RELATION BETWEEN AREA AND VOLUME FOR λ–CONVEX SETS IN HADAMARD MANIFOLDS

A.A. BORISENKO, E. GALLEGO, AND A. REVENTÓS

Abstract. It is known that for a sequence $\{\Omega_t\}$ of convex sets expanding over the whole hyperbolic space $\mathbb{H}^{n+1}$ the limit of the quotient $\mathrm{vol}(\Omega_t)/\mathrm{vol}(\partial\Omega_t)$ is less or equal than $1/n$, and exactly $1/n$ when the sets considered are convex with respect to horocycles. When convexity is with respect to equidistant lines, i.e. curves with constant geodesic curvature λ less than one, the above limit has λ/n as lower bound. Looking how the boundary bends, in this paper we give bounds of the above quotient for a compact λ-convex domain in a complete simply-connected manifold of negative and bounded sectional curvature, a Hadamard manifold. Then we see that the limit of $\mathrm{vol}(\Omega_t)/\mathrm{vol}(\partial\Omega_t)$ for sequences of λ-convex domains expanding over the whole space lies between the values λ/nk_2^2 and $1/nk_1$.

1. Introduction

When we consider a circumference passing through a point in the hyperbolic space $\mathbb{H}^{n+1}$ and make the center of it to go to infinity, the resulting curve is called an *horocycle*. This curve is characterized by having geodesic curvature equal ± 1. Given two points in $\mathbb{H}^{n+1}$ there is a family of horocycles joining them. We say that a set is *h-convex* if for every couple of points in it, every horocycle joining them is completely contained in the set.

In 1972 Santaló and Yañez ([SYn72]) proved the following result. Let $\{\Omega(t)\}_{t\in\mathbb{R}}$ be a family of compact h-convex domains in $\mathbb{H}^2$ expanding over the whole space. Then

$$\lim_{t\to\infty} \frac{\mathrm{area}(\Omega(t))}{\mathrm{length}(\partial\Omega(t))} = 1. \tag{1}$$

For $\mathbb{H}^{n+1}$ it was proven in [BM99] the generalization of this result. Let $\{\Omega(t)\}_{t\in\mathbb{R}}$ be a family of compact h-convex domains expanding over the

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1991 *Mathematics Subject Classification.* Primary 52A55; Secondary 52A10.

Key words and phrases. Hyperbolic space, Hadamard manifold, normal curvature, volume, λ-geodesic, horocycle, λ-convex set.

Work partially supported by DGYCIT grant number PB96 – 1178.

whole space, then

$$\lim_{t\to\infty} \frac{\mathrm{vol}(\Omega(t))}{\mathrm{vol}(\partial\Omega(t))} = \frac{1}{n}.$$

On the other hand, the following linear isoperimetric inequality holds for a domain Ω in a complete simply-connected manifold with negative least upper bound K of the sectional curvatures (cf. [Yau75])

$$n\sqrt{-K}\mathrm{vol}(\Omega) \leq \mathrm{vol}(\partial\Omega).$$

This give us an upper bound for the quotient of volumes, $\mathrm{vol}(\Omega)/\mathrm{vol}(\partial\Omega) \leq 1/n\sqrt{-K}$.

An *h-convex* domain in a simply connected riemannian space M of non-positive curvature is a domain $\Omega \subset M$ with boundary $\partial\Omega$ such that, for every $p \in \partial\Omega$, there is a horosphere $\mathcal{H}$ of M through p such that Ω is locally contained in the horoball of M bounded by $\mathcal{H}$. When M is a Lobachevsky space, then this definition is equivalent to the above definition.

For simply-connected riemannian manifolds with sectional curvature satisfying $-k_2^2 \leq K \leq -k_1^2$ it was proved in [BV99] that

$$\frac{1}{nk_2} \leq \liminf_{t\to\infty} \frac{\mathrm{vol}(\Omega(t))}{\mathrm{vol}(\partial\Omega(t))} \leq \limsup_{t\to\infty} \frac{\mathrm{vol}(\Omega(t))}{\mathrm{vol}(\partial\Omega(t))} \leq \frac{1}{nk_1} \tag{2}$$

where $\Omega(t)$ are h-convex bodies expanding over the whole space.

In [GR85] it was shown that equation (1) is not true for general convex sets. This limit can take, in the hyperbolic plane, any value between 0 and 1. Since horocycles are curves of geodesic curvature ± 1 and geodesics are curves of geodesic curvature 0, they can be considered as particular cases of curves of constant geodesic curvature λ, $0 \leq |\lambda| \leq 1$.

Thus if convexity is defined with respect to horocycles this limit is 1 and when convexity is defined with respect to geodesics the limit of the quotient $\mathrm{area}(\Omega(t))/\mathrm{length}(\Omega(t))$ is less or equal than 1. In [BM99] it was introduced the notion of λ-convexity and the question of the influence of λ in this limit was posed. When convexity is defined with respect λ-geodesic curves it was proved in [GR99] that for each $\alpha \in [\lambda, 1]$, there exists a sequence of λ-convex polygons $\{K_n\}$ expanding over the whole hyperbolic plane such that

$$\lim_{t\to\infty} \frac{\mathrm{area}(\Omega(t))}{\mathrm{length}(\Omega(t))} = \alpha.$$

and if the sequence is formed by λ-convex sets with piecewise C^2 boundary, then the lim sup and lim inf of these ratios lie between λ and 1. For Lobachevsky space $\mathbb{H}^{n+1}$ it was proved in [BV99] that

$$\frac{\lambda}{n} \leq \liminf_{t\to\infty} \frac{\mathrm{vol}(\Omega(t))}{\mathrm{vol}(\partial\Omega(t))} \leq \limsup_{t\to\infty} \frac{\mathrm{vol}(\Omega(t))}{\mathrm{vol}(\partial\Omega(t))} \leq \frac{1}{n}.$$

for a family $\{\Omega(t)\}_{t\in\mathbb{R}^+}$ of λ-convex domains expanding over the whole space.

It is possible to generalize in a natural way the notion of λ-convexity for riemannian manifolds. A domain Ω with regular boundary is λ-convex when all the normal curvatures are bounded below by λ (see section 2 for a precise definition). The main result of this work is

Theorem 2. *Let M be a $(n+1)$-dimensional Hadamard manifold with sectional curvature K such that*

$$-k_2^2 \leq K \leq -k_1^2 \qquad k_1, k_2 > 0.$$

Let Ω be a compact λ-convex domain in M with $\lambda \leq k_2$. Then there are functions $\alpha(r)$ of the inradius and $\beta(R)$ of the circumradius such that $\alpha(r) \to 1/(nk_2)$ and $\beta(R) \to 1/(nk_1)$ when r and R grow to infinity and that

$$\alpha(r)\frac{\lambda}{k_2} \leq \frac{\mathrm{vol}(\Omega)}{\mathrm{vol}(\partial\Omega)} \leq \beta(R).$$

As a consequence we see that

Theorem 3. *If M is a $(n+1)$-dimensional Hadamard manifold with sectional curvature K such that $-k_2^2 \leq K \leq -k_1^2$ with $k_1, k_2 > 0$*

$$\frac{\lambda}{nk_2^2} \leq \liminf_{t\to\infty} \frac{\mathrm{vol}(\Omega(t))}{\mathrm{vol}(\partial\Omega(t))} \leq \limsup_{t\to\infty} \frac{\mathrm{vol}(\Omega(t))}{\mathrm{vol}(\partial\Omega(t))} \leq \frac{1}{nk_1}.$$

for a family $\{\Omega(t)\}_{t\in\mathbb{R}^+}$ of compact λ-convex domains with $\lambda \leq k_2$ expanding over the whole space.

The case $\lambda = k_2$ corresponds to a sequence of h-convex sets.

The main tool for proving these results will be an estimation of the angle between the radial direction from an interior point of Ω and the normal of $\partial\Omega$. This will we proved in section 4. We also prove an interesting formula relating the variation of this angle and the normal curvature in a direction of the boundary.

Acknowledgements. This work was done while the first author was invited professor at the Universitat Autònoma de Barcelona in the first trimester of 2000 with an UAB-CIRIT grant and the financial support of the CRM. He wants to thank the University and the CRM by the facilities they gave him. We would like to thank specially G. Solanes for many helpful conversations during the preparation of this work.

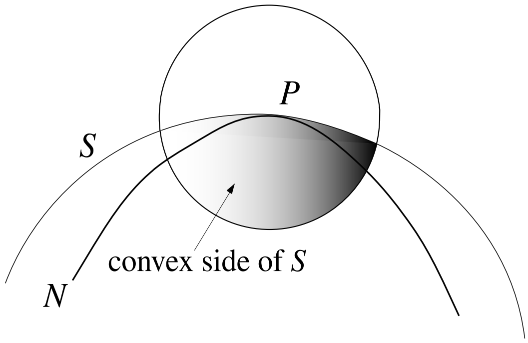

FIGURE 1

2. Definitions and preliminary results

Definition 2.1. A *Hadamard manifold* is a simply-connected complete Riemannian manifold of non-positive sectional curvature.

In this paper we shall deal with $(n+1)$-dimensional pinched Hadamard manifolds, this means the sectional curvature K satisfies the relation $-k_2^2 \leq K \leq -k_1^2$ with $0 < k_1 \leq k_2$.

Definition 2.2. A C^2 hypersurface $N \subset M$ such that in every point all the normal curvatures are greater or equal than a non-negative λ is said a *regular λ-convex hypersurface*. When N is the boundary of a domain Ω it is said that Ω is a *regular λ-convex domain* when its normal curvature with respect to the inward normal direction is greater than λ.

This definition can be generalized to the non-regular case

Definition 2.3. A *λ-convex hypersurface* is an hypersurface $N \subset M$ such that for every point P there is a regular λ-convex hypersurface S leaving a neighborhood of P in N in the convex side of S. A domain Ω of M is *λ-convex* if its boundary is a λ-convex hypersurface (see figure 1).

Remark. It can be seen that a 0-convex hypersurface is an ordinary locally convex hypersurface and a 0-convex domain is an ordinary convex domain. Also note that λ-convex implies 0-convex.

We shall need the fact, proved for instance in [Pet98], that if (M, g) is a Hadamard manifold with sectional curvature K satisfying $-k_2^2 \leq K \leq -k_1^2$ then the normal curvature k_n in any direction of a geodesic sphere of radius r satisfies

$$k_1 \coth(k_1 r) \leq k_n \leq k_2 \coth(k_2 r). \tag{3}$$

Note that he value $k\coth(kr)$ is the geodesic curvature of a circumference of radius r in Lobachevsky plane of curvature $-k^2$.

Remark. Since $k_1 \leq k_1\coth(k_1 r) \leq k_n$ we deduce that for every $\lambda \leq k_1$, geodesic spheres are λ-convex hypersurfaces.

Notice also that, if Ω is a λ-convex set with $\lambda > k_2$ then every inscribed ball $B(r)$ must satisfy that $r \leq \frac{1}{k_2}\operatorname{arctanh}\left(\frac{k_2}{\lambda}\right)$. Indeed there are points in $\partial\Omega$ such that the normal curvature is less or equal than the curvature of $\partial B(r)$, therefore $\lambda \leq k_2\coth(k_2 r)$ and the inequality for r follows. We conclude that λ-convex sets of any radius exists only if $\lambda \leq k_2$.

Definition 2.4. An *horosphere* in a Hadamard manifold is the limit of a geodesic sphere as the radius tends to infinity

Given a point P and a complete geodesic ray γ starting on P, the limit of the sequence of geodesic spheres centered in $\gamma(t)$ an passing by P when t tends to infinity is an horosphere. Using (3) we see that horospheres have normal curvature between k_1 and k_2 when the sectional curvature K of ambient space satisfies $-k_2^2 \leq K \leq -k_1^2$.

Definition 2.5. A locally convex hypersurface N of a Hadamard manifold is said *h-convex* if every point has a locally supporting horosphere.

Remark. This means that for every x in N there is an horosphere H such that x belongs to H and N is locally contained in the convex side defined by H. A convex domain Ω is *h-convex* if its boundary is an h-convex hypersurface. Note also that every λ-convex domain with $\lambda \geq k_2$ is h-convex.

3. Normal curvature on riemannian manifolds

In this section we want to find an estimation of the normal curvature in a point P of N, an hypersurface of a riemannian manifold M. Consider N defined by the equation $t = \rho(\theta)$ of class C^2, the distance to a point O. N can be seen as the 0-level set of the function $F = t - \rho$. Remember that for a function f in M the gradient, $\mathrm{grad} f$, is the unique vector field in M such that $< \mathrm{grad} f, v >= df(v) = v(f)$. ∇ will denote always covariant derivative in M.

With respect to the point O we consider polar coordinates $(t, \theta^1, \ldots, \theta^n)$. The arc element is given by $ds^2 = dt^2 + g_{ij}(t,\theta)d\theta^i d\theta^j$. If we write $\boldsymbol{n} = \mathrm{grad}F/\|\mathrm{grad}F\|$ for the normal unit vector to N and φ for the angle between the radial direction and the unit normal we have that $\cos\varphi =< \boldsymbol{n}, \partial/\partial_t >$. Then $1/\|\mathrm{grad}F\| = \cos\varphi$. Let $f = t$ as a function on M. If $Z \in T_pN$ then $Z(f) =< \partial/\partial_t, Z >$. It follows that $\mathrm{grad}_N\rho$ is the orthogonal projection

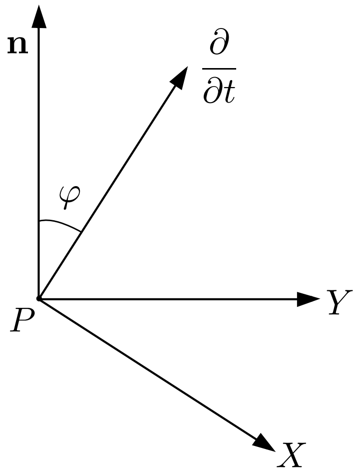

FIGURE 2

of ∂/∂_t onto N and the vectors $\boldsymbol{n}$, ∂/∂_t and $Y = \frac{\text{grad}_N \rho}{\|\text{grad}_N \rho\|}$ belong to a 2-dimensional plane (see figure 2). Let denote by X the unit vector in this plane and orthogonal to ∂/∂_t. The normal curvature at $P \in N$ in the direction given by Y is

$$k_n = < \nabla_Y Y, \boldsymbol{n} > .$$

Next proposition was announced by A.A. Borisenko who gave a first version of its proof.

Proposition 3.1. *If μ_n is the normal curvature in the direction of X of the sphere centered in O with radius ρ and $\frac{d\varphi}{ds}$ the derivative of φ with respect the arc parameter of the integral curve of Y by P, then*

$$k_n = \mu_n \cos\varphi + \frac{d\varphi}{ds} \tag{4}$$

Remark. This is a kind of Liouville formula. It must be noticed that when this formula is applied to the boundary of a convex domain containing the point O, k_n and μ_n are both negative.

Proof. We have that

$$\left.\begin{array}{rcl} \boldsymbol{n} & = & \cos\varphi \cdot \partial/\partial_t - \sin\varphi \cdot X \\ Y & = & \cos\varphi \cdot X + \sin\varphi \cdot \partial/\partial_t \end{array}\right\} .$$

Hence

$$k_n = \sin\varphi \; < \nabla_{\partial/\partial_t} Y, \boldsymbol{n} > + \cos\varphi \; < \nabla_X Y, \boldsymbol{n} > .$$

A straightforward calculation shows that the first term vanishes. Let us decompose the second term.

$$< \nabla_X Y, \boldsymbol{n} >= \cos\varphi \ < \nabla_X \cos\varphi \ X, \partial/\partial_t > - \sin\varphi \ < \nabla_X \cos\varphi \ X, X > + \\ \cos\varphi \ < \nabla_X \sin\varphi \ \partial/\partial_t, \partial/\partial_t > - \sin\varphi \ < \nabla_X \sin\varphi \ \partial/\partial_t, X > .$$

But

$$< \nabla_X \cos\varphi \ X, \partial/\partial_t >= \cos\varphi \ < \nabla_X X, \partial/\partial_t >= \mu_n \cos\varphi$$

with μ_n the normal curvature in the direction X of the n-dimensional sphere centered in O with radius ρ.

$$< \nabla_X \cos\varphi \ X, X >= -X(\varphi)\sin\varphi \, ,$$

$$< \nabla_X \sin\varphi \ \partial/\partial_t, \partial/\partial_t >= X(\varphi)\cos\varphi \, ,$$

and

$$< \nabla_X \sin\varphi \ \partial/\partial_t, X >= -\mu_n \sin\varphi \, .$$

Therefore we obtain

(5) $$k_n = \mu_n \cos\varphi \ + X(\varphi)\cos\varphi \, .$$

Using that $X = Y/\cos\varphi \ + (\tan\varphi)\,\partial/\partial_t$ we obtain

(6) $$k_n = \mu_n \cos\varphi \ + Y(\varphi).$$

But differentiation in direction Y of φ is the derivative with respect the arc parameter of the integral curve of Y by P. This finishes the proof. □

4. Lower bound for $\cos\varphi \ =< \boldsymbol{n}, \partial/\partial_t >$

In this section we shall study the angle φ between the radial direction and the normal direction to the hypersurface. We divide the proof in the regular and the non-regular case.

4.1. **Regular case.** We shall prove the following

Theorem 1. *Let M be a $(n+1)$-dimensional Hadamard manifold with sectional curvature K such that $-k_2^2 \leq K \leq -k_1^2$ with $k_1, k_2 > 0$. Let Ω be a λ-convex domain with C^2 boundary N, $\lambda < k_2$ and O an interior point of Ω. If φ denote the angle of the normal to N an the exterior radial direction, when $d(O, N) \leq \frac{1}{k_2}\operatorname{arctanh}(\frac{\lambda}{k_2})$ we have*

$$\cos\varphi \geq \frac{1}{k_2}\sqrt{\lambda^2\cosh^2 k_2 s - k_2^2 \sinh^2 k_2 s}.$$

If $d(O, \partial N) \geq \frac{1}{k_2}\operatorname{arctanh}(\frac{\lambda}{k_2})$ we have

$$\cos\varphi \geq \frac{\lambda}{k_2}.$$

We start studying what happens in the hyperbolic space.

Lemma 4.1 ([BV99]). *Let γ be a λ-geodesic line in the Lobachevsky plane of constant curvature $-k^2$. Let O be a point in the convex side of γ. Let r be the distance between γ and O. For each point in γ we define β as the angle between the radial field from O and the outwards normal field of γ. If*

$$r < d := \frac{1}{k}\operatorname{arctanh}\frac{\lambda}{k} \qquad \left(= \log\sqrt{\frac{k+\lambda}{k-\lambda}}\right)$$

then

$$\cos\beta \geq \frac{2\sqrt{\rho(\lambda-k\rho)(k-\lambda\rho)}}{k(1-\rho^2)} \tag{7}$$

where $\rho = \tanh\frac{1}{2}kr$. Alternatively, if $r \geq d$ then

$$\cos\beta \geq \frac{\lambda}{k}. \tag{8}$$

Remark. The estimate (7) can be given in the following equivalent form

$$\cos\beta \geq \frac{1}{k}\sqrt{\lambda^2\cosh^2 ks - k^2\sinh^2 ks} \tag{9}$$

where $s = d - r$.

We shall see now in a synthetic way a new proof of those expressions. Assume that we are in the conformal Poincaré disk model and that O is the origin. We can also suppose that γ is the intersection with the disk of a circle C centered at $Q = (0, q)$ with $q < 0$. Now, at any point $P \in \gamma$, β is the angle $\widehat{QPO}$. Consider the curves defined as the locus of the point from which OQ is in a given angle. It is known that these level curves are arcs of circles joining O and Q. Two of such arcs are tangent to C. Thus, the maximum of $\widehat{QPO}$ for $P \in C$ is attained when P is one of these tangency points. That is, when $\widehat{POQ} = \pi/2$.

Now, by definition γ is the equidistant curve at distance d to some geodesic σ. If $r < d$ then O is in the region bounded by γ and σ. So, γ meets the boundary of the model at points with negative second coordinate. Thus, the points $P \in C$ where $\widehat{QPO}$ is maximum are in γ. Then, the maximum of β is also attained in P. If O' and P' are the points in σ at minimum distance, respectively, from O and P, then $O'OPP'$ is a quadrilateral with three right angles and an acute angle equal to β. Using a hyperbolic trigonometric formula for quadrilaterals (cf. [Rat94]),

$$\sin\beta = \frac{\cosh k\overline{OO'}}{\cosh k\overline{PP'}}.$$

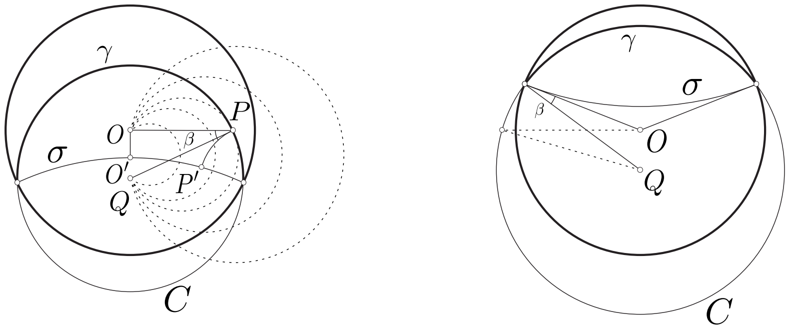

FIGURE 3

From this we obtain easily the expression (9). A straightforward computation shows that it is equivalent to (7).

In the case that $r \geq d$, the points $P \in C$ with the greatest angle $\widehat{QPO}$ are outside the disk. Then, at every point of γ, β is less than the angle between the λ-geodesic and the boundary of the disk and this angle has cosine λ/k. □

Proof of theorem 1. Let γ be an integral curve of the field $Y = \text{grad}_N \rho$ through a point P of the boundary. Following γ in the direction that ρ decreases we arrive at a point Q (maybe at infinite time of the parameter). In this point $Y = 0$, hence $\varphi = 0$. Let $d(O,Q) = d\,(\geq d(O,N))$. If $d' = d(O,P)$ we can parametrize the segment of γ between P an Q with the distance $t \in (d, d']$ of O to the corresponding point in the segment. If s is the arc parameter we have by lemma 3.1

$$k_n(\gamma(t)) = \cos\varphi(\gamma(t))\mu_n(\gamma(t)) + \frac{d\varphi}{dt}\frac{dt}{ds}$$

but

$$\frac{dt}{ds} = \frac{Y}{\|Y\|}(\rho) = \frac{<\text{grad}_N\rho, \text{grad}_N\rho>}{\|\text{grad}_N\rho\|} = \sin\varphi.$$

As N is λ-convex and using the comparison formula (3) we have

$$-\lambda \geq -k_2 \coth(k_2 \cdot t)\cos\varphi + \sin\varphi\frac{d\varphi}{dt}. \tag{10}$$

Now consider in $\mathbb{H}^2(-k_2^2)$ an arbitrary λ-geodesic line $\overline{\gamma}$ and a point $\overline{Q}$ in it. Consider an orthogonal geodesic from $\overline{Q}$ to a point $\overline{O}$ at distance d from $\overline{Q}$. In $\overline{\gamma}$ consider a point $\overline{P}$ at distance $d' = d(O,P)$ from $\overline{O}$. We have the same situation as before, but now in the hyperbolic plane of constant curvature

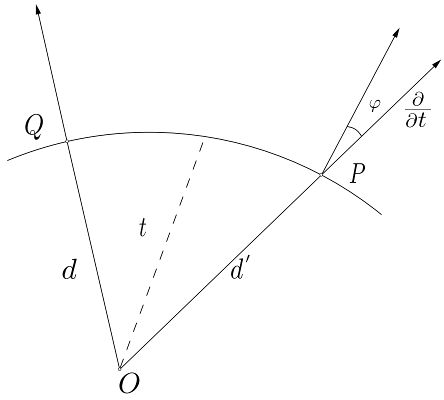

FIGURE 4

$-k_2^2$. If β is the angle between the normal to $\overline{\gamma}$ in the direction of the ray vector from $\overline{O}$ and this ray vector, we have the exact formula

$$-\lambda = -k_2 \coth(k_2 \cdot t) \cos\beta + \sin\beta \frac{d\beta}{dt}, \tag{11}$$

where t is again the distance from $\overline{O}$ to the corresponding point in $\overline{\gamma}$ (see figure 4).

Suppose that $\gamma(t) > \beta(t)$. As $\gamma(d) = \beta(d) = 0$ we must have $\gamma' > \beta'$ at some point. From equations (10) and (11) we deduce

$$\begin{aligned} &- k_2 \coth(k_2 \cdot t) \cos\beta + \sin\beta \frac{d\beta}{dt} \geq \\ &\qquad - k_2 \coth(k_2 \cdot t) \cos\varphi + \sin\varphi \frac{d\varphi}{dt} > \\ &\qquad\qquad - k_2 \coth(k_2 \cdot t) \cos\beta + \sin\beta \frac{d\beta}{dt} \end{aligned}$$

which is a contradiction. Therefore we must have $\varphi \leq \beta$ hence $\cos\varphi(t) \geq \cos\beta(t)$ and the bound follows. □

It is possible to prove in an easier way a less strong result

Proposition 4.1. *Let M be a Hadamard manifold with sectional curvature $-k_2^2 \leq K \leq -k_1^2$. Suppose Ω be a C^2 λ-convex set with $\lambda < k_2$ and $\partial\Omega$ a connected boundary component. Let O a point in the interior of Ω. Then the angle φ between geodesic rays from O and the unit normal to $\partial\Omega$ satisfies the inequality*

$$\cos\varphi \geq \frac{\lambda}{k_2} \tanh(k_2\, r)$$

where r is the minimum distance from O to $\partial\Omega$.

Proof. Note that the field $\mathrm{grad}_N \rho$ is zero if and only if $\cos\varphi = 1$ and in this case $\partial/\partial t = \mathrm{grad} F$.

The angle φ takes its value in the interval $[0, \pi/2]$ then there is a supremum φ_0 of it. Consider any integral curve γ of $Y/\|Y\|$. If at some point $\gamma(s_0)$ the value φ_0 is achieved we have in this point that $\varphi' = 0$ and so

$$\cos\varphi = \frac{k_n}{\mu_n}$$

concluding that

$$\cos\varphi \geq \frac{\lambda}{k_2 \coth(k_2 \rho_o)}. \tag{12}$$

If the maximum value is not achieved we have two different possibilities, there exists a value s_0 such that $\varphi(\gamma(s))$ increases when $s > s_0$ in this case $\varphi' > 0$ and then $(-k_n)\cos\varphi \geq -\mu_n$, it follows (12) again. The other case is that $\varphi(\gamma(s))$ goes to φ_0 in a non monotone way, in this case there is a increasing sequence s_n such that $\varphi'(\gamma(s_n)) = 0$ and $\varphi(\gamma(s_n)) \to \varphi_0$. Again we obtain (12).

□

4.2. **Non regular case.** Now we shall consider a general λ-convex domain Ω. Let N_ϵ be the outer parallel set at distance ϵ to $N = \partial\Omega$. Then it is a general fact that N_ϵ is of class of regularity $C^{1,1}$. When N is λ-convex, N_ϵ is λ_ϵ-convex with $\lambda_\epsilon \geq \lambda - C\epsilon$. It is true also that

$$\lim_{\epsilon\to 0} N_\epsilon = N, \qquad \lim_{\epsilon\to 0} \varphi_\epsilon = \varphi.$$

Here φ corresponds to the angle of the normal of the limit supporting tangent plane with the radial direction $\partial/\partial t$ (see figure 5).

If we found a bound for φ_ϵ then we will obtain an evaluation for φ. Now we consider the gradient of the distance function for N_ϵ, this field has integral curves of class of regularity $C^{1,1}$. In fact in almost all points the class is C^2. Therefore the function $\varphi_\epsilon(t)$ giving the angle is C^1 in those points. Applying proposition 3.1 to φ_ϵ and using that

$$\varphi(s) = \varphi(s_0) + \int_{s_0}^{s} \frac{d\varphi}{ds} dt \tag{13}$$

we obtain that the same evaluation for $\cos\varphi$ as in the regular case is valid now. Taking limits with respect to ϵ we obtain the proof of theorem 1 for the general case.

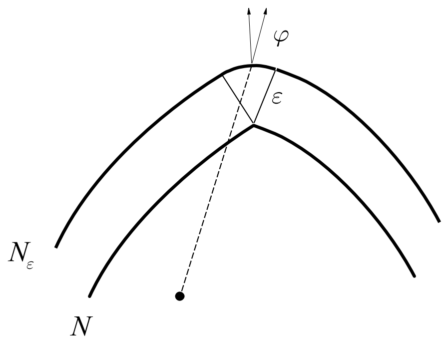

FIGURE 5

5. Estimates for the ratio of volumes

First of all we state the following lemma (see for instance [BZ94]).

Lemma 5.1. *Suppose that on the geodesic line $\gamma : [0, s] \to M$ of a manifold M there are no conjugate points to $\gamma(0)$ and at every point of γ all the sectional curvatures K_σ are bounded by*

$$k_2 \leq K_\sigma \leq k_1.$$

Then, for $t < s$

$$\frac{J_{k_2}(t)}{J_{k_2}(s)} \leq \frac{J(t)}{J(s)} \leq \frac{J_{k_1}(t)}{J_{k_1}(s)}$$

where $J(t)$ and $J_k(t)$ denote the jacobians at the points corresponding to $\gamma(t)$ by the exponential maps of M and of the space with constant curvature k, respectively.

Theorem 2. *Let M be a $(n+1)$-dimensional Hadamard manifold with sectional curvature K such that*

$$-k_2^2 \leq K \leq -k_1^2 \qquad k_1, k_2 > 0.$$

Let Ω be a compact λ-convex domain in M. Then if $\lambda < k_2$

$$f(r) \cdot C(r)\frac{\lambda}{k_2} \leq \frac{\mathrm{vol}(\Omega)}{\mathrm{vol}(\partial\Omega)} \leq h(R)$$

where r is the inradius of Ω, R is the circumradius,

$$f(r) := \frac{1}{(1 - e^{-2k_2 r})^n}\left[\frac{1}{k_2 n}(1 - \mathrm{e}^{-k_2 n r}) - \frac{n}{k_2(n-2)}(\mathrm{e}^{-2k_2 r} - \mathrm{e}^{-k_2 n r})\right]$$

$$h(R) := \frac{1}{k_1 n}(1 - \mathrm{e}^{-k_1 n R})$$

and

$$C(r) := \begin{cases} \frac{1}{k_2}\sqrt{\lambda^2 \cosh^2 k_2 s - k_2^2 \sinh^2 k_2 s} & \text{if} \quad r \leq \frac{1}{k_2}\text{arctanh}\frac{\lambda}{k_2} \\ 1 & \text{if} \quad r > \frac{1}{k_2}\text{arctanh}\frac{\lambda}{k_2}. \end{cases}$$

Proof. Let O be any point interior to Ω. Consider the exponential map in O, $\exp : T_O M \longrightarrow M$. For each unitary vector $u \in T_O M$ we define $l(u)$ as the positive real number such that

$$\exp(l(u)u) \in \partial\Omega.$$

Let r and R be respectively the minimum and the maximum of l. Let $A = \{(u, t \in S^n \times \mathbb{R}; 0 < t \leq l(u)\}$. Identifying $S^n \times \mathbb{R}$ with $T_O M - \{O\}$ we have $\Omega = \exp(A)$. Hence

$$\text{vol}(\Omega) = \int_\Omega \eta = \int_{esp(A)} \eta = \int_A \exp^* \eta = \int_{S^n} \int_0^{l(u)} J(\exp) t^n \text{d}t\text{d}S.$$

where η and $\text{d}S$ are, respectively, the volume elements of M and S^n.

Analogously, if we define $\phi : S^n \longrightarrow \partial\Omega$ by $\phi(u) = \exp(l(u))u$, then

$$\text{vol}(\partial\Omega) = \int_{\partial\Omega} \mu = \int_{\phi(S^n)} \mu = \int_{Sn} \phi^* \mu = \int_{S^n} \text{Jac}_u(\phi)\text{d}S.$$

where μ is the volume element of $\partial\Omega$. Now, we compute the jacobian of ϕ at a point $u \in S^n$. Let $e_1, \ldots, e_n$ be an orthonormal basis of $T_u S^n$. By definition, we have

$$\text{Jac}_u(\phi) = \mu(\phi_* e_1, \ldots, \phi_* e_n) = \eta(N, \phi_* e_1, \ldots, \phi_* e_n)$$

where N is orthogonal to $\partial\Omega$. If ∂_t is the radial field from O, we can write

$$\text{Jac}_u(\phi) = \eta(\frac{\partial_t}{< \partial_t, N >}, \phi_* e_1, \ldots, \phi_* e_n).$$

Now, $\phi_*(e_i) = \exp_*(\text{d}l(e_i)u + l(u)e_i)$, so

$$\text{Jac}_u(\phi) = \frac{1}{< \partial_t, N >}\eta(< \partial_t, N >, \exp_*(l(u)e_1), \ldots, \exp_*(l(u)e_n)) =$$

$$\frac{l^n(u)}{< \partial_t, N >}\eta(\exp^*(u), \exp_*(l(u)e_1), \ldots, \exp_*(l(u)e_n)) = \frac{l^n(u)}{< \partial_t, N >}\text{Jac}_{l(u)u}(\exp).$$

Therefore,

$$\frac{\text{vol}(\Omega)}{\text{vol}(\partial\Omega)} = \frac{\int_{S^n} \int_0^{l(u)} \text{Jac}_{l(u)u}(\exp) t^n \text{d}t\text{d}S}{\int_{S^n} \frac{l^n(u)}{<\partial_t, N>} \text{Jac}_{l(u)u}(\exp)\text{d}S}.$$

Setting

$$g(u) = \int_0^{l(u)} \frac{\text{Jac}_{tu}(\exp) t^n}{\text{Jac}_{l(u)u}(\exp) l(u)^n}\text{d}t$$

we can write

$$\mathrm{vol}(\Omega) = \int_{S^n} g(u) l(u)^n \mathrm{Jac}_{l(u)u}(\exp) \mathrm{d}S.$$

Now, from lemma 5.1, comparing with the spaces of constant curvature $-k_1^2$ and $-k_2^2$ we can state that

$$\frac{\mathrm{Jac}_{tu}(\exp^{-k_2^2})}{\mathrm{Jac}_{su}(\exp^{-k_2^2})} \leq \frac{\mathrm{Jac}_{tu}(\exp)}{\mathrm{Jac}_{su}(\exp)} \leq \frac{\mathrm{Jac}_{tu}(\exp^{-k_1^2})}{\mathrm{Jac}_{su}(\exp^{-k_1^2})} \qquad \text{for } t < s$$

where $\exp^{-k_i^2}$ denotes the exponential map at any point of the space of curvature $-k_i^2$. It is known that $\mathrm{Jac}_{tu}(\exp^{-k_i^2}) = (\frac{1}{k_i} \sinh k_i t)^n t^{-n}$. Hence

$$\int_0^{l(u)} \frac{(\sinh k_2 t)^n}{(\sinh k_2 s)^n} \mathrm{d}t \leq g(u) \leq \int_0^{l(u)} \frac{(\sinh k_1 t)^n}{(\sinh k_1 s)^n} \mathrm{d}t.$$

We can estimate the first integral by using the fact that $(1-a)^n \geq 1 - na$ for $0 \leq a \leq 1$.

$$\int_0^s \frac{\sinh(k_2 t)^n}{\sinh(k_2 s)^n} \mathrm{d}t = \frac{1}{(1-e^{-2k_2 s})^n} \int_0^s (1 - \mathrm{e}^{-2k_2 t})^n \mathrm{e}^{k_2 n(t-s)} \mathrm{d}t \geq$$

$$\geq \frac{1}{(1-e^{-2k_2 s})^n} \int_0^s (1 - n\mathrm{e}^{-2k_2 t}) \mathrm{e}^{k_2 n(t-s)} \mathrm{d}t =$$

$$= \frac{1}{(1-e^{-2k_2 s})^n} \left[\frac{1}{k_2 n} (1 - \mathrm{e}^{-k_2 n s}) - \frac{n}{k_2(n-2)} (\mathrm{e}^{-2k_2 s} - \mathrm{e}^{-k_2 n s}) \right] =: f(s)$$

On the other hand,

$$\int_0^s \frac{\sinh(k_1 t)^n}{\sinh(k_1 s)^n} \mathrm{d}t \leq \int_0^s \mathrm{e}^{k_1 n(t-s)} \mathrm{d}t = \frac{1}{k_1 n} (1 - \mathrm{e}^{-k_1 n s}) =: h(s)$$

Therefore, since $r \leq l(u) \leq R$ for every $u \in S^n$,

$$f(r) \int_{S^n} l(u)^n \mathrm{Jac}_{l(u)u}(\exp) \mathrm{d}S \leq \mathrm{vol}(\Omega) \leq h(R) \int_{S^n} l(u)^n \mathrm{Jac}_{l(u)u}(\exp) \mathrm{d}S.$$

Finally, using theorem 1, we find that

$$f(r) \cdot C(r) \frac{\lambda}{k_2} \leq \frac{\mathrm{vol}(\Omega)}{\mathrm{vol}(\partial\Omega)} \leq h(R).$$

Now, choosing O to be the incenter and the circumcenter of Ω, we have proved the two inequalities with r and R the inradius and the circumradius respectively. □

Note that the theorem would be true, with the same proof, if r and R were the radius of any geodesic ball contained and containing, respectively, Ω.

Now, we get the main result of the paper

Theorem 3. *Let M be a $(n+1)$-dimensional Hadamard manifold with sectional curvature K such that*

$$-k_2^2 \leq K \leq -k_1^2 \qquad k_1, k_2 > 0.$$

Let $\{\Omega(t)\}_{t\in\mathbb{R}^+}$ be a family of λ-convex compact domains expanding over the whole space. Then, if $\lambda \leq k_2$

$$\frac{\lambda}{nk_2^2} \leq \liminf \frac{\mathrm{vol}(\Omega(t))}{\mathrm{vol}(\partial\Omega(t))} \leq \limsup \frac{\mathrm{vol}(\Omega(t))}{\mathrm{vol}(\partial\Omega(t))} \leq \frac{1}{nk_1}.$$

Proof. Since $\Omega(t)$ expands over the whole hyperbolic space, r and R go to infinity. Then $h(R)$ goes to $1/nk_1$ and $f(r)$ goes to $1/nk_2$. When $\lambda = k_2$ the domains are h-convex and the inequality follows from [BV99]. □

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