# RECIPROCITY LAWS À LA IWASAWA-WILES

FRANCESC BARS, IGNAZIO LONGHI

Abstract. This paper is a brief survey on explicit reciprocity laws of Artin-Hasse-Iwasawa-Wiles type for the Kummer pairing on local fields.

## 1. Introduction

Let $K$ be a complete discrete valuation field, $\mathcal{O}_K$ its ring of integers, $\mathfrak{m}_K$ its maximal ideal and $k_K$ the residue field. Suppose that $K$ is an $\ell$-dimensional local field: this means that there is a chain of fields $K_\ell = K, K_{\ell-1}, \ldots, K_0$ where $K_i$ is a complete discrete valuation field with residue field $K_{i-1}$ and $K_0$ is a finite field. We shall always assume that $char(K_0) = p$.

Suppose $char(k_K) = p > 0$: then we have the reciprocity law map

$$( \ , K^{ab}/K) : K_\ell^M(K) \to Gal(K^{ab}/K), \tag{1}$$

where $K^{ab}$ is the maximal abelian extension of $K$, $Gal$ denotes the Galois group and $K_\ell^M$ is the Milnor $K$-theory.

Assume $char(K) = 0$ and $\zeta_{p^m} \in K$, where $\zeta_{p^m}$ is a primitive $p^m$-th root of unity. The classical Hilbert symbol $( \ , \ )_m : K^* \times K_\ell^M(K) \to <\zeta_{p^m}> =: \boldsymbol{\mu}_{p^m}$ is:

$$(\alpha_0, \{\alpha_1, \ldots, \alpha_\ell\})_m := \frac{\beta^{(\{\alpha_1,\ldots,\alpha_\ell\}, K^{ab}/K)}}{\beta}, \tag{2}$$

where $\beta$ is a solution of $X^{p^m} = \alpha_0$ and $(\{\alpha_1, \ldots, \alpha_\ell\}, K^{ab}/K)$ is the element given by the reciprocity law map.

When $\ell = 1$, $K$ is the completion at some place of a global field (i.e., a finite extension of $\mathbb{F}_p((T))$ or of the $p$-adic field $\mathbb{Q}_p$), $K_1^M(K) \cong K^*$ and $( \ , K^{ab}/K)$ is the classical norm symbol map of local class field theory.

Historically there was deep interest to compute this Hilbert symbol (or, better, Kummer pairing) in terms of analytic objects, as a step in the program of making local class field theory completely explicit. Vostokov [31] suggests the existence of two different branches of explicit reciprocity formulas: Kummer's type and Artin-Hasse's type (later extended by Iwasawa and Wiles). Kummer's reciprocity law [21] is

**Theorem 1** (Kummer 1858)**.** *Let $K = \mathbb{Q}_p(\zeta_p)$, $p \neq 2$, and $\alpha_0, \alpha_1$ principal units. Then*

$$(\alpha_0, \alpha_1)_1 = \zeta_p^{res(\log \tilde{\alpha_0}(X) \mathrm{dlog}\, \tilde{\alpha_1}(X) X^{-p})}$$

*where $\tilde{\alpha_0}(X), \tilde{\alpha_1}(X) \in \mathbb{Z}_p[[X]]^*$ are power series such that $\tilde{\alpha_1}(\zeta_p - 1) = \alpha_1$, $\tilde{\alpha_0}(\zeta_p - 1) = \alpha_0$, res means the residue and* dlog *is the logarithmic derivative.*

Artin-Hasse's reciprocity law [2] is:

**Theorem 2** (Artin-Hasse 1928)**.** *Let* $K = \mathbb{Q}_p(\zeta_{p^m})$*,* $p \neq 2$*, and* $\alpha_1 \in K^*$ *a principal unit. Take* $\pi = \zeta_{p^m} - 1$ *a prime of* $K$*: then*

$$(\zeta_{p^m}, \alpha_1)_m = \zeta_{p^m}^{Tr_{K/\mathbb{Q}_p}(-\log \alpha_1)/p^m}, \quad (\pi, \alpha_1)_m = \zeta_{p^m}^{Tr_{K/\mathbb{Q}_p}(\pi^{-1}\zeta_{p^m} \log \alpha_1)/p^m},$$

*where* log *is the* $p$*-adic logarithm. Later Iwasawa* [17] *gave a formula for* $(\alpha_0, \alpha_1)_m$ *with* $\alpha_0$ *any principal unit such that* $val_K(\alpha_0 - 1) > \frac{2val_K(p)}{p-1}$*.*

Roughly speaking the difference between these two branches is that Kummer's type refers to residue formulas involving a power series for each component of the pairing, while Artin-Hasse-Iwasawa's type are non-residue formulas evaluating some generic series at the $K$-theory component of the Hilbert pairing.

There is a big amount of articles in the literature that contribute to and extend the above seminal works of Kummer and Artin-Hasse-Iwasawa. The Hilbert symbol can be extended to Lubin-Tate formal groups, and also to $p$-divisible groups. These extensions are defined from Kummer theory: hence one often speaks of Kummer pairing instead of Hilbert symbol. Wiles extended Iwasawa's result to Lubin-Tate formal groups.

In this survey we intend to review some of the results on Artin-Hasse-Iwasawa-Wiles' type reciprocity laws. We list different variants of the Kummer pairing; afterwards we sketch some of the main points in the proof of the Artin-Hasse-Iwasawa-Wiles reciprocity law for 1-dimensional local fields. Finally we review Kato's generalization of Wiles' reciprocity law, which is done in a cohomological setting. Kato's work also extends the explicit reciprocity law to higher dimensional local fields and to schemes.

For Kummer's type, one should cite also a lot of contributions by Vostokov, and many others, for example Shafarevich, Kneser, Brückner, Henniart, Fesenko, Demchenko, and Kato and Kurihara in the cohomological setting. We refer to Vostokov's paper [31] for a list of results and references for reciprocity laws in this case, adding to it the recent works of Benois [5] (for cyclotomic extensions), Cao [8] (for Lubin-Tate formal groups) and Fukaya [12] and the works of Ankeny and Berrizbeitia [6]. Given $\alpha$ a unit of $\mathcal{O}_K$, $K$ a totally ramified finite extension of $\mathbb{Q}_p$ and $\pi$ a uniformizer of $K$, $\alpha$ has a factorization modulo $K^{*p^n}$ by a product of $E(\pi^k)$, $k \in \mathbb{N}$, where $E$ is the Artin-Hasse exponential. The contribution by Shafarevich is to compute $(\alpha, \beta)_n$ in terms of the above factorization for $\alpha$ and $\beta$. Berrizbeitia [6] recovers Shafarevich results with different methods but also using the above factorization.

**Notations**

Let $L$ be a valuation field: we denote $\mathcal{O}_L$ the ring of integers, $\mathfrak{m}_L$ its maximal ideal, $k_L$ the residue field and $val_L$ the valuation function. We write $\overline{L}$ for the algebraic closure of $L$ if $char(L) = 0$ and the separable closure otherwise;

$\hat{\overline{L}}$ will be the completion (which is algebraically closed). Denote $Gal(\overline{L}/L)$ by $G_L$ and the continuous Galois cohomology group $H^i(G_L, A)$ by $H^i(L, A)$.

For any ring $R$, $R^*$ means the invertible elements of $R$. As usual $\mathbb{F}_q$ is the finite field of cardinality $q$.

The symbol $M$ will always denote a finite extension of $\mathbb{Q}_p$: $M_0$ is the subfield of $M$ such that $M/M_0$ is totally ramified and $M_0/\mathbb{Q}_p$ is unramified.

For $N$ a $\mathbb{Z}_p$-module, $N(r)$ will be its $r$-th Tate twist, $r \in \mathbb{Z}$ (recall that $\mathbb{Z}_p(1) := \varprojlim_n \boldsymbol{\mu}_{p^n}$)

## 2. The Kummer pairing

Let $K$ be an $\ell$-dimensional local field. Then $K$ is isomorphic to one of

- $\mathbb{F}_q((X_1))\dots((X_\ell))$ if $char(K) = p$
- $M((X_1))\dots((X_{\ell-1}))$ if $char(K_1) = 0$
- a finite extension of $M\{\{T_1\}\}\dots\{\{T_n\}\}((X_{n+2}))\dots((X_\ell))$ if $char(K_{n+1}) = 0$ and $char(K_n) = p > 0$

where $L\{\{T\}\}$ is

$$\left\{\sum_{-\infty}^{+\infty} a_i T^i : a_i \in L, \inf(val_L(a_i)) > -\infty, \lim_{i\to-\infty} val_L(a_i) = +\infty\right\}.$$

Defining $val_{L\{\{T\}\}}(\sum a_i T^i) := \min\{val_L(a_i)\}$ makes $L\{\{T\}\}$ a discrete valuation field with residue field $k_L((t))$ (see Zhukov [33]).

Since we assume $char(k_K) = p > 0$ it follows that the $\ell$-dimensional local field $K$ has residue field isomorphic to $\mathbb{F}_q((X_1))\dots((X_{\ell-1}))$. Then we have the reciprocity map (1) obtained by Kato [18] (see the exposition [23]), which had already been proved by Parshin [25] when $char(K) = p > 0$.

Now we introduce the Kummer pairing through the classical Hilbert symbol (2). Assume first $char(K) = 0$ and $\zeta_{p^m} \in K$ and restrict the Hilbert pairing (2) to $(1 + \mathfrak{m}_K) \times K_\ell^M(K)$: then

$$(1 + \delta_0, \{\alpha_1, \dots, \alpha_\ell\})_m = \frac{\beta^{(\{\alpha_1,\dots,\alpha_\ell\}, K^{ab}/K)}}{\beta} = \zeta_{p^m}^c,$$

where $\beta$ is a solution of $X^{p^m} = 1 + \delta_0$ and $c \in \mathbb{Z}/p^m\mathbb{Z}$ is determined by $\delta_0$ and the $\alpha_i$'s. Recall that Milnor $K$-groups $K_n^M(K)$ are defined as $(K^*)^{\otimes^n}$ modulo the subgroup generated by $a \otimes (1 - a)$, $a \in K^* - 1$.

We rewrite the above restricted pairing as

$$\mathfrak{m}_K \times K_\ell^M(K) \to W_{p^m} := \{\zeta_{p^m}^j - 1 | j \in \mathbb{Z}\} = \{w \in \hat{\overline{K}} | (w+1)^{p^m} - 1 = 0\}$$

$$(\delta_0, \{\alpha_1, \dots, \alpha_\ell\})_m := \zeta_{p^m}^c - 1. \tag{3}$$

Recall that $\widehat{\mathbb{G}}_m$ is the formal group given by $X +_{\widehat{\mathbb{G}}_m} Y := X + Y + XY$ and $-_{\widehat{\mathbb{G}}_m} X = \sum_{i=1}^{\infty} (-1)^i X^i$. Summation $p^m$-times in $\widehat{\mathbb{G}}_m$ is given by $[p^m](X) =$

$(1+X)^{p^m}-1$. Observe that

$$(\delta_0, \{\alpha_1, \dots, \alpha_\ell\})_m = ((\{\alpha_1, \dots, \alpha_\ell\}, K^{ab}/K) -_{\widehat{\mathbb{G}}_m} 1)(\beta) \tag{4}$$

where $\beta$ is a solution of $(1+X)^{p^m}-1=\delta_0$.

This reformulation of the classical Kummer pairing for $\widehat{\mathbb{G}}_m$ can be extended to classical Lubin-Tate formal groups, to Drinfeld modules and in greater generality to $p$-divisible groups.

2.1. **Classical Lubin-Tate formal groups.** Lubin-Tate formal groups are defined for 1-dimensional local fields: classically the emphasis is on the unequal characteristic case and in this survey we shall restrict to such setting when considering Lubin-Tate formal groups. We refer the interested reader to [15] and [28] for proofs and detailed explanations.

Recall that a formal 1-dimensional commutative group law $F$ over $\mathcal{O}_M$ is $F(X,Y) \in \mathcal{O}_M[[X,Y]]$, satisfying:
(i) $F(X,Y) \equiv X+Y \ mod\ deg\ 2$;
(ii) $F(X,F(Y,Z)) = F(F(X,Y),Z)$;
(iii) $F(X,Y)=F(Y,X)$.

We fix $\pi$ a uniformizer of $M$. Define

$$\mathcal{F}_\pi := \{f \in \mathcal{O}_M[[X]] \big| f \equiv \pi X \ mod\ deg\ 2, f \equiv X^{p^l} \ mod\ \mathfrak{m}_M\},$$

where $p^l$ is the number of elements of $k_M$.

**Theorem 3** (Lubin-Tate)**.** *We have:*
*1. for every $f \in \mathcal{F}_\pi$ it exists a unique 1-dimensional commutative formal group law $F_f$ defined over $\mathcal{O}_M$ and called of Lubin-Tate, such that $f \in End(F_f)$ (i.e. $F_f(f(X),f(Y)) = f(F_f(X,Y))$).*
*2. Given $f,g \in \mathcal{F}_\pi$ then $F_g$ and $F_f$ are isomorphic over $\mathcal{O}_M$.*
*3. Given $f \in \mathcal{F}_\pi$ and $a \in \mathcal{O}_M$, it exists a unique $[a]_f(X) \in \mathcal{O}_M[[X]]$ with $[a]_f \in End(F_f)$ and $[a]_f(X) \equiv aX \ mod\ deg\ 2$; the map $a \mapsto [a]_f$ gives an embedding $\mathcal{O}_M \to End(F_f)$. We denote also $[a]_{F_f}$ by $[a]_f$.*

**Example** The formal multiplicative group $\widehat{\mathbb{G}}_m$ over $\mathbb{Q}_p$ corresponds to the Lubin-Tate formal group with $\pi = p$ and $f = (1+X)^p - 1 \in \mathcal{F}_p$.

We denote by $F_f(B)$ the $B$-valued points of $F_f$. Consider

$$W_{F_f,m} := \{zeroes\ of\ [\pi^m]_{F_f}\ in\ F_f(\mathfrak{m}_{\hat{\overline{K}}})\}.$$

**Remark 4.** We want to emphasize that we have an "embedding" of $\mathcal{O}_M$ into $End(F_f)$, from which we get a tower of field extensions $M(W_{F_f,m})$ in $\hat{\overline{M}}$. For example take $F_f = \widehat{\mathbb{G}}_m$ and consider $[p^m]_{\widehat{\mathbb{G}}_m}$ as $m$ varies in $\mathbb{N}$: the roots of $[p^m]_{\widehat{\mathbb{G}}_m}(X) = (1+X)^{p^m}-1$ are $\zeta_{p^m}-1$, thus $M(W_{\widehat{\mathbb{G}}_m,m})$ is the cyclotomic extension $M(\zeta_{p^m})$. The groups $W_{F_f,m}$ are the Lubin-Tate analogs of $\boldsymbol{\mu}_{p^m}$.

Let $K$ be a finite extension of $M$ such that $W_{F_f,m} \subseteq K$. The Kummer pairing for $F_f$ is:

$$( \, , \, )_m : F_f(\mathfrak{m}_K) \times K^* \to W_{F_f,m}$$

$$(a,u)_m := ((u, K^{ab}/K) -_{F_f} 1)\beta \tag{5}$$

where $\beta$ is a solution of $[\pi^m]_{F_f}(X) = a$.

There is a more general notion of Lubin-Tate formal group, called relative Lubin-Tate formal group, which involves the unique unramified extension of $M$ of degree $d$. The corresponding formulation of the Kummer pairing differs slightly from (5). We refer to [10, Chapter 1, §1.1,§1.4,§4.1] for the precise statement.

We also remark that there is a notion of $n$-dimensional Lubin-Tate formal group [15]. Also in this case we have an embedding of $\mathcal{O}_M$ into $End(F_f)$, and some generalized Kummer pairing appears. Since Kummer pairing on $p$-divisible groups will also include this case we do not discuss it any further here.

2.2. **1-dimensional local fields in** $char = p > 0$**: Drinfeld modules.** The key property of Lubin-Tate formal groups is the embedding: $\mathcal{O}_M \to End(F_f)$. This suggests to define a Kummer pairing for 1-dimensional local fields $K$ with $char(K) = p > 0$ in the following way.

For simplicity we take $F = \mathbb{F}_q(T)$ and put $A = \mathbb{F}_q[T]$. Observe that $End_{\mathbb{F}_q}(\mathbb{G}_a/F) \cong F\{\tau\}$, where $\tau a = a^q \tau$ for $a \in F$, $\tau^0 = id$. We can think of $F\{\tau\}$ as a subset of $F[X]$, via $\tau_0 \leftrightarrow X, \tau \leftrightarrow X^q$: then $F\{\tau\}$ consists of the additive polynomials and the product in $F\{\tau\}$ corresponds to composition.

Drinfeld [11] defined elliptic modules (now called Drinfeld modules) as embeddings

$$\Phi : A \to End_{\mathbb{F}_q}(\mathbb{G}_a/F)$$

$$a \mapsto \Phi_a = a + (\ldots)\tau,$$

$\mathbb{F}_q$-linear and non-trivial (i.e., there is $a \in A$ such that $\Phi_a$ is not equal to $a$).

For $a \in A$ we write $\Phi[a] := \{zeroes\ of\ \Phi_a(X)\}$. We have that $\Phi[a] \cong (A/(a))^d$ and define $rank(\Phi) := d$. Let $\mathfrak{p} = (\pi)$ be a place of $A$: under some technical assumptions (see [3] for a reference) we can extend $\Phi$ to

$$A_{\mathfrak{p}} \to End_{\mathbb{F}_q}(\widehat{\mathbb{G}}_a) \cong F_{\mathfrak{p}}\{\{\tau\}\}$$

which we also call $\Phi$, $a \mapsto \Phi_a$, where $F_{\mathfrak{p}}$ is the completion of the field $F$ at $\mathfrak{p}$ and $F_{\mathfrak{p}}\{\{\tau\}\}$ is the ring of skew power series.

Denote by $W_{\Phi,\pi^m}$ the set of roots of $\Phi_{\pi^m}(X)$ in $\overline{F_{\mathfrak{p}}}$. For any finite extension $K/F_{\mathfrak{p}}$ containing $W_{\Phi,\pi^m}$ we define the Kummer pairing:

$$( \, , \, )_m : \mathfrak{m}_K \times K^* \to W_{\Phi,\pi^m}$$

$$(a,u)_m := ((u, K^{ab}/K) - 1)(\beta) \tag{6}$$

where $\beta$ is a root of $\Phi_{\pi}^m(X) = a$.

For example for rank 1, the simplest Drinfeld module is the Carlitz module defined by $\Phi_T(\tau) = Tid + \tau$. Then $\Phi_{T^2}(\tau) = \Phi_T(\tau) \circ \Phi_T(\tau) = (Tid + \tau) \circ (Tid + \tau) = T^2 id + \tau T + T\tau + \tau^2 = T^2 id + (T^q + T)\tau + \tau^2$ and similarly $\Phi_{T^3}(\tau) = T^3 id + (T^{2q} + T^{q+1} + T^2)\tau + (T^{q^2} + T^q + T)\tau^2 + \tau^3$. If we take $\mathfrak{p} = (T)$ $(\pi = T)$ then $F_{\mathfrak{p}} \cong \mathbb{F}_q((T))$ and $W_{\phi,T} = \{zeroes\ of\ TX + X^q\}$, $W_{\phi,T^2} = \{zeroes\ of\ T^2X + (T^q + T)X^q + X^{q^2}\}$, $W_{\phi,T^3} = \{zeroes\ of\ T^3X + (T^{2q} + T^{q+1} + T^2)X^q + (T + T^q + T^{q^2})X^{q^2} + X^{q^3}\}$, ...

2.3. **$p$-divisible groups.** A vast extension of the theory above considers $p$-divisible groups (also called Barsotti-Tate groups). For the definition and main properties see [29].

Let $G$ be a $p$-divisible group scheme over $\mathcal{O}_K$ of dimension $d$ and finite height $h$, where $K$ is any $\ell$-dimensional local field with $char(k_K) = p > 0$. We denote by $[p^m]_G$ the $p^m$-th power map $G \to G$ and let $G[p^m]$ be the group scheme kernel. As usual, $\mathfrak{X}(B)$ denotes the $B$-points of the scheme $\mathfrak{X}$. We put $W_{G,p^m} := G[p^m](\mathcal{O}_{\overline{K}})$ and we impose $W_{G,p^m} = G[p^m](\mathcal{O}_K)$, i.e. $W_{G,p^m} \subset K$. Then Kummer pairing is defined by:

$$( \ , \ )_m : G(\mathfrak{m}_K) \times K_\ell^M(K) \to W_{G,p^m}$$

(7) $$(a, u)_m := ((u, K^{ab}/K) -_G 1)(\beta)$$

where $\beta$ is a root of $[p^m]_G(X) = a$. Finally we observe that $p$-groups with an action of $\mathcal{O}_K$ have also been studied [29]. In this case (7) can be reformulated with respect to $[\pi^m]_G$, $\pi$ a uniformizer of $K$.

2.4. **Cohomological interpretation.** Let $K$ be an $\ell$-dimensional local field with $char(k_K) = p > 0$ and $char(K) = 0$ . The Kummer pairing also admits an interpretation in terms of Galois cohomology. We do this in the generality of $p$-divisible groups. Consider the exact sequence

$$0 \to G[p^m](\mathfrak{m}_{\overline{K}}) \to G(\mathfrak{m}_{\overline{K}}) \to G(\mathfrak{m}_{\overline{K}}) \to 0.$$

This induces:

$$\delta_{1,G,m} : G(\mathfrak{m}_K) \to H^1(K, G[p^m](\mathcal{O}_{\overline{K}})).$$

We assume $W_{G,p^m} \subset K$, i.e. $G[p^m](\mathcal{O}_{\overline{K}}) = G[p^m](\mathcal{O}_K)$. The Galois symbol map:

$$h_K^r : K_r^M(K) \to H^r(K, \mathbb{Z}_p(r))$$

is obtained by cup product $h_K^1 \cup \ldots \cup h_K^1$, where $h_K^1$ is the connecting morphism $h_K^1 : K^* \to H^1(K, \mathbb{Z}_p(1))$ from the usual Kummer sequence.

There is a canonical isomorphism [18] $H^{\ell+1}(K, \mathbb{Z}_p(\ell)) \cong \mathbb{Z}_p$ defining the following pairing:

$$\widehat{( \ , \ )}_m : G(\mathfrak{m}_K) \times K_\ell^M(K) \to H^{\ell+1}(K, \mathbb{Z}_p(\ell) \otimes G[p^m](\mathcal{O}_K)) \cong G[p^m](\mathcal{O}_K)$$

(8) $$\widehat{(a, u)}_m := (-1)^\ell \delta_{1,G,m}(u) \cup h_K^\ell(a).$$

Then by [12, Proposition 6.1.1] $\widehat{(a,u)}_m = ((u, K^{ab}/K) -_G 1)(\beta) = (a,u)_m$ where $\beta$ is a root of $[p^m]_G(X) = a$.

If one could define a good analog in characteristic $p$ of the Galois symbol maps for the cohomological groups from Illusie cohomologies used by Kato [18] to obtain the reciprocity law map $(\ ,K^{ab}/K)$, then a cohomological interpretation should appear when $char(K) = p > 0$ (following arguments like the proof of [12, Proposition 6.1.1]). We refer to [23, §5] for a very quick review of the definitions of these cohomological groups and of Kato's higher local class field theory.

2.5. **Limit forms of the Kummer pairing.** In this subsection $(\ ,\ )_m$ denotes any of the pairings (4) to (7).

Let $W_{\sharp,m}$ be one of $W_{F_f,\pi^m}$, $W_{\Phi,\pi^m}$ or $W_{G,p^m}$. We shorten $K(W_{\sharp,m})$ to $K_{\sharp,m}$ and write $\mathcal{O}_{\sharp,m}$ (resp. $\mathfrak{m}_{\sharp,m}$) for the ring of integers (resp. the maximal ideal). By definition the Tate module is $T_p(\sharp) := \varprojlim_m W_{\sharp,m}$ where the limit is taken with respect to the map $[\sharp]$ (which means $[\pi]_f, \Phi_\pi$ or $[p]_G$). Consider $W_{\sharp,\infty} := \varinjlim_m W_{\sharp,m} = \cup_m W_{\sharp,m}$.

The symbol $\mathcal{M}_{\sharp,m}$ denotes one of $F_f(\mathfrak{m}_{\sharp,m})$, $\mathfrak{m}_{\sharp,m}$ or $G(\mathfrak{m}_{\sharp,m})$. Consider $\varinjlim_m \mathcal{M}_{\sharp,m}$ as the direct limit of $[\sharp]$ : it consists of sequences $(a_n)_{n\geq N}$ (for some $N \in \mathbb{N}$) such that $a_n \in K_{\sharp,n}$ and $a_{n+1} = [\sharp]a_n$.

We need to impose that the $K_{\sharp,m}$'s are **abelian** extensions of $K$ (this is satisfied for example by 1-dimensional classical Lubin-Tate formal groups and rank 1 Drinfeld modules). Then we have a limit version of the Kummer pairing:

$$(\ ,\ ) : \varinjlim_m \mathcal{M}_{\sharp,m} \times \varprojlim_m K_\ell^M(K_{\sharp,m}) \to W_{\sharp,\infty} \tag{9}$$

where $\varprojlim_m K_\ell^M(K_{\sharp,m})$ is with respect to the Norm map. The limit pairing is well defined: by the abelian assumption we have (Kato-Parshin's reciprocity law) $(N_{K_{\sharp,n}/K_{\sharp,n-1}}(u'), K_{\sharp,n-1}^{ab}/K_{\sharp,n-1}) = (u', K_{\sharp,n}^{ab}/K_{\sharp,n})$ acting on the roots of $[\sharp]^{n-1}(X) = a$, when $u' \in K_\ell^M(K_{\sharp,n})$.

We remark that the above is often formulated without taking the whole limit tower. That is, suppose that one wants to compute explicitly $(a,u)_m$ and $u' \in K_\ell^M(K_{\sharp,m+k})$ is given so that $N_{K_{\sharp,m+k}/K_{\sharp,m}}(u') = u$: then by a similar argument as needed for defining (9) we have

$$(a,u)_m = ([\sharp]a, N_{K_{\sharp,m+k}/K_{\sharp,m+1}}(u'))_{m+1} = \ldots = ([\sharp]^k a, u')_{m+k}. \tag{10}$$

We can also write the above limit form (9) as:

$$\varprojlim_n K_\ell^M(K_{\sharp,n}) \to Hom(\mathcal{M}_{\sharp,m}, T_p(\sharp)) \tag{11}$$

sending $u = (u_n)_n$ to $a \mapsto \lim_{n\to\infty}(a, u_n)_n$ with fixed $m$. These homomorphisms are continuous because of the continuity of the Kummer pairing: see [3, Lemma 15] for the case $\ell = 1$.

## 3. Explicit reciprocity law formulas à la Wiles' for 1-dimensional local fields

In this section we fix $\ell = 1$ and sketch some of the main ideas to obtain explicit reciprocity laws for a 1-dimensional local field $K$ in the context of classical Lubin-Tate formal groups and rank 1 Drinfeld modules, introducing Coleman power series. See the last section for different approaches to this explicit reciprocity law through the exponential or dual exponential map.

Let $K$ be either $M$ or $F_{\mathfrak{p}}$. In the Drinfeld module case $\Phi$ is required to be of rank 1 (in order to obtain abelian extensions) and sign-normalized. We refrain from explaining this last technical condition (the reader is referred to [3, §2.1] and the sources cited there) and just notice that when $A = \mathbb{F}_q[T]$ as in §2.2 this means that $\Phi$ is the Carlitz module, i.e. $\Phi_T(X) = TX + X^q$. Furthermore we take $\pi \in A$ a monic polynomial.

We lighten the notation introduced in §2.5 by shortening $K_{\sharp,m}$ to $K_m$.

The extensions $K_m/K$, are totally ramified and abelian: they are generated by roots of Eisenstein polynomials and $Gal(K_m/K) \cong (\mathcal{O}_K/(\pi)^n)^*$. The Tate module $T_p(\sharp)$ is a rank 1 $\mathcal{O}_K$-module with $\mathcal{O}_K$-action given by $\alpha\cdot\gamma := [\alpha]_f\gamma$ or $\Phi_\alpha(\gamma)$. Let $(\varepsilon_n)_n$ be an $\mathcal{O}_K$-generator of $T_p(\sharp)$: then $K_{\sharp,m} = K(\varepsilon_m)$, since $\varepsilon_m$ generates $W_{\sharp,m}$, and one has $[\sharp]\varepsilon_{n+1} = \varepsilon_n$, $[\sharp]^{n+1}\varepsilon_n = 0$ where $[\sharp]$ is respectively $[\pi]_f$ or $\Phi_\pi$. Moreover the $\varepsilon_n$'s form a norm compatible system. Denote $\cup_m K(W_{\sharp,m})$ by $K_{\sharp,\infty}$.

In this section $(\ ,\ )_m$ refers to (5) or (6) and $(\ ,\ )$ refers to (9).

First, notice that $(\ ,\ )_m$ is bilinear, additive in the first variable and multiplicative in the second variable. In particular $(\ ,\zeta)_m = 0$ for any root of unity $\zeta$.

Consider the character

$$\chi : Gal(K_{\sharp,\infty}/K) \to \mathcal{O}_K^*$$

$\sigma \mapsto \chi_\sigma$ defined by $\sigma((\varepsilon_n)_n) = \chi_\sigma\cdot((\varepsilon_n)_n)$; it can be thought of as a character of $Gal(K^{ab}/K)$. We remark that for $\widehat{\mathbb{G}}_m$ this $\chi$ corresponds to the cyclotomic character $\chi_{cycl}$. The main point is the following: $\chi^{-1} : \mathcal{O}_K^* \to Gal(K_{\sharp,\infty}/K)$ coincides with the inverse of the local norm symbol map. Therefore:

$$(\varepsilon_m, u)_m = \begin{cases} ([N_{K_m/K}u]_f^{-1} -_{F_f} 1)(\varepsilon_{2m}) & K = M \\ \Phi_{(N_{K_m/K}u)^{-1}}(\varepsilon_{2m}) - \varepsilon_{2m} & K = F_{\mathfrak{p}}. \end{cases} \tag{12}$$

Inspired by (12) one can ask: could we find $h$ such that $(a_m, u_m)_m = [h(a,u)]_f(\varepsilon_m)$ for classical Lubin-Tate formal groups or $= \Phi_{h(a,u)}(\varepsilon_m)$ for rank 1 Drinfeld modules? Observe that $h(a,u)$ modulo $\pi^m$ defines the same action over $\varepsilon_m$ (because $W_{\sharp,m}$ is isomorphic to $\mathcal{O}_K/(\pi^m)$).

In this direction one obtains the following results. From class field theory it follows $(a,a)_n = 0$. Exploiting linearity and continuity of the pairing $(\ ,\ )_m$ one obtains (assuming $v(c)$ greater than a fixed value which depends linearly of $n$),

$$(c,w)_n = (c\varepsilon_n \frac{\mathrm{dlog}\, w}{\mathrm{d}\varepsilon_n}, \varepsilon_n)_n \tag{13}$$

where

$$\mathrm{dlog}\ : \mathcal{O}^*_{\sharp,n} \to \Omega^1_{\mathcal{O}_{\sharp,n}/\mathcal{O}_K}$$

is the map $x \mapsto \frac{dx}{x}$ (recall that the module of Kähler differentials $\Omega^1_{\mathcal{O}_{\sharp,n}/\mathcal{O}_K}$ is free with generator $\mathrm{d}\varepsilon_n$ over $\mathcal{O}_{\sharp,n}/\mathfrak{d}_{K_n/K}$, where $\mathfrak{d}_{K_n/K}$ is the different of $K_n$ over $K$). Notice that one can pick a power series $g \in \mathcal{O}_K((X))$ such that $g(\varepsilon_n) = w$ and $\mathrm{dlog}\, w/\mathrm{d}\varepsilon_n = g'(\varepsilon_n)/g(\varepsilon_n)$.

Finally one proves that exists $m > n$ such that

$$(a_n, \varepsilon_n)_n = -(\varepsilon_m, 1 + a_m\varepsilon_m^{-1})_m$$

giving a positive answer to the question above.

This suggests the following definition of an analytic pairing:

$$[a,u]_n := Tr_{K_n/K}\left(\pi^{-n}\lambda(a)\frac{\mathrm{dlog}\, u}{\mathrm{d}\varepsilon_n}\right)\cdot \varepsilon_n\,.$$

Here $\cdot$ is the action on the Tate module at level $n$ and $\lambda$ is the logarithm map defined by $\lambda(a) := \lim \frac{1}{\pi^n}[\sharp]^n a$ (the limit exists for $val_K(a)$ sufficiently big).

In order to compare $(a,u)_n$ and $[a,u]_n$, Iwasawa (theorem 2) imposes the condition that there exists $m$ such that $u = N_{\mathbb{Q}_p(\zeta_{p^m})/\mathbb{Q}_p(\zeta_{p^n})}(u')$: then $(a,u)_n = ([p]^{m-n}_{\widehat{\mathbb{G}}_m} a, u')_m$ and it is at level $m$ that he compares the two pairings obtaining $([p]^{m-n}_{\widehat{\mathbb{G}}_m} a, u')_m = [[p]^{m-n}_{\widehat{\mathbb{G}}_m} a, u']_m$.

The general case follows Iwasawa's argument. Here we will state a limit version, hence we suppose that $(u_n) \in \varprojlim K_n^*$ (limit w.r.t. the norm).

To express in compact form the limit of the pairings $[\ ,\ ]_n$ it is convenient to introduce Coleman's power series.

**Theorem 5.** *Let $K$, $F_f$, $\Phi$ be as above. Then*

(1) [9] *(case $K = M$) There exists a unique operator $\mathcal{N}$ (the Coleman norm) defined by the property*

$$(\mathcal{N}h)\circ f(X) = \prod_{w\in W_{F_f,0}} h(X +_{F_f} w)$$

*for any $h \in M((x))_1$ (the set of those Laurent series which are convergent in the unit ball).*

(2) [3] *(case $K = F_\mathfrak{p}$) There exists a unique operator $\mathcal{N}$ such that*

$$(\mathcal{N}h)\circ \Phi_\pi = \prod_{v\in\Phi[\mathfrak{p}]} h(x+v)$$

*for any* $h \in F_{\mathfrak{p}}((x))_1$.

*Moreover, in both cases the evaluation map* $ev : f \mapsto (f(\varepsilon_n))_n$ *gives an isomorphism*

$$(\mathcal{O}_K((x))^*)^{\mathcal{N}=id} \cong \varprojlim K^*_{\sharp,n}\,.$$

Denote by $Col_u$ the power series associated to $u \in \varprojlim K^*_{\sharp,n}$. Then we define the limit form of the analytic pairing by

$$[\ ,\ ] : \varinjlim_n \mathcal{M}_{\sharp,m} \times \varprojlim_n K^*_{\sharp,m} \to W_{\sharp,\infty}$$

$$[a,u] := Tr_{K_{\sharp,n}/K}(\pi^{-n}\lambda(a_n)\mathrm{dlog}\,Col_u(\varepsilon_n)) \cdot \varepsilon_n\,.$$

Notice that $\mathrm{dlog}\,Col_u(\varepsilon_n) = \frac{\mathrm{dlog}\,u_n}{d\varepsilon_n}$.

Finally one observes that $[a,u]$ has similar properties to $(a,u)$; in particular $[a,u] = [a_n\varepsilon_n\mathrm{dlog}\,Col_u(\varepsilon_n), \varepsilon_n]_n$ (compare with (13)) and to prove the reciprocity law one is reduced to show $[\ ,\varepsilon_n]_n = (\ ,\varepsilon_n)_n$ for $n$ big enough.

**Theorem 6.** *Under the above notation, we have*

(1) [32] $(\ ,\ ) = [\ ,\ ]$ *for classical Lubin-Tate formal groups* $F_f$.
(2) [3] $(\ ,\ ) = [\ ,\ ]$ *for* $F_{\mathfrak{p}}$ *and* $\Phi$.

In [32] Wiles proved this result for classical Lubin-Tate formal groups without using Coleman power series: he takes $m$ big enough in order to compare $(\ ,\ )_m$ and $[\ ,\ ]_m$, like Iwasawa had done for $\widehat{\mathbb{G}}_m$. This strategy requires a very precise valuation calculation. For a detailed explanation one can look also at [24, §8,9]. The above limit version with Coleman power series for classical Lubin-Tate formal groups can be found in [10, I,§4].

For the Carlitz module, Anglès in [1] obtained Wiles' version of theorem 6, i.e. without Coleman power series.

## 4. Explicit reciprocity laws and higher $K$-theory groups

The starting point for the results in §3 is explicit local class field theory applied to $\varepsilon$ and the dlog homomorphism. In [19] and [20] Kato reinterpreted Wiles' reciprocity law as an equality between two maps obtained by composition of natural maps from cohomology groups and gave generalized reciprocity laws in higher $K$-theory. Here we introduce his approach to the classical Hilbert symbol for $\ell$-dimensional local fields, $\ell > 1$, and reformulate parts of it for rank 1 Drinfeld modules.

### 4.1. Exponential map in the Hilbert symbol.

In this paragraph we assume $char(K) = 0$ and $char(k_K) = p > 0$. Take:

$$exp_\delta : \mathcal{O}_K \to \mathcal{O}_K^*$$

given by $exp_\delta(a) = exp(\delta a)$ if $val_K(\delta) > \frac{val_K(p)}{(p-1)}$ where $exp(X) = \sum_{m\geq 0}\frac{X^m}{m!}$ is the exponential. Assuming $val_K(\delta) \geq \frac{2val_K(p)}{(p-1)}$, the map $exp_\delta$ extends to a group morphism [22]:

$$exp_{\delta,r} : \Omega^{r-1}_{\mathcal{O}_K} \to \widehat{K^M_r(K)}$$
$$a\mathrm{dlog}\, b_1 \wedge \cdots \wedge \mathrm{dlog}\, b_{r-1} \mapsto \{exp(\delta a), b_1, \ldots, b_{r-1}\},$$

where $\widehat{K^M_r(K)} := \varprojlim_n K^M_r(K)/p^n K^M_r(K)$, $\Omega^r_R := \bigwedge^r_R \Omega^1_R$, $\Omega^1_R$ are the Kähler differentials and $r$ is a strict positive integer. The function $exp_{\delta,r}$ factors through $\bigwedge^{r-1}_{\mathcal{O}_K} \hat{\Omega}^1_{\mathcal{O}_K}$ where $\hat{\Omega}^1_{\mathcal{O}_K}$ is the $p$-adic completion of $\Omega^1_{\mathcal{O}_K}$.

Sen [27] generalizes theorem 2:

**Theorem 7.** *Assume $\zeta_{p^n} \in M$. Fix $\pi$ a uniformizer of $M$, $\alpha_1 \in \mathcal{O}_M - \{0\}$, and $\alpha \in K$ with $val_K(\alpha - 1) \geq \frac{2val_K(p)}{p-1} + val_K(\alpha_1)$. Then*

$$(\alpha, \alpha_1)_n = \zeta^c_{p^n}, \text{ with } c = \frac{-1}{p^n} Tr_{M/\mathbb{Q}_p}\left(\frac{\zeta_{p^m}}{h'(\pi)} \frac{g'(\pi)}{\alpha_1} \log \alpha\right)$$

*where $g, h \in \mathcal{O}_{M_0}[T]$ are such that $g(\pi) = \alpha_1$ and $h(\pi) = \zeta_{p^n}$, and the pairing is (2).*

From now on take $val_M(\eta) = \frac{2val_M(p)}{p-1}$, $\eta \in M_0(\zeta_{p^n})$. The proof of theorem 7 reduces to theorem 2 because $Tr_{M/M_0(\zeta_{p^m})}(ad\pi) = Tr_{M/M_0(\zeta_{p^m})}(\frac{a}{h'(\pi)})d\zeta_{p^m}$ and the commutativity of the following diagram ([22, p.217]):

$$\begin{array}{ccccc} \hat{\Omega}^1_{\mathcal{O}_M} & \xrightarrow{exp_{\eta,2}} & \widehat{K^M_2}(M) & \xrightarrow{h_M} & \mathbb{Z}/p^n(1) \\ \downarrow Tr_{M/M_0(\zeta_{p^m})} & & \downarrow N_{M/M_0(\zeta_{p^m})} & & \downarrow id \\ \hat{\Omega}^1_{\mathcal{O}_{M_0(\zeta_{p^n})}} & \xrightarrow{exp_{\eta,2}} & \widehat{K}^M_2(M_0(\zeta_{p^n})) & \xrightarrow{h_{M_0(\zeta_{p^n})}} & \mathbb{Z}/p^n(1) \end{array}$$

where $h_*$ is the Hilbert symbol $\{a, b\} \mapsto (a, b)_n$.

We rewrite Sen's theorem as a commutative diagram. Take $\gamma \in \mathcal{O}_M$ with $val_M(\gamma) = \frac{val_M(p)}{p-1}$. Fontaine proved that we have the following isomorphism

$$\Psi_M : \hat{\Omega}^1_{\mathcal{O}_M}/p^n\gamma^{-1} \to \gamma^{-1}\mathfrak{d}^{-1}_{M/M_0}/\gamma^{-1}\mathfrak{d}^{-1}_{M/M_0(\zeta_{p^n})}(1)$$

induced from the map $a\mathrm{dlog}\, \zeta_{p^m} \mapsto ap^{-m} \otimes (\zeta_{p^m})_{m>0}$ where $\mathfrak{d}_{L_1/L_2}$ is the different of the finite extension of fields $L_1/L_2$. Commutativity of the diagram

$$\begin{array}{ccc} \hat{\Omega}^1_{\mathcal{O}_M}/p^n\gamma^{-1} & \xrightarrow{\Psi_M} & \gamma^{-1}\mathfrak{d}^{-1}_{M/M_0}/\gamma^{-1}\mathfrak{d}^{-1}_{M/M_0(\zeta_{p^n})}(1) \\ {\scriptstyle -exp_\eta}\downarrow & & {\scriptstyle Tr_\eta \otimes id}\downarrow \\ K^M_2(M)/p^n & \xrightarrow{h_M} & \mathbb{Z}/p^n(1) \end{array}$$

(where $Tr_\eta$ is the map induced by $x \mapsto Tr_{M/\mathbb{Q}_p}(\eta x)$) is Sen's theorem [22, §4].

Now we consider $\ell > 1$. Assume $\zeta_{p^n} \in K$. Consider $K_0$ with $K/K_0$ a finite and totally ramified extension such that $p$ is a prime element of $\mathcal{O}_{K_0}$.

Recall $k_K = \mathbb{F}_q((t_1))\dots((t_{\ell-1}))$. Take $M$ such that $M = M_0$, $M \subseteq K_0$ and the residue field is $\mathbb{F}_q$: then $K_0 = M\{\{T_1\}\}\cdots\{\{T_{\ell-1}\}\}$. Kurihara extends Sen's result to higher Milnor $K$-theory (theorem 8) by means of the commutativity of the diagram:

$$\begin{array}{ccccc}
\hat{\Omega}^{\ell}_{\mathcal{O}_K} & \xrightarrow{exp_{\eta,\ell+1}} & \widehat{K^M_{\ell+1}}(K) & \xrightarrow{h_K} & \mathbb{Z}/p^n(1) = \boldsymbol{\mu}_{p^n} \\
\downarrow{\scriptstyle Tr_{K/K_0(\zeta_{p^m})}} & & \downarrow{\scriptstyle N_{K/K_0(\zeta_{p^m})}} & & \downarrow{\scriptstyle id} \\
\hat{\Omega}^{\ell}_{\mathcal{O}_{K_0(\zeta_{p^n})}} & \xrightarrow{exp_{\eta,\ell+1}} & \widehat{K^M_{\ell+1}}(K_0(\zeta_{p^n})) & \xrightarrow{h_{K_0(\zeta_{p^n})}} & \mathbb{Z}/p^n(1) \\
\downarrow{\scriptstyle Res} & & \downarrow{\scriptstyle \underline{Res}} & & \downarrow{\scriptstyle id} \\
\hat{\Omega}^{1}_{\mathcal{O}_{M(\zeta_{p^n})}} & \xrightarrow{exp_{\eta,2}} & \widehat{K^M_{2}}(M(\zeta_{p^n})) & \xrightarrow{h_{M(\zeta_{p^n})}} & \mathbb{Z}/p^n(1)
\end{array}$$

where $Res : \hat{\Omega}^{\ell}_{\mathcal{O}_{K_0(\zeta_{p^n})}} \to \hat{\Omega}^1_{\mathcal{O}_M}$ is defined by $Res(w\mathrm{dlog}\, T_1 \wedge\cdots\wedge \mathrm{dlog}\, T_{\ell-1}) = w$ where $w \in \hat{\Omega}^1_{\mathcal{O}_{M(\zeta_{p^n})}}$, $\underline{Res}$ is the Kato's residue homomorphism in Milnor $K$-groups and $h_L$ is the Hilbert symbol $\{a_1,\dots,a_{\ell+1}\} \mapsto (a_1, \{a_2,\dots,a_{\ell+1}\})_n$.

**Theorem 8** ([22])**.** *Take $\alpha_1,\dots,\alpha_\ell \in \mathcal{O}^*_K$ and $\alpha \in \mathcal{O}^*_K$ with $val_K(\alpha-1) \geq \frac{2val_K(p)}{p-1}$. Take $f_i(T,T_2,\dots,T_\ell) \in \mathcal{O}_{K_0}[T]$ such that $f_i(\pi,T_2,\dots,T_\ell) = \alpha_i$, and $h(T) \in \mathcal{O}_{K_0}[T]$ with $h(\pi) = \zeta_{p^n}$ where $\pi$ is a fixed uniformizer of $K$. Then $(\alpha, \{\alpha_1,\dots,\alpha_\ell\}) = \zeta^c_{p^n}$ where*

$$c = -\frac{1}{p^n}\mathcal{T}\left(\log\alpha \frac{T_2\cdots T_\ell}{\alpha_1\cdots\alpha_\ell}\frac{\zeta_{p^n}}{h'(\pi)}\left[\det(\frac{\partial f_i}{\partial T_j})_{1\leq i,j\leq \ell}\right]_{|T_1=\pi}\right).$$

*Here $\mathcal{T} := Tr_{M(\zeta_{p^n})/\mathbb{Q}_p}\circ c_{K_0(\zeta_{p^n})/M(\zeta_{p^n})}\circ Tr_{K/K_0(\zeta_{p^n})}$, with $c_{L\{\{T\}\}/L}(\sum a_i T^i) := a_0$ and $c_{L\{\{T_2\}\}\{\{T_3\}\}\cdots\{\{T_k\}\}/L}$ defined recursively as composition of the maps $c_{L\{\{T_2\}\}\{\{T_3\}\}\cdots\{\{T_{i+1}\}\}/L\{\{T_2\}\}\{\{T_3\}\}\cdots\{\{T_i\}\}}$.*

**Remark 9.** Benois in [4] extends theorem 7 to formal groups.

4.2. **Kato's generalized explicit reciprocity laws.** Kato generalizes Wiles' reciprocity law for an unequal characteristic local field $K$ giving a natural interpretation in the context of cohomology groups. Here we introduce two generalizations.

4.2.1. *Local approach.* Let us first rewrite (11) in the context of Wiles' reciprocity law: let $\pi$ be a fixed uniformizer of $M$ , $F_f$ a formal Lubin-Tate group and $\varepsilon$ a fixed generator for the Tate module. For simplicity we assume that $p$ is prime in $K$. Recall [20, Remark 4.1.3] that a Lubin-Tate formal group $G$ over $\mathcal{O}_M$ has the following characterization in $p$-divisible groups: $dim(G) = 1$, and the canonical map $End(G) \to End_{\mathcal{O}_M}(LieG) \cong \mathcal{O}_M$ is an isomorphism, where $Lie(G)$ is the tangent of $G$ at the origin and $coLie(G)$ is $Hom_{\mathcal{O}_M}(Lie(G), \mathcal{O}_M)$. Put $G' := G \times_{\mathcal{O}_M} \mathcal{O}_{K_{F_f,m}}$; then

$Lie(G') \otimes_{\mathcal{O}_M} \mathbb{Q} \cong K_{F_f,m}$ and $coLie(G') \otimes_{\mathcal{O}_M} \mathbb{Q} \cong K_{F_f,m}$. Consider the map $\varrho_\varepsilon : \varprojlim_n K^*_{F_f,n} \to K_{F_f,m}$ given by the composition

$$\varprojlim_n K^*_{F_f,n} \xrightarrow{(\ ,\ )(11)} Hom_{\mathcal{O}_M,cont}(F_f(\mathfrak{m}_{f,m}),\mathcal{O}_M) \xrightarrow{\circ exp} Hom_{\mathcal{O}_M}(Lie(G'),M)$$

$$\xrightarrow{Tr} \quad K_{F_f,m} \cong \mathbb{Q} \otimes_{\mathcal{O}_M} coLie(G')$$

where the last isomorphism is Kato's trace pairing [19, II,§2] and $exp$ is the exponential map.

Wiles' reciprocity law affirms that $\varrho_\varepsilon(a)$ has an expression in terms of dlog, which can be reformulated by defining a natural map $\delta_\varepsilon$. Here we define $\varrho_\varepsilon$ and $\delta_\varepsilon$ in Kato's generality [20, §6]: this essentially includes Lubin-Tate formal groups and the generalized Wiles' reciprocity law obtained in §3 ([20, 6.1.10]).

Let $K$ be an $\ell$-dimensional local field with $char(K) = 0$ and $char(k_K) = p > 0$. Take $G$ a $p$-divisible group over $\mathcal{O}_K$ with $dim(G) = 1$. Suppose $\Lambda \hookrightarrow End(G)$ with $\Lambda$ an integral domain over $\mathbb{Z}_p$ which is free of finite rank as $\mathbb{Z}_p$-module. Suppose $T_pG$ is a free $\Lambda$-module of rank 1 and fix a generator $\varepsilon$. Let $K_n/K$ be the field extension corresponding to $Ker(G_K \to Aut_\Lambda(T_pG/p^n))$. Denote the $p$-adic $G_K$-representation $T_pG \otimes \mathbb{Q}_p$ by $V_pG$: to it one can apply Fontaine's theory. By Fontaine's ring $B_{dR,K}$ (here we just recall that it is a filtered $G_K$-module - see [19, §2] for more) one constructs the filtered module $D_{dR,K}(V_pG) := H^0(K, B_{dR,K} \otimes_{\mathbb{Q}_p} V_pG)$. One has that $gr^{-1}D_{dR,K}(V_pG)$ is canonically isomorphic to $\mathbb{Q} \otimes Lie(G)$ and $dim_K(D_{dR,K}(V_pG)) = dim_{\mathbb{Q}_p}(V_pG)$. Then one defines a map [20, Proposition(2.3.3),p.118]

$$F_{DR} : H^{\ell-1}(K, \mathbb{Z}_p(\ell) \otimes Hom_\Lambda(T_pG, \Lambda)) \to \hat{\Omega}^{\ell-1}_K \otimes_{\mathcal{O}_K} coLie(G)$$

where $\hat{\Omega}^r_K$ is the $p$-adic completion of $\Omega^r_{\mathcal{O}_K/\mathbb{Z}}$ tensored by $\mathbb{Q}$.

Kato extends the above $\varrho_\varepsilon$ to

$$\varrho_\varepsilon : \varprojlim_n K^M_\ell(K_n) \xrightarrow{GSM} H^\ell(K, \mathbb{Z}_p(\ell) \otimes Hom_\Lambda(T_pG, \Lambda))$$

$$\xrightarrow{F_{DR}} \qquad \hat{\Omega}^{\ell-1}_K \otimes_{\mathcal{O}_K} coLie(G)\,.$$

The recipe to construct $GSM$ is: fix $n$, then compose the Galois symbol map defined in §2.4 (but with coefficients in $(\mathbb{Z}/p^n\mathbb{Z})(\ell)$) instead of $\mathbb{Z}_p(\ell)$) with $\cup\varepsilon_n^{-1} : H^\ell(K_n, (\mathbb{Z}/p^n\mathbb{Z})(\ell)) \to H^\ell(K_n, (\mathbb{Z}/p^n\mathbb{Z})(\ell) \otimes Hom_\Lambda(T_pG, \Lambda))$ and take the trace map to have the cohomology group over $K$, and finally take the limit with respect to $n$.

As for $\delta_\varepsilon$: up to some technical details which we do not reproduce here (see [20, p.119]) it is essentially the map dlog $: K^M_\ell(K_n) \to \hat{\Omega}^\ell_K$ given by $\{\alpha_1, \ldots, \alpha_\ell\} \mapsto \mathrm{dlog}\,(\alpha_1) \wedge \cdots \wedge \mathrm{dlog}\,(\alpha_\ell)$.

**Theorem 10.** *One has* $\varrho_\varepsilon = (-1)^{\ell-1}\delta_\varepsilon$ [20, Theorem 6.1.9].

**Remark 11.** Kato obtains a generalized reciprocity law when $dim(G) = 1$, $T_pG$ is $\Lambda$-free with $rank_\Lambda T_pG = \ell \geq 1$ and some technical conditions from Fontaine's theory are satisfied [20, Theorem 4.3.4]. He also defines $\varrho^s$ (from Fontaine's theory) and $\delta^s$ (on the dlog side) with $s \in \mathbb{N}^\ell$ as maps $\varprojlim_n K_\ell^M(Y_n) \to \mathcal{O}(Y_m) \otimes_{\mathcal{O}_K} coLie(G)^{\otimes(\ell+r(s))}$ where $Y_n$ is a scheme representing a functor related with the $p^n$-torsion points associated to $G$ and $r(s) \in \mathbb{N}$.

**Remark 12.** Kato's generalized reciprocity laws (theorem 10 and remark 11) did not include a generalization of Coleman power series. Fukaya [13] obtains a $K_2$-analog for the Coleman isomorphism.

**Remark 13.** In the previous paper [19, II] Kato had obtained a reciprocity law for a map $\hat{\varrho}_\varepsilon$ whose definition involves the Kummer map (11) for the tower of fields given by any classical Lubin-Tate formal group (i.e. $\ell = 1$ and $K = M$) and the dual exponential map (see below) associated to a certain representation of the formal group (more precisely, to a power of a Hecke character obtained from a CM elliptic curve). He relates $\hat{\varrho}_\varepsilon$ with a map $\hat{\delta}_\varepsilon$ constructed mainly from dlog. This reciprocity law is expressed in terms of Coleman power series [19, II]. Tsuji in [30, I] extends Kato's reciprocity law [19, II] to representations coming from more general Hecke characters.

4.2.2. *Global approach.* Now the base field is $M$, in particular $\ell = 1$ and $car(k_M) = p > 0$. As explained in [26, §3.3], Kato proves that Wiles' reciprocity (or rather Iwasawa's, since it is the case $\mathcal{F}_f = \mathbb{G}_m$) is equivalent to the commutativity of the following diagram (which follows from cohomological properties):

$$\begin{array}{ccc}
\mathbb{Q} \otimes \varprojlim_{n,Norm} \mathcal{O}^*_{M(\zeta_{p^n})} & \xrightarrow[Kummer]{} & \mathbb{Q} \otimes \varprojlim_n Hom_{cont}(G_{M(\zeta_{p^n})}, \boldsymbol{\mu}_{p^n}) \\
\downarrow \text{dlog} & & \downarrow \zeta_{p^n} \mapsto 1 \\
\mathbb{Q} \otimes \varprojlim_{n,Trace} \Omega^1_{\mathcal{O}_{M(\zeta_{p^n})}/\mathcal{O}_M} & & \mathbb{Q} \otimes \varprojlim_n H^1(M(\zeta_{p^n}), \mathbb{Z}/p^n) \\
\downarrow \text{dlog}\,\zeta_{p^n} \mapsto 1 & & \downarrow cor \\
\mathbb{Q} \otimes \varprojlim_{t_{-,-}} \mathcal{O}_M[\zeta_{p^n}]/p^n & & \mathbb{Q} \otimes \varprojlim_n H^1(K_m, \mathbb{Z}/p^n) \\
\downarrow (t_{n,m})_n & & \downarrow inc \\
M(\zeta_{p^m}) & \xrightarrow[\cup \frac{1}{p^m} \log \chi_{cycl}]{\cong} & H^1(M(\zeta_{p^m}), \hat{\overline{M}})
\end{array}$$

where $t_{n,m} := \frac{1}{p^{n-m}} Tr_{M(\zeta_{p^n})/M(\zeta_{p^m})}$, $cor$ is the corestriction map, $\chi_{cycl}$ is the cyclotomic character as in §3, $\cup \log \chi_{cycl}$ is the cup product by the element

$\log \chi_{cycl} \in Hom_{cont}(G_M, \mathbb{Z}_p)$, $inc$ is the map induced by $\varprojlim_n \mathbb{Z}/p^n = \mathbb{Z}_p \hookrightarrow \hat{\overline{M}}$ and $Hom_{cont}(G_{M(\zeta_{p^n})}, \boldsymbol{\mu}_{p^n}) \cong H^1(M(\zeta_{p^n}), \mathbb{Z}/p^n(1))$ by $\zeta_{p^n} \mapsto 1$.

Inspired by this cohomological approach, Kato formulates a new reciprocity law. To state it we need to introduce some more notation. Take a smooth $\mathcal{O}_M$-scheme $U$, complement of a divisor $Z$ with relatively normal crossings in a smooth proper $\mathcal{O}_M$-scheme $X$. We assume there is a theory of Chern classes for higher Quillen $K$-theory giving functorial homomorphisms

$$ch_r : K_r(U \otimes_{\mathcal{O}_M} \mathcal{O}_{M(\zeta_{p^n})}) \to H^r_{et}(U \otimes_M M(\zeta_{p^n}), \mathbb{Z}_p(r))$$

where the $K_r$'s are Quillen $K$-theory groups and $H_{et}$ denotes continuous ètale cohomology. By the Hochschild-Serre spectral sequence we obtain a map

$$HS \circ ch_r : K_r(U \otimes_{\mathcal{O}_M} \mathcal{O}_{M(\zeta_{p^n})}) \to H^r(M(\zeta_{p^n}), H^{r-1}(U \otimes_M \overline{M}, \mathbb{Z}_p(r))).$$

Now $V := H^{r-1}(U \otimes_M \overline{M}, \mathbb{Z}_p(r))$ is a $p$-adic $G_M$-representation and Fontaine's theory applies. By Fontaine's ring $B_{dR,M}$ (here we just recall that it is a filtered $G_M$-module - see [19, §2] for more) one constructs the filtered $M$-vector space $D_{dR,M}(V) := (B_{dR,M} \otimes_{\mathbb{Q}_p} V)^{G_M}$. Suppose that $V$ is de Rham, that is $dim_M(D_{dR,M}(V)) = dim_{\mathbb{Q}_p}(V)$. Then the dual exponential map $exp^* : H^1(M, V) \to D^0_{dR,M}(V)$ is given by the composition of

$$H^1(M,V) \xrightarrow{inclusion} H^1(M, \hat{\overline{M}} \otimes_{\mathbb{Q}_p} V)$$
$$\xrightarrow{\cong} H^1(M, \hat{\overline{M}}) \otimes_M D^0_{dR,M}(V) \xrightarrow{\cong \cup \log \chi_{cycl}} D^0_{dR,M}(V),$$

where $\cup \log \chi_{cycl}$ is the Tate isomorphism $M = H^0(M, \hat{\overline{M}}) \to H^1(M, \hat{\overline{M}})$ and the first isomorphism is induced by the Hodge-Tate decomposition $\hat{\overline{M}} \otimes_{\mathbb{Q}_p} V \cong \oplus_{i \in \mathbb{Z}} \hat{\overline{M}}(-i) \otimes_M gr^i D_{dR,M}(V)$ [26, p.407] (for more on $exp^*$ and how it fits into a motivic Tamagawa number framework see [7]).

On the dlog side we need a Chern character into de Rham cohomology (cohomology of differentials). When $X = Spec(R)$ is noetherian it exists a map $\text{dlog} : K_q(Spec(R)) \to \Omega^q_{R/\mathbb{Z}}$ satisfying $\text{dlog}\,(a \cup b) = \text{dlog}\,(a) \wedge \text{dlog}\,(b)$, and other properties [26, p.393]. Denote $H^i_{dR}(U/\mathcal{O}_M) := H^i(X, \Omega^{\cdot}_{X/\mathcal{O}_M}(Z))$ the hypercohomology of the de Rham complex of differentials on $X$ with logarithmic singularities over $Z$: it has a natural filtration. We recall that

$$D^0_{dR,M}(V) = H^0(X, \Omega^1_{X/\mathcal{O}_M}(\log Z)) \otimes_{\mathcal{O}_M} M = Fil^1 H^1_{dR}(U/\mathcal{O}_M) \otimes_{O_M} M.$$

For simplicity assume $M = M_0$, so that $\Omega^1_{\mathcal{O}_{M(\zeta_{p^n})}/\mathcal{O}_M}$ is generated by $\text{dlog}\,\zeta_{p^n}$ and $\mathfrak{d}_{M(\zeta_{p^n})/M}$ is $p^n(\zeta_p - 1)^{-1}\mathcal{O}_M[\zeta_{p^n}]$.

**Theorem 14.** *(Explicit reciprocity law) Take $p > 2$. Let $X$ be a smooth and proper curve over $\mathcal{O}_M$ and take an affine $U \subset X$ as above. Then the* *following diagram commutes:*

$$
\begin{array}{ccc}
\varprojlim_n (K_2(U \otimes \mathcal{O}_{M(\zeta_{p^n})})) \otimes \boldsymbol{\mu}_{p^n}^{\otimes -1} & \xrightarrow[HS\circ ch_2]{} & \varprojlim_n H^1(M(\zeta_{p^n}), H^1_{et}(U \times_M \overline{M}, \boldsymbol{\mu}_{p^n})) \\
\downarrow \text{dlog} & & cor \downarrow \\
\varprojlim_n H^0(X \otimes \mathcal{O}_{M(\zeta_{p^n})}, \Omega^2_{X \otimes \mathcal{O}_{M(\zeta_{p^n})}/\mathcal{O}_M}(\log Z))(-1) & & \varprojlim_{n \geq m} H^1(M(\zeta_{p^m}), H^1(U \times_M \overline{M}, \boldsymbol{\mu}_{p^n})) \\
\downarrow = & & \cong \downarrow \\
\varprojlim_n \Omega^1_{\mathcal{O}_{M(\zeta_{p^n})}/\mathcal{O}_M} \otimes Fil^1(H^1_{dR}(U/\mathcal{O}_M)) & & H^1(M(\zeta_{p^m}), H^1(U \times_M \overline{M}, \mathbb{Z}_p)(1)) \\
\downarrow & & exp^* \downarrow \\
\varprojlim_n \mathcal{O}_{M(\zeta_{p^n})}/\mathfrak{d}_{M(\zeta_{p^n})/M} \otimes_{\mathcal{O}_M} Fil^1 H^1_{dR}(U/\mathcal{O}_M) & \xrightarrow[tr]{} & M(\zeta_{p^m}) \otimes_{\mathcal{O}_M} Fil^1 H^1_{dR}(U/\mathcal{O}_M)
\end{array}
$$

*where tr denotes* $(\frac{1}{p^n} tr_{M(\zeta_{p^n})/M(\zeta_{p^m})})_{n \geq m}$.

**Remark 15.** Take $X = \mathbb{P}^1$ and $U = \mathbb{A}^1 - \{0\} = Spec(\mathcal{O}_M[t, t^{-1}])$. Take $(u_n)_n \in \varprojlim_n \mathcal{O}^*_{M(\zeta_{p^n})}$ and consider $\{u_n, t\} \in K_2(U \otimes \mathcal{O}_{M(\zeta_{p^n})})$. Then from theorem 14 one recovers Iwasawa's theorem 2 [26, Remark p.411]. See also remark 13.

**Remark 16.** Theorem 14 has been vastly extended by Härkönen [16]: he shows that, under some technical assumptions, the map $exp^* \circ corestriction \circ HS \circ ch_r : \varprojlim_n K_r(U \otimes_{\mathcal{O}_M} M(\varepsilon_n)) \to M(\varepsilon_m) \otimes D^0_{dR,M}(V)$ can be computed in terms of a dlog map, for any $r$ with $1 \leq r \leq p-2$ (as before $\varepsilon$ is a fixed generator of $T_pF_f$ for a Lubin-Tate formal group $F_f$ over $\mathcal{O}_M$). We refer to [16] for precise statements.

4.3. **Rank 1 Drinfeld modules.** Kato's approach inspired the construction of the diagram presented in this subsection, which is the only new material of this survey paper. As in the rest of this work, we restrict ourselves to the case of the Carlitz module; however, we remark that theorem 18 can easily be extended to any rank 1 sign-normalized Drinfeld module, with exactly the same proof (*mutatis mutandis*: the changes are the same necessary to pass from the function field results exposed in this paper to the more general statements of [3]).

Remember that the Galois action on the module of Kähler differentials $\Omega_{\mathcal{O}_{\Phi,n}/\mathcal{O}}$ is given by $\sigma(\alpha d\beta) = \sigma(\alpha) d\sigma(\beta)$, $\sigma \in Gal(K_{\Phi,n}/K)$, $\alpha, \beta \in \mathcal{O}_{\Phi,n}$. In particular, one has

$$\sigma(d\varepsilon_n) = d(\sigma\varepsilon_n) = d(\Phi_{\chi(\sigma)}(\varepsilon_n)) = \chi(\sigma) d\varepsilon_n .$$

Besides, $d\varepsilon_n = d\Phi_{\pi^k}(\varepsilon_{n+k}) = \pi^k d\varepsilon_{n+k}$.

By $\varprojlim \Omega_{\mathcal{O}_{\Phi,n}/\mathcal{O}}$ we denote the limit with respect to the trace map, defined as usual by $Tr^n_m(\omega) := \sum_{\sigma \in Gal(K_{\Phi,n}/K_{\Phi,m})} \sigma(\omega)$ where $Tr^n_m = Tr_{K_{\Phi,n}/K_{\Phi,m}}$.

**Lemma 17.** *Let* $(\omega_n)_n \in \varprojlim \Omega_{\mathcal{O}_{\Phi,n}/\mathcal{O}}$ *and for each* $n$ *choose* $x_n \in \mathcal{O}_n$ *so that* $\omega_n = x_n d\varepsilon_n$*. Then the sequence* $y_n := \pi^{-n} Tr^n_m(x_n)$ *converges to a limit in* $K_{\Phi,m}$ *for any fixed* $m$*.*

*Proof.* To lighten notation put $G^r_n := Gal(K_{\Phi,r}/K_{\Phi,n})$, $r > n$. The diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G^r_n & \longrightarrow & Gal(K_{\Phi,r}/K) & \longrightarrow & Gal(K_{\Phi,n}/K) & \longrightarrow & 0 \\ & & \simeq\downarrow & & \simeq\downarrow & & \simeq\downarrow & & \\ 0 & \longrightarrow & (1+\mathfrak{p}^n)/(1+\mathfrak{p}^r) & \longrightarrow & (A/\mathfrak{p}^r)^* & \longrightarrow & (A/\mathfrak{p}^n)^* & \longrightarrow & 0 \end{array}$$

(where vertical maps are induced by $\chi$) shows that $\chi(\sigma)-1 \in \mathfrak{p}^n$ for $\sigma \in G^r_n$. The equality $\omega_n = Tr^{n+k}_n \omega_{n+k}$ can be rewritten as

$$x_n \pi^k d\varepsilon_{n+k} = x_n d\varepsilon_n = \sum_{\sigma \in G^{n+k}_n} \sigma(x_{n+k})\chi(\sigma) d\varepsilon_{n+k} \,,$$

that is, $\pi^k x_n = \sum \sigma(x_{n+k})\chi(\sigma) + \delta_{n,n+k}$ for some $\delta_{n,n+k} \in \mathfrak{d}_{K_{\Phi,n+k}/K}$. Let

$$z_{n,n+k} := \sum_{\sigma \in G^{n+k}_n} \sigma(x_{n+k})(\chi(\sigma)-1) + \delta_{n,n+k} \,.$$

By [3, Lemma 3] we know $v(\delta_{n,n+k}) > n+k-1$; together with the observation above, this implies that $v(z_{n,n+k}) \geq n$. It is computed in [3, cor.4] that $v(Tr^n_m(a)) > v(a) + n - m - 1$: applying it to

$$y_n - y_{n+k} = \frac{1}{\pi^{n+k}} Tr^n_m \left( x_n \pi^k - \sum_{\sigma \in G^{n+k}_n} \sigma(x_{n+k}) \right) = Tr^n_m \left( \frac{z_{n,n+k}}{\pi^{n+k}} \right)$$

we see that $v(y_n - y_{n+k}) \geq n - (m+k+1)$, proving that the $y_n$'s form a Cauchy sequence. ∎

By abuse of notation, we denote the limit in lemma 17 as $\lim_{n\to\infty} \frac{1}{\pi^n} Tr^n_m \left( \frac{\omega_n}{d\varepsilon_n} \right)$.

Inspired by [20, §1.1] and [26, Theorem 3.3.15], we construct the following diagram:

$$\begin{array}{ccc} \varprojlim K^*_{\Phi,n} & \xrightarrow{(1)} & Hom_{cont}(\mathfrak{m}_{\Phi,m}, T_p\Phi) \\ \downarrow(2) & & (4)\downarrow \\ \varprojlim \mathbb{F}_p \frac{d\varepsilon_n}{\varepsilon_n} \oplus \Omega^1_{\mathcal{O}_{\Phi,n}/\mathcal{O}_{F_\mathfrak{p}}} & \xrightarrow{(3)} & K_{\Phi,m} \simeq Hom_{F_\mathfrak{p}}(K_{m,\Phi}, F_\mathfrak{p}) \end{array}$$

Arrow (1) is the Kummer map: it sends $u = (u_n)_n$ to $a \to \lim_{n\to\infty} (a, u_n)_n$. (This limit exists in $T_p\Phi$:

$$\Phi_\pi(a, u_n)_n = ((u_n, K^{ab}_{\Phi,n}/K_{\Phi,n}) - 1)\Phi_\pi(\sqrt[n]{a}_\pi) = (a, u_{n-1})_{n-1}$$

because $\Phi_\pi \in \mathcal{O}\{\tau\}$ commutes with the action of $G_K$; here $\sqrt[n]{a}_\pi$ is a root of $\Phi^n_\pi(X) = a$.)

As for (2), it is just dlog $: u \mapsto \frac{du}{u}$, extended to $K^*_{\Phi,n}$ by putting $\operatorname{dlog}\varepsilon_n^i := i\frac{d\varepsilon_n}{\varepsilon_n}$, as described in [3, §4.2.1]; the limit of differentials is taken with respect to the trace and $\frac{d\varepsilon}{\varepsilon}$ denotes the inverse system $\frac{d\varepsilon_n}{\varepsilon_n}$.
The isomorphism $K_{\Phi,m} \simeq Hom_{F_\mathfrak{p}}(K_{\Phi,m}, F_\mathfrak{p})$ is given by the trace pairing: $b \in K_{\Phi,m}$ is sent to $a \mapsto Tr_{K_{\Phi,m}/F_\mathfrak{p}}(ab)$. The map (3) is

$$(\omega_n)_n \mapsto \lim_{n\to\infty} \frac{1}{\pi^n} Tr_{K_{\Phi,n}/K_{\Phi,m}}\left(\frac{\omega_n}{d\varepsilon_n}\right) .$$

The definition of (4) needs more explanation. The logarithm $\lambda$ has locally an inverse $e : \mathfrak{m}^{t_n}_{\Phi,n} \to \mathfrak{m}_{\Phi,n}$ for $t_n \gg 0$; this can be extended to $\tilde{e} : K_{\Phi,n} \to F_\mathfrak{p} \otimes_\Phi \mathfrak{m}_{\Phi,n}$ (the tensor product is taken on $A_\mathfrak{p}$, which acts on $\mathfrak{m}_{\Phi,n}$ via $\Phi$) by putting $\tilde{e}(\pi^i z) := \pi^i \otimes e(z)$ for $i \gg 0$. In order to define (4), first remember the isomorphism $T_p\Phi \simeq A_\mathfrak{p}$, via $a \cdot \varepsilon \leftrightarrow a$; then use composition with $\tilde{e}$, $f \mapsto f \circ \tilde{e}$, to get $Hom(\mathfrak{m}_{\Phi,m}, A_\mathfrak{p}) \to Hom(K_{\Phi,m}, F_\mathfrak{p})$.

**Theorem 18.** *The diagram above is commutative.*

*Proof.* Working out definitions, one sees that this is equivalent to part 2 of theorem 6. More precisely (3) $\circ$ (2) sends $u = (u_n) \in \lim\limits_{\leftarrow} K^*_{\Phi,n}$ to the map

$$w \mapsto Tr_{K_{\Phi,m}/K}\left(\frac{w}{\pi^m} \operatorname{dlog} Col_u(\varepsilon_m)\right) .$$

On the other hand, the image of $u$ under (4) $\circ$ (1) is the morphism mapping $\pi^i z$, $v(z) \gg 0$, to $\pi^i g_u(e(z))$, where $g_u \in Hom(\mathfrak{m}_{\Phi,m}, A_\mathfrak{p})$ is uniquely determined by the condition $g_u(a) \cdot \varepsilon_n = (a, u_n)_n$ for all $n \geq m$. Recalling that

$$[e(z), u_n]_n := Tr_{K_{\Phi,n}/K} \quad \frac{\lambda(e(z))}{\pi^n} \operatorname{dlog} Col_u(\varepsilon_n) \quad \cdot \varepsilon_n = Tr_{K_{\Phi,m}/K} \quad \frac{z}{\pi^m} \operatorname{dlog} Col_u(\varepsilon_m) \quad \cdot \varepsilon_n ,$$

it is clear that (3) $\circ$ (2) = (4) $\circ$ (1) iff $(\cdot,\cdot) = [\cdot,\cdot]$. □

**Remark 19.** The similarity of our diagram with the first diagram of §4.2.2 is rather vague. It would be nice to express (and prove) the reciprocity law in the cohomological setting, as in [26, §3.3]; the big problem here is to find a good analogue of $H^1(G_{\mathbb{Q}_p}, \mathbb{C}_p)$ in characteristic $p > 0$ (a naive approach cannot work: Y. Taguchi proved that $H^1(G_K, \mathbf{C}_\mathfrak{p}) = 0$). Recent developments in extending Fontaine's theory to the equal characteristic case (for a survey see [14]) might be helpful.

## References

[1] Anglès, B.: *On explicit reciprocity laws for the local Carlitz-Kummer symbols.* J. Number Theory **78** (1999), no. 2, 228–252.

[2] Artin, E., and Hasse, H.: *Die beiden Ergänzungssätze zum Reziprozitätsgesetz der $l^n$-ten Potenzreste im Körper der $l^n$-ten Einheitswurzeln.* Abh. Math. Sem. Univ. Hamburg **6** (1928), 146-162.

[3] Bars, F. and Longhi, I.: *Coleman's power series and Wiles' reciprocity for Drinfeld modules.* Corrected version at http://mat.uab.cat/~francesc/publicac.html

[4] Benois, D.: *Périodes p-adiques et lois de réciprocité explicites.* J.reine angew. Math. **493** (1997), 115-151.
[5] Benois, D.: *On Iwasawa theory of crystalline representations.* Duke Math. J. **104** (2000), 211-267.
[6] Berrizbeitia, P.J.: *An explicit reciprocity theorem in finite extensions of* $\mathbb{Q}_p$. Thesis, Massachussets Institute of Technology (1986).
[7] Bloch, S. and Kato, K.: *L-functions and Tamagawa numbers of motives.* In P. Cartier et al. eds.: Grothendieck Festschrift Vol. I., Birkhäuser (1990).
[8] Cao, L.: *Explicit reciprocity law for Lubin-Tate formal groups.* Acta Math. Sinica, english series **22**, No.5 (2006), 1399-1412.
[9] Coleman, R.: *Division values in local fields.* Inven.Math. **53** (1979), 91-116.
[10] de Shalit, E.: Iwasawa Theory of Elliptic Curves with Complex Multiplication. Persp. in Mathematics **3**, Acad.Press, Boston, MA 1987.
[11] Drinfeld, V.G.: *Elliptic modules.* Math.USSR Sbornik **23** (1974), No.4, 561-592.
[12] Fukaya, T.: *Explicit reciprocity laws for p-divisible groups over higher dimensional local fields.* J.Reine Angew. Math. **531** (2001), 61-119.
[13] Fukaya, T.: *The theory of Coleman power series for* $K_2$. Journal of Alg.Geometry**12** (2003), 1-80.
[14] Hartl, U.:*A dictionary between fontaine-theory and its analogue in equal characteristic.* Arxiv preprint.
[15] Hazewinkel, M.: Formal groups and applications. Academic Press, New York (1978).
[16] Härkönen, H.: *Explicit reciprocity laws and cohomology of log schemes* Preprint June 2007.
[17] Iwasawa, K.: *On explicit formulas for the norm residue symbol.* J. Math. Soc. Japan **20** (1968), 151-164.
[18] Kato, K.: *A generalization of local class field theory by using* $K$*-groups* I J.Fac.Sci.Univ.Tokyo**26** (1979), 303-376; II ibid.**27** (1980), 603-683; III ibid.**29** (1982), 31-43.
[19] Kato, K.: *Lectures on the approach to Iwasawa theory for Hasse-Weil L-functions via* $B_{dR}$. In Arithmetic Algebraic Geometry (Trento, 1991), 50-163. LNM 1553. Springer (1993).
[20] Kato, K.: *Generalized explicit reciprocity laws.* Adv. Stud. Contemp. Math. (Pusan) **1** (1999), 57-126.
[21] Kummer, E.: *Über die allgemeinen Reziprozitätsgesetze der Potenzreste.* J. Reine Angew. Math. **56** (1858), 270-279.
[22] Kurihara, M.: *The exponential homomorphisms for the Milnor* $K$*-groups and an explicit reciprocity law.* J. Reine Angew. Math. **498** (1998), 201-221.
[23] Kurihara, M.: *Kato's higher local class field theory.* In Invitation to higher local fields., 53-60. Geometry and Topology Monographs Vol **3**, 2000, International Press.
[24] Lang, S.: Cyclotomic Fields I and II. Graduate Texts in Mathematics **121**. Springer-Verlag, New York, 1990.
[25] Parshin, A.N.: *Class field theory and algebraic* $K$*-theory.* Uspekhi Mat.Nauk **30**, no.1 (1975), 253-254, English transl.:Russian Math.Surv.
[26] Scholl, A.J.: *An introduction to Kato's Euler systems* In Galois representations in Arithmetic Algebraic Geometry (A.J.Scholl and R.L.Taylor eds.), 379-460. London Math.Soc. Lecture Notes **254**, 1998.
[27] Sen, S.: *On explicit reciprocity laws.* J.Reine Angew. Math.**313** (1980), 1-26.
[28] Serre, J.P.: *Local class field theory.* In Algebraic Number Fields. J.W.S.Cassels and A. Fröhlich, eds., Academic Press 1967
[29] Tate, J.: *p-divisible groups.* In Proc. of a conf. on local fields., 158-183. Driebergen 1966, Springer (1967).

[30] Tsuji, T.: *Explicit reciprocity law and formal moduli for Lubin-Tate formal groups*. J.reine angew. Math.**569** (2004), 103-173.
[31] Vostokov, S.: *Explicit formulas for the Hilbert symbol*. In Invitation to higher local fields., 81-89. Geometry and Topology Monographs Vol **3**, 2000, International Press.
[32] Wiles, A.: *Higher explicit reciprocity laws*. Ann. of Math. **107** (1978), no. 2, 235-254.
[33] Zhukov, I.B.: *Structure theorems for complete fields* In Concerning the Hilbert 16th problem. Proceedings of ths St.Petersburg Mathematical Society., 175-192. Amer. Math. Soc. Transl. (Ser. 2) **165**, 1995.

Francesc Bars Cortina, Depart. Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra. Catalonia. Spain.
E-mail: francesc@mat.uab.cat

Ignazio Longhi, Dipartimento di Matematica "Federigo Enriques", Università degli Studi di Milano. Via Cesare Saldini 50, 20133 Milan. Italy.
E-mail: longhi@mat.unimi.it