Foliation

on an n-dimensional manifold M^n

Main Information

A decomposition of M^n into path-connected subsets, called leaves, such that M^n can be covered by coordinate neighbourhoods U_α with local coordinates $x_\alpha^1, ..., x_\alpha^n$, in terms of which the local leaves, that is, the connected components of the intersection of the leaves with U_α, are given by the equations $x_\alpha^{p+1} = \text{const}, ..., x_\alpha^n = \text{const}$. A foliation in this sense is called a topological foliation. If one also requires that M^n has a piecewise-linear, differentiable or analytic structure, and that the local coordinates are piecewise-linear, differentiable (of class C^r) or analytic, then one obtains the definition of a piecewise-linear, differentiable (of class C^r) or analytic foliation, respectively. The definition of a differentiable foliation of class C^r also makes sense when $r = 0$, and coincides with the definition of a topological foliation. When speaking about a differentiable foliation, it is usually understood that $r \geq 1$. The leaves are naturally equipped with a structure of p-dimensional manifolds (topological, piecewise-linear, differentiable, or analytic) and are submanifolds (in the wide sense of the word) of M^n. The number p (the dimension of the leaves) is called the dimension of the foliation and $q = n - p$ is called its codimension. When considering foliations on a manifold with boundary one usually requires either transversality of the leaves to the boundary, or that a leaf which meets the boundary is completely contained within it. Complex-analytic foliations are defined in the obvious way. In what follows, it is, as a rule, assumed that foliations and mappings are differentiable, since this is the most important case.

The mapping (in the situation above)

$$\phi_\alpha : U_\alpha \to \mathbf{R}^q, \qquad u \to (x_\alpha^{p+1}(u), ..., x_\alpha^n(u))$$

is a **submersion**. The local leaves are $\phi_\alpha^{-1}(c)$, $c \in \mathbf{R}^q$. The system of local submersions $\{(U_\alpha, \phi_\alpha)\}$ is compatible (or coherent) in the sense that if $u \in U_\alpha \cap U_\beta$, then near u one can pass from $\phi_\alpha(v)$ to $\phi_\beta(v)$ by means of a certain local diffeomorphism $\phi_{\beta\alpha,u}$ (of class C^r) of $\mathbf{R}^q$, that is, for all v sufficiently close to u one has $\phi_\beta(v) = \phi_{\beta\alpha,u}\,\phi_\alpha(v)$. Conversely, if M^n is covered by domains U_α and one is given submersions $\phi_\alpha : U_\alpha \to \mathbf{R}^q$ that are compatible in the above sense, then by suitably "glueing together" the $\phi_\alpha^{-1}(c)$ among themselves, one obtains a foliation such that every $\phi_\alpha^{-1}(c)$ is contained in some leaf.

By assigning to every point $u \in M^n$ the tangent space to the leaf passing through this point one obtains a field of p-dimensional tangent subspaces (in other words, a p-dimensional distribution), called the tangent field of the foliation. When $p = 1$, any field of p-dimensional tangent subspaces, under the most minimal requirements of differentiability, is the tangent field of a uniquely determined foliation. When $p > 1$ this is not so. This question is of a local nature (see **Frobenius theorem**). The direct application of Frobenius' theorem to an **involutive distribution** shows that, when the corresponding conditions hold, there is a system of compatible local submersions ϕ_α whose tangent field is tangent to $\phi_\alpha^{-1}(c)$, passage to a foliation being realized by means of appropriate "glueing together" (for an alternative, more detailed, description of this see [3]).

The notion of a foliation evolved in the 1940s in a series of papers of G. Reeb and Ch. Ehresmann, culminating in the book [1] (for its history, see [2]), and was concerned with passing to the global point of view. This was stimulated in part by the theory of smooth dynamical systems (cf. **Dynamical system**), where the decomposition of the phase manifold (with the exception of the equilibrium positions, cf. **Equilibrium position**) into trajectories is a one-dimensional foliation. Flows on surfaces occupy a special place in this theory (see **Poincaré-Bendixson theory**; **Differential equations on a torus**; **Kneser theorem**), in which the trajectories locally partition the space; this helped to focus attention on foliations of codimension 1. Another example of a foliation, also studied in the 1940s, is the decomposition of a Lie group into cosets by an analytic subgroup (not necessarily closed) (see [3]). Finally, in the complex domain, the solutions of a differential equation $dw/dz = f(z, w)$ with analytic right-hand side form (from a real point of view) a two-dimensional foliation.

After this early work there was a gap in the development of the theory of foliations, which still lacked significant results. Intensive development began with the work of A. Haefliger [4] and S.P. Novikov [7], the most well-known results of which are as follows (see [17]): A foliation of codimension 1 on a three-dimensional sphere has a compact leaf [7] and cannot be analytic [4], although Reeb constructed a foliation of class C^∞. At that time, in the study of certain dynamical systems (Y-systems, cf. Y**-system**, and others related to them), there arose certain auxiliary foliations (not of codimension 1) which also stimulated the study of foliations (see [7],

[8]). All these papers and a series of later ones were concerned with the "geometric" or "qualitative" aspect of the theory [16]. In this much attention was paid to foliations of codimension 1, the existence of compact leaves, stability theorems (establishing that, under certain conditions, foliations with a compact leaf are structured like fibrations, in a neighbourhood of the leaf or globally; the first theorems of this type were already proved by Reeb, see [17]), the "growth" characteristic of the leaves (that is, the dependence on r of the p-dimensional volume of a geodesic sphere of radius r on a leaf), and their fundamental groups. Consider the following, recently solved, question: If a closed manifold M^n has a p-dimensional foliation all leaves of which are compact, then is the p -dimensional volume of the leaves necessarily bounded? D. Epstein, D. Sullivan and others have shown that the answer is positive only when $q \leq 2$ (see [9]).

There later arose a "homotopy" direction of the theory, whose prototype was the homotopy theory of fibre bundles (cf. **Fibre space**). One of the difficulties here stemmed from the fact that for foliations there is in general no analogue of the **induced fibre bundle**. This necessitates the study of more general objects called Haefliger structures (cf. **Haefliger structure**) (a kind of foliation with singularities), for which such an analogue exists. Two foliations F_0 and F_1 on M are called concordant if there is a foliation on the "cylinder" $M \times [0,1]$ (having the same codimension) whose leaves are transversal to the "floor" and "roof" of the cylinder and "cut out" on them the foliations F_0 and F_1, respectively. Concordance of Haefliger structures is defined in a similar way. Every Haefliger structure is concordant to one which, outside a set of "singular points" on M, corresponds to some foliation, and certain conditions hold for the behaviour of the leaves of the latter around these points. In this sense, a Haefliger structure can be thought of as a foliation with singularities. There is a natural bijection between the classes of concordant Haefliger structures and the homotopy classes of continuous mappings of M into a so-called classifying space $B\Gamma_q^r$ (where q denotes the codimension and r the smoothness class of the Haefliger structure).

Homotopy theory establishes which homotopy objects determine concordance of foliations: two foliations are concordant if and only if they are concordant as Haefliger structures and their tangent fields are homotopic (see [6], [10], [11]). A related result is the proof of the existence of p-dimensional foliations on all open M (see [6]) and on those closed M on which there is a continuous field of p-dimensional tangent subspaces (which is an obvious necessary condition, see [10], [11]). Several authors had earlier proved existence theorems for foliations on various manifolds by direct construction [12]. The idea (see [10], [11]) is to begin with a foliation with singularities, and then liquidate them by modifying the foliation in a certain way. The case $q > 1$ is relatively simple (see [10], [13]) and the liquidation of singularities can be carried out in the spirit of the "geometric" theory [14]; the case $q = 1$ is more complicated [11].

A mapping $f: M^n \to B\Gamma_q^r$ induces a mapping of cohomology groups that leads to the concept of a **characteristic class** of a foliation. The "homological" or "quantitative" theory of foliations arising from this (see [13], [15], [16]) also includes certain results obtained earlier without reference to $B\Gamma_q^r$, such as the Godbillon-Weil invariant (for $n = 3$ see [17]), or R. Bott's necessary conditions for a continuous field of tangent subspaces to be homotopic to the tangent field of a foliation.

References

[1] G. Reeb, *Sur certains propriétés topologiques des variétés feuilletées,* Hermann, 1952

[2] G. Reeb, P.A. Schweitzer, Une théorème de Thurston établi au moyen de l'analyse non standard, *Differential topology, foliations and Gel'fand-Fuks cohomology, Lect. notes in math.,* Vol. 652, Springer, 1978 , pp. 318.

[3] C. Chevalley, *Theory of Lie groups,* Vol. 1, Princeton Univ. Press, 1946

[4] A. Haefliger, "Variétés feuilletées", *Ann. Scuola Norm. Sup. Pisa. Ser. 3* **16** (1962) , 367-397.

[5] Ed. N.H. Kuiper, *Manifolds (Amsterdam, 1970), Lect. notes in math.,* Vol. 197, Springer, 1971

[6] A. Haefliger, "Feuilletages sur les variétés ouvertes", *Topology* **9** No. 2(1970) , 183-194.

[7] S.P. Novikov, "Topology of foliations", *Trans. Amer. Math. Soc.* **14** (1967) , 268-304.*Trudy Moskov. Mat. Obshch.* **14** (1965) , 248-278.

[8] *Smooth dynamical systems,* Moscow, 1977 (In Russian; translated from the English)

[9] A.L. Besse, *Manifolds all of whose geodesics are closed,* Springer, 1978

[10] W. Thurston, "The theory of foliations of codimension greater than one", *Comm. Math. Helv.* **49** (1974) , 214-231.

[11] W. Thurston, "Existence of codimension one foliations", *Ann. of Math.* **104** No. 2(1976) , 249-268.

[12] H. Lawson, "Foliations", *Bull. Amer. Math. Soc.* **80** (1974) , 369-418.

[13] H. Lawson, *The quantitative theory of foliations,* Amer. Math. Soc., 1977

[14] M.M. Mishachev, Ya.M. Eliashberg, "Surgery on singularities of foliations", *Funct. Anal. Appl.* **11** No. 3(1977) , 197-205. *Funktsional. Anal. Prilozhen.* **11** No. 3(1977) , 43-53.

[15] D.B. Fuks, "Cohomology of infinite-dimensional Lie algebras and characteristic classes of foliations", *J. Soviet Math.* **11** No. 6(1979) , 922-980.*Itogi Nauk. Tekhn. Sovrem. Probl. Mat.* **10** (1978) , 179-285.

[16] D.B. Fuks, "Foliations", *J. Soviet Math.* **18** No. 2(1982) , 255-291.*Itogi Nauk. Tekhn. Algebra Topol. Geom.* **18** (1981) , 151-213.

[17] I. Tamura, *Topology of foliations,* Iwanami Shoten, 1976 (In Japanese)

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Expert Comments

A Y-system is better known as an Anosov system in the West.

References

[A1] B. Reinhart, *Differential geometry of foliations,* Springer, 1983

[A2] D.V. Anosov, "Geodesic flows on closed Riemannian manifolds with negative curvature", *Proc. Steklov Inst. Math.* **90** (1969) *Trudy Mat. Inst. Steklov.* **90** (1967)