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Width of convex bodies in spaces of constant curvature

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Abstract. We consider the measure of points, the measure of lines and the measure of planes intersecting a given convex body K in a space form. We obtain some integral formulas involving the width of K and the curvature of its boundary ∂K. Also we study the special case of constant width. Moreover we obtain a generalisation of the Heintze–Karcher inequality to space forms.

1. Introduction

In this paper we consider convex bodies in spaces of constant curvature, from the viewpoint of integral geometry.

Let X_c^n denote the n-dimensional complete and simply connected riemannian space of constant curvature c, i.e. the euclidean space $\mathbb{E}^n$ for $c = 0$, the sphere $\mathbb{S}_c^n$ for $c > 0$, or the hyperbolic space $\mathbb{H}_c^n$ for $c < 0$. We shall focus our interest in dimensions 2 and 3, but the ideas here contained can be extended to arbitrary dimensions.

We give the measure, with some weight, of points, lines, planes, and λ-planes intersecting a convex body K in function of its width and the curvature of its boundary ∂K.

With respect to the measure of points in K, in the case of constant width, we obtain for instance

$$2V = a_0(w)M_0 + a_1(w)M_1 + a_2(w)M_2,$$

where V is the volume, a_i are functions of the width w of K and M_i are the ith integrals of mean curvature of the boundary ∂K, see Theorem 2. This is a particular

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case of a formula relating the number of critical points of the distance function to the boundary with some integrals of the curvature functions of ∂K, see Theorem 1. Using similar tools, we prove for convex sets in X_c^3 that

$$V \leq \int_M \frac{V(\rho_H)}{A(\rho_H)} dM_x$$

where $V(\rho_H)$ and $A(\rho_H)$ are the volume and area of the sphere of radius $\rho_H(x)$, the mean curvature radius of M at x, see Theorem 4. When $c = 0$ this is the classical Heintze–Karcher formula

$$3V \leq \int_M \frac{1}{H} dM_x$$

(see for instance [14]).

With respect to the measure of lines intersecting $K \subset \mathbb{E}^3$ we obtain

$$\int_{\{\xi \in \mathscr{L}_1 : K\cap\xi \neq \emptyset\}} \nu(h_\xi|_M)\, d\xi \leq \int_M \Big(c_0(x) + c_1(x)\, w(x) + c_2(x)\, w^2(x) \Big)\, dM_x$$

where $c_i(x)$ are functions of the mean and Gaussian curvature in ∂K, $\mathscr{L}_1$ is the space of lines with measure $d\xi$ and $\nu(h_\xi|_M)$ denote the number of critical points of the distance function to lines ξ, see Theorem 5. Equality holds if and only if K has constant width.

With respect to the measure of planes intersecting $K \subset X_c^3$ we obtain

$$2 \int_{\{\xi \in \mathscr{L}_2 : K\cap\eta \neq \emptyset\}} d\eta = \int_M (\alpha_0(x) + \alpha_1(x)\, H(x) + \alpha_2(x)\, K(x))\, dM_x$$

where $\alpha_i(x)$ are functions of the width $w(x)$ for $x \in \partial K$ and $\mathscr{L}_2$ is the space of planes with measure $d\eta$, see Theorem 6. As a consequence of this result, when K is of constant width w, we deduce the well known formula (cf. [18])

$$2cV + M_1 = 2\pi w.$$

In dimension 2 we obtain a generalisation of Barbier theorem (cf. [2,8,12,24])

$$\mathrm{sn}_c(w) M_0 = \frac{1}{c}(1 - \mathrm{cn}_c(w)) M_1.$$

2. Preliminaries

Definition 1. A domain $K \subset X_c^n$ with regular boundary $M = \partial K$ is said to be *convex* (resp. *strongly convex*) if the normal curvatures in every point of M are non-negative (resp. positive). In the spherical case $c > 0$ we assume that K lies in some halfsphere of $\mathbb{S}_c^n$.

If $c < 0$ we have

Definition 2. A domain $K \subset \mathbb{H}^n_c$ with regular boundary $M = \partial K$ is said to be *h-convex* (resp. *strongly h-convex*) if the normal curvatures in every point of M are greater or equal than $\sqrt{-c}$ (resp. greater than $\sqrt{-c}$).

For instance, balls of radius r in $\mathbb{H}^n_c$ have curvature equal to $\sqrt{-c}\coth(\sqrt{-c}r)$, thus they are strongly h-convex.

Remark 1. It must be noticed that the notion of convexity given here is equivalent to the usual notion of geodesic convexity. Sometimes h-convex bodies are also called horocyclically convex bodies because in this case the horocycles joining points in K are contained in K.

Remark 2. We can extend the notion of convexity (resp. h-convexity) to non-regular domains. A domain K is said to be convex (resp. h-convex) if every point of the boundary is locally supported by a regular convex (resp. h-convex) hypersurface S leaving K on the convex side of S. These notions are also valid in Hadamard manifolds (see [4]).

Let N denote the inward pointing unit normal field along M. For each $x \in M$ there are exactly two supporting hyperplanes (i.e. complete totally geodesic hypersurfaces in X^n_c) orthogonal to the normal geodesic $\exp_x(tN(x))$, $t \in \mathbb{R}$, to M through x. Then K lies inside the strip defined by these two supporting hyperplanes.

In euclidean spaces ($c = 0$) the classical width is directly related to the support function. The width of K is a function depending on directions, namely the distance between two parallel support hyperplanes, or the sum of the values of the support function at opposite directions respectively. For $c \neq 0$ there are neither parallelity nor natural support functions. Therefore, our definition of width of K is based on K itself. We will consider the width as a function defined on $M = \partial K$.

In case $c \neq 0$ there are other concepts of width, based on support functions defined after the election of an arbitrary point("origin") (cf. [8,12,18]).

All these concepts have their pros and cons. In euclidean spaces ($c = 0$), due to parallelity, these concepts essentially coincide.

Definition 3. The *width* $w(x)$ of K at $x \in \partial K$ is the smallest positive number r such that the hyperplane through the point $\exp_x(rN(x))$ that is orthogonal to the geodesic $\exp_x(tN(x))$ is tangent to ∂K.

In this paper we shall deal with convex bodies specially with convex bodies of constant width. The condition $w(x)$ constant is equivalent to the property of double normals (the geodesic $\exp_x(tN(x))$ is orthogonal to M in two points); see [6]. Other good references on this subject are [5,9] and [17].

Let $\{x; e_1, \ldots, e_n\}$ be a moving orthonormal frame along M with x belonging to M, e_1, ..., e_{n-1} tangent to M at x, and $e_n = N(x)$ the inward pointing normal to M. Consider $M_w = \cup_{x \in M} (\{x\} \times [0, w(x)]) \subset M \times \mathbb{R}$ and let $\{y; f_1, \ldots, f_n\}$ be a moving orthonormal frame on X^n_c defined over M_w by $y := \exp_x(\rho e_n)$, $\rho \in [0, w(x)]$, and f_1, ..., f_n given through geodesic parallel translation of e_1, ..., e_n along the geodesic $\exp_x(te_n)$, $t \in \mathbb{R}$, from x to y. By ω_i, ω_{ij} and σ_i, σ_{ij} we denote the corresponding connection forms on M and M_w.

It is well known that the growth of the Jacobi fields in X_c^n is described by the functions

$$\operatorname{sn}_c(\rho) := \begin{cases} \frac{1}{\sqrt{-c}}\sinh(\sqrt{-c}\,\rho), & c<0\\ \rho, & c=0\\ \frac{1}{\sqrt{c}}\sin(\sqrt{c}\,\rho), & c>0 \end{cases}$$

and

$$\operatorname{cn}_c(\rho) := \begin{cases} \cosh(\sqrt{-c}\,\rho), & c<0\\ 1, & c=0\\ \cos(\sqrt{c}\,\rho), & c>0. \end{cases}$$

Note that for these functions $c\,\operatorname{sn}_c^2(\rho)+\operatorname{cn}_c^2(\rho)=1$, $\operatorname{sn}_c'(\rho)=\operatorname{cn}_c(\rho)$ and $\operatorname{cn}_c'(\rho)=-c\,\operatorname{sn}_c(\rho)$.

The relation between the connection forms in M and M_w considered before is given by

Lemma 1.

$$\begin{aligned} \sigma_n &= d\rho\\ \sigma_i &= \operatorname{cn}_c(\rho)\,\pi_1^*\omega_i + \operatorname{sn}_c(\rho)\,\pi_1^*\omega_{ni}\\ \sigma_{ni} &= -c\ \operatorname{sn}_c(\rho)\,\pi_1^*\omega_i + \operatorname{cn}_c(\rho)\,\pi_1^*\omega_{ni} \end{aligned} \tag{1}$$

where $\pi_1 : M_w \to M$ is the canonical projection and $1 \le i \le n-1$.

Proof. The formulas follow considering variation through geodesics, taking into account the curvature of equidistants $-c\ \operatorname{sn}_c(\rho)/\operatorname{cn}_c(\rho)$ and the curvature of distance circles $\operatorname{cn}_c(\rho)/\operatorname{sn}_c(\rho)$. □

Lemma 2.

$$\begin{aligned} \int \operatorname{sn}_c^2(\rho)\,d\rho &= \frac{1}{2c}(\rho - \operatorname{sn}_c(\rho)\ \operatorname{cn}_c(\rho))\\ \int \operatorname{cn}_c^2(\rho)\,d\rho &= \frac{1}{2}(\rho + \operatorname{sn}_c(\rho)\ \operatorname{cn}_c(\rho))\\ \int \operatorname{sn}_c(\rho)\ \operatorname{cn}_c(\rho)\,d\rho &= \frac{1}{2}\operatorname{sn}_c^2(\rho). \end{aligned} \tag{2}$$

Remark 3. When c tends to 0, the right-hand side of the first formula becomes $\rho^3/3$.

Let us now consider a height function h, i.e. a submersion $h : X_c^n \to \mathbb{R}$ (defined at least locally). Let $p \in M$ be a critical point of the induced height function $h|_M$ along M, then some level hypersurface S of h is tangent to M at p, and hence grad $h(p) = \lambda\, N(p)$. We have

Lemma 3.

$$\text{hess } h|_M(p) = \lambda \, II_M(p) - |\lambda| II_S(p) \tag{3}$$

where $II_M(p)$ is the second fundamental form of M at p with respect to its unit normal $N(p)$, and $II_S(p)$ is the second fundamental form of S at p with respect to its unit normal grad $h(p)/ \parallel \text{grad } h(p) \parallel$.

Proof. Locally along M around p we write grad $h|_M = \text{grad } h - \lambda N$ with an appropriate function λ. Then

$$\begin{aligned}\text{hess } h|_M(p)(X, Y) &= g(\nabla_X \text{ grad } h|_M, Y)|_p \\ &= g(\nabla_X \text{ grad } h, Y)|_p - d\lambda(X)\, g(N, Y)|_p \\ &\quad -\lambda(p)\, g(\nabla_X N, Y)|_p \\ &= -|\lambda(p)|\, \text{II}_S(X, Y) + \lambda(p)\, \text{II}_M(X, Y)\end{aligned}$$

with $X, Y \in T_p M$ and g the first fundamental form of M. □

Definition 4. Given a function $\rho > 0$ on the boundary $M = \partial K$ of a regular convex body K, the *ρ-parallel set* of M is the set $F_\rho(M) = \{\exp_x(\rho(x)N(x)) : x \in M\} \subset X^n_c$. The *focal set* $F(M)$ of M will be the union of the ρ_i-parallel sets when ρ_i are the principal radii of curvature ($i = 1, \ldots, n-1$).

Remark 4. In the two dimensional case the focal set is the evolute of the boundary. Note also that $F(M)$ is locally smooth and that the normal geodesics going to the interior of K are tangent to $F(M)$.

Definition 5. The *winding number* wind(S, y) of an oriented hypersurface $S \subset X^n_c$ with respect to a point $y \in X^n_c \setminus S$ is the mapping degree of the radial projection via the exponential map of S into the tangent unit-sphere $T^1_y X^n_c$: to each point x in S we associate the unit tangent vector in y of the unique geodesic joining y and x.

Equivalently, wind(S, y) equals the algebraic intersection number of S with an arbitrary geodesic ray emanating from y.

3. Measure of points

For every y in X^n_c we consider the distance function $h_y(x) := d(x, y), x \in X^n_c$. A point $x \in M$ is a critical point of $h_y|_M$ if and only if the normal geodesic $\exp_x(tN(x))$ to M at x runs through y. Let $\nu(h_y|_M)$ denote the number of critical points of $h_y|_M$. Then

Theorem 1. *Let K be a strongly convex set in X^3_c, if $c \geq 0$, or strongly h-convex set if $c < 0$ with regular boundary surface $M = \partial K$. Then*

$$\int_K \nu(h_y|_M)\, dy \leq \int_M (a_0(x) + a_1(x)\, H(x) + a_2(x)\, K(x))\, dM_x \tag{4}$$

where

$$a_0(x) = (\rho_2 - \rho_1 + \mathrm{sn}_c(\rho_2 - \rho_1)\ \mathrm{cn}_c(\rho_2 + \rho_1) + \frac{1}{2}(w + \mathrm{sn}_c(w)\ \mathrm{cn}_c(w)))(x)$$
$$a_1(x) = (-2(\mathrm{sn}_c^2(\rho_2) - \mathrm{sn}_c^2(\rho_1)) - \mathrm{sn}_c^2(w))(x)$$
$$a_2(x) = \left(2\frac{1}{2c}(\rho_2 - \rho_1 - \mathrm{sn}_c(\rho_2 - \rho_1)\ \mathrm{cn}_c(\rho_2 + \rho_1)) + \frac{1}{2c}(w - \mathrm{sn}_c(w)\ \mathrm{cn}_c(w))\right)(x),$$

and $H(x) = \frac{1}{2}(\kappa_1(x) + \kappa_2(x))$ *is the mean curvature of* M *at* x *with respect to* $N(x)$, $K(x) = \kappa_1(x)\kappa_2(x)$ *is the Gauss curvature,* $\rho_1(x)$ *and* $\rho_2(x)$ *are the principal curvature radii given by* $\kappa_1(x) = \mathrm{cn}_c(\rho_1)/\mathrm{sn}_c(\rho_1)$ *and* $\kappa_2(x) = \mathrm{cn}_c(\rho_2)/\mathrm{sn}_c(\rho_2)$. *Equality holds if and only if* K *has constant width.*

Proof. We consider the map $\Phi : M_w \to X_c^3$ defined by $\Phi(x, \rho) := y = \exp_x(\rho N(x))$, in order to parametrize all points of K, in general not injectively.

Using the adapted frame $\{x; e_1, e_2, e_3\}$ with e_1 and e_2 the principal directions of curvature and $e_3 = N$, Lemma 1 gives

$$\begin{aligned}\Phi^* dy &= \sigma_1 \wedge \sigma_2 \wedge \sigma_3 \\ &= (\mathrm{cn}_c(\rho) - \kappa_1\ \mathrm{sn}_c(\rho))(\mathrm{cn}_c(\rho) - \kappa_2\ \mathrm{sn}_c(\rho))\,\pi_1^*\omega_1 \wedge \pi_1^*\omega_2 \wedge d\rho \\ &= p(x, \rho)\,\pi_1^* dM \wedge d\rho. \end{aligned} \tag{5}$$

Applying the co-area formula (cf. for instance [11]) to Φ we have

$$\int_{\Phi(M_w)} \#(\Phi^{-1}(y))dy = \int_{M_w} |\Phi^* dy|.$$

Because of its construction Φ catches at least each point $y \in K$ exactly $\nu(h_y|_M)$ times, i.e. $K \subset \Phi(M_w)$ and $\#(\Phi^{-1}(y)) = \nu(h_y|_M)$ for $y \in K$. Therefore the co-area formula gives

$$\int_K \nu(h_y|_M)dy \le \int_{\Phi(M_w)} \nu(h_y|_M)dy = \int_{M_w} |\Phi^* dy|. \tag{6}$$

Let us compute the right side of this inequality. Observe that the function

$$p(x, \rho) = (\mathrm{cn}_c(\rho) - \kappa_1\ \mathrm{sn}_c(\rho))(\mathrm{cn}_c(\rho) - \kappa_2\ \mathrm{sn}_c(\rho))$$

changes sign at the principal curvature radii $\rho_1(x)$ and $\rho_2(x)$. We may assume, without loss of generality, $0 < \kappa_1 \le \kappa_2$, hence $0 < \rho_2 \le \rho_1$. Lemma 2 shows

$$\begin{aligned}\int p(x, \rho)\, d\rho &= \int \left(\mathrm{cn}_c^2(\rho) - (\kappa_1 + \kappa_2)\ \mathrm{sn}_c(\rho)\ \mathrm{cn}_c(\rho) + \kappa_1\kappa_2\ \mathrm{sn}_c^2(\rho)\right) d\rho \\ &= \frac{1}{2}(\rho + \mathrm{sn}_c(\rho)\ \mathrm{cn}_c(\rho)) - \frac{\kappa_1 + \kappa_2}{2}\ \mathrm{sn}_c^2(\rho) + \frac{\kappa_1\kappa_2}{2c}(\rho - \mathrm{sn}_c(\rho)\ \mathrm{cn}_c(\rho)).\end{aligned} \tag{7}$$

Therefore

$$\begin{aligned}\int_0^{w(x)} |p|\,d\rho &\le \int_0^{w(x)} p\,d\rho - 2\int_0^{\rho_1} p\,d\rho + 2\int_0^{\rho_2} p\,d\rho \\ &\quad \text{(equality iff } \rho_1(x) \le w(x)) \\ &= \rho_2 + \mathrm{sn}_c(\rho_2)\ \mathrm{cn}_c(\rho_2) - (\rho_1 + \mathrm{sn}_c(\rho_1)\ \mathrm{cn}_c(\rho_1)) \\ &\quad -(\kappa_1+\kappa_2)(\mathrm{sn}_c^2(\rho_2) - \mathrm{sn}_c^2(\rho_1)) \\ &\quad +\kappa_1\kappa_2\,\frac{1}{c}\ (\rho_2 - \mathrm{sn}_c(\rho_2)\ \mathrm{cn}_c(\rho_2) - (\rho_1 - \mathrm{sn}_c(\rho_1)\ \mathrm{cn}_c(\rho_1))) \\ &\quad +\frac{1}{2}(w + \mathrm{sn}_c(w)\ \mathrm{cn}_c(w)) \\ &\quad -\frac{1}{2}(\kappa_1+\kappa_2)\ \mathrm{sn}_c^2(w) \\ &\quad +\kappa_1\kappa_2\,\frac{1}{2c}(w - \mathrm{sn}_c(w)\ \mathrm{cn}_c(w))\end{aligned}$$

and

$$\begin{aligned}\int_0^{w(x)} |p|\,d\rho &\le \rho_2 - \rho_1 + \mathrm{sn}_c(\rho_2-\rho_1)\ \mathrm{cn}_c(\rho_2+\rho_1) \\ &\quad -(\kappa_1+\kappa_2)(\mathrm{sn}_c^2(\rho_2) - \mathrm{sn}_c^2(\rho_1)) \\ &\quad +\kappa_1\kappa_2\,\frac{1}{c}\ (\rho_2 - \rho_1 - \mathrm{sn}_c(\rho_2-\rho_1)\ \mathrm{cn}_c(\rho_2+\rho_1)) \\ &\quad +\frac{1}{2}(w + \mathrm{sn}_c(w)\ \mathrm{cn}_c(w)) \\ &\quad -\frac{1}{2}(\kappa_1+\kappa_2)\ \mathrm{sn}_c^2(w) \\ &\quad +\kappa_1\kappa_2\,\frac{1}{2c}(w - \mathrm{sn}_c(w)\ \mathrm{cn}_c(w)).\end{aligned}$$

Taking into account the inequality (6) and Fubini theorem, the result follows.

Note that Φ catches exactly (with multiplicity) the points of K if and only if each normal to M is a double normal, i.e. if and only if K has constant width; and in this case the focal points are inside K, hence $0 < \rho_2 \le \rho_1 < w$. □

Considering the case of constant width we can state the following theorem.

Theorem 2. *Let $K \subset X_c^3$ be convex of constant width w, with regular boundary surface $M = \partial K$. Then*

$$2\,V = a_0(w)\,M_0 + a_1(w)\,M_1 + a_2(w)\,M_2, \tag{8}$$

where

$$a_0(w) = \frac{1}{2}(w + \operatorname{sn}_c(w)\ \operatorname{cn}_c(w)),$$
$$a_1(w) = -\operatorname{sn}_c^2(w),$$
$$a_2(w) = \frac{1}{2c}(w - \operatorname{sn}_c(w)\ \operatorname{cn}_c(w)),$$

and V is the volume of K, M_0 the area of M, $M_1 = \int_M H\,dM$ the total mean curvature of M and $M_2 = \int_M K\,dM$ the total extrinsic Gauss curvature of M.

Proof. As in Theorem 1 the proof works through application of the co-area formula to the map Φ, but now taking into account orientations. Let $x \in M$ be a critical point of $h_y|_M$, i.e. $y = \exp_x(\rho N(x))$ for some ρ. Then grad $h_y(x) = -N(x)$, and using the value of the curvature of distance circles $\operatorname{cn}_c(\rho)/\operatorname{sn}_c(\rho)$, the second fundamental form of the level surface at x is equal to $\operatorname{cn}_c(\rho)/\operatorname{sn}_c(\rho) \cdot I_M(x)$ ($I_M =$ first fundamental form of M). Therefore, according to Lemma 3, the signum of the critical point x of $h_y|_M$ is equal to sign $p(x, \rho)$. The sum of critical points of $h_y|_M$ weighted by their signs is equal to the Euler characteristic $\chi(M)$ of M. But K is convex, hence $\chi(M) = \chi(\mathbb{S}^2) = 2$, and therefore on the left-hand side of (8) we get twice the volume of K. □

In dimension 2 we have (cf. [15])

Theorem 3. *Let K be a strongly convex set in X_c^2, if $c \geq 0$, or strongly h-convex set if $c < 0$ with regular boundary curve $M = \partial K$. Then*

$$\int_K \nu(h_y|_M)\,dy \leq \int_M (a_0(x) + a_1(x)\,\kappa(x))\ dM_x \tag{9}$$

where

$$a_0(x) = 2\ \operatorname{sn}_c(\rho(x)) - \operatorname{sn}_c(w(x)),$$
$$a_1(x) = -2\frac{1}{c}(1 - \operatorname{cn}_c(\rho(x))) + \frac{1}{c}(1 - \operatorname{cn}_c(w(x)))$$

and $\rho(x)$ the curvature radius, i.e. $\kappa(x) = \operatorname{cn}_c(\rho(x))/\operatorname{sn}_c(\rho(x))$.

Equality holds if and only if K has constant width. Moreover, if K has constant width w, then

$$0 = -\operatorname{sn}_c(w)\,M_0 + \frac{1}{c}(1 - \operatorname{cn}_c(w))\,M_1, \tag{10}$$

where M_0 is the length of M, $M_1 = \int_M \kappa\,dM$ the total curvature of M.

Proof. The proof runs as in Theorem 1 and Theorem 2. Now with

$$\begin{aligned}\Phi^* dy &= \sigma_1 \wedge \sigma_2 = (\operatorname{cn}_c(\rho) - \kappa\ \operatorname{sn}_c(\rho))\,\pi_1^*\omega_1 \wedge d\rho \\ &= p(x, \rho)\,\pi_1^* dM \wedge d\rho,\end{aligned} \tag{11}$$

and

$$\int p(x,\rho)\,d\rho = \mathrm{sn}_c(\rho) - \kappa(x)\frac{1}{c}(1-\mathrm{cn}_c(\rho)). \tag{12}$$

Since

$$\int_0^{w(x)} |p| d\rho = \int_0^{\rho(x)} p d\rho - \int_{\rho(x)}^{w(x)} p d\rho$$

we obtain the result.

Finally, in the case of constant width, we use $\chi(M) = \chi(\mathbb{S}^1) = 0$. □

Remark 5. For $c = 0$ the term $\frac{1}{c}(1-\mathrm{cn}_c(w))$ in formula (10) becomes $w^2/2$ and we have

$$0 = -wM_0 + \frac{w^2}{2}2\pi$$

i.e. $M_0 = w\pi$ which is the classical Barbier theorem. Hence for $c \neq 0$ formula (10) can be considered as a generalisation of Barbier formula [2,8].

Remark 6. For convex K of constant width, formulas (8), (10) are due to L. A. Santaló [18,17] and W. Blaschke [3], see also remark in Sect. 11 of [5].

There is an alternative description of the left-hand side of (9) in terms of the volume bounded by M and the focal set $F(M)$ of M weighted with some winding numbers.

Proposition 1. *Let K be a strongly convex set in X_c^n ($n = 2$ or $n = 3$), if $c \geq 0$, or strongly h-convex set if $c < 0$ with regular boundary $M = \partial K$. Then*

$$\int_K \nu(h_y|_M)\,dy = 2\int_K (1 + \mathrm{wind}(F(M), y))\,dy, \tag{13}$$

where $\mathrm{wind}(F(M), y)$ *is the winding number of $F(M)$ with respect to the point y.*

Proof. The regular parts of the focal set $F(M)$ (i.e. up to its cusps or folding curves respectively) are oriented through the unit normal vector such that the generating enveloping normals of M locally lie on the normal vector side of $F(M)$. This orientation coincides with the suitably chosen orientation on M ($n = 2$), or on the two copies of M ($n = 3$) parameterizing $F(M)$ respectively.

For every point $y \in X_c^n$ we consider the number $\nu^*(y)$ of normal lines to M through y. The function ν^* on X_c^n is locally constant on $X_c^n \setminus F(M)$, integer-valued and jumps at $F(M)$ with jumps of magnitude ± 2. In detail, following y along a path crossing $F(M)$ into the normal vector side, y wins two hitting normal lines of M. On the other side, the function $2\ \mathrm{wind}(F(M), y)$ has exactly the same jump behaviour. Now, for points y far away from K and $F(M)$ (note that K strongly convex and h-convex when $c < 0$, in case $c > 0$ choose y as the center of the halfsphere complementary to some halfsphere containing K), we have $\mathrm{wind}(F(M), y)) = 0$ and $\nu^*(y) = 2$ (normal lines through the points of M where $h_y|_M$ attains its maximum or minimum respectively). Therefore $\nu^*(y) = 2(1 + \mathrm{wind}(F(M), y))$.

Now, the distance spheres with center y are orthogonal to the geodesic lines through y (Gauss lemma). Hence the number $\nu(h_y|_M)$ of critical points of the distance function $h_y|_M$ is equal to the number $\nu^*(y)$ of normal lines to M through y, i.e. $\nu(h_y|_M) = \nu^*(y)$. This proves (13). □

Remark 7. For K of constant width all normals are double normals. Hence running around M once implies running through $F(M)$ twice.

In case $c > 0$, concerning ν^* see [1].

Remark 8. In the euclidean case $c = 0$ formula (4) can be written

$$\int_K \nu(h_y|_M)\, dy \leq \int_M (b_0(x) + b_1(x)\, H(x) + b_2(x)\, K(x))\, dM_x, \tag{14}$$

where

$$\begin{aligned} b_0(x) &= w(x), \\ b_1(x) &= -w^2(x), \\ b_2(x) &= \frac{1}{3}\left(\frac{1}{\kappa_1(x)} - \frac{1}{\kappa_2(x)}\right)^3 + \frac{1}{3}w^3(x). \end{aligned}$$

Analogously formula (9) can be written

$$\int_K \nu(h_y|_M)\, dy \leq \int_M \left(\frac{1}{\kappa(x)} - w(x) + \frac{1}{2}w^2(x)\kappa(x)\right) dM_x. \tag{15}$$

Specially for convex K with constant width w in the euclidean plane, using Barbier's theorem $L = \pi w$ and $\int_M \kappa\, dM = 2\pi$, (15) gives

$$\int_K \nu(h_y|_M)\, dy = \int_M \frac{1}{\kappa(x)}\, dM_x. \tag{16}$$

The integral of the curvature radius has been studied in [7].

Remark 9. In the spherical case $c > 0$, we can use the map $\Phi : M \times [0, \pi/\sqrt{c}]$ to catch all points of $\mathbb{S}^n_c$. Similar to the proofs of (8), (10), this leads just to the classical Gauss–Bonnet formulas.

4. On the Heintze and Karcher inequality

The Heintze and Karcher inequality states

$$\int_S \frac{1}{H} dA \geq 3V,$$

where $H > 0$ is the mean curvature of a compact embedded surface S in $\mathbb{R}^3$ bounding a domain D of volume V. Equality holds if and only if S is a standard sphere, see [13,14].

The expression of the function

$$p(x, \rho) = (\mathrm{cn}_c(\rho) - \kappa_1(x)\ \mathrm{sn}_c(\rho))(\mathrm{cn}_c(\rho) - \kappa_2(x)\ \mathrm{sn}_c(\rho))$$

given in the proof of Theorem 1 enables us to obtain a version of the Heintze and Karcher's inequality in X_c^3.

First recall that the volume and area of the sphere of radius ρ in X_c^3 are given by

$$V(\rho) = \frac{2\pi}{c}(\rho - \mathrm{sn}_c(\rho)\,\mathrm{cn}_c(\rho)) \tag{17}$$

$$A(\rho) = 4\pi\ \mathrm{sn}_c^2(\rho) \tag{18}$$

see, for instance, [20], p. 308.

Let K be a strongly convex set in X_c^3, if $c \geq 0$, or strongly h-convex set if $c < 0$. Let $H = H(x)$ de mean curvature at $x \in K$. We define $\rho_H = \rho_H(x)$, the *mean curvature radius* at $x \in K$, by the equation

$$\cot_c(\rho_H) = \frac{\mathrm{cn}_c(\rho_H)}{\mathrm{sn}_c(\rho_H)} = H.$$

Note that ρ_H is well defined. In the hyperbolic case ($c < 0$) we have $H > \sqrt{-c}$, since $\kappa_i > \sqrt{-c}$, and the equation

$$\cot_c(\rho_H) = \sqrt{-c}\coth(\sqrt{-c}\rho_H) = H$$

defines ρ_H because $\coth(t)$ is a decreasing function with $1 < \coth(t) < \infty$ for $t > 0$.

In the spherical case ($c > 0$) we have $H > 0$, since $\kappa_i > 0$, and the equation

$$\cot_c(\rho_H) = \sqrt{c}\cot(\sqrt{c}\rho_H) = H$$

defines ρ_H because $\cot(t)$ is a decreasing function with $0 < \cot(t) < \infty$ for $0 < t < \pi/2$.

In the Euclidean case ($c = 0$) we have $H > 0$, since $\kappa_i > 0$, and the equation

$$\cot_c(\rho_H) = \frac{1}{\rho_H} = H$$

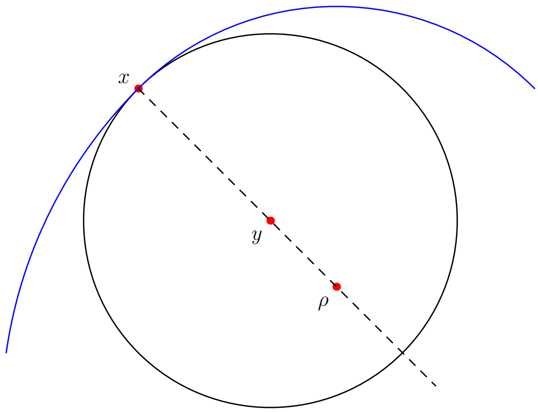
defines ρ_H.

Theorem 4. *Let K be a strongly convex set in X_c^3 (strongly h-convex if $c < 0$) with regular boundary $M = \partial K$ and volume V. Then*

$$V \leq \int_M \frac{V(\rho_H)}{A(\rho_H)} dM_x \tag{19}$$

where $V(\rho_H)$ and $A(\rho_H)$ are the volume and area of the sphere of radius $\rho_H(x)$, the mean curvature radius of M at x. Equality holds if and only if M is a sphere.

Proof. Consider the principal curvatures $\kappa_1(x) \leq \kappa_2(x)$, then $\rho_2(x) \leq \rho_1(x)$. We see that every point in K is covered (at least once) when we follow a distance ρ_2 the normal inward geodesic given by the the normal vector direction $N(x)$, in each point $x \in M = \partial K$. Indeed, if we consider, for each point $y \in K$, the biggest sphere centered at y and interior to K, which is tangent to M in a certain point $x \in M$, the normal curvatures of this sphere are greater than the curvature of M at x, in particular $\cot_c d(x, y) \geq \cot_c \rho_H$, and hence $d(x, y) \leq \rho_H$. This implies that each point $y \in K$ is counted at least once in this parallel body.



Hence we have

$$V \leq \int_M \int_0^{\rho_2(x)} (\mathrm{cn}_c(\rho) - \kappa_1(x)\ \mathrm{sn}_c(\rho))(\mathrm{cn}_c(\rho) - \kappa_2(x)\ \mathrm{sn}_c(\rho))\ d\rho\ dM_x.$$

Using that $ab \leq (\frac{a+b}{2})^2$ we obtain

$$\begin{aligned} V &\leq \int_M \int_0^{\rho_2(x)} (\mathrm{cn}_c(\rho) - H\,\mathrm{sn}_c(\rho))^2\ d\rho\ dM_x \\ &\leq \int_M \int_0^{\rho_H(x)} (\mathrm{cn}_c(\rho) - H\,\mathrm{sn}_c(\rho))^2\ d\rho\ dM_x. \end{aligned} \tag{20}$$

But the integral

$$\int_0^{\rho_H(x)} (\mathrm{cn}_c(\rho) - H\,\mathrm{sn}_c(\rho))^2\, d\rho$$

is a function that depends only on the value of $\rho_H(x)$. In order to compute this function note that for the case of the sphere S of radius r we have

$$V(r) = \int_S \int_0^r (\mathrm{cn}_c(\rho) - H\,\mathrm{sn}_c(\rho))^2\, d\rho\, dS_x = A(r) \int_0^r (\mathrm{cn}_c(\rho) - H\,\mathrm{sn}_c(\rho))^2\, d\rho.$$

Using this equality in Eq. (20) we have

$$V \le \int_M \frac{V(\rho_H)}{A(\rho_H)} dM_x.$$

Finally notice that equality holds if and only if the arithmetic mean is equal to the geometric mean. And this occurs if and only if $\kappa_1 = \kappa_2$. □

Remark 10. If $c = 0$ the inequality (19) becomes

$$V \le \int_M \frac{\rho_H}{3} dM_x,$$

which is the classical Heintze and Karcher formula [10,13].

5. Measure of lines

Let $\mathcal{L}_1$ be the homogeneous space of lines in X_c^n. For each ξ in $\mathcal{L}_1$ we define the distance function h_ξ on X_c^n by $h_\xi(x) = d(\xi, x),\ \xi \in \mathcal{L}_1$. The level surfaces of h_ξ are just the tube surfaces around ξ. A point $x \in M$ is a critical point of $h_\xi|_M$ if and only if the normal geodesic $\exp_x(tN(x))$ to M through x hits ξ orthogonally. Let $\nu(h_\xi|_M)$ denote the number of critical points of $h_\xi|_M$.

Let us consider the euclidean case.

Theorem 5. *Let $K \subset \mathbb{E}^3$ be a strongly convex set with regular boundary surface $M = \partial K$. Then*

$$\int_{\{\xi \in \mathcal{L}_1 : K \cap \xi \neq \emptyset\}} \nu(h_\xi|_M)\, d\xi \le \int_M \left(c_0(x) + c_1(x)\, w(x) + c_2(x)\, w^2(x)\right) dM_x \tag{21}$$

where

$$c_0(x) = \pi \left(2\frac{H(x)}{\sqrt{K(x)}} - 1\right),$$
$$c_1(x) = -\pi\, H(x),$$
$$c_2(x) = \frac{\pi}{4}\left(H^2(x) + K(x)\right).$$

Equality holds if and only if K has constant width.

Proof. We consider the map

$$\Phi : \cup_{x\in M}\left(T_x^1 M \times [0, w(x)]\right) \to \mathcal{L}_1$$

defined by $\Phi(x, v, \rho) = \xi$ where ξ is the geodesic through $\exp_x(\rho N(x))$ with direction given by the parallel translation of v along the normal geodesic $\exp_x(tN(x))$.

Using adapted frames as before, with $x \in M$, $e_1 = v$ and $e_3 = N(x)$, Lemma 1 shows

$$\begin{aligned}
\Phi^* d\xi &= \sigma_{12} \wedge \sigma_{13} \wedge \sigma_2 \wedge \sigma_3 \\
&= \pi_1^*\omega_{12} \wedge \pi_1^*\omega_{31} \wedge (\pi_1^*\omega_2 + \rho\, \pi_1^*\omega_{32}) \wedge d\rho \\
&= (1 - \kappa_n(e_2)\,\rho)\,\kappa_n(e_1)\,\pi_1^*\omega_1 \wedge \pi_1^*\omega_2 \wedge \pi_1^*\omega_{12} \wedge d\rho \\
&= p(x, v, \rho)\,\pi_1^* dM \wedge dv \wedge d\rho,
\end{aligned} \tag{22}$$

where $\kappa_n(e_1), \kappa_n(e_2)$ are the normal curvatures of M in the directions e_1, e_2.

We shall apply the co-area formula to Φ.

Since

$$p(x, v, \rho) \;=\; \kappa_n(e_1) - \kappa_n(e_2)\,\kappa_n(e_1)\;\rho,$$

we have

$$\int p(x, v, \rho)\, d\rho \;=\; \rho\,\kappa_n(e_1) - \frac{1}{2}\rho^2\,\kappa_n(e_2)\,\kappa_n(e_1).$$

Therefore

$$\begin{aligned}
\int_0^w |p|\, d\rho &\le -\int_0^w p\, d\rho + 2\int_0^{1/\kappa_n(e_2)} p\, d\rho \\
&= (\text{equality iff } 1/\kappa_n(e_2) \le w) \\
&= -w\,\kappa_n(e_1) + \frac{1}{2}\,w^2\kappa_n(e_2)\,\kappa_n(e_1) + \frac{\kappa_n(e_1)}{\kappa_n(e_2)}.
\end{aligned} \tag{23}$$

In order to integrate (23) with respect to v, for a fixed x in M we parametrize v by its angle φ with respect to the principal curvature directions. We use Euler's formula

$$\begin{aligned}
\kappa_n(e_1) &= \kappa_1\,\cos^2\varphi + \kappa_2\,\sin^2\varphi \\
\kappa_n(e_2) &= \kappa_1\,\sin^2\varphi + \kappa_2\,\cos^2\varphi.
\end{aligned}$$

Hence

$$\kappa_n(e_1)\,\kappa_n(e_2) = (\kappa_1^2 + \kappa_2^2)\,\sin^2\varphi\,\cos^2\varphi + \kappa_1\,\kappa_2\,(\sin^4\varphi + \cos^4\varphi),$$

$$\frac{\kappa_n(e_1)}{\kappa_n(e_2)} = \frac{\kappa_1 + \kappa_2}{\kappa_1\,\sin^2\varphi + \kappa_2\,\cos^2\varphi} - 1.$$

Taking into account that $\kappa_1^2 + \kappa_2^2 = 4\,H^2 - 2K$ and that

$$\int_0^\pi \sin^2\varphi\,\cos^2\varphi\,d\varphi = \frac{\pi}{8},$$

$$\int_0^\pi (\sin^4\varphi + \cos^4\varphi)\,d\varphi = \frac{6\pi}{8},$$

$$\int_0^\pi \frac{\kappa_1 + \kappa_2}{\kappa_1\,\sin^2\varphi + \kappa_2\,\cos^2\varphi}\,d\varphi = \pi\,\frac{\kappa_1 + \kappa_2}{\sqrt{\kappa_1\,\kappa_2}},$$

integration of (23) over M leads to the right-hand side of (21).

Because of its construction Φ catches at least the lines ξ intersecting K, each exactly $\nu(h_\xi|_M)$ times. Therefore application of the co-area formula yields (21).

Φ catches exactly the lines ξ intersecting K if and only if each normal to M is a double normal, i.e. if and only if K has constant width; and in this case the focal points are inside K. □

Remark 11. Using Cauchy–Crofton's formula (cf. [20]) and taking into account that $2 \leq \nu(h_\xi|_M)$, the left hand side of (21) verifies

$$\pi\,M_0 \;=\; 2 \int_{\{\xi\in\mathscr{L}_1:K\cap\xi\neq\emptyset\}} d\xi \;\leq\; \int_{\{\xi\in\mathscr{L}_1:K\cap\xi\neq\emptyset\}} \nu(h_\xi|_M)\,d\xi,$$

where M_0 is the area of M. The equality is valid if and only if K is a ball.

6. Measure of planes

Let $\mathscr{L}_{n-1}$ be the homogeneous space of hyperplanes in X_c^n. For each η in $\mathscr{L}_{n-1}$ we define the distance function h_η on X_c^n by $h_\eta(x) = d(\eta, x)$, $\eta \in \mathscr{L}_{n-1}$. The level hypersurfaces of h_η are just the equidistants to η. A point $x \in M$ is a critical point of $h_\eta|_M$ if and only if the normal geodesic $\exp_x(tN(x))$ to M through x hits η orthogonally. Let $\nu(h_\eta|_M)$ denote the number of critical points of $h_\eta|_M$. Then

Theorem 6. *Let K be a convex set in X_c^3 with regular boundary surface $M = \partial K$. Then*

$$2 \int_{\{\xi\in\mathscr{L}_2:K\cap\eta\neq\emptyset\}} d\eta \;=\; \int_M (\alpha_0(x) + \alpha_1(x)\,H(x) + \alpha_2(x)\,K(x))\,dM_x \qquad (24)$$

where

$$\alpha_0(x) = c\,\frac{1}{2}(w - \mathrm{sn}_c(w)\,\mathrm{cn}_c(w))(x),$$
$$\alpha_1(x) = c\,\mathrm{sn}_c^2(w)(x),$$
$$\alpha_2(x) = \frac{1}{2}(w + \mathrm{sn}_c(w)\,\mathrm{cn}_c(w))(x).$$

Proof. We consider the map $\Phi : M_w \to \mathscr{L}_2$ defined by $\Phi(x,\rho) = \eta$ where η is the plane through $\exp_x(\rho N(x))$ orthogonal to the normal geodesic $\exp_x(tN(x))$.

Using frames adapted to the principal curvature directions of M, Lemma 1 shows

$$\begin{aligned}\Phi^* d\eta &= \sigma_{13} \wedge \sigma_{23} \wedge \sigma_3 \\ &= (c\ \mathrm{sn}_c(\rho) + \kappa_1\ \mathrm{cn}_c(\rho))(c\ \mathrm{sn}_c(\rho) + \kappa_2\ \mathrm{cn}_c(\rho))\, \pi_1^*\omega_1 \wedge \pi_1^*\omega_2 \wedge d\rho \\ &= p(x,\rho)\, \pi_1^* dM \wedge d\rho. \end{aligned} \tag{25}$$

We shall apply the co-area formula to Φ, taking into account orientations. Let $x \in M$ be a critical point of $h_\eta|_M$, i.e. $\Phi(x,\rho) = \eta$ for some ρ, then grad $h_\eta(x) = -N(x)$. Taking into account the curvature of equidistants $-c\ \mathrm{sn}_c(\rho)/\ \mathrm{cn}_c(\rho)$, the second fundamental form of the level surface at x is equal to $c\ \mathrm{sn}_c(\rho)/\ \mathrm{cn}_c(\rho)\ I_M(x)$. Then, according to Lemma 3, the sign of the critical point x of $h_\eta|_M$ is equal to sign $p(x,\rho)$.

Since

$$p(x,\rho) = c^2\ \mathrm{sn}_c^2(\rho) + (\kappa_1 + \kappa_2)\, c\ \mathrm{sn}_c(\rho)\ \mathrm{cn}_c(\rho) + \kappa_1\kappa_2\ \mathrm{cn}_c^2(\rho),$$

using Lemma 2 we have

$$\begin{aligned}\int p(x,\rho)\, d\rho &= c\,\frac{1}{2}(\rho - \mathrm{sn}_c(\rho)\ \mathrm{cn}_c(\rho)) \\ &\quad + (\kappa_1+\kappa_2)\, c\, \frac{1}{2}\ \mathrm{sn}_c^2(\rho) + \kappa_1\kappa_2\, \frac{1}{2}(\rho + \mathrm{sn}_c(\rho)\ \mathrm{cn}_c(\rho)). \end{aligned} \tag{26}$$

Because of its construction Φ catches each plane intersecting K exactly $\nu(h_\eta|_M)$ times. The sum of the critical points of $h_\eta|_M$ weighted by their signs is equal to the Euler characteristic $\chi(M) = \chi(\mathbb{S}^2) = 2$ (note that $\chi(\eta \cap M) = \chi(\mathbb{S}^1) = 0$). Therefore application of the co-area formula yields (24). □

Remark 12. When K is of constant width w, (24) becomes

$$2 \int_{\{\eta \in \mathscr{L}_2 : K \cap \eta \neq \emptyset\}} d\eta \ = \ \beta_0(w)\, M_0 + \beta_1(w)\, M_1 + \beta_2(w)\, M_2, \tag{27}$$

with

$$\begin{aligned}\beta_0(w) &= c\,\frac{1}{2}(w - \mathrm{sn}_c(w)\ \mathrm{cn}_c(w)), \\ \beta_1(w) &= c\ \mathrm{sn}_c^2(w), \\ \beta_2(w) &= \frac{1}{2}(w + \mathrm{sn}_c(w)\ \mathrm{cn}_c(w)). \end{aligned}$$

Remark 13. The left-hand side of (24) and (27) are just Quermaßintegrale of K, cf. [20,21,23].

Recall the Gauss–Bonnet formula

$$M_2 + c\, M_0 = 4\pi \tag{28}$$

and the representation of Quermaßintegrale of K,

$$\int_{\{\eta\in\mathscr{L}_2 : K\cap\eta\neq\emptyset\}} d\eta = M_1 + c\,V.$$

(cf. [20]).

Using these formulas and computing (27)+c(8), we get for K of constant width w

$$4\,c\,V \,+\, 2\,M_1 = 4\pi\, w. \tag{29}$$

Similarly, computing (27)−c(8), we get for K of constant width w

$$c\,\mathrm{sn}_c(w)M_0 + \mathrm{cn}_c(w)M_1 = 2\pi\,\mathrm{sn}_c(w). \tag{30}$$

The relations (28), (29), and (30) already appeared in [18], and they form a complete system of equalities for convex sets of constant width. Therefore, all other relations involving V, M_0, M_1, and M_2 can be obtained from them.

Remark 14. For strongly convex domains K in X_c^2 (strongly h-convex when $c < 0$) we can obtain a result similar to Theorem 6. Indeed, taking into account Cauchy–Crofton's formula (cf. [20]), we get

$$2\,L \;=\; \int_M (\alpha_0(x) + \alpha_1(x)\,\kappa(x))\; dM_x, \tag{31}$$

where

$$\alpha_0(x) = 1 - \mathrm{cn}_c(w(x)),$$
$$\alpha_1(x) = \mathrm{sn}_c(w(x)),$$

(cf. [16]). Specially for K with constant width w, (31) is just Barbier's theorem.

For negative values of c, Theorem 6 can be generalized by considering umbilical surfaces instead of planes.

Definition 6. An oriented complete totally umbilical hypersurface of X_c^n will be called a λ-*hyperplane*, where λ refers to the (constant) normal curvature (with respect to the orientation). We denote by $\mathscr{L}_{n-1}^{\lambda}$ the set of all λ-hyperplanes of X_c^n.

For $0 \neq |\lambda| < \sqrt{-c}$ every λ-hyperplane is equidistant from some geodesic hyperplane. For $|\lambda| \geq \sqrt{-c}$ the λ-hyperplanes are metric balls. In the limit case $\lambda = \pm\sqrt{-c}$ one has horospheres. From now on we restrict to the case $|\lambda| \leq \sqrt{-c}$.

Given λ, a point $p \in X_c^n$, and a vector $v \in T_pX_c^n$ there is a unique λ-hyperplane through p, normal to v and with the orientation according to v. If K is a compact convex set in X_c^n with regular boundary $M = \partial K$, we consider the map Φ_λ : $M \times \mathbb{R} \to \mathscr{L}_{n-1}^{\lambda}$ sending (x, ρ) to the λ-hyperplane defined by $\exp_x(\rho N(x))$ and $-(\exp_x(tN))'_{t=\rho}$.

Definition 7. For a a given $\lambda \in [-\sqrt{-c}, \sqrt{-c}]$, we define the λ-*width* of K as

$$\mathrm{w}_\lambda(x) = \sup\{\rho : \Phi_\lambda(x, \rho) \cap K \neq \emptyset\}.$$

Remark 15. It can be seen that $w_\lambda(x) = w$ for every $x \in M$ if and only if $w(x) = w$ for every $x \in M$.

It was seen in [22] that $\mathscr{L}_{n-1}^\lambda$ is a homogeneous space of the isometry group and admits an invariant measure $d\eta_\lambda$.

Theorem 7. *Let K be an h-convex set in X_c^3 $(c < 0)$ with regular boundary surface $M = \partial K$. Then*

$$\int_{\{\eta_\lambda \in \mathscr{L}_2^\lambda : K \cap \eta_\lambda \neq \emptyset\}} d\eta_\lambda = \int_M (\alpha_0^\lambda(x) + \alpha_1^\lambda(x) H(x) + \alpha_2^\lambda(x) K(x)) dM_x$$

where

$$\begin{aligned}
\alpha_0^\lambda(x) &= \frac{c}{2}(w_\lambda - \mathrm{sn}_c(w_\lambda)\,\mathrm{cn}_c(w_\lambda)) + \lambda c\,\mathrm{sn}_c^2(w_\lambda) + \frac{\lambda^2}{2}(w_\lambda + \mathrm{sn}_c(w_\lambda)\,\mathrm{cn}_c(w_\lambda)) \\
\alpha_1^\lambda(x) &= (c - \lambda^2)\,\mathrm{sn}_c^2(w_\lambda) + 2\lambda\,\mathrm{sn}_c(w_\lambda)\,\mathrm{cn}_c(w_\lambda) \\
\alpha_2^\lambda(x) &= \frac{1}{2}(w_\lambda + \mathrm{sn}_c(w_\lambda)\,\mathrm{cn}_c(w_\lambda)) - \lambda\,\mathrm{sn}_c^2(w_\lambda) + \frac{\lambda^2}{2c}(w_\lambda - \mathrm{sn}_c(w_\lambda)\,\mathrm{cn}_c(w_\lambda))
\end{aligned}$$

Proof. The measure $d\eta_\lambda$ of λ-hyperplanes is given by (cf. [22])

$$\Phi_\lambda^* d\eta_\lambda = (\sigma_{31} - \lambda\sigma_1) \wedge (\sigma_{32} - \lambda\sigma_2) \wedge \sigma_3,$$

using (1) we get after some manipulations

$$\begin{aligned}
\Phi_\lambda^* d\eta_\lambda = {} & ((c\,\mathrm{sn}_c\,\rho + \lambda\,\mathrm{cn}_c\,\rho)^2 + 2(c\,\mathrm{sn}_c\,\rho + \lambda\,\mathrm{cn}_c\,\rho)(\mathrm{cn}_c\,\rho - \lambda\,\mathrm{sn}_c\,\rho) H(x) \\
& + (\mathrm{cn}_c\,\rho - \lambda\,\mathrm{sn}_c\,\rho)^2 K(x)) dM_x \wedge d\rho.
\end{aligned}$$

By integrating with respect to ρ from 0 to w_λ we obtain the desired formula. □

Remark 16. In particular, taking $\lambda = \sqrt{-c}$ we find that the measure of horospheres intersecting an h-convex body is

$$\int_{\{K\cap\eta_\lambda\neq\emptyset\}} d\eta_\lambda = \int_M \mathrm{sn}_c\, w(x)(\mathrm{cn}_c\, w(x) - \sqrt{-c}\,\mathrm{sn}_c\, w(x)) \times(-c + 2\sqrt{-c}H(x) + K(x))dM_x.$$

On the other hand applying the results of [19] we get

$$2M_1 = \int_M \mathrm{sn}_c\, w(x)(\mathrm{cn}_c\, w(x) - \sqrt{-c}\,\mathrm{sn}_c\, w(x))(-c + 2\sqrt{-c}H(x) + K(x))dM_x.$$

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