the local Tamagawa number conjecture with K-coefficients,

Theorem A.2. *Let p be a prime different from 2 and 3 (hence, in particular, $p \nmid \#|\mathcal{O}_K^*|$), and $p > N_{K/\mathbb{Q}}\mathfrak{f}$. Suppose that ψ_θ has infinity type (a_θ, b_θ) with a_θ, b_θ non-negative integers, such that $a_\theta \not\equiv b_\theta \mod|\mathcal{O}_K^*|$ and $w = a_\theta + b_\theta \geq 1$ verifies $-w-2l \leq -3$ with $l < 0$. Suppose that $\mathcal{O}_K^* \to (\mathcal{O}_K/\mathfrak{f}_\theta)^*$ is injective.*

Suppose moreover that the representation χ of Δ in $Hom_{\mathcal{O}_p}(H^w(M_\theta \times_K \overline{K}, \mathbb{Z}_p(w+l)), \mathcal{O}_p)$ is a good representation (see the definition in 4.12) which is not the cyclotomic character.

If we denote by $M_{\theta\mathbb{Z}_p}(w+m) = H^w(M_\theta \times_K \overline{K}, \mathbb{Z}_p(w+m))$, then, there is an $\mathcal{O}_K$-submodule $\mathcal{R}_\theta \subset H_{\mathcal{M}}^{w+1}(M_\theta, \mathbb{Q}(w+l+1))$ of rank 1 such that:

1. *$det_{\mathcal{O}_K}(r_{\mathcal{D}}(\mathcal{R}_\theta)) \cong L_S^*(\overline{\psi}_\theta, -l)det_{\mathcal{O}_K}(H_B^w(M_{\theta\mathbb{C}}, \mathbb{Z}(w+l)))$ in $det_{\mathcal{O}_K\otimes\mathbb{R}}(H_B^w(M_{\theta\mathbb{C}}, \mathbb{Z}(w+l)) \otimes \mathbb{R})$.*
2. *The map r_p induces an isomorphism*

$$det_{\mathcal{O}_K\otimes\mathbb{Z}_p}(\mathcal{R}_\theta) \cong det_{\mathcal{O}_K\otimes\mathbb{Z}_p}(R\Gamma(\mathcal{O}_K[1/S], M_{\theta\mathbb{Z}_p}(w+l+1))^{-1}.$$

Here

$$L_S^*(\overline{\psi}_\theta, -l) = \lim_{s\to -l}\frac{L_S(\overline{\psi}_\theta, s)}{s+l},$$

and S is the set of primes of K dividing p and the ones dividing $\mathfrak{f}_\theta$.

Moreover, if r_p is injective on $\mathcal{R}_\theta$, the second part can be written as

$$det_{\mathcal{O}_K\otimes\mathbb{Z}_p}(H^1(\mathcal{O}_K[1/S], M_{\theta\mathbb{Z}_p}(w+l+1))/r_p(\mathcal{R}_\theta)) \cong$$

$$det_{\mathcal{O}_K\otimes\mathbb{Z}_p}H^2(\mathcal{O}_K[1/S], M_{\theta\mathbb{Z}_p}(w+l+1)).$$

And the local Tamagawa number conjecture with $\mathbb{Q}$-coefficients,

Theorem A.3. *Let p be a prime different from 2 and 3 (hence, in particular, $p \nmid \#|\mathcal{O}_K^*|$), and $p > N_{K/\mathbb{Q}}\mathfrak{f}$. Suppose that ψ_θ has infinity type (a_θ, b_θ) with a_θ, b_θ non-negative integers, such that $a_\theta \not\equiv b_\theta \mod|\mathcal{O}_K^*|$ and $w = a_\theta + b_\theta \geq 1$ verifies $-w - 2l \leq -3$ with $l < 0$. Suppose $\mathcal{O}_K^* \to (\mathcal{O}_K/\mathfrak{f}_\theta)^*$ is injective.*

Moreover, suppose that χ, the representation of Δ in $Hom_{\mathcal{O}_p}(M_{\theta\mathbb{Z}_p}(w + l), \mathcal{O}_p)$, is a good representation which is not the cyclotomic character.

Then, there is a submodule $\mathcal{R}_\theta$ in $H_{\mathcal{M}}^{w+1}(M_\theta, \mathbb{Q}(w + l + 1))$ such that:

1. *The map $r_{\mathcal{D}} \otimes \mathbb{R}$ is an isomorphism restricted to $\mathcal{R}_\theta \otimes \mathbb{R}$.*
2. *$dim_{\mathbb{Q}}(H_B^w(M_{\theta\mathbb{C}}, \mathbb{Z}(w + l)) \otimes \mathbb{Q}) = ord_{s=-l} L_S(H^w(M_\theta, \mathbb{Q}_p), s) = 2$.*
3. *We have the equality*
$$r_{\mathcal{D}}(det_{\mathbb{Z}}(\mathcal{R}_\theta)) = L_S^*(H_{et}^w(M_\theta, \mathbb{Q}_p), -l) det_{\mathbb{Z}}(H_B^w(M_\theta, \mathbb{Z}(w + l)))$$
where
$$L_S^*(H_{et}^w(M_\theta, \mathbb{Q}_p), -l) = \lim_{s \to -l} \frac{L_S(H_{et}^w(M_\theta, \mathbb{Q}_p), s)}{(s + l)^2}$$
and S is the set of places of K that divides p and the places dividing the conductor $\mathfrak{f}_\theta$.
4. *We have that*
$$det_{\mathbb{Z}_p}(\mathcal{R}_\theta) = det_{\mathbb{Z}_p}(R\Gamma(\mathcal{O}_K[1/S], M_{\theta\mathbb{Z}_p}(w + l + 1)))^{-1}.$$
If r_p is injective on $\mathcal{R}_\theta$, then $r_p(det_{\mathbb{Z}}(\mathcal{R}_\theta))$ is a basis of the $\mathbb{Z}_p$-lattice
$$det_{\mathbb{Z}_p}(R\Gamma(\mathcal{O}_K[1/S], M_{\theta\mathbb{Z}_p}(w + l + 1)))^{-1}$$
$$\subset det_{\mathbb{Q}_p}(R\Gamma(\mathcal{O}_K[1/S], M_{\theta\mathbb{Z}_p}(w + l + 1) \otimes \mathbb{Q})[-1]).$$

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