HOROSPHERES AND CONVEX BODIES IN n-DIMENSIONAL HYPERBOLIC SPACE

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September 26, 2002

To the memory of professor L.A. Santaló

Abstract. In n-dimensional euclidean space, the measure of hyperplanes intersecting a convex domain is proportional to the $(n-2)$-mean curvature integral of its boundary. This question was considered by Santaló in hyperbolic space. In non-euclidean geometry the best analogue of linear subspaces are not always the totally geodesic hypersurfaces. In some situations horospheres play the role of euclidean hyperplanes.

In dimensions $n = 2$ and 3, Santaló proved that the measure of horospheres intersecting a convex domain is also proportional to the $(n-2)$-mean curvature integral of its boundary.

In this paper we show that this analogy does not generalize to higher dimensions. We express the measure of horospheres intersecting a convex body as a linear combination of the mean curvature integrals of its boundary.

1. Introduction

One of the first results in *integral geometry* is the Cauchy-Crofton formula. It states that the length ℓ of a piecewise differentiable plane curve Γ is the measure of the set of lines L intersecting Γ, counted with multiplicity:

$$\int_{L\cap\Gamma\neq\emptyset} \sharp(L\cap\Gamma)dL = 2\,\ell(\Gamma)$$

where dL is a measure for lines which is invariant under the group of rigid motions. When lines L are given by the equation $x\cos\theta + y\sin\theta = p$, this measure is written as $dL = dp\wedge d\theta$ (see Figure 1).

As a consequence, the measure of all lines intersecting a convex domain is the length of its boundary.

This formula has several generalizations to higher dimensional euclidean spaces $\mathbb{E}^n$ (see [San76]). Firstly, the total measure of hyperplanes L intersecting a convex domain D with C^2-regular boundary is

$$\int_{L\cap D\neq\emptyset} dL = M_{n-2}(\partial D) \tag{1}$$

where M_{n-2} is the integral of the $(n-2)$-function of curvature of ∂D (see (4)) and dL is the invariant measure for hyperplanes.

Secondly, for a hypersurface Σ we have

$$\int_{L\cap\Sigma\neq\emptyset} \mathrm{vol}_{n-2}(L\cap\Sigma)dL = \frac{O_nO_{n-2}}{O_{n-1}O_0}\mathrm{vol}_{n-1}(\Sigma) \tag{2}$$

1991 *Mathematics Subject Classification.* Primary 52A55; Secondary 52A10.

Key words and phrases. Hyperbolic space, volume, horocycle, horosphere, convex set, h-convex set.

Work partially supported by DGI grants number BFM2000-0007 (first and third authors) and BFM2001-3548 (second author).

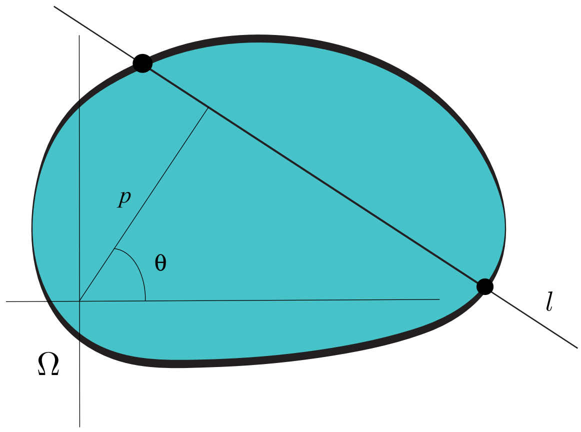

FIGURE 1

where O_r denotes the volume of the r-dimensional unit sphere. Notice that in the plane case $\mathrm{vol}_0(L \cap \Sigma)$ is the number of intersection points $\sharp(L \cap \Sigma)$.

Thirdly, for lines L in $\mathbb{E}^n$

$$\int_{L\cap\Sigma\neq\emptyset} \sharp(L \cap \Sigma) dL = \frac{O_n}{O_1} \mathrm{vol}_{n-1}(\Sigma). \tag{3}$$

In hyperbolic space $\mathbb{H}^n$ formulas (2) and (3) hold without change, but formula (1) is no longer valid. In fact, it is known (see [San76, p. 310]) that the total measure of hyperplanes (complete totally geodesic hypersurfaces) intersecting a convex domain is a linear combination of some curvature integrals and the volume of the domain. For instance, in $\mathbb{H}^3$ it is the difference of the mean curvature integral of the boundary and the volume of the domain, $M_1(\partial D) - \mathrm{vol}(D)$.

In some cases the natural analogue for hyperbolic space of the euclidean hyperplanes are the *horospheres* (limit spheres). Note for instance that the intrinsic geometry of horospheres is euclidean. In [San67] and [San68], Santaló proved that equality (1) holds in $\mathbb{H}^2$ and in $\mathbb{H}^3$, if one makes horospheres play the role of hyperplanes.

In this paper we obtain an expression for the measure of the set of horospheres intersecting an h-convex domain (convex with respect to horospheres) in $\mathbb{H}^n$. Indeed the main theorem is more general.

Theorem. *If D is a domain in $\mathbb{H}^n$ bounded by an embedded C^2 hypersurface Σ then*

$$\int_{D\cap H\neq\emptyset} \chi(D \cap H) dH = 2 \sum_{k=0}^{[(n-2)/2]} \binom{n-2}{2k} \frac{1}{2k+1} M_{n-2-2k}(\Sigma)$$

where χ is the Euler-Poincaré characteristic and dH the isometry invariant measure for horospheres.

Thus, the measure of horospheres intersecting an h-convex domain is a linear combination of the mean curvature integrals of its boundary. Therefore we see that, in the hyperbolic case, formula (1) is only valid for horospheres in dimensions 2 and 3.

In section 2 we review some general notions and fix the notation for the rest of the paper. Section 3 is devoted to find an invariant measure for horospheres in

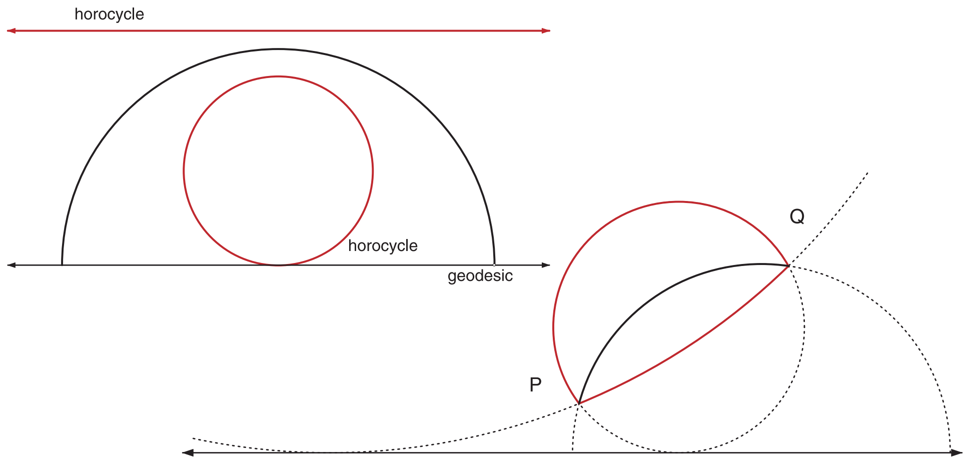

FIGURE 2

hyperbolic space. Next, we get an analogue of formula (2) for horospheres. Finally, in section 4 we prove the main theorem, give examples and discuss some related problems.

2. Preliminaries

Hyperbolic space $\mathbb{H}^n$ is the unique n-dimensional simply connected and complete Riemannian manifold of constant curvature -1. We do not consider any particular model but sometimes it will be useful to have in mind the *half-space model*, the upper half-space $\{x_n > 0\}$ of $\mathbb{R}^n$ with the metric $(dx_1^2 + \cdots + dx_n^2)/x_n^2$. In this model, geodesic lines are euclidean semicircles and straight lines orthogonal to the boundary.

In geodesic polar coordinates $\{\rho, \theta_1, \dots \theta_{n-1}\}$ the hyperbolic arc element has the form

$$ds^2 = d\rho^2 + \sinh^2 \rho (d\theta_1^2 + \cdots + d\theta_{n-1}^2)$$

and the volume element is

$$dV = \sinh^{n-1} \rho d\rho \wedge dO_{n-1}$$

where dO_{n-1} is the spherical volume element.

We consider two particular families of hypersurfaces in hyperbolic space. *Hyperplanes* are complete totally geodesic hypersurfaces. In the half-space model they are either euclidean half spheres with center in $\{x_n = 0\}$ or euclidean hyperplanes orthogonal to $\{x_n = 0\}$. *Horospheres* are limit spheres: given a geodesic $c(t)$ and a fixed point $c(t_0) = p$, consider the hyperbolic sphere with center $c(t)$ passing through p; when t tends to infinity we obtain a horosphere. They can also be defined to be hypersurfaces orthogonal to a family of parallel geodesics. In the half-space model, horospheres are either euclidean spheres tangent to the boundary or horizontal hyperplanes $\{x_n = c\}$. With the metric induced by the immersion in $\mathbb{H}^n$, hyperplanes are isometric to $\mathbb{H}^{n-1}$ and horospheres are isometric to $\mathbb{E}^{n-1}$.

We define the notion of convexity recursively: a set D in $\mathbb{H}^n$ is said to be *convex* if $L \cap D$ is a convex set in L for every hyperplane L; in dimension one, convex sets are connected segments. This definition of convexity is equivalent to the usual one: every geodesic segment joining two points in D is contained in D.

Similarly, a set D in $\mathbb{H}^n$ is *h-convex* or *horociclycally convex* if $D \cap H$ is a convex set in H for every horosphere H intersecting D. Since H is isometric to an euclidean space, convexity in H is the usual convexity in euclidean spaces. Note that h-convex sets are convex but the converse is not true.

We call a hypersurface convex (resp. h-convex) if it is the boundary of a convex (resp. h-convex) domain. Spheres of any radius are examples of h-convex hypersurfaces.

Let Σ be a compact hypersurface of class C^2 oriented by a unit normal $\mathbf{n}$. The second fundamental form h_Σ is defined by $h_\Sigma(X,Y) = \langle \nabla_X Y, \mathbf{n}\rangle$. It is a bilinear symmetric form and its eigenvalues $\kappa_1, \ldots, \kappa_{n-1}$ are the *principal curvatures.* The *mean curvature functions* $\sigma_i^\Sigma(x)$ are given by

$$\det(I + t\, h_\Sigma)(x) = \sum_{i=0}^{n-1} \binom{n-1}{i} \sigma_i^\Sigma(x) t^i.$$

Define the k-th *mean curvature integral* M_k to be

$$M_k(\Sigma) = \int_\Sigma \sigma_k^\Sigma(x) dx. \tag{4}$$

Notice that M_0 is the volume, and M_{n-1} is the integral of the Gauss curvature $K = \kappa_1 \ldots \kappa_{n-1}$.

We shall need the Gauss-Bonnet formula in n-dimensional euclidean space. Let Σ be a compact, orientable hypersurface of class C^2 in $\mathbb{E}^n$. When n is odd,

$$M_{n-1}(\Sigma) = \frac{1}{2} O_{n-1} \chi(\Sigma)$$

where χ is the Euler characteristic. Furthermore (see [Hop27]), in arbitrary dimension n, when Σ is the boundary of a domain D, we have

$$M_{n-1}(\partial D) = O_{n-1}\chi(D). \tag{5}$$

3. Measure for horospheres

In this section we find a measure for horospheres in $\mathbb{H}^n$ invariant under isometries. The isometry group of $\mathbb{H}^n$ admits a bi-invariant measure dK which is usually known as the *kinematic measure* of $\mathbb{H}^n$. From now on, only direct rigid motions are considered. Every rigid motion is determined by giving a point V and a rotation $\mathfrak{R}$ around this point. So, dK can be expressed as $dK = dV \wedge d\mathfrak{R}$. These and the subsequent differential forms must be taken up to the sign because we consider all measures to be positive.

The following result is proved in [San76, p. 323].

Theorem. *Let M^q be a fixed q-dimensional compact submanifold in $\mathbb{H}^n$ and let N^r be an r-dimensional compact submanifold moving with kinematic measure dK. Assume $r+q-n \geq 0$ and let $\mathrm{vol}_{r+q-n}(M\cap N)$ denote the $(r+q-n)$-dimensional volume of the intersection. Then we have*

$$\int_{M\cap N\neq\emptyset} \mathrm{vol}_{r+q-n}(M\cap N) dK = \frac{O_n O_{n-1} \ldots O_1 O_{r+q-n}}{O_q O_r} \mathrm{vol}_q(M)\mathrm{vol}_r(N) \tag{6}$$

where $\mathrm{vol}_q(M)$ and $\mathrm{vol}_r(N)$ are the volumes of M and N respectively, and O_k denotes the volume of the k-dimensional euclidean sphere.

When $r+q-n=0$, vol_0 is the number of intersection points $\sharp(M\cap N)$. For instance, when M is a curve Γ of length ℓ and N a compact hypersurface, formula (6) can be written as

$$\int_{\Gamma\cap N\neq\emptyset} \sharp(\Gamma\cap N) dK = \frac{O_n O_{n-1}\ldots O_1 O_0}{O_{n-1}O_1} \ell\, \mathrm{vol}_{n-1}(N).$$

Now, let us consider N to be a sphere S_R of radius R. The last formula becomes

$$\int_{\Gamma\cap S_R\neq\emptyset} \sharp(\Gamma\cap S_R)\, dK = \frac{O_n O_{n-1}\dots O_1 O_0}{O_{n-1}O_1}\ell\, O_{n-1}\sinh^{n-1}(R). \tag{7}$$

On the other hand, since $dK = dV\wedge d\mathfrak{R}$ and rotations around the center V of the sphere leave S_R invariant, we have

$$\begin{aligned}
&\int_{\Gamma\cap S_R\neq\emptyset} \sharp(\Gamma\cap S_R)\, dK = \\
&\quad = \mathrm{vol}(SO(n))\int_{\Gamma\cap S_R\neq\emptyset} \sharp(\Gamma\cap S_R)\, dV \\
&\quad = O_{n-1}\dots O_1\int_{\Gamma\cap S_R\neq\emptyset} \sharp(\Gamma\cap S_R)\, dV.
\end{aligned}$$

Comparing with (7) we can write

$$\int_{\Gamma\cap S_R\neq\emptyset} \sharp(\Gamma\cap S_R)\, dV = \frac{O_n O_0}{O_1}\ell\,\sinh^{n-1} R. \tag{8}$$

Spheres are determined by its center V. Then dV is an invariant measure for spheres of fixed radius in $\mathbb{H}^n$. However, we will consider the normalized measure $dS_R = dV/\sinh^{n-1} R$. This normalization is motivated by the formula

$$\int_{\Gamma\cap L_{n-1}\neq\emptyset} \sharp(\Gamma\cap L_{n-1})\, dL_{n-1} = \frac{O_n\, O_0}{O_1}\ell \tag{9}$$

where L_{n-1} are hyperplanes and dL_{n-1} is its usual invariant measure ([San76, p. 310]). The measure dS_R is such that formula (9) remains valid when we replace hyperplanes by spheres of fixed radius.

Now fix a point P in $\mathbb{H}^n$ and denote by ρ the signed distance from P to a sphere (the sign of ρ will be negative when P is interior to S_R). Every sphere S_R not centered at P is identified with a point in $\mathbb{R}\times S^{n-1}$. We can express dV as

$$dV = \sinh^{n-1}(\rho+R)d\rho\wedge dO_{n-1}$$

where dO_{n-1} denotes the spherical measure for directions from P to the center V. Thus the measure for spheres of fixed radius in $\mathbb{H}^n$ corresponds to the following volume element in $\mathbb{R}\times S^{n-1}$:

$$dS_R = \left(\frac{\sinh^{n-1}(\rho+R)}{\sinh^{n-1}(R)}\right) d\rho\wedge dO_{n-1}.$$

As R goes to infinity the sphere corresponding to an element in $\mathbb{R}\times S^{n-1}$ tends to a horosphere. Consequently, $\lim_{R\to\infty} dS_R$ corresponds to a measure in the space of horospheres $\mathcal{H}$. This measure is isometry-invariant and is written

$$dH = e^{(n-1)\rho}d\rho\wedge dO_{n-1}.$$

From (8) it follows that

$$\int_{\Gamma\cap H\neq\emptyset} \sharp(\Gamma\cap H)dH = \frac{O_n\, O_0}{O_1}\ell.$$

This formula was given in [San67] and [San68] for dimensions two an three respectively.

The same argument, applying formula (6) to a compact manifold M and a sphere of radius R, leads to a more general result.

Proposition 3.1. *Let M^q be a fixed q-dimensional compact manifold in $\mathbb{H}^n$ and H a moving horosphere, then*

$$\int_{H\cap M\neq\emptyset} \mathrm{vol}_{q-1}(H\cap M)dH = \frac{O_n O_{q-1}}{O_q}\mathrm{vol}_q(M)$$

with dH the invariant measure for horospheres.

4. Horospheres which intersect a given domain

Suppose D is a domain in $\mathbb{H}^n$ with boundary Σ of class C^2. Given a horosphere H, the intersection $\Sigma\cap H$ is denoted by C. When H and Σ are in general position, C is an embedded hypersurface both of H and Σ.

Horospheres have an intrinsic euclidean structure. Using Gauss-Bonnet formula (5) for $D\cap H$ as a domain of H we have

$$\chi(D\cap H) = \frac{1}{O_{n-2}}M_{n-2}(C).$$

Integrating $\chi(D\cap H)$ over the space $\mathcal{H}$ of horospheres we get

$$\begin{aligned}\int_{D\cap H\neq\emptyset}\chi(D\cap H)dH &= \frac{1}{O_{n-2}}\int_{D\cap H\neq\emptyset} M_{n-2}(C)dH \\ &= \frac{1}{O_{n-2}}\int_{\mathcal{H}}\int_C \sigma^C_{n-2}\, dx_{n-2}dH.\end{aligned} \tag{10}$$

Using the next proposition we shall be able to change the order of integration in the last integral.

Proposition 4.1. *Let Σ be a hypersurface in $\mathbb{H}^n$ and H a horosphere intersecting Σ in a $(n-2)$-dimensional submanifold C. Denote by dx_{n-1} and dx_{n-2} the volume elements of Σ and C respectively. Given a point x in C, let dO_{n-1} be the spherical volume element corresponding to a unit normal of H in x. The angle between Σ and H will be denoted by θ. Then,*

$$dx_{n-2}\wedge dH = \sin\theta\, dO_{n-1}\wedge dx_{n-1}.$$

Proof. Let us replace the horosphere H in the statement of the proposition by a sphere of radius R. Let $dT_C = dx_{n-2}\wedge d\mathfrak{R}_C$ where $d\mathfrak{R}_C$ is the volume element of orthogonal transformations in T_xC. One can define analogously $dT_\Sigma = dx_{n-1}\wedge d\mathfrak{R}_\Sigma$ for Σ and $dT_S = dx_S\wedge d\mathfrak{R}_S$ for S_R. An important formula on kinematic measure (see [San76, p. 262]) states that

$$dT_C\wedge dK = \sin^{n-1}\theta d\theta\wedge dT_\Sigma\wedge dT_S. \tag{11}$$

As before, $dK = dV\wedge d\mathfrak{R}$ where $d\mathfrak{R}$ measures rotations in $\mathbb{H}^n$ around the center V of the sphere S_R. Rotations in T_xS_R are naturally identified with rotations in $\mathbb{H}^n$ around V fixing the point x. Then $d\mathfrak{R} = d\mathfrak{R}_S\wedge dO_{n-1}$ being dO_{n-1} the measure for directions from V to x. With this equality, (11) becomes

$$dT_C\wedge dV\wedge d\mathfrak{R}_S\wedge dO_{n-1} = \sin^{n-1}\theta d\theta\wedge dT_\Sigma\wedge dx_S\wedge d\mathfrak{R}_S.$$

Integrating over rotations in T_xS_R we can cancel $d\mathfrak{R}_S$ in both sides. Expressing dx_S in polar coordinates centered at V we have

$$dT_C\wedge dV\wedge dO_{n-1} = \sin^{n-1}\theta\,\sinh^{n-1}R\, d\theta\wedge dT_\Sigma\wedge dO_{n-1}.$$

Integrating again along the fiber we cancel dO_{n-1} to obtain

$$dT_C\wedge dV = \sin^{n-1}\theta\,\sinh^{n-1}R\, d\theta\wedge dT_\Sigma.$$

Also we have

$$dx_{n-2}\wedge d\mathfrak{R}_C\wedge dV = \sin^{n-1}\theta\,\sinh^{n-1}R\, d\theta\wedge dx_{n-1}\wedge d\mathfrak{R}_C\wedge dO_{n-2}.$$

Here we have used that $d\mathfrak{R}_\Sigma = d\mathfrak{R}_C \wedge dO_{n-2}$ where dO_{n-2} corresponds to directions in Σ normal to C. As before, we integrate to cancel $d\mathfrak{R}_C$. The spherical volume element dO_{n-1} can be written in polar coordinates as $dO_{n-1} = \sin^{n-2}\theta d\theta \wedge dO_{n-2}$. Therefore

$$dx_{n-2} \wedge dV = \sin\theta \, \sinh^{n-1} R \, dO_{n-1} \wedge dx_{n-1}.$$

Normalizing and making R go to infinity we get

$$dx_{n-2} \wedge dH = \sin\theta \, dO_{n-1} \wedge dx_{n-1}$$

which is the desired formula. □

Now we can change the order of integration in (10):

$$\int_{\mathcal{H}} \int_C \sigma_{n-2}^C \, dx_{n-2} dH = \int_\Sigma \int_{S^{n-1}} \sigma_{n-2}^C \, \sin\theta \, dO_{n-1} dx_{n-2}. \tag{12}$$

We shall write σ_{n-2}^C in terms of the mean curvature functions σ_i^Σ of Σ.

Given a submanifold M of a riemannian manifold N, the *vectorial second fundamental* form B_M^N is the normal part of ∇^N, the covariant derivative in N. For X, Y tangent to M it verifies

$$\nabla_X^N Y = \nabla_X^M Y + B_M^N(X, Y). \tag{13}$$

If M is a hypersurface oriented by a unit normal $\mathbf{n}_M$, then we put $B_M^N(X,Y) = h_M^N(X,Y)\,\mathbf{n}_M$. The normal curvature in a direction given by a unit vector v is $k_M^N(v) = h_M^N(v,v)$. In the following, when the ambient space N is $\mathbb{H}^n$ it will be omitted in the notation.

Lemma 4.1. *Let v be a tangent vector in C. Then*

$$k_\Sigma(v) = \cos\theta \, k_H(v) + \sin\theta \, k_C^H(v)$$

where θ is the angle between Σ and H, $k_\Sigma(v)$ (resp. $k_H(v)$) is the normal curvature of Σ (resp. H) and $k_C^H(v)$ is the normal curvature of C as a hypersurface of H.

Proof. From (13), it is easily seen that $B_C^H = B_C - B_H$. Then

$$B_C^H(X,Y) = B_M(X,Y) - B_H(X,Y) + \nabla_X^M Y - \nabla_X^C Y. \tag{14}$$

Let $\mathbf{n}$ be the normal of C in H, then $\mathbf{n} \cdot \mathbf{n}_\Sigma = \sin\theta$ and $\mathbf{n}_H \cdot \mathbf{n}_\Sigma = \cos\theta$. We multiply both sides of (14) by $\mathbf{n}_\Sigma$ and the lemma follows. □

Horospheres are totally umbilical hypersurfaces with normal curvature equal to 1, so the matrix expression of h_H in a orthonormal basis is equal to the identity matrix I. Consequently, for tangent vectors in C we have the equality

$$h_C^H = \frac{h_\Sigma}{\sin\theta} - \frac{I}{\tan\theta}.$$

Now we relate σ_{n-2}^C, the $(n-2)$-symmetric curvature function of C (as a hypersurface of H), and the symmetric curvature functions of Σ restricted to the tangent space of C. We have that $\sigma_{n-2}^C = \det(h_C^H)$, then

$$\begin{aligned} \sigma_{n-2}^C &= \det\left(\frac{h_\Sigma}{\sin\theta} - \frac{I}{\tan\theta}\right) = \\ &= \sum_{i=0}^{n-2} \binom{n-2}{i} (-1)^{n-2-i} \frac{\cos^{n-2-i}\theta}{\sin^{n-2}\theta} (\sigma_i^\Sigma|_C). \end{aligned} \tag{15}$$

Here $\sigma_i^\Sigma|_C$ means the i-symmetric function of the restriction to C of the second fundamental form h_Σ. Using this expression we compute $\int_{S^{n-1}} \sigma_{n-2}^C \sin\theta dO_{n-1}$ in formula (12). The point x in Σ is fixed, then the direction $\mathbf{n}_\Sigma$ normal to Σ is also fixed. Elements of S^{n-1} give directions defining H. Hence, if we use polar coordinates in S^{n-1} centered at $\mathbf{n}_\Sigma$, by virtue of formula (15) we can write

$$\int_{S^{n-1}} \sigma_{n-2}^C \sin\theta dO_{n-1} = \int_{S^{n-2}} \int_0^\pi \sigma_{n-2}^C \sin\theta \sin^{n-2}\theta d\theta dO_{n-2}$$
$$= \int_{S^{n-2}} \int_0^\pi \left(\sum_{i=0}^{n-2} \binom{n-2}{i} (-1)^{n-2-i} \frac{\cos^{n-2-i}\theta}{\sin^{n-2}\theta} (\sigma_i^\Sigma|_C) \right) \sin\theta \sin^{n-2}\theta d\theta dO_{n-2}.$$

For a fixed point in S^{n-2}, when θ varies, C can change but not its tangent space. Then the value of $\sigma_i^\Sigma|_C$ does not depend on θ. Therefore

$$\int_{S^{n-1}} \sigma_{n-2}^C \sin\theta dO_{n-1} =$$
$$= \sum_{i=0}^{n-2} (-1)^{n-2-i} \binom{n-2}{i} \int_{S^{n-2}} (\sigma_i^\Sigma|_C) \left(\int_0^\pi \cos^{n-2-i}\theta \sin\theta d\theta \right) dO_{n-2}$$
$$= \sum_{i=0}^{n-2} (-1)^{n-2-i} \binom{n-2}{i} \frac{2\epsilon(n-1-i)}{n-1-i} \int_{S^{n-2}} (\sigma_i^\Sigma|_C) dO_{n-2}$$

where $\epsilon(n-1-i)$ equals 0 if $n-1-i$ is even and 1 if it is odd. From [Lan80] we have that

$$\int_{\mathbb{R}P^{n-2}} (\sigma_i^\Sigma|_C) dO_{n-2} = \mathrm{vol}(\mathbb{R}P^{n-2})\, \sigma_i^\Sigma.$$

This is a generalization of the well known formula $\int_0^\pi k_n(\theta)d\theta = \pi\sigma$ for the mean curvature σ of a surface. Using this relation we have that

$$\int_{S^{n-1}} \sigma_{n-2}^C \sin\theta d\theta dO_{n-1} =$$
$$= 2 \sum_{i=0}^{n-2} (-1)^{n-2-i} \binom{n-2}{i} \frac{\epsilon(n-1-i)}{n-1-i} O_{n-2}\, \sigma_i^\Sigma.$$

Consequently,

$$\int_{D\cap H\neq\emptyset} \chi(D\cap H) dH = \frac{1}{O_{n-2}} \int_{\mathcal{H}} \int_C \sigma_{n-2}^C \, dx_{n-2} dH$$
$$= 2 \sum_{i=0}^{n-2} (-1)^{n-2-i} \binom{n-2}{i} \frac{\epsilon(n-1-i)}{n-1-i} \int_\Sigma \sigma_i^\Sigma dx_{n-2}.$$

Reordering indices we have proved our main theorem.

Theorem. *If D is a domain in $\mathbb{H}^n$ bounded by an embedded hypersurface Σ then*

$$\int_{D\cap H\neq\emptyset} \chi(D\cap H) dH = 2 \sum_{h=0}^{[(n-2)/2]} \binom{n-2}{2h} \frac{1}{2h+1} M_{n-2-2h}(\Sigma). \tag{16}$$

Remark. According to the parity of n we can write different formulas.

For $n = 2m + 1$,

$$\int_{D\cap H\neq\emptyset} \chi(D\cap H)dH = \frac{2}{n-1}\sum_{k \text{ odd}} \binom{2m}{k} M_k(\Sigma).$$

For $n = 2m$,

$$\int_{D\cap H\neq\emptyset} \chi(D\cap H)dH = \frac{2}{n-1}\sum_{k>2 \text{ even}} \binom{2m}{k+1} M_k(\Sigma).$$

When the domain D is h-convex we have $\chi(D\cap H) = 1$, therefore we can find the measure of horospheres intersecting an h-convex set.

Corollary 4.1. *The total measure of horospheres intersecting an h-convex hypersurface Σ can be expressed as a linear combination of its mean curvature integrals.*

For $\mathbb{H}^2$ and $\mathbb{H}^3$, formula (16) becomes

$$\int_{D\cap H\neq\emptyset} \chi(D\cap H)dH = 2M_0(\Sigma) = 2\,\ell(\Sigma)$$

and

$$\int_{D\cap H\neq\emptyset} \chi(D\cap H)dH = 2M_1(\Sigma)$$

respectively. These formulas were given in [San67] and [San68].

Remark. For a convex domain D in $\mathbb{H}^n$, the expression of the measure of hyperplanes intersecting D contains the volume of D ([San76, p. 310]). For instance, in $\mathbb{H}^3$ this measure equals $M_1(\partial D) - \mathrm{vol}(D)$. For an h-convex domain D, we have shown that the expression of the measure of horospheres intersecting D involves only curvature integrals of its boundary ∂D. This is also the case of hyperplanes in euclidean space.

In order to compare the measures of horospheres and hyperplanes, consider a sphere S_R of radius R going to infinity. We have

$$m(L : L\cap S_R \neq \emptyset) = O_{n-1}\int_0^R \cosh^{n-1} r dr$$

and

$$m(H : H\cap S_R \neq \emptyset) \approx \frac{2^{n-2}}{n-1}\mathrm{vol}(S_R).$$

When R tends to infinity $m(H : H\cap S_R \neq \emptyset)/m(L : L\cap S_R \neq \emptyset)$ tends to 2^{n-2}, so these measures are of the same order.

Finally, let us give some possible further developments. Hyperplanes are hypersurfaces with vanishing normal curvature in every direction and horospheres have normal curvature equal to 1. Hypersurfaces with constant normal curvature equal to some λ between 0 and 1 are called *equidistants*. It seems an interesting problem to find the measure of equidistants of a given curvature λ that intersect a domain.

References

[Hop27] Heinz Hopf, *Vektorfelder in n-dimensionalen mannigfaltigkeiten*, Math. Annalen **96** (1927), 225–250.

[Lan80] Rémi Langevin, *Courbures, feuilletages et surfaces*, Université de Paris-Sud Département de Mathématique, Orsay, 1980, Mesures et distributions de Gauss. [Gaussian measures and distributions], Dissertation, Université Paris-Sud, Orsay, 1980, With an English summary.

[San67] L. A. Santaló, *Horocycles and convex sets in hyperbolic plane*, Arch. Math. (Basel) **18** (1967), 529–533.

[San68] L. A. Santaló, *Horospheres and convex bodies in hyperbolic space*, Proc. Amer. Math. Soc. **19** (1968), 390–395.

[San76] Luis A. Santaló, *Integral geometry and geometric probability*, Addison-Wesley Publishing Co., Reading, Mass.-London-Amsterdam, 1976, With a foreword by Mark Kac, Encyclopedia of Mathematics and its Applications, Vol. 1.

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