ASYMPTOTIC BEHAVIOUR OF λ-CONVEX SETS IN THE COMPLEX HYPERBOLIC SPACE

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Abstract. In a Riemann manifold a regular convex domain is said to be λ-convex if its normal curvature at any point is bigger or equal than λ. In Hadamard manifolds, the asymptotic behaviour of the quotient $\mathrm{vol}(\Omega(t))/\mathrm{vol}(\partial\Omega(t))$ for a family of λ-convex domains $\Omega(t)$ expanding over the whole space has been studied and general bounds for this quotient are known. The aim of this paper is to study the asymptotic behaviour of the same quotient in complex hyperbolic space $\mathbb{CH}^n(-4k^2)$, a Hadamard manifold with constant holomorphic curvature equal to $-4k^2$. First, we give some specific properties of convex domains in complex hyperbolic space, $\mathbb{CH}^n(-4k^2)$. Indeed, we prove that λ-convex domains of arbitrary radius exists if $\lambda \leq k$. Finally, we prove that the general bounds can be improved for complex hyperbolic space and we give a sharp upper bound.

1. Introduction

In the Euclidean space, given a family of convex domains expanding over the whole space, the quotient between the volume and the area tends to infinity. This behaviour does not hold in hyperbolic plane $\mathbb{H}^2(-1)$ where the quotient tends to a value less or equal than 1.

The first result about the asymptotic behaviour of convex domains in $\mathbb{H}^2(-1)$ was given in 1972 by Santaló and Yañez ([SY72]) for a family of h-convex domains (a convex domain is said to be h-convex if for each pair of points belonging to the convex, the entire segments of the two horocycles joining them also belong to the convex contains a segment). If $\{\Omega(t)\}_{t\in\mathbb{R}}$ is a family of compact h-convex domains in $\mathbb{H}^2$ expanding over the whole plane then

$$\lim_{t\to\infty}\frac{\mathrm{area}(\Omega(t))}{\mathrm{length}(\partial\Omega(t))}=1.$$

In hyperbolic space, given a geodesic line the set of equidistant points to l are two curves called equidistants. These curves have constant geodesic curvature λ such that $0<\lambda<1$. From this it is defined the

1991 *Mathematics Subject Classification.* Primary 52A55; Secondary 52A10.
Key words and phrases. Complex hyperbolic space, volume, λ-convex set.
Work partially supported by DGICYT grant # BFM2003-03458.

notion of λ-convexity. It is said that a convex domain is λ-convex if for each pair of points belonging to the convex, the entire segment of the two curves with constant geodesic curvature λ joining them also belong to the convex. In $\mathbb{H}^2(-1)$ it is proved in [GR99] that a convex is λ-convex if and only if the geodesic curvature of the boundary is bigger than λ. Roughly speaking λ-convex sets ($\lambda > 0$) have the boundary more curved than convex sets. The concept of λ-convex domain is also defined for any manifold (cf. definition 3.3).

The result about the behaviour of the quotients was generalized for families of λ-convex domains in $\mathbb{H}^n$ expanding over the whole space (cf. [GR85], [BM99], [GR99], [BV99]) and for families of λ-convex domains in any Hadamard manifold (cf. [BGR01]). Recall that a Hadamard manifold is a simply connected manifold with nonpositive sectional curvature.

The result obtained in [BGR01] for a family of λ-convex domains in a Hadamard manifold with sectional curvature K such that $-k_2^2 \leq K \leq -k_1^2$ is

(1)
$$\frac{\lambda}{(n-1)k_2^2} \leq \liminf_{t\to\infty} \frac{\mathrm{vol}(\Omega(t))}{\mathrm{vol}(\partial\Omega(t))} \leq \limsup_{t\to\infty} \frac{\mathrm{vol}(\Omega(t))}{\mathrm{vol}(\partial\Omega(t))} \leq \frac{1}{(n-1)k_1},$$

for $0 \leq \lambda \leq k_2$.

As $\mathbb{H}^n$ is a Hadamard manifold we can apply this result to $\mathbb{H}^n$. We obtain the same bounds as the ones obtained in the previous results. Moreover, it is known that in this space the bounds are sharp (cf. [Sol03]).

The purpose of this paper is to study the λ-convexity in $\mathbb{CH}^n(-4k^2)$ (see definition 2.1) and to show that the bounds in (1) can be improved for $\mathbb{CH}^n(-4k^2)$. The obtained result is:

Theorem 1. *Let $\{\Omega(t)\}_{t\in\mathbb{R}^+}$ be a family of compact λ-convex domains, $\lambda \leq k$, expanding over the whole space $\mathbb{CH}^n(-4k^2)$, $n \geq 2$. Then,*

$$\frac{\lambda}{4nk^2} \leq \liminf_{t\to\infty} \frac{\mathrm{vol}(\Omega(t))}{\mathrm{vol}(\partial\Omega(t))} \leq \limsup_{t\to\infty} \frac{\mathrm{vol}(\Omega(t))}{\mathrm{vol}(\partial\Omega(t))} \leq \frac{1}{2nk}.$$

Moreover, the upper bound is sharp.

Note that the bounds improve the ones given in the general result in [BGR01]. That is, the complex hyperbolic space $\mathbb{CH}^n(-4k^2)$ has real dimension $2n$ and its sectional curvature takes values between $-4k \leq K_{sec} \leq -k$ (cf. next section), so that with the obvious changes in (1) we get that the lower bound of the quotient volume/area is $\dfrac{\lambda}{4(2n-1)k^2}$

which is lesser than the given in theorem 1, $\dfrac{\lambda}{4nk^2}$, and the upper bound is $\dfrac{1}{(2n-1)k}$ which is greater than $\dfrac{1}{2nk}$.

The result obtained for convexity in $\mathbb{CH}^n(-4k^2)$ is the following:

Theorem 2. *In complex hyperbolic space,* $\mathbb{CH}^n(-4k^2)$*, it can only exists families of compact convex domains piecewise* $\mathcal{C}^2$ *expanding over the whole* $\mathbb{CH}^n(-4k^2)$ *if they are* λ*-convex with* $\lambda \leq k$*.*

This theorem gives us more information than the general one for Hadamard manifolds. That is, for any Hadamard manifold with sectional curvature K such that $-k_2^2 \leq K \leq -k_1^2$ we can assure that cannot exists families of λ-convex domains expanding over the whole space with $\lambda > k_2$ but in $\mathbb{CH}^n(-4k^2)$ this bound is restricted to k_1. This fact is consequence of the properties of the normal curvatures of the spheres, which takes values in all of its possible range (see corollary 2.4).

2. Preliminaries

Definition 2.1. The *complex hyperbolic space* is the only (up to holomorphic isometries) complete simply connected Kähler manifold of constant holomorphic curvature $-4k^2$. We denote it by $\mathbb{CH}^n(-4k^2)$.

We can see $\mathbb{CH}^n(-4k^2)$ as a subspace of $\mathbb{CP}^n$. In $\mathbb{C}^{n+1}$ we consider the hermitian product

$$\langle z, w\rangle = -z_0\overline{w_0} + \sum_{j=1}^{n} z_j\overline{w_j}$$

and the subset

$$M = \{z \in \mathbb{C}^{n+1} \ : \ \langle z, z\rangle = -k\}.$$

The projection of M in $\mathbb{CP}^n$ with the induced metric corresponds to complex hyperbolic space $\mathbb{CH}^n(-4k^2)$.

Then geodesics are projection of complex planes in $\mathbb{C}^{n+1}$ and the isometries come from the applications preserving the hermitian metric.

As $\mathbb{CH}^n(-4k^2)$ is a Kähler manifold it has complex structure. We will denote it by J. Then, holomorphic curvature is defined as the sectional curvature of the planes generated by a vector v and Jv. In spaces of constant holomorphic curvature, seccional curvature is expressed as (cf. [KN69])

$$g(R(u,v)v,u) = -\frac{K_{hol}}{4}(1 + 3(g(u,Jv))^2).$$

So, in $\mathbb{CH}^n(-4k^2)$ since holomorphic curvature is $-4k^2$ sectional curvatures are such that $-4k^2 \leq K \leq -k^2$.

In the proof of theorem 1 we use an expression for the volume of compact convex domains. We parametrize a compact convex domain Ω from the exponential map of the set

$$A = \{(u,t) \in S^{2n-1} \times \mathbb{R} \mid 0 < t \leq l(u)\}$$

where $l(u)$ is the distance between p, a fixed interior point, and $\partial\Omega$ in the direction u.

Then

$$\mathrm{vol}(\Omega) = \int_{\Omega} \tau = \int_{A} \exp_p^* \tau$$

where τ is the volume element of $\mathbb{CH}^n(-4k^2)^n$.

If

$$\exp_p^* \tau = J(t,u) t^{2n-1} dt dS_{2n-1}$$

then $J(t,u)$ is the Jacobian of the exponential map. Using an orthonormal basis and the Jacobi fields along the geodesic given by the direction u we have

$$J(t,u) = \frac{\sinh^{2n-1}(kt)\cosh(kt)}{(kt)^{2n-1}}.$$

Thus, the volume of a compact convex domain Ω:

$$\mathrm{vol}(\Omega) = \frac{1}{2nk^{2n}} \int_{S^{2n-1}} \sinh^{2n}(kl(u)) dS_{2n-1}, \tag{2}$$

and the intrinsic volume of $\partial\Omega$ is:

$$\mathrm{vol}(\partial\Omega) = \int_{S^{2n-1}} \frac{\sinh^{2n-1}(kl(u))\cosh(kl(u))}{k^{2n-1}\langle \partial_t, N\rangle} dS_{2n-1} \tag{3}$$

where N is the outward unit normal vector of Ω and ∂_t the radial field from p. If we denote by ϕ the angle between ∂_t and N we have that $\langle \partial_t, N\rangle = \cos\phi$.

Another fact we use about $\mathbb{CH}^n(-4k^2)$ concerns about principal curvatures of a geodesic sphere. Spheres in $\mathbb{CH}^n(-4k^2)$ are not umbilical hypersurfaces as in real space forms.

Proposition 2.2 ([Mon85]). *The principal curvatures of a geodesic sphere of radius r in $\mathbb{CH}^n(-4k^2)$ are:*

- $2k\coth(2kr)$ *with multiplicity 1 and principal direction* $-JN$ *(where N is the outward unit normal vector).*
- $k\coth(kr)$ *with multiplicity* $2n-2$.

Therefore,

Corollary 2.3. *Let z be a point in a geodesic sphere of $\mathbb{CH}^n(-4k^2)$, N the outward unit normal vector and v a principal direction. Then the submanifolds generated by the exponential map of $\{N, v\}$ at the point z are totally geodesic.*

Proof. Since v is a principal direction we have $-\nabla_v N = \lambda v$. Then,

$$\langle -J\nabla_v N, N\rangle = \lambda\langle v, JN\rangle.$$

Moreover, if $v \neq -JN$ then

$$\langle -J\nabla_v N, N\rangle = \langle \nabla_v(-JN), N\rangle = 0$$

from the properties of the second fundamental form. So, $\langle v, JN\rangle = 0$, which is the condition for a pair of vectors to generate a totally real plane. But, a totally real plane is isomorphic to $\mathbb{H}^2(-k^2)$ and it is totally geodesic.

If $v = -JN$ then we consider the submanifold generated by $\{N, JN\}$ and it is known that it is isomorphic to $\mathbb{H}^2(-4k^2)$ and it is also totally geodesic. □

From the proposition 2.2 it also follows:

Corollary 2.4. *The normal curvature of spheres lies between* $k\coth(kr)$ *and* $2k\coth(2kr)$. *Moreover, all the possible values are taken.*

3. Convexity on complex hyperbolic space

Definition 3.1. A $\mathcal{C}^2$ *hypersurface* of a Riemann manifold is said to be *regular* λ*-convex* if at every point all the normal curvatures (with respect the invariant normal unit vector) are greater or equal than $\lambda \geq 0$.

This definition can be generalized to the non-regular case.

Definition 3.2. A λ*-convex hypersurface* is a hypersurface such that for every point p there is a regular λ-convex hypersurface S leaving a neighborhood of p in the hypersurface in the convex side of S.

Definition 3.3. A domain is said to be *(regular)* λ*-convex* if its boundary is a regular λ-convex hypersurface.

From the definition we get that a λ-convex hypersurface is also λ'-convex for any $\lambda' \leq \lambda$.

Remark 3.4. In spaces of constant curvature and in complete simply connected manifolds with non-positive sectional curvature the notion of 0-convexity is equivalent to the convexity with respect to geodesics (cf. [Ale77]).

The notion of λ-convexity gives some relations on how the boundary bends. Indeed, we have

Proposition 3.5 ([BGR01])**.** *Let* M *be a* $(n+1)$*-dimensional Hadamard manifold with sectional curvature* K *such that* $-k_2^2 \leq K \leq -k_1^2$. *Let* Ω *be a* λ*-convex domain with* $\mathcal{C}^2$ *boundary,* $\lambda < k_2$ *and* O *an interior* *point of Ω. If φ denotes the angle of the normal to $\partial\Omega$ and the exterior radial direction, when $d(O, \partial\Omega) \leq (1/k_2)\mathrm{arctanh}(\lambda/k_2)$ we have*

$$\cos\varphi \geq \frac{1}{k_2}\sqrt{\lambda^2\cosh^2 k_2 s - k_2^2 \sinh^2 k_2 s}$$

If $d(O, \partial\Omega) \geq (1/k_2)\mathrm{arctanh}(\lambda/k_2)$ then

$$\cos\varphi \geq \frac{\lambda}{2k}.$$

Now we study which values can take $\lambda \geq 0$ for a family of λ-convex domains in $\mathbb{CH}^n(-4k^2)$ expanding over the whole space. We prove that λ-convex domains of any inradius (the radius of the biggest inscribed sphere) with piecewise $\mathcal{C}^2$ boundary have $\lambda \leq k$. To get this result we study the normal curvature at some points of the boundary of the domain.

Definition 3.6. Given a point p of class $\mathcal{C}^2$ in the boundary of a convex domain Ω, we call *inscribed sphere* at p the biggest sphere tangent to $\partial\Omega$ at p contained in the domain. We call *circumscribed sphere* at p the smallest sphere tangent to $\partial\Omega$ at p containing the domain.

Lemma 3.7. *Let $\Omega \in \mathbb{CH}^n(-4k^2)$ be a $\mathcal{C}^2$ convex domain. If at point $p \in \partial\Omega$ there exists an inscribed sphere S_i and a circumscribed sphere S_c tangents to $\partial\Omega$ at p, then*

$$K_{n,S_c}(X) \leq K_{n,\Omega}(X) \leq K_{n,S_i}(X) \tag{4}$$

for any $X \in T_pM$.

Proof. Let N be unit normal vector to $\partial\Omega$ at p. Let $\{e_1, ..., e_{2n-1} = -JN\}$ be a basis of principal directions of the inscribed sphere at p. Note that it is also a basis of principal directions of the circumscribed sphere at p. That is, the direction $-JN$ is always a principal direction of a sphere and the directions perpendicular to $-JN$ are all principal directions (cf. proposition 2.2). So, if two spheres are tangent at a point then they have the same principal directions (although they do not have the same principal curvatures).

From corollary 2.3, the exponential map of the vectors $\{N, e_i\}$ at p are totally geodesic submanifolds. When $i = 2n-1$ it is isometric to $\mathbb{H}^2(-4)$. In the other cases they are isometric to $\mathbb{H}^2(-1)$.

As it is known that this lemma is true stated in $\mathbb{H}^2(-k^2)$ instead of $\mathbb{CH}^n(-4k^2)$ we can assert that it is also valid for planes generated by a principal direction and normal direction. Since the other directions are linear combination of principal directions the result follows. □

Then, from the previous lemma and using the properties of the principal curvatures of a sphere (see proposition 2.2) we have:

Proposition 3.8. *Let $\Omega \in \mathbb{CH}^n(-4k^2)$ be a piecewise $\mathcal{C}^2$ compact convex domain. If at point $p \in \partial\Omega$ there exists inscribed and circumscribed sphere, then the normal curvature satisfies:*

$$\coth(R) \leq K_{n,\Omega} \leq 2k \coth(2kr)$$

for any direction, with r denoting the radius of the inscribed sphere at p and R the radius of the circumscribed sphere at p.

This proposition is valid for any direction but for some directions we can improve the inequalities since we know for which directions the normal curvature in a sphere takes the maximum and the minimum value.

Corollary 3.9. *Let $\Omega \in \mathbb{CH}^n(-4k^2)$ be a piecewise $\mathcal{C}^2$ compact convex domain and N the normal vector of $\partial\Omega$. If at a point $p \in \partial\Omega$ of class $\mathcal{C}^2$ there exists inscribed and circumscribed sphere, then the normal curvature in the directions X such that $\{N, X\}$ span a totally real plane satisfies:*

$$k \coth(kR) \leq K_{n,\Omega}(X) \leq k \coth(kr)$$

and in the direction $-JN$ it satisfies:

$$2k \coth(2kR) \leq K_{n,\Omega}(-JN) \leq 2k \coth(2kr)$$

with r denoting the radius of the inscribed sphere at p and R the radius of the circumscribed sphere at p.

Now, we can study the λ-convexity of families of convex domains expanding over the whole $\mathbb{CH}^n(-4k^2)$. In [BGR01] it is proved that in a Hadamard manifold with sectional curvature K such that $-k_2^2 \leq K \leq -k_1^2$ it can only exists families of λ-convex domains expanding over the whole manifold if $\lambda \leq k_2$.

The specific geometry of $\mathbb{CH}^n(-4k^2)$ allows us to prove that it can only exists families of λ-convex domains expanding over the whole space if $\lambda \leq k_1 = k$.

The result obtained for convexity in $\mathbb{CH}^n(-4k^2)$ is the following:

Theorem 2. *In complex hyperbolic space, $\mathbb{CH}^n(-4k^2)$, it can only exists families of compact convex domains piecewise $\mathcal{C}^2$ expanding over the whole $\mathbb{CH}^n(-4k^2)$ if they are λ-convex with $\lambda \leq k$.*

Proof. From the definition of λ-convexity, we have that for any convex domain $\Omega(t)$ of the family, for any point $p \in \partial\Omega(t)$ of class $\mathcal{C}^2$ and for some direction X tangent to $\partial\Omega(t)$ at p it is satisfied

$$\lambda \leq K_{n,\,\Omega(t)}(X) \leq K_{n,S_i}(X) = k \coth(kr) \tag{5}$$

where S_i is the inscribed sphere of radius r in Ω at point p.

This argument must hold for every convex domain in the family. Since the radius of the inscribed ball tends to infinity when the convex domains grow, for a convex big enough, $k \coth(kr)$ is close to k and from the inequalities in (5) it is necessary that $\lambda \leq k$. ☐

Remark 3.10. The argument followed to prove the last proposition uses strongly the properties of $\mathbb{CH}^n(-4k^2)$. That is, we use that for some direction of the tangent space at every point of a sphere of radius r, the normal curvature is exactly $k\coth(kr)$. In arbitrary Hadamard manifolds the analogue cannot be assured. Given a Hadamard manifold with sectional curvature K such that $-k_2^2 \leq K \leq -k_1^2$ it can only be assured that the normal curvature K_n of a sphere of radius r is such that $k_1\coth(k_1 r) \leq K_n \leq k_2\coth(k_2 r)$ (cf. [Pet98]).

Examples. In any Riemann manifold *geodesic spheres* are convex domains. As it is known that in $\mathbb{CH}^n(-4k^2)$ normal curvatures of a geodesic sphere of radius r satisfy $k\coth(kr) \leq K_n \leq 2k\coth(2kr)$ (cf. proposition 2.2) they are $k\coth(kr)$-convex domains. If we fix a point in a sphere and let the radius tends to infinity we get a non-compact convex domain called *horosphere*. It is k-convex.

Let $\mathbb{CH}^p(-4k^2)$ be a isometrically embedded complex hyperbolic space in $\mathbb{CH}^n(-4k^2)$ of dimension less than n and let $\mathbb{RH}^n$ be a real hyperbolic space isometrically embedded in $\mathbb{CH}^n(-4k^2)$. Then the *equidistant hypersurface* from a fixed $\mathbb{CH}^p(-4k^2)$, $1 \leq p < n$, or from a fixed $\mathbb{RH}^n$, that is, the set of points which are at a fixed constant distance from the submanifold are also convex hypersurfaces bounding a non-compact domain. Its λ-convexity is $k\tanh(kr)$. These facts follow from [Mon85].

It is known that in $\mathbb{CH}^n(-4k^2)$ there not exists totally geodesic hypersurfaces. The hypersurfaces which can be considered as the substitute of the totally geodesic hypersurfaces in $\mathbb{CH}^n(-4k^2)$ are the ones generated by the exponential map of $2n-1$ tangents vectors at a point, that is, *bisectors*. However, they are not convex hypersurfaces and do not bound convex domains (cf. [Gol99]). So, we cannot construct any convex domain with a part of its boundary contained in a bisector.

Given a geodesic we can construct a convex domain from this taking its tube of radius $r > 0$.

Definition 3.11. Let (M, g) be a Hadamard manifold and γ a complete geodesic. We define the *tube* of radius $r \geq 0$ about γ as

$$\begin{aligned}\tau(\gamma, r) &= \{x \in M \ : \ \text{there exists a geodesic } \xi \text{ of length } L(\xi) \leq r \\ &\qquad \text{from } x \text{ meeting } \gamma \text{ orthogonally}\}.\end{aligned}$$

This definition is equivalent to (cf. [Gra04])

$$\tau(\gamma, r) = \bigcup_{y\in\gamma}\{\exp_y(v) \ : \ v \in (\gamma_y)^\perp \text{ and } ||v|| \leq r\}$$

and also to

$$\tau(\gamma, r) = \bigcup_{y\in\gamma}\{\exp_y(v) \ : \ v \in T_yM \text{ and } ||v|| \leq r\}.$$

That is, a tube about a geodesic γ can be defined as the set of points which are at a distance less or equal than $r \geq 0$ (the radius of the tube) of γ or as the union of the balls of radius r with center in γ.

In the next lemma we prove that the tube about a geodesic is convex in any Hadamard manifold and in the next one we give a way to modify this to obtain a convex compact hypersurface.

Lemma 3.12. *Let (M, g) be a Hadamard manifold. Let γ be a complete geodesic. Then the tube of radius r about γ is a convex domain.*

Proof. To prove that the boundary of the tube of radius r about γ, $\tau(\gamma, r)$, is a convex hypersurface we study its second fundamental form. If it is semi-definite at every point then it is convex. Let us parametrize the boundary of the tube in the following way:

$$\begin{array}{rccc} \phi: & \mathbb{R} \times \mathbb{R}^+ \times S^{2n-1} & \longrightarrow & \mathbb{CH}^n(-4k^2) \\ & (r,\, t,\, v) & \mapsto & \exp_{\gamma(t)}(rv) \end{array}$$

where we identify S^{2n-1} with the normal space at every point of $\gamma(t)$.

From this parametrization we consider the following fields over $\tau(\gamma, r)$

$$\begin{aligned} N &= d\phi\,(\partial/\partial r) \\ T &= d\phi\,(\partial/\partial t) \\ T_i &= d\phi\,(\partial/\partial v_i)\,. \end{aligned}$$

By Gauss lemma, N is normal on every hypersurface $\{r = \text{constant}\}$.

Using the fact that T and T_i are orthogonal to N and ϕ-related we have:

$$\begin{aligned} (6)\quad 2\langle -\nabla_{T_i} N, T_i\rangle &= T_i\langle N, T_i\rangle + N\langle T_i, T_i\rangle - T_i\langle T_i, N\rangle - \\ &\quad -\langle T_i, [N, T_i]\rangle + \langle N, [T_i, T_i]\rangle + \langle T_i, [T_i, N]\rangle \\ &= N\langle T_i, T_i\rangle. \end{aligned}$$

So, to study the sign of $\langle -\nabla_{T_i} N, T_i\rangle$ we can study the sign of $N\langle T_i, T_i\rangle$, which tells us how the norm of T_i change along the normal field.

Since

$$T_i = d\phi_{(r,t,v)}(\partial/\partial v_i) = \left.\frac{\partial \exp_{\gamma(t)}(r(v + s\partial/\partial v_i))}{\partial/\partial s}\right|_{(r,0)}$$

and

$$T = d\phi_{(r,t,v)}(\partial/\partial t) = \left.\frac{\partial \exp_{\gamma(t+s)}(rv))}{\partial/\partial s}\right|_{(r,0)},$$

T and T_i are Jacobi fields. Now, using that in a Hadamard manifold the norm of Jacobi fields is also non-negative we have that $N\langle T_i, T_i\rangle > 0$.

In the same way we have that T is a Jacobi field and $N\langle T, T\rangle > 0$. $\square$

As we are studying examples of convex domains to know better the asymptotic behaviour of convex domains in complex hyperbolic space we are interested in having compact convex domains. Perhaps the easiest way to obtain a convex domain is intersecting the tube with a compact convex domain, for instance, a ball. Since the intersection of two convex domains is a convex domain we obtain a compact convex domain. We consider that the ball has its center in the geodesic which defines the tube and its radius is bigger than the radius of the tube. This domain is also convex in any Hadamard manifold.

Note that in complex hyperbolic space we cannot modify the tube to obtain a compact convex domain just taking the tube about a geodesic segment because it should have a part of the boundary contained in a bisector, and hence it would not be convex.

Anyway, we can modify the tube $\tau(\gamma, r)$ in order to obtain a compact convex domain in another way. Let γ_L be a geodesic segment of lenght L. At each of the endpoints of the segment we attach a ball of radius r and center the endpoint. We denote the union of these three convex domains by $\tau_L(\gamma, r)$. This way to obtain a compact convex domain from the tube will be useful to calculate explicitly its volume.

Lemma 3.13. *Let (M, g) be a Hadamard manifold and γ a complete geodesic. In the hypothesis of the previous lemma we have that the modified tube $\tau_L(\gamma, r)$ defined in the paragraph above is convex.*

Proof. The convexity of $\tau_L(\gamma, r)$ can only fails at the boundary intersection points between the ball and the tube. In these points the boundary is not of class $\mathcal{C}^2$ so that we have to use the definition 3.2 in order to prove the convexity. The regular convex hypersurface we consider is the infinite tube $\tau(\gamma, r)$. From the last lemma this is convex. Moreover, since we have that a tube about a geodesic can also be described as the union of all balls of radius the radius of the tube and center in any point of the geodesic which defines the tube, we can assert that all the points of the ball are inside this tube so, it leaves a neighborhood of the boundary intersection points we are taking into account in the convex side. □

4. Asymptotic behaviour

The purpose of this section is to prove the main result for the asymptotic behaviour of λ-convex domains in $\mathbb{CH}^n(-4k^2)$, that is

Theorem 1. *Let $\{\Omega(t)\}_{t\in\mathbb{R}^+}$ be a family of compact λ-convex domains, $\lambda \leq k$, expanding over the whole space $\mathbb{CH}^n(-4k^2)$, $n \geq 2$. Then,*

$$\frac{\lambda}{4nk^2} \leq \liminf_{t\to\infty} \frac{\mathrm{vol}(\Omega(t))}{\mathrm{vol}(\partial\Omega(t))} \leq \limsup_{t\to\infty} \frac{\mathrm{vol}(\Omega(t))}{\mathrm{vol}(\partial\Omega(t))} \leq \frac{1}{2nk}. \tag{7}$$

Moreover, the upper bound is sharp.

Proof. The upper bound follows just using expressions (2) and (3), that $\cos\varphi \leq 1$ (φ is the angle between the exterior normal to $\partial\Omega(t)$ and the radial direction from a fixed point O inside the convex) and that

$$\int_{S^{2n-1}} \sinh^{2n}(kl(u))dS_{2n-1} \leq \int_{S^{2n-1}} \sinh^{2n-1}(kl(u))\cosh(kl(u))dS_{2n-1}.$$

Hence,

$$\begin{aligned} \frac{\mathrm{vol}(\Omega_t)}{\mathrm{vol}(\partial\Omega_t)} &= \frac{\dfrac{1}{2nk^{2n}}\displaystyle\int_{S^{2n-1}} \sinh^{2n}(kl(u))dS_{2n-1}}{\displaystyle\int_{S^{2n-1}} \frac{\sinh^{2n-1}(kl(u))\cosh(kl(u))}{k^{2n-1}\cos\varphi}dS_{2n-1}} \\ &\leq \frac{\displaystyle\int_{S^{2n-1}} \sinh^{2n}(kl(u))dS_{2n-1}}{2nk\displaystyle\int_{S^{2n-1}} \sinh^{2n-1}(kl(u))\cosh(kl(u))dS_{2n-1}} \leq \frac{1}{2nk}. \end{aligned}$$

In order to obtain the lower bound we use that $\cos\varphi \geq \lambda/2k$ for a convex big enough (cf. proposition 3.5). From this we obtain the following inequality:

$$\begin{aligned} \frac{\mathrm{vol}(\Omega(t))}{\mathrm{vol}(\partial\Omega(t))} &= \frac{\dfrac{1}{2nk^{2n}}\displaystyle\int_{S^{2n-1}} \sinh^{2n}(kl(u))dS_{2n-1}}{\displaystyle\int_{S^{2n-1}} \frac{\sinh^{2n-1}(kl(u))\cosh(kl(u))}{k^{2n-1}\cos\varphi}dS_{2n-1}} \\ &= \frac{\displaystyle\int_{S^{2n-1}} \sinh^{2n-1}(kl(u))\cosh(kl(u))\tanh(kl(u))dS_{2n-1}}{2nk\displaystyle\int_{S^{2n-1}} \frac{\sinh^{2n-1}(kl(u))\cosh(kl(u))}{\cos\varphi}dS_{2n-1}} \\ &\geq \tanh(kr)\frac{\lambda\displaystyle\int_{S^{2n-1}} \sinh^{2n-1}(l(u))\cosh(l(u))dS_{2n-1}}{4nk^2\displaystyle\int_{S^{2n-1}} \sinh^{2n-1}(l(u))\cosh(l(u))dS_{2n-1}} \\ &= \frac{\lambda}{4nk^2}\tanh(r) \end{aligned}$$

where r is the distance between O (a fixed interior point from which we parametrize the convex domain) and the boundary of the convex domain. Then, as $\tanh(r)$ tends to 1 when $\Omega(t)$ tends to expand over the whole space we can assert that

$$\liminf_{t\to\infty} \frac{\mathrm{vol}(\Omega(t))}{\mathrm{vol}(\partial\Omega(t))} \geq \frac{1}{4kn^2}.$$

In order to show the sharpness of an inequality it is enough to give an example. Balls are λ-convex domains for every $\lambda \in [0, k\coth(kr)]$ because normal curvatures are bigger or equal than $k\coth(kr)$. For a family of balls with the same center we have

$$\begin{aligned}\limsup_{r\to\infty}\frac{\mathrm{vol}(B_r)}{\mathrm{vol}(\partial B_r)} &= \limsup_{r\to\infty}\frac{\sinh^{2n}(kr)\pi^n n!}{2nk\sinh^{2n-1}(kr)\cosh(kr)\pi^n n!}\\ &= \limsup_{r\to\infty}\frac{\tanh(kr)}{2nk}=\frac{1}{2nk},\end{aligned}$$

which gives the upper bound given on (7). □

To complete the study of this asymptotic behaviour in complex hyperbolic space it remains to decide if the lower bound is sharp. In the study of the analogous problem in real hyperbolic space it is given another family of convex domains expanding over the whole space which tends to the lower bound (see [Sol03]). If we try to construct a family of domains in a similar way we get into trouble because of the richer trigonometry of complex hyperbolic space. Instead of these examples we studied the modified tubes described in lemma 3.13. Anyway, they do not give an example of the lower bound but another example of the upper bound.

Using the formula for the volume of a tube about a curve σ given in [GV82],

$$\mathrm{vol}(\tau_L(\sigma,r))=\frac{L\mathrm{vol}(S^{2n-2})}{(4k^2)^{n-1}}\int_0^r\sinh^{2n-2}(ks)\left(1+\frac{2n}{2n-1}\sinh^2(ks)\right)ds,$$

it can be proved by a simple calculation that

if $\{\tau_L(\gamma,r)\}_{L,r}$ is a family of the defined modified tubes (cf. lemma 3.13) then

$$\limsup_{r\to\infty,L\to\infty}\frac{\mathrm{vol}(\tau_L(\gamma,r))}{\mathrm{vol}(\partial\tau_L(\gamma,r))}=\frac{1}{2nk},$$

which is also equal to the value of the upper bound of (7). The result of the limit of the quotient does not depend on the relation between the grow of the length of the segment and the radius of the tube.

These examples allows us to assert that the family of convex domains obtained from the intersection between a tube about a geodesic and a ball centered at a point of the geodesic gives another example of a family of convex domains expanding over the whole space which its quotient tends to the upper bound. That is, we expand the convex over the whole complex hyperbolic space increasing the length of the segment and the radius of the ball. The relation between the growing of the length and the radius does not effect the result. The explicit calculation of the quotient *volume/area* is more complicated for these domains since we should know the volume of a piece of a ball. Anyway, using a decomposition of the domain into two parts with a part the tube contained in the domain and using the result obtained in the other example of modified tubes, we can assert that the value of the limit of this quotient is also $1/2nk$, the upper bound of (7).

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