Actions of the braid group, and new algebraic proofs of results of Dehornoy and Larue.

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To Martin Dunwoody on the occasion of his 70th birthday.

Abstract

This article surveys many standard results about the braid group, with emphasis on simplifying the usual algebraic proofs.

We use van der Waerden’s trick to illuminate the Artin-Magnus proof of the classic presentation of the braid group considered as the algebraic mapping-class group of a disc with punctures.

We give a simple, new proof of the $\sigma_1$-trichotomy for the braid group, and, hence, recover the Dehornoy right-ordering of the braid group.

We give three proofs of the Birman-Hilden theorem concerning the fidelity of braid-group actions on free products of finite cyclic groups, and discuss the consequences derived by Perron-Vannier and the connections with Artin groups and the Wada representations.

The first, very direct, proof, is due to Crisp-Paris and uses the $\sigma_1$-trichotomy and the Larue-Shpilrain technique. The second proof arises by studying ends of free groups, and gives interesting extra information. The third proof arises from Larue’s study of polygonal curves in discs with punctures, and gives extremely detailed information.

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1 General Notation

Let $\mathbb{N}$ denote the set of finite cardinals, $\{0, 1, 2, \ldots\}$.

Throughout, we fix an element $n$ of $\mathbb{N}$.

Let $G$ be a multiplicative group.

For elements $a$, $b$ of $G$, we write $\overline{a} := a^{-1}$, $a^b := bab$, $[a] := \{a^g \mid g \in G\}$, the conjugacy class of $a$ in $G$, and $a^{\overline{n}} := \overline{b}a^n b$. We let $\operatorname{Aut} G$ denote the group of all automorphisms of $G$, acting on $G$ on the right with exponent notation.
For two subsets $A, B$ of a set $X$, the complement of $A \cap B$ in $A$ will be denoted by $A - B$ (and not by $A \setminus B$ since we let $G \setminus Y$ denote the set of $G$-orbits of a left $G$-set $Y$).

An *ordering* of a set will mean a total ordering for the set. An *ordered* set is a set endowed with a specific ordering.

We will make frequent use of sequences, usually with vector notation. We shall use the language of sequences to introduce indexed symbols and to realize free monoids. Formally, we define a *sequence* as a set endowed with a specified listing of its elements. Thus a sequence has an underlying set; with vector notation, the coordinates are the elements of (the underlying set of) the sequence. For two sequences $A, B$, their concatenation will be denoted $A \lor B$. By a sequence $A$ in a given set $X$, we mean a sequence endowed with a specified map of sets $A \to X$; to avoid extra notation, we shall use the same symbol to denote an element of $A$ and its image in $X$ even when the map is not injective. We often treat $A$ as an element in the free monoid on $X$ with concatenation as binary operation, and then the elements of $A$ are its atomic factors.

Let $i, j \in \mathbb{Z}$.

We write $[i \uparrow j] = \begin{cases} (i, i + 1, \ldots, j - 1, j) \in \mathbb{Z}^{j-i+1} & \text{if } i \leq j, \\ () \in \mathbb{Z}^0 & \text{if } i > j. \end{cases}$ Also, $[i \uparrow \infty] = (i, i + 1, i + 2, \ldots)$. We define $[j \downarrow i]$ to be the reverse of the sequence $[i \uparrow j]$, $(j, j - 1, \ldots, i + 1, i)$.

Let $v$ be a symbol.

For each $k \in \mathbb{Z}$, we let $v_k$ denote the ordered pair $(v, k)$.

We let $v_{[i \uparrow j]} = \begin{cases} (v_i, v_{i+1}, \ldots, v_j) & \text{if } i \leq j, \\ () & \text{if } i > j. \end{cases}$ Also, $v_{[i \uparrow \infty]} = (v_i, v_{i+1}, v_{i+2}, \ldots)$. We define $v_{[j \downarrow i]}$ to be the reverse of the sequence $v_{[i \uparrow j]}$.

Now suppose that $v_{[i \uparrow j]}$ is a sequence in the multiplicative group $G$, that is, there is specified a map of sets $v_{[i \uparrow j]} \to G$, and we treat the elements of $v_{[i \uparrow j]}$ as elements of $G$ (possibly with repetitions). We let

\[
\Pi v_{[i \uparrow j]} := \begin{cases} v_i v_{i+1} \cdots v_{j-1} v_j \in G & \text{if } i \leq j, \\ 1 \in G & \text{if } i > j. \end{cases}
\]

\[
\Pi v_{[j \downarrow i]} := \begin{cases} v_j v_{j-1} \cdots v_{i+1} v_i \in G & \text{if } j \geq i, \\ 1 \in G & \text{if } j < i. \end{cases}
\]

2 Outline

Recall that $n \in \mathbb{N}$.

Let $\Sigma_{0,1,n} := \langle z_1, t_{[1 \uparrow n]} \mid z_1 \Pi t_{[1 \uparrow n]} = 1 \rangle$. Here, $z$ and $t$ are symbols, and $\Sigma_{0,1,n}$ is presented as a one-relator group with generating sequence $(z_1, t_1, \ldots, t_n) =$
In particular, $t_{[1],n} \vee \tilde{t}_{[1],n}$ is a sequence in $\Sigma_{0,1,n}$, and we see that $\Sigma_{0,1,n}$ is freely generated by $t_{[1],n}$.

Let $\text{Out}_{0,1,n}^+$ denote the subgroup of $\text{Aut} \Sigma_{0,1,n}$ consisting of all the automorphisms of $\Sigma_{0,1,n}$ which respect the sets $\{z_1\}$ and $\{[t_i]\}_{i \in [1],n}$. Let $\text{Out}_{0,1,0}$ denote a group of order two, and, for $n \geq 1$, let $\text{Out}_{0,1,n}$ denote the group of all automorphisms of $\Sigma_{0,1,n}$ which respect the sets $\{z_1, \bar{z}_1\}$ and $\{[t_i] \cup \tilde{[t_i]}\}_{i \in [1],n}$. Then $\text{Out}_{0,1,n}^+$ is a subgroup of index two in $\text{Out}_{0,1,n}$. We call $\text{Out}_{0,1,n}$ the algebraic mapping-class group of the surface of genus 0 with 1 boundary component and $n$ punctures; see [18] for background on algebraic mapping-class groups.

Frequently, $\text{Out}_{0,1,n}^+$ will be denoted $\mathcal{B}_n$ and called the $n$-string braid group. (The similar symbol $B_n$ denotes a Coxeter diagram.)

In Section 3, we define a sequence $\sigma_{1, [n-1]}$ in $\text{Out}_{0,1,n}^+$; we review Artin’s 1925 proof that $\sigma_{1, [n-1]}$ generates $\text{Out}_{0,1,n}^+$, and we present related results that we shall apply in subsequent sections. In Section 4, we recall the definition of Artin groups, specifically $\text{Artin}(A_n)$, $\text{Artin}(B_n)$ and $\text{Artin}(D_n)$. In Section 5, we verify Artin’s 1925 result that $\text{Out}_{0,1,n}^+ \simeq \text{Artin}(A_{n-1})$, by combining Magnus’ 1934 proof, Manfredini’s observation that $\text{Out}_{0,1,(n-1),1}^+ \simeq \text{Artin}(B_{n-1})$, and the van der Waerden trick, to condense the calculations involved.

In Section 6, we use results of Section 4 to recover the celebrated $\sigma_1$-trichotomy and the Dehornoy right-ordering of $\mathcal{B}_n$. This free-group-action approach represents a substantial simplification over previous arguments. Let us emphasize that we verify directly that $\text{Out}_{0,1,n}^+$ satisfies the $\sigma_1$-trichotomy, which is the reverse of the route taken by Larue [22], where the $\sigma_1$-trichotomy for $\text{Artin}(A_{n-1})$ is used to verify that $\text{Artin}(A_{n-1})$ acts faithfully on $\Sigma_{0,1,n}$.

In Section 7, we review the action of $\mathcal{B}_n$ on the set of ends of $\Sigma_{0,1,n}$. We recall the argument of Thurston [29] that yields the Dehornoy right-ordering of $\mathcal{B}_n$, but not the $\sigma_1$-trichotomy. By analysing further, we obtain new results about the $\mathcal{B}_n$-orbit of $t_1$ in $\Sigma_{0,1,n}$.

In Section 8, for each $m \geq 2$, we introduce $\text{Out}_{0,1,n(m)}$, the algebraic mapping-class group of the disc with $n \ C_m$-points. We recall the Larue-Shpilrain-type proof by Crisp-Paris of the Birman-Hilden result that the natural map from $\text{Out}_{0,1,n}$ to $\text{Out}_{0,1,n(m)}$ is injective. We then modify an argument of Steve Humphries to show that there is a natural identification $\text{Out}_{0,1,n(m)} = \text{Out}_{0,1,n}$. The results previously obtained in Section 7 then provide additional information in this context.

In Section 9, we review some applications by Perron-Vannier [27] of the above Birman-Hilden result, and discuss connections with the actions given by Wada [31] and studied by Shpilrain [30] and Crisp-Paris [10], [11].

Following a kind suggestion of Patrick Dehornoy, we studied the analysis of the $\mathcal{B}_n$-orbit of $t_1$ in $\Sigma_{0,1,n}$ given by David Larue [21]. Larue’s approach is combinatorial and uses polygonal curves in the punctured disc. By combining Larue’s approach with Whitehead’s use of graphs, we were able to simplify Larue’s main arguments; we record our combinatorial approach in an appendix.
We also show how Larue’s results imply the results we had obtained in Section 7 by studying ends.

3 Artin’s generators of \( \mathcal{B}_n \)

In this section we recall Artin’s generating sequence \( \sigma_{[1]^{(n-1)}} \) of \( \mathcal{B}_n \).

Let us first fix more notation related to \( \Sigma_{0,1,n} = \langle z_1, t_{[1]^n} \mid z_1 \Pi t_{[1]^n} = 1 \rangle \) and \( \mathcal{B}_n \leq \text{Aut} \Sigma_{0,1,n} \).

### 3.1 Notation

Let \( m \in \mathbb{N} \). Consider an element \( w \) of \( \Sigma_{0,1,n} \) and a sequence \( a_{[1]^m} \) in \( t_{[1]^n} \vee T_{[1]^n} \). We also view \( a_{[1]^m} \) as a sequence in \( \Sigma_{0,1,n} \).

If \( \Pi a_{[1]^m} = w \) in \( \Sigma_{0,1,n} \), we say that \( a_{[1]^m} \) is a monoid expression for \( w \), in \( t_{[1]^n} \vee T_{[1]^n} \), of length \( m \). We say that \( a_{[1]^m} \) is reduced if, for all \( j \in [1^m(n-1)] \), \( a_j \neq a_{j+1} \) in \( t_{[1]^n} \vee T_{[1]^n} \). Each element of \( \Sigma_{0,1,n} \) has a unique reduced expression, called the normal form.

Suppose that \( a_{[1]^m} \) is the normal form for \( w \). We define the length of \( w \) to be \( |w| := m \). The set of elements of \( \Sigma_{0,1,n} \) whose normal forms have \( a_{[1]^m} \) as an initial segment is denoted \( (w*) \); and, the set of elements of \( \Sigma_{0,1,n} \) whose normal forms have \( a_{[1]^m} \) as a terminal segment is denoted \( (*w) \). The elements of \( (w*) \) are said to begin with \( w \), and the elements of \( (*w) \) are said to end with \( w \).

Let \( \text{Sym}_n \) denote the group of permutations of (the set underlying) \( [1^n] \), acting with exponent notation.

Let \( \phi \in \mathcal{B}_n \). There exists a unique permutation \( \pi \in \text{Sym}_n \), and a unique sequence \( w_{[0^n+1]} \) in \( \Sigma_{0,1,n} \) such that \( w_0 = 1 \) and \( w_{n+1} = 1 \), and, for each \( i \in [1^n] \), \( w_i \notin (t_i*) \cup (t_i* \cdot \cdot \cdot t_i*) \) and

\[
t^\phi_i = t^w_i.
\]

For each \( i \in [0^n] \), let \( u_i = w_i w_{i+1} \). If \( j \in [i^n] \), then \( \Pi u_{[i,j]} = w_i w_{j+1} \). In particular, \( \Pi u_{[i^n]} = w_i \). We define \( \pi(\phi) := \pi, w_i(\phi) := w_i \) for \( i \in [0^n+1] \), and \( u_i(\phi) := u_i \) for \( i \in [0^n] \). We write \( ||\phi|| := \sum_{i \in [1^n]} |t^\phi_i| = n + 2 \sum_{i \in [1^n]} |w_i(\phi)| \).

Let \( \sigma_{[1]^m\text{m-1}} \) be the sequence in \( \mathcal{B}_n \) defined as follows: for all \( i \in [1^n(n-1)] \)

\[
t^\sigma_{k} = \begin{cases} 
t_k & \text{if } k \in [1^n(i-1)] \cup [(i+2)^n], \\
t_{i+1} & \text{if } k = i, \\
t_{i+1}^\prime & \text{if } k = i + 1.
\end{cases}
\]

In the literature, \( \sigma_i \) is sometimes represented in \( 2 \times n \)-matrix notation, for example, in the format

\[
\sigma_i = \begin{pmatrix} 
t_1 & \ldots & t_{i-1} & t_i & t_{i+1} & t_{i+2} & \ldots & t_n \\
t_1 & \ldots & t_{i-1} & t_{i+1} & t_{i+1}^\prime & t_{i+2} & \ldots & t_n
\end{pmatrix}.
\]

We find it convenient to avoid dots and we say that \( \sigma_i \) and \( \sigma_i \) are determined by the expressions
Suppose that there exists some \(i \in [1|(n-1)]\) such that \(u_i \in (\star t_{(i+1)^n})\).

Then \(\|\sigma_i \phi\| \leq \|\phi\| - 2\). Moreover, for each \(j \in [1|i]\), \(t_j^{\sigma_i \phi}\) and \(t_j^{\phi}\) both begin with the same element of \(t_{[1|n]} \cup t_{(i+1)^n}\).

(ii). Suppose that there exists some \(i \in [1|(n-1)]\) such that \(u_i \in (t_i, t_{i+1} \star)\). Then \(\|\sigma_i \phi\| \leq \|\phi\| - 2\). Moreover, for each \(j \in [1|i-1]\), \(t_j^{\sigma_i \phi}\) and \(t_j^{\phi}\) both begin with the same element of \(t_{[1|n]} \cup t_{(i+1)^n}\).

(iii). Suppose that, for each \(i \in [1|(n-1)]\), \(u_i \not\in (t_i, t_{i+1} \star) \cup (\star t_{(i+1)^n})\). Then \(\phi = 1\).

Proof. (i). There exists some \(v \in \Sigma_{0,1,n} - (\star t_{(i+1)^n})\) such that \(u_i = \overline{v}t_{(i+1)^n}\). Since \(w_i(\phi) = u_iw_{i+1}(\phi)\), we have

\[
(3.2.1) \quad w_i(\phi) = v\overline{t}_{(i+1)^n}w_{i+1}(\phi).
\]

Since \(v \not\in (\star t_{(i+1)^n})\) and \(w_{i+1}(\phi) \not\in (t_{(i+1)^n} \star)\), there is no cancellation in the expression \(t_{(i+1)^n}w_{i+1}(\phi)\) for \(t_i^{\phi}\); hence

\[
(3.2.2) \quad t_i^{\phi} \in (\overline{w}_{i+1}(\phi)t_{(i+1)^n} \star) \text{ and } |t_i^{\phi}| = 1 + 2|v| + 2 + 2|w_{i+1}(\phi)|.
\]

For all \(j \in [1|(i-1)] \cup [(i+2)|n]\), \(t_j^{\sigma_i \phi} = t_j^{\phi}\); hence, \(t_j^{\sigma_i \phi}\) has the same first letter as \(t_j^{\phi}\), and, \(|t_j^{\sigma_i \phi}| = |t_j^{\phi}|\).

Since \(t_i^{\sigma_i \phi} = t_i^{\phi} \in (\overline{w}_{i+1}(\phi)t_{(i+1)^n} \star)\), we see, from (3.2.2), that \(t_i^{\sigma_i \phi}\) has the same first letter as \(t_i^{\phi}\). Also, \(|t_i^{\sigma_i \phi}| = |t_i^{\phi}|\).

By (3.2.1), \(w_i(\phi)\overline{w}_{i+1}(\phi)t_{(i+1)^n} = v\); hence

\[
(3.2.3) \quad t_i^{\sigma_i \phi} = (t_i^{t_i^{\phi} + 1})^{\phi} = (\overline{w}_{i+1}(\phi)w_{i+1}(\phi))^{(3.2.2)} t_i^{\overline{w}_{i+1}(\phi)}.
\]

Hence, \(|t_i^{\sigma_i \phi}| \leq 1 + 2|v| + 2|w_{i+1}(\phi)|\) \((3.2.2)\)\)

It now follows that \(\|\sigma_i \phi\| \leq \|\phi\| - 2\), and (i) is proved.

(ii). There exists some \(v \in \Sigma_{0,1,n} - (t_i \star)\) such that \(u_i = \overline{t}_i \star v\). Since \(w_{i+1}(\phi) = \overline{u}_i w_i(\phi)\), we have

\[
(3.2.3) \quad w_{i+1}(\phi) = \overline{v}t_i \star w_i(\phi).
\]
Since $\tau \not\in (\tilde{t}_i \ast)$ and $w_i(\phi) \not\in (\tilde{t}_n \ast)$, there is no cancellation in the expression $t_{(i+1)^n}^w(\phi)$ for $t_{i+1}^\phi$; hence

\[(3.2.4) \quad |t_{i+1}^\phi| = 1 + 2|\tau| + 2 + 2|w_i(\phi)|.\]

For all $j \in [1\,\ast (i-1)] \cup [(i+2)\,\ast n]$, $t_j^\tau = t_j^\varphi$; hence, $t_j^\tau = t_j^\varphi$ has the same first letter as $t_j^\varphi$ and, $|t_j^\tau| = |t_j^\varphi|$. Since $t_{i+1}^\phi = t_i^\phi$, we see that $|t_{i+1}^\tau| = |t_i^\varphi|$. By (3.2.3), $w_{i+1}(\phi)w_i(\phi)t_i^n = \tau$; hence

\[t_i^\tau = (t_{i+1}^\phi)^{t_i^n(\phi)} = (t_{(i+1)^n}^{w_i(\phi)})^{t_i^{w_i(\phi)}} = t_{i+1}^\phi.\]

Hence, $|t_{i+1}^\tau| \leq 1 + 2|\tau| + 2|w_i(\phi)| \leq |t_{i+1}^\phi| - 2$.

It now follows that $||\tau|| \leq ||\phi||$ and (ii) is proved.

(iii). Since $u_0 = w_1(\phi) \not\in (\tilde{t}_i \ast)$ and $u_n = w_n(\phi) \not\in (\tilde{t}_n \ast)$, we see that there is no cancellation anywhere in the expression $u_0 \prod_{i \in [1\,\ast n]} (t_i \ast u_i)$. Hence,

\[|u_0 \prod_{i \in [1\,\ast n]} (t_i \ast u_i)| = \sum_{i \in [0\,\ast n]} |u_i| + n, \text{ that is, } \sum_{i \in [0\,\ast n]} |u_i| = |u_0 \prod_{i \in [1\,\ast n]} (t_i \ast u_i)| - n.\]

Recall that $u_0 \prod_{i \in [1\,\ast n]} (t_i \ast u_i) = \prod_{i \in [1\,\ast n]} (t_i \ast u_i) = \prod_{i \in [1\,\ast n]} t_i \ast u_i$. Hence

\[|u_0 \prod_{i \in [1\,\ast n]} (t_i \ast u_i)| = n \text{ and } \sum_{i \in [0\,\ast n]} |u_i| = n - n = 0.\]

Hence, all the elements of $u_{[0\,\ast n]}$ are trivial.

For each $i \in [0\,\ast (n+1)]$, $w_i = \Pi w_{[i\,\ast n]}$; hence, all the elements of $w_{[1\,\ast n]}$ are trivial. Also, $\prod_{i \in [1\,\ast n]} t_i = u_0 \prod_{i \in [1\,\ast n]} (t_i \ast u_i) = \prod_{i \in [1\,\ast n]} t_i$. Hence $\pi$ is trivial. Thus $\phi = 1$.

The following is then immediate.

3.3 Proposition (Artin [3]). For each $\phi \in B_n$, either $\phi = 1$, or there exists some $\sigma_i^* \in \sigma_{[1\,\ast (n-1)]} \cup \sigma_{[1\,\ast (n-1)]}$ such that $||\sigma_i^* \phi|| \leq ||\phi|| - 2$. Hence, $\langle \sigma_{[1\,\ast (n-1)]} \rangle = B_n$. □

3.4 Remarks. If $w \in \Sigma_{0,1,n}$ has odd length, then $w^{\sigma_i}$ has odd length, and $|w^{\sigma_i}| \leq 2|w| + 1$, with equality being achieved only if every odd letter of $w$ equals $t_{i+1}$ or $\tilde{t}_{i+1}$. Similar statements hold with $\tilde{\sigma}_i$ in place of $\sigma_i$.

Let $\phi \in B_n$ and let $|\phi|$ denote the minimum length of a monoid expression for $\phi$ in $\sigma_{[1\,\ast (n-1)]} \cup \tilde{\sigma}_{[1\,\ast (n-1)]}$. Thus, $|t_{i+1}^\phi| \leq 2|\phi| - 1 + 1$. Hence, $||\phi|| \leq n2^{|\phi| + 1} - n$. Proposition 3.3 gives an algorithm which yields a monoid expression for $\phi$ in $\sigma_{[1\,\ast (n-1)]} \cup \tilde{\sigma}_{[1\,\ast (n-1)]}$ of length at most $\frac{|\phi| - n}{2}$, and we have now seen that

\[
\frac{|\phi| - n}{2} \leq n2^{|\phi| + 1} - 2n = n2^{|\phi|} - n.
\]

□
4 Coxeter diagrams and Artin groups

4.1 Definition. A Coxeter diagram $X$ consists of a set $V$ together with a function $V \times V \to \mathbb{N} \cup \{\infty\}$, $(x, y) \mapsto m_{x,y}$, such that, for all $x, y \in V$, $m_{x,x} = 0$ and $m_{x,y} = m_{y,x}$. The elements of $V$ are called the vertices of $X$, and, for all $x, y \in V$, we say that $m_{x,y}$ is the number of edges joining $x$ and $y$; thus we can represent $X$ diagrammatically. We then define the Artin group of $X$, denoted $\text{Artin} \langle X \rangle$, to be the group presented with generating set $V$ and relations saying that, for all $x, y \in V$, $xy = yx$ if $m_{x,y} = 0$, $xyx = yxy$ if $m_{x,y} = 1$, $xyxy = yxyx$ if $m_{x,y} = 2$, etc.

Notice that if $m_{x,y} = \infty$, then no relation is imposed. Notice also that if $V$ is empty, then $\text{Artin} \langle X \rangle$ is the trivial group. $\square$

4.2 Notation. (i). Let $A_n$ denote the Coxeter diagram

\[ a_1 \rightarrow a_2 \rightarrow \cdots \rightarrow a_{n-1} \rightarrow a_n. \]

It is understood that $A_0$ is empty. We define $A_{-1}$ to be empty also.

Thus, in $A_n$, the vertex set is $a_{[1\ up to n]}$, and, for $i, j \in [1\ up to n]$, the number of edges joining $a_i$ to $a_j$ is

\[ \begin{cases} 1 & \text{if } |i - j| = 1, \\ 0 & \text{if } |i - j| \neq 1. \end{cases} \]

Hence, $\text{Artin} \langle A_n \rangle$ has a presentation with generating set $a_{[1\ up to n]}$ and relations saying that, for $i, j \in [1\ up to n]$, $a_ia_j = a_ja_i$ if $|i - j| \neq 1$, $a_ia_ia_i = a_ia_ia_j$ if $|i - j| = 1$.

(ii). Let $B_n$ denote the Coxeter diagram

\[ b_1 \rightarrow b_2 \rightarrow \cdots \rightarrow b_{n-1} \rightarrow b_n. \]

Here, the vertex set is $b_{[1\ up to n]}$, and, for $i, j \in [1\ up to n]$, the number of edges joining $b_i$ to $b_j$ is

\[ \begin{cases} 2 & \text{if } \{i, j\} = \{n-1, n\}, \\ 1 & \text{if } |i - j| = 1 \text{ and } \{i, j\} \neq \{n-1, n\}, \\ 0 & \text{if } |i - j| \neq 1. \end{cases} \]

(iii). For $n \geq 2$, let $D_n$ denote the Coxeter diagram

\[ d_n \]

\[ d_1 \rightarrow d_2 \rightarrow \cdots \rightarrow d_{n-3} \rightarrow d_{n-2} \rightarrow d_{n-1}. \]

Here, the vertex set is $d_{[1\ up to n]}$, and, for $i, j \in [1\ up to n]$, the number of edges joining $d_i$ to $d_j$ is

\[ \begin{cases} 1 & \text{if } \{i, j\} \in \{\{1, 2\}, \{2, 3\}, \ldots, \{n-2, n-1\}, \{n-2, n\}\}, \\ 0 & \text{otherwise.} \end{cases} \]
5 Artin's presentation of $\mathcal{B}_n$

In this section, we verify Artin's result that there exists an isomorphism

$$\gamma_n: \text{Artin}(A_{n-1}) \rightarrow \mathcal{B}_n$$

determined by

$$a_i \mapsto (\sigma_i)^{\gamma_n}.$$  \hspace{1cm} (a)

We express this result by

writing $\mathcal{B}_n = \text{Artin}(\sigma_1 \cdots \sigma_n)$.

5.1 Proposition. There exists a homomorphism $\gamma_n: \text{Artin}(A_{n-1}) \rightarrow \mathcal{B}_n$ determined by

$$a_i \mapsto (\sigma_i)^{\gamma_n},$$

and $\gamma_n$ is surjective.

Proof. (a). Suppose that $1 \leq i \leq i+2 \leq j \leq n-1$. We have the following.

$$\begin{array}{c|ccc|ccc|c}
& t_i & t_{i+1} & t_k & t_j & t_{j+1} & t_k \gamma_i \\
\hline
k \in [1/(i-1)] & t_k & t_{i+1} & t_k & t_j & t_{j+1} & t_k \gamma_i \\
(k \in [i+2]/(j-1)] & t_k & t_{i+1} & t_k & t_j & t_{j+1} & t_k \gamma_i \\
(k \in [j+2]/n] & t_k & t_{i+1} & t_k & t_j & t_{j+1} & t_k \gamma_i \\
\end{array}$$

(b). Suppose that $1 \leq i \leq n-2$. We have the following.

$$\begin{array}{c|ccc|ccc|c}
& t_i & t_{i+1} & t_{i+2} & t_k \gamma_i \\
\hline
k \in [1/(i-1)] & t_k & t_{i+1} & t_{i+2} & t_k \gamma_i \\
(k \in [i+2]/n] & t_k & t_{i+1} & t_{i+2} & t_k \gamma_i \\
(k \in [j+2]/n] & t_k & t_{i+1} & t_{i+2} & t_k \gamma_i \\
\end{array}$$

By (a) and (b), there exists a homomorphism $\gamma_n: \text{Artin}(A_{n-1}) \rightarrow \mathcal{B}_n$ determined by

$$a_i \mapsto (\sigma_i)^{\gamma_n}.$$  \hspace{1cm} (b)

By Proposition 3.3, $\langle \sigma_i \rangle = \mathcal{B}_n$, and, hence, $\gamma_n$ is surjective.

In the remainder of this section, we shall use induction on $n$ to show that the surjective homomorphism $\gamma_n: \text{Artin}(A_{n-1}) \rightarrow \mathcal{B}_n$ of Proposition 5.1 is an isomorphism. Notice that $\gamma_n$ endows $\text{Artin}(A_{n-1})$ with a canonical action on $\Sigma_{0,1,n}$.

The following is precisely [25, Proposition 1] and, also, [10, Proposition 2.1(2)].
5.2 Lemma (Manfredini [25]). If \( n \geq 1 \), then
\[
\text{Artin}(A_{n-1}) \ltimes \Sigma_{0,1,n} = \text{Artin}(a_1 \longrightarrow a_2 \longrightarrow \cdots \longrightarrow a_{n-1} \equiv \bar{t}_n) \simeq \text{Artin}(B_n).
\]

Proof. For \( n = 1 \), the result is clear.
For \( n = 2 \), we have the following.
\[
\text{Artin}(A_1) \ltimes \Sigma_{0,1,2} = \langle a_1, t_{[1,2]} | t_1^{\sigma_1} = t_2, t_2^{\sigma_1} = \bar{t}_1 t_2 \rangle
\]
\[
= \langle a_1, t_2 | t_2^{\sigma_1} = \bar{t}_2 t_1 t_2 \rangle = \langle a_1, t_2 | (\bar{t}_1 t_2)(a_1) = (\bar{t}_2 a_1)(t_2 \bar{t}_1 t_2) \rangle
\]
\[
= \langle a_1, t_2 | (a_1)(\bar{t}_2 a_1) = (\bar{t}_2 a_1)(\bar{t}_2 a_1) \rangle = \text{Artin}(a_1 \equiv \bar{t}_2).
\]

From the case \( n = 2 \), we see that there exists a homomorphism
\[
\mu: \text{Artin}(B_n) \rightarrow \text{Artin}(A_{n-1}) \ltimes \Sigma_{0,1,n} \text{ determined by } \begin{pmatrix} b_i & b_n \mu \\ a_i & \bar{t}_n \end{pmatrix}.
\]

For each \( k \in [1 \uparrow n] \), let \( t_k \) denote the element \( \bar{b}_{n \downarrow (n-1)}^{\uparrow k} \) of \( \text{Artin}(B_n) \). For each \( i \in [1 \uparrow (n-1)] \) and \( k \in [1 \uparrow n] \), let us formally define
\[
\bar{t}_k^i := \begin{cases} 
  t_k & \text{if } k \in [1 \uparrow i - 1] \lor [i + 2 \uparrow n], \\
  \bar{t_i}^{i+1} & \text{if } k = i, \\
  t_i & \text{if } k = i + 1.
\end{cases}
\]

We shall see that \( \bar{t}_k^i = \bar{t}_k^\sigma \); this immediately implies that there exists a homomorphism
\[
\bar{\sigma}: \text{Artin}(A_{n-1}) \ltimes \Sigma_{0,1,n} \rightarrow \text{Artin}(B_n) \text{ determined by } \begin{pmatrix} a_i & t_k \bar{\sigma} \\ \bar{t_i} & t_k \end{pmatrix}
\]
which is then clearly inverse to \( \mu \), and the result will be proved.

For each \( m \in [n \downarrow 1] \), we shall show, by decreasing induction on \( m \), that, for each \( k \in [n \downarrow m] \) and each \( i \in [(n-1) \downarrow m] \), \( \bar{t}_k^i = \bar{t}_k^\sigma \). For \( m = n \), this is trivial, and, for \( m = n - 1 \), it follows from the case \( n = 2 \). Suppose that \( m \in [(n-2) \downarrow 1] \).

(a) For each \( k \in [n \downarrow (m+1)] \) and each \( i \in [(n-1) \downarrow (m+1)] \), \( \bar{t}_k^i = \bar{t}_k^\sigma \), by hypothesis.

(b) For each \( k \in [n \downarrow (m+2)] \), \( t_k \in (b_{[n \downarrow (m+2)]}) \) and, hence, \( \bar{t}_k^i = \bar{t}_k^\sigma \).

(c) \( \bar{t}_{m+1}^m = \bar{b}_{m}^{(n \downarrow (n-1 \downarrow (m+1)))} = \bar{t}_m = \bar{t}_m^{\sigma_{m+1}} \).

(d) For each \( i \in [(n-1) \downarrow (m+2)] \), \( \bar{t}_m^i \equiv \bar{t}_m^{\sigma_{m+1}} = \bar{t}_m^{\sigma_{m+1}}(a) \equiv \bar{t}_{m+1}^{\sigma_{m+1}}(a) = \bar{t}_m = \bar{t}_m^{\sigma_{m+1}} \).

(e) \( \bar{t}_{m+1}^{\sigma_{m+1}}(a) \equiv \bar{t}_m \equiv \bar{t}_m^{\sigma_{m+1}} = \bar{t}_{m+1}^{\sigma_{m+1}}(a) \equiv \bar{t}_m = \bar{t}_m^{\sigma_{m+1}} = \bar{t}_m \equiv \bar{t}_m^{\sigma_{m+1}} = \bar{t}_m^{\sigma_{m+1}} \).

(f) \( \bar{t}_m = \bar{t}_m \equiv \bar{t}_m = \bar{t}_m^{\sigma_{m+1}}(a) \equiv \bar{t}_m = \bar{t}_m^{\sigma_{m+1}}(a) \equiv \bar{t}_m = \bar{t}_m^{\sigma_{m+1}} = \bar{t}_m \).

Now the result follows by induction. \( \square \)
We write \( \text{Stab}(\text{Artin}(A_n); [t_{n+1}]) \) to denote the \( \text{Artin}(A_n) \)-stabilizer of the conjugacy class \([t_{n+1}] \) under the \( \text{Artin}(A_n) \)-action on \( \Sigma_{0,1,n+1} \). The Reidemeister-Schreier rewriting technique automatically gives a useful presentation of \( \text{Stab}(\text{Artin}(A_n); [t_{n+1}]) \) but the resulting exposition is tedious. Once the presentation has been found, we can verify it directly using the van der Waerden trick, as in the following proof.

5.3 Theorem (Magnus [24]). Let \( n \geq 1 \).

(i). There exists a homomorphism \( \phi_n : \text{Artin}(A_{n-1}) \times \Sigma_{0,1,n} \to \text{Artin}(A_n) \) determined by \( (a_i, t_{n})^{\phi_n} = (a_i, t_{\bar{a}_i^n}) \).

(ii). \( \phi_n \) is injective.

(iii). For each \( i \in [1|^n] \), \( t_i^{\phi_n} = \bar{a}_i^{2a_i(i+1)n} \) in \( \text{Artin}(A_n) \).

(iv). The image of \( \phi_n \) is \( \text{Stab}(\text{Artin}(A_n); [t_{n+1}]) \).

Proof. Let us write \( G = \text{Artin}(A_n) \) and \( H = \text{Artin}(A_{n-1}) \times \Sigma_{0,1,n} \).

In \( G \),

\[(a_{n-1}a_{n-1}a_{n-1})^n = (a_{n-1}a_{n-1}a_{n-1}) = (a_{n-1}a_{n-1}a_{n-1}) = a_{n-1}a_{n}^2a_{n-1}, \]

and, hence, \( a_{n-1}a_{n-1}a_{n-1}^2 = a_{n}^2a_{n-1}a_{n-1}^2 \). By Lemma 5.2, \( H \cong \text{Artin}(B_n) \), and we see that there exist a homomorphism \( \phi_n : H \to G \) determined by

\[(a_i, t_{n})^{\phi_n} = (a_i, a_{n-1}^2) \text{ and (i) is proved.} \]

Let \( v \) be a symbol and let \( H \times v_{[1|n+1]} \) denote a free left \( H \)-set with left \( H \)-transversal \( v_{[1|n+1]} \). We construct a right \( G \)-action on \( H \times v_{[1|n+1]} \) such that \( H \times v_{[1|n+1]} \) becomes an \((H, G)\)-bi-set. For each \( i \in [1|^n] \), the element \( a_i \) of \( G \) acts on the right on \( H \times v_{[1|n+1]} \) as the left \( H \)-map that is determined by the following.

\[
\begin{array}{c|c|c|c|c|c}
\hline
k \in [1|^i-1)] & k \in [i+1](n+1)] & k \in [(i+2)](n+1)] \\
\hline
(a_i, v_i) & v_i & v_{i+1} & v_k & v_j & v_{j+1} \\
\hline
(a_i, v_i) & v_j & v_{j+1} & v_k & v_i & v_{i+1} \\
\hline
(a_i, v_i) & v_{j+1} & v_k & v_i & v_{i+1} & v_j \\
\hline
(a_i, v_i) & v_k & v_{i+1} & v_j & v_{j+1} & v_{k+1} \\
\hline
\end{array}
\]

We now verify that the relations of \( G \) are respected.

(a). Suppose that \( 1 \leq i < j + 2 \leq n \). We have the following.

\[
\begin{array}{c|c|c|c|c|c|c|c|c|c}
\hline
k \in [1|^i-1)] & k \in [(i+2)](n+1)] & k \in [(j+2)](n+1)] \\
\hline
(a_i, v_i) & v_i & v_{i+1} & v_k & v_j & v_{j+1} & v_{k+1} & v_{k+2} & v_{k+3} & v_{k+4} \\
\hline
(a_i, v_i) & v_j & v_{j+1} & v_k & v_i & v_{i+1} & v_{i+2} & v_{i+3} & v_{i+4} & v_{i+5} \\
\hline
(a_i, v_i) & v_{j+1} & v_k & v_i & v_{i+1} & v_{i+2} & v_{i+3} & v_{i+4} & v_{i+5} & v_{i+6} \\
\hline
(a_i, v_i) & v_k & v_{i+1} & v_j & v_{j+1} & v_{k+1} & v_{k+2} & v_{k+3} & v_{k+4} & v_{k+5} \\
\hline
\end{array}
\]

There is no full answer available.
(b) Suppose that $1 \leq i \leq n - 1$. We have the following.

\[
\begin{align*}
\text{if } k \in [1, (i-1)] \quad & (v_k, v_i, v_{i+1}, v_{i+2}, v_k) \xrightarrow{a_{i-1} a_i} (v_k, v_i, v_{i+1}, v_{i+2}, v_i) \\
\text{if } k \in [(i+2), (n+1)] \quad & (v_k, v_i, v_{i+1}, v_{i+2}, v_k) \xrightarrow{a_i a_{i+1}} (v_k, v_i, v_{i+1}, v_{i+2}, v_k)
\end{align*}
\]

By (a) and (b), the relations of $G$ are respected. Hence, we have a right $G$-action on $H \times v_{[1,(n+1)]}$ by left $H$-maps.

Notice that $\tau_n^\phi v_n = \tau_n \tau_v \sigma_n = \tau_n \tau_v \sigma_n = \tau_n \tau_v \sigma_n$, and, for each $i \in [1,(n-1)]$, $v_{n+1}^\phi = v_{n+1} \sigma_n = a_i v_{n+1}$. It follows that, for each $h \in H$, $v_{n+1}^\phi h = hv_{n+1}$.

Hence, $\phi_n$ is injective. This proves (ii).

Recall that $G = \text{Artin}(A_n)$.

Let $i \in [1,n]$. We shall show by decreasing induction on $i$ that

\[
(5.3.1) \quad a_i \pi_{n(i+1)\mid i} = a_i \pi_{n(i-1)\mid i}.
\]

If $i = n$, then (5.3.1) holds. Now suppose that $i \geq 2$, and that (5.3.1) holds. Conjugating (5.3.1) by $\tau_{n-1}$ yields

\[
a_n \pi_{n(i-1)\mid i} = a_i \pi_{n(i+1)\mid i} \pi_{n(i-1)\mid i} = a_i \pi_{n(i+1)\mid i} = a_i \pi_{n(i+1)\mid i} = a_i \pi_{n(i+1)\mid i}.
\]

By induction, (5.3.1) holds.

Now $\tau_{\phi_n} = (\tau_n \pi_{n(i+1)\mid i})^{-1} \pi_{n(i-1)\mid i} = a_i \pi_{n(i+1)\mid i}$. This proves (iii).

Also, $\tau_{\phi_n} \pi_{n(i+1)\mid i} = (\pi_{n(i+1)\mid i}) a_i \pi_{n(i+1)\mid i}$.

If $k \in [1,(i-1)]$, then

\[
a_i \pi_{n(k\mid i)} = a_i \pi_{n(k\mid i)} \pi_{n(i+1)\mid i} = a_i \pi_{n(k\mid i)} \pi_{n(i+1)\mid i} = a_i \pi_{n(i+1)\mid i} = a_i \pi_{n(i+1)\mid i}.
\]

Hence, $a_i \pi_{n(k\mid i)} = (\pi_{n(k\mid i)}) a_i \pi_{n(i+1)\mid i}$.

Let $\psi_n$ denote the map of sets

\[
\psi_n : H \times v_{[1,n+1]} \rightarrow G, \quad hv_k \mapsto h^\phi \pi_{n(k\mid i)} \pi_{n(i+1)\mid i} \pi_{n(i-1)\mid i} \pi_{n(i-1)\mid i} = (h, v_k) \in H \times v_{[1,(n+1)]}.
\]

Hence, for each $h \in H$, we have the following, in $G$.

\[
\begin{align*}
\text{if } k \in [1, (i-1)] \quad & (h \ (v_k, v_i, v_{i+1}, v_{i+2}, v_k)) \xrightarrow{a_i \pi_{n(i+1)\mid i}} (h \ (v_k, v_i, v_{i+1}, v_{i+2}, v_i)) \\
\text{if } k \in [(i+2), (n+1)] \quad & (h \ (v_k, v_i, v_{i+1}, v_{i+2}, v_k)) \xrightarrow{a_i \pi_{n(i+1)\mid i}} (h \ (v_k, v_i, v_{i+1}, v_{i+2}, v_k))
\end{align*}
\]

\[
\]
This proves that \( \psi_n \) is a map of right \( G \)-sets, and hence, \( \psi_n \) must be surjective. Thus, \( G = \bigcup_{k \in [1, n+1]} H^{\phi_n} \), and hence, the index of \( H^{\phi_n} \) in \( G \) is at most \( n+1 \).

Consider the action of \( G \) on the set of conjugacy classes \( \{[t_k]\}_{k \in [1, n+1]} \) in \( \Sigma_{0, n} \). For any \( i \in [1, n] \), \( a_i \) acts as the transposition \( ([t_k], [t_{k+1}]) \). In particular, the index of \( \text{Stab}(G; [t_{n+1}]) \) in \( G \) is \( n+1 \). Also, the elements of \( a_{[1, n]} \setminus \{a_n^2\} \) fix \( [t_{n+1}] \), and, hence, \( H^{\phi_n} \leq \text{Stab}(G; [t_{n+1}]) \). By comparing indices in \( G \), we see that \( H^{\phi_n} = \text{Stab}(G; [t_{n+1}]) \). This proves (iv). \( \square \)

5.4 Theorem (Artin). \( \mathcal{B}_n = \text{Artin}(\sigma_1 \cdots \sigma_{n-1}) \).

Proof. This is trivial for \( n \leq 1 \).

Hence, we may assume that \( n \geq 1 \) and that the homomorphism
\[
\gamma_n: \text{Artin}(A_{n-1}) \to \mathcal{B}_n, \text{ of Proposition 5.1, determined by } (a_i)^{\gamma_n} = (\sigma_i)
\]

is an isomorphism. By induction, it remains to show that the surjective homomorphism
\[
\gamma_{n+1}: \text{Artin}(A_n) \to \mathcal{B}_{n+1}
\]
is injective.

Consider an element \( w \) of the kernel of \( \gamma_{n+1} \). In particular, \( w \) fixes \( t_{n+1} \) in the \( \text{Artin}(A_n) \)-action on \( \Sigma_0, n+1 \). By Theorem 5.3(iv), \( w \) lies in the image of the homomorphism \( \phi_n: \text{Artin}(A_{n-1}) \ltimes \Sigma_0, n \to \text{Artin}(A_n) \) determined by \( i \in [1, n-1] \)
\[
(a_i \ t_{n+1})^{\phi_n} = (a_i \ t_{n+1}^2) \quad \text{and there is a resulting factorization of the form } w = w(a_{[1, n-1]})w_2(t_{[1, n]}^{\phi_n}).
\]

(5.4.1) in \( \text{Artin}(A_n) \ltimes \Sigma_0, n+1 \), \( t_{n+1} = \frac{w}{w_1} = \frac{w_1(\sigma_{[1, n-1]})(a_n^{\phi_n})}{w_2(t_{[1, n]}^{\phi_n})} = \frac{w_2(t_{[1, n]}^{\phi_n})}{w_2(t_{[1, n]}^{\phi_n})} \).

Consider the homomorphism \( \phi_{n+1}: \text{Artin}(A_n) \ltimes \Sigma_0, n+1 \to \text{Artin}(A_{n+1}) \) determined by \( (a_i \ t_{n+1})^{\phi_{n+1}} \). Let \( i \in [1, n] \). By Theorem 5.3(iii),
\[
(a_i \ t_{n+1}^2) = (a_i \ t_{n+1}^{\phi_{n+1}})
\]

In particular, the two sequences \( t_{[1, n]} \) and \( t_{[1, n-1]} \) (in \( \text{Artin}(A_n) \ltimes \Sigma_0, n+1 \)) become conjugate (in \( \text{Artin}(A_{n+1}) \)) under \( \phi_{n+1} \). By Theorem 5.3(ii), \( \phi_{n+1} \) is injective. Since \( t_{[1, n+1]} \) freely generates the free subgroup \( \Sigma_0, n+1 \) of \( \text{Artin}(A_n) \ltimes \Sigma_0, n+1 \), we see that \( t_{[1, n]} \) also freely generates a free subgroup of \( \text{Artin}(A_n) \ltimes \Sigma_0, n+1 \). From (5.4.1), we see that \( w_2 \) must be trivial.
Hence, \( w = w_1(a_{[1\uparrow(n-1)]}) \) in Artin\( (A_n) \). By the induction hypothesis, \( w_1(a_{[1\uparrow(n-1)]}) = 1 \) in Artin\( (A_{n-1}) \). Hence \( w = 1 \) in Artin\( (A_n) \).

Now the result holds by induction. \( \square \)

Combining Lemma 5.2, Theorem 5.3 and Theorem 5.4, we have the following.

5.5 Corollary (Artin-Magnus-Manfredini). If \( n \geq 2 \), then

\[
\mathcal{B}_n = \text{Artin}(\sigma_1 \cdot \sigma_2 \cdot \cdots \cdot \sigma_{n-2} \cdot \sigma_{n-1}) \cong \text{Artin}(A_{n-1}),
\]

\[
\text{Stab}(\mathcal{B}_n; [t_n]) = \text{Artin}(\sigma_1 \cdot \sigma_2 \cdot \cdots \cdot \sigma_{n-2} = \sigma_{n-1}^2) \cong \text{Artin}(B_{n-1}),
\]

\[
\mathcal{B}_{n-1} \times \Sigma_{0,1,n-1} = \text{Artin}(\sigma_1 \cdot \sigma_2 \cdot \cdots \cdot \sigma_{n-2} = \tilde{t}_{n-1}) \cong \text{Artin}(B_{n-1}).
\]

5.6 Historical Remarks. In 1925, Artin [3] found the above presentation of \( \mathcal{B}_n \) by an intuitive topological argument; later [4], he indicated that there were difficulties that could be corrected. In 1934, Magnus [24] gave an algebraic proof that the relations suffice. In 1945, Markov [26] gave a similar algebraic proof. In 1947, Bohnenblust [7] gave a similar algebraic proof; in 1948, Chow [8] simplified the latter proof. All these algebraic proofs of the sufficiency of the relations involve the Reidemeister-Schreier rewriting process for the subgroup of index \( n \).

Larue [22] gave a new algebraic proof of the sufficiency of the relations, by using the \( \sigma_1 \)-trichotomy [14] for braid groups. We shall proceed in the opposite direction. Proofs of the \( \sigma_1 \)-trichotomy for Artin\( (A_{n-1}) \) have tended to be more difficult than proofs that Out\( ^+_{0,1,n} = \text{Artin}(A_{n-1}) \), and we shall now see that Artin’s generation argument easily gives the \( \sigma_1 \)-trichotomy for Out\( ^+_{0,1,n} \). \( \square \)

6 Three trichotomies

6.1 Definitions. Let \( \phi \in \mathcal{B}_n \).

We say that \( \phi \) is \( \sigma_1 \)-neutral if \( \phi \) lies in the subgroup of \( \mathcal{B}_n \) generated by \( \sigma_{[2\uparrow(n-1)]} \). This holds automatically if \( n \leq 1 \).

We say that \( \phi \) is \( \sigma_1 \)-positive if \( n \geq 2 \) and \( \phi \) has a monoid expression in \( \sigma_{[1\uparrow(n-1)]} \lor \bar{\sigma}_{[2\uparrow(n-1)]} \) such that at least one term of the expression is \( \sigma_1 \). We say that \( \phi \) is \( \sigma \)-positive if \( n \geq 2 \) and, for some \( i \in [1\uparrow(n-1)] \), \( \phi \) has a monoid expression in \( \sigma_{[1\uparrow(n-1)]} \lor \bar{\sigma}_{[i\uparrow(i+1\uparrow)(n-1)]} \) such that at least one term of the expression is \( \sigma_i \).

We say that \( \phi \) is \( \sigma_1 \)-negative if \( \bar{\phi} \) is \( \sigma_1 \)-positive, that is, \( n \geq 2 \) and \( \phi \) has a monoid expression in \( \sigma_{[2\uparrow(n-1)]} \lor \bar{\sigma}_{[1\uparrow(n-1)]} \) such that at least one term of the expression is \( \sigma_1 \).

If \( \phi \) satisfies exactly one of the properties of being \( \sigma_1 \)-neutral, \( \sigma_1 \)-positive \( \sigma_1 \)-negative, we say that \( \phi \) satisfies the \( \sigma_1 \)-trichotomy.

If every element of \( \mathcal{B}_n \) satisfies the \( \sigma_1 \)-trichotomy, then we say that \( \mathcal{B}_n \) satisfies the \( \sigma_1 \)-trichotomy. \( \square \)
6.2 Historical Remarks. View Artin(\(A_n\)) as a subgroup of Artin(\(A_{n+1}\)) in a natural way, and let Artin(\(A_{\infty}\)) denote the union of the resulting chain; thus Artin(\(A_{\infty}\)) = \(\langle a_{[1,\infty]} \rangle\). Dehornoy [14, Theorem 6] gave a one-sided ordering of Artin(\(A_{\infty}\)); the positive semigroup for this ordering is the set of ‘\(a\)-positive’ elements of Artin(\(A_{\infty}\)).

Let \(\phi \in \mathcal{B}_n\). By replacing \(\phi\) with \(\bar{\phi}\) if necessary, we can apply Dehornoy’s result to deduce that there exists some \(n' \geq n\) such that \(\phi\) is \(\sigma\)-negative in \(\mathcal{B}_{n'}\), or \(\phi = 1\). Larue [21] showed that this implies that \(t_1^\phi \in (t_1^*)\) and that this in turn implies that \(\phi\) has a monoid expression in \(\sigma_{[2\mid(n-1)]} \vee \sigma_{[1\mid(n-1)]}\), of length at most \(|\phi| + \frac{1}{2}n^2|\phi|\). Thus, \(\mathcal{B}_n\) satisfies the \(\sigma_1\)-trichotomy. Larue’s work is surveyed in [16, Chapter 6]. Fenn-Greene-Rolfsen-Rourke-Wiest [19] gave a direct topological proof of the \(\sigma_1\)-trichotomy for \(\mathcal{B}_n\) without being aware of Larue’s work and without applying Dehornoy’s result. Their work is surveyed in [16, Chapter 6].

We shall give elementary direct proofs of the foregoing results and replace Larue’s bound \(|\phi| + \frac{1}{2}n^2|\phi|\) with the much smaller bound \(n|\phi| - n\). Larue’s proof contains much interesting information that we shall rework in the Appendix. □

Part (iii) of the following is new.

6.3 Lemma. Let \(n \geq 1\) and let \(\phi\) be an element of \(\mathcal{B}_n\) such that \(t_1^\phi \in (t_1^*)\). Let \(\pi = \pi(\phi)\) and, for each \(i \in [1\mid n]\), let \(u_i = u_i(\phi)\).

(i). Suppose that there exists some \(i \in [1\mid(n-1)]\) such that \(u_i \in (\ast \bar{t}_i(i+1)^*)\). Then \(\|\pi_i^\phi\| \leq \|\phi\| - 2\) and \(t_1^\phi t_i^\phi \in (t_1^*)\). Moreover, if \(t_1^\phi = t_1\), then \(i \in [2\mid(n-1)]\).

(ii). Suppose that there exists some \(i \in [2\mid(n-1)]\) such that \(u_i \in (\bar{t}_i^* \ast \ast)\). Then \(\|\pi_i^\phi\| \leq \|\phi\| - 2\) and \(t_1^\phi t_i^\phi \in (t_1^*)\).

(iii). Suppose that, for each \(i \in [1\mid(n-1)]\), \(u_i \notin (\ast \bar{t}_i(i+1)^*)\) and, for each \(i \in [2\mid(n-1)]\), \(u_i \notin (\bar{t}_i^* \ast \ast)\). Then \(\phi = 1\).

Proof. For each \(i \in [0\mid(n+1)]\), let \(w_i = w_i(\phi)\).

(i). The first conclusion follows from Artin’s Lemma 3.2(i). Notice that, if \(t_1^\phi = t_1\), then \(w_1 = 1\) and \(u_1 = \bar{w}_2 \notin (\ast \bar{t}_2^*)\).

(ii) follows from Lemma 3.2(ii).

(iii). Recall that \(u_0 \prod_{i \in [1\mid n]} (t_i \ast u_i) = \prod_{i \in [1\mid n]} (t_i^{w_i}) = (\prod_{i \in [1\mid n]} t_i)^\phi = \prod_{i \in [1\mid n]} t_i\). Hence,

\[ u_0 t_1^* \prod_{i \in [1\mid n]} (t_i \ast u_i) = t_1 \prod_{i \in [1\mid n]} t_i \text{ and, hence,} \]

\[ |u_1 \prod_{i \in [2\mid n]} (t_i \ast u_i)| = |\bar{t}_1^* \bar{u}_0 t_1 \prod_{i \in [2\mid n]} t_i| \leq |\bar{t}_1^* \bar{u}_0 t_1| + n - 1. \]
Since \( u_n = w_n \not\in (\tilde{t}_n^\ast) \), the hypotheses imply that there is no cancellation anywhere in the expression \( u_1 \prod_{i \in [2\mid n]} (t_{i^\ast} u_i) \). Hence,

\[
(6.3.2) \quad \sum_{i \in [1\mid n]} |u_i| + n - 1 = |u_1 \prod_{i \in [2\mid n]} (t_{i^\ast} u_i)| \leq |\tilde{t}_1 u_0 t_1| + n - 1.
\]

Since \( t_1^{\mu_0} = t_1^{\mu_1} = t_1^{\phi} \in (t_1^\ast) \), we see that \( u_0 t_1^\ast \in (t_1^\ast) \), and

\[
(6.3.3) \quad |\tilde{t}_1 u_0 t_1^\ast| = -1 + |u_0 t_1^\ast| \leq -1 + |u_0| + 1 = |u_0|.
\]

Since \( \prod_{i \mid n} u_{[1\mid n]} = w_0 \overline{w}_{n+1} = 1 \), we see that

\[
(6.3.4) \quad \prod_{i \mid n} u_{[1\mid n]} = \overline{\mu_0} = w_1 \not\in (\tilde{t}_1^\ast).
\]

Now, \( \sum_{i \in [1\mid n]} |u_i| \leq |\tilde{t}_1 u_0 t_1| \leq |\overline{\mu_0}| = |\prod_{i \mid n} u_{[1\mid n]}| \). Therefore, there is no cancellation in \( \prod_{i \mid n} u_{[1\mid n]} \), and, by (6.3.4), \( u_1 \not\in (\tilde{t}_1^\ast) \). By Lemma 3.2(iii), \( \phi = 1 \).

As in Remarks 3.4, we deduce the following from Lemma 6.3 by induction on \( \|\phi\| \).

6.4 Corollary (Larue [21]). Let \( n \geq 1 \) and let \( \phi \in \mathcal{B}_n \).

(i). If \( t_1^{\phi} \in (t_1^\ast) \), then \( \phi \) has a monoid expression in \( \sigma_{[2\mid (n-1)]} \lor \sigma_{[1\mid (n-1)]} \) of length at most \( \frac{\|\phi\| - n}{2} \leq n2^{\|\phi\|} - n \). In particular, \( \phi \) is \( \sigma_1 \)-negative or \( \sigma_1 \)-neutral.

(ii). \( \phi \) is \( \sigma_1 \)-neutral if and only if \( t_1^{\phi} = t_1 \).

6.5 Notation. For each \( i \in [1\mid (n-1)] \), let \( \sigma_i' \) and \( \sigma_i'' \) be the automorphisms of \( \Sigma_{0,1,n} \) determined by

\[
\begin{align*}
\sigma'_{k \in [1\mid i]} &= (t_k \ t_{i+1} \ t_k)_{i+1} = (t_k \ t_{i+1} \ t_k), \\
\sigma''_{k \in [1\mid i]} &= (t_k \ t_i \ t_{i+1} \ t_k)_{i+1} = (t_k \ t_{i+1} \ t_i \ t_k).
\end{align*}
\]

Then \( \sigma_i = \sigma_i' \sigma_i'' \). Any normal form in \( t_{[1\mid n]} \) factorizes into an alternating product with factors which are normal forms of non-trivial elements of \( \langle t_{[1\mid (i+1)]} \rangle \) alternating with factors which are normal forms of non-trivial elements of \( \langle t_{[1\mid (i-1)]} \rangle \). On \( t_{[1\mid (i+1)]} \), \( \sigma_i' \) acts as conjugation by \( t_i \), while \( \sigma_i'' \) interchanges the two free generators. On \( t_{[1\mid (i-1)]} \), \( \sigma_i' \) and \( \sigma_i'' \) act as the identity map.

The next result gives three trichotomies, called (a), (b) and (c), which hold for elements of \( \mathcal{B}_n \). Attribution is not sharply defined, but it is reasonable to attribute (b) to Dehorney [14], and (c) to Larue [21].
6.6 Theorem (Dehornoy-Larue [14], [21]). Let \( n \geq 1 \), let \( \phi \in \mathcal{B}_n \), and consider the following nine assertions:

\[
\begin{align*}
(a1). & \ t_1^\phi = t_1. \\
(b1). & \ \phi \text{ is } \sigma_1\text{-neutral.} \\
(c1). & \ (t_1^*)^\phi = (t_1^*) \quad (c2). \ (t_1^*)^\phi \subset (t_1^*) \\
(a2). & \ t_1^{\phi^0} \in (t_1^*) - \{t_1\}. \\
(b2). & \ \phi \text{ is } \sigma_1\text{-negative.} \\
(c2). & \ (t_1^*)^{\phi^0} \subset (t_1^*) \\
(a3). & \ t_1^{\phi^0} \not\in (t_1^*) \\
(b3). & \ \phi \text{ is } \sigma_1\text{-positive.} \\
(c3). & \ (t_1^*)^{\phi^0} \supset (t_1^*) \\
\end{align*}
\]

Then the following column-equivalences hold:

\[
\begin{align*}
(a1) \iff (b1) \iff (c1); & \quad (a2) \iff (b2) \iff (c2); & \quad (a3) \iff (b3) \iff (c3).
\end{align*}
\]

Hence, exactly one of (b1), (b2), (b3), holds; that is, \( \phi \) satisfies the \( \sigma_1 \)-trichotomy. Hence, \( \mathcal{B}_n \) satisfies the \( \sigma_1 \)-trichotomy.

Proof. (a1) \( \iff \) (b1) by Corollary 6.4(ii). We shall use (a1) and (b1) interchangeably in the remainder of the proof.

(b1) \( \Rightarrow \) (c1). If \( \phi \) is \( \sigma_1 \)-neutral, then so is \( \overline{\phi} \). It follows that \( (t_1^*)^\phi \subseteq (t_1^*) \) and \( (t_1^*)^{\overline{\phi}} \subseteq (t_1^*) \). Thus, \( (t_1^*)^\phi = (t_1^*) \).

(a2) \( \Rightarrow \) (b2). If (a2) holds, then Corollary 6.4(i) shows that (b1) or (b2) holds. Since (a1) fails, (b1) fails. Thus (b2) holds.

(b2) \( \Rightarrow \) (c2). Using Notation 6.5, we see that

\[
(t_1^*)^{\mathcal{T}_1^1} = (t_1^*)^{\mathcal{T}_1^1} = (t_1^*)^{\mathcal{T}_1^1} \subseteq (t_1^*) \subseteq (t_1^*).
\]

Since the composition of injective self-maps of \( (t_1^*) \) can be bijective only if all the factors are bijective, we see that (b2) \( \Rightarrow \) (c2).

(a3) \( \Rightarrow \) (b3). We translate into algebra the crucial reflection argument of [16, Corollary 5.2.4].

Suppose that (a3) holds.

With Notation 3.1, let \( w_1 = w_1(\phi) \) and \( \pi = \pi(\phi) \). Then \( \overline{w_1}t_1w_1 = t_1^\phi \not\in (t_1^*) \).

It follows that \( \overline{w_1}t_1w_1 = t_1^\phi \not\in (t_1^*) \). Hence, \( \overline{w_1}t_1w_1 \not\in (t_1^*) \). Hence, \( t_1^\phi = \overline{w_1}t_1w_1 \not\in (t_1^*) \cup \{1\} \). On conjugating by \( t_1 \), we see that \( t_1^\phi \mathcal{T}_1^1 \in (\mathcal{T}_1^1) \).

Let \( \zeta \) be the automorphism of \( \Sigma_{0,1,n} \) determined by

\[
(\begin{array}{c}
(\prod_{k \in [1\mid n]} t_k^\mathcal{P}_0) \\
(\prod_{k \in [2\mid n]} t_k^\mathcal{P}_0)
\end{array})^\zeta = (\begin{array}{c}
(\prod_{k \in [1\mid n]} t_k^\mathcal{P}_0) \\
(\prod_{k \in [2\mid n]} t_k^\mathcal{P}_0)
\end{array})
\]

For each \( k \in [1\mid n] \), \( (\prod_{k \in [1\mid k]} t_k^\mathcal{P}_0) = \prod_{k \in [1\mid k]} t_k^\mathcal{P}_0 \). It follows that \( \zeta^2 = 1 \). Notice that \( \zeta \) belongs to \( \text{Out}^{+}_{0,1,n} := \text{Out}_{0,1,n} - \text{Out}^{-}_{0,1,n} \). Also, \( (t_1 t_k) \mathcal{T}_1^\zeta = (t_1 t_k) \mathcal{T}_1^\zeta \). Hence,

\[
(\begin{array}{c}
(\prod_{k \in [1\mid n]} t_k^\mathcal{P}_0) \\
(\prod_{k \in [2\mid n]} t_k^\mathcal{P}_0)
\end{array})^\zeta = (\begin{array}{c}
(\prod_{k \in [1\mid n]} t_k^\mathcal{P}_0) \\
(\prod_{k \in [2\mid n]} t_k^\mathcal{P}_0)
\end{array})
\]

By Corollary 6.4(i), \( \phi^\zeta \) has a monoid expression in \( \sigma_{[2\mid n-1]} \cup \overline{\sigma}_{[1\mid n-1]} \). It is not difficult to check that, for each \( i \in [1\mid (n-1)] \), \( \sigma_i^\zeta = \overline{\sigma}_i \) in \( \text{Out}_{0,1,n} \). Hence
\( \phi^k \) has a monoid expression in \( \sigma_{2[n]} \vee \overline{\sigma}_{1[n]} \).
Hence, (b3) or (b1) holds. Since (a3) holds, (a1) fails, and (b1) fails. Thus (b3) holds.

(b3) \( \Rightarrow \) (c3). If \( \phi \) is \( \sigma_1 \)-positive, then \( \phi \) is \( \sigma_1 \)-negative, and, by (b2) \( \Rightarrow \) (c2),

(c1) \( \Rightarrow \) (a1). Suppose that (a1) fails. Then (a2) or (a3) holds. Hence (c2)
or (c3) holds. Hence (c1) fails.

(c2) \( \Rightarrow \) (a2) and (c3) \( \Rightarrow \) (a3) are proved similarly.

Thus the desired equivalences hold.

Since exactly one of (a1), (a2), (a3) holds, exactly one of (b1), (b2), (b3) holds.

Recall the definition of \( \sigma \)-positive from Definitions 6.1.

6.7 Theorem (Dehornoy [14]). For each \( \phi \in \mathcal{B}_n \) exactly one of the following holds: \( \phi = 1 \); \( \phi \) is \( \sigma \)-positive; \( \phi \) is \( \sigma \)-negative. The set of \( \sigma \)-positive elements of \( \mathcal{B}_n \) is the positive cone of a right-ordering of \( \mathcal{B}_n \), called the Dehornoy right-ordering of \( \mathcal{B}_n \).

Proof. Suppose that \( \phi \neq 1 \).

Let \( i \) be the largest element of \( \mathcal{I}[n-1] \) such that \( \phi \in \mathcal{I}[n-1] \). The natural subscript-shifting isomorphism from \( \mathcal{I}[n] \) to \( \Sigma_{0,n-i+1} \) induces an isomorphism from \( \mathcal{I}[n-1] \) to \( \mathcal{B}_{n-i+1} \). Notice that \( \phi \) is mapped to an element of \( \mathcal{B}_{n-i+1} \) which is not \( \sigma_1 \)-neutral; by Theorem 6.6, this image is \( \sigma_1 \)-positive or \( \sigma_1 \)-negative but not both. Hence exactly one of \( \phi, \overline{\phi} \) is \( \sigma \)-positive.

It is easy to see that the product of two \( \sigma \)-positive elements of \( \mathcal{B}_n \) is \( \sigma \)-positive.

Hence the set of \( \sigma \)-positive elements of \( \mathcal{B}_n \) is the positive cone for a right-ordering of \( \mathcal{B}_n \).

\[ \square \]

7 Ends, right-orderings and squarefreeness

7.1 Review. An end of \( \Sigma_{0,n} \) is a sequence \( a_{1[n]} \) in \( t_{1[n]} \vee \overline{t}_{1[n]} \) such that, for each \( i \in [1][n] \), \( a_i \neq \overline{a}_i \). We represent \( a_{1[n]} \) as a formal right-infinite reduced product, \( a_1 a_2 \cdots \) or \( \Pi a_{1[n]} \).

We denote the set of ends of \( \Sigma_{0,n} \) by \( \mathcal{E}(\Sigma_{0,n}) \), or simply by \( \mathcal{E} \) if there is no risk of confusion.

An element of \( \Sigma_{0,n} \cup \mathcal{E}(\Sigma_{0,n}) \) is said to be squarefree if, in its reduced expression, no two consecutive terms are equal; for example: \( t_1 t_2 \) is a squarefree end; \( t_1 t_2 t_3 \) is non-squarefree.

For each \( w \in \Sigma_{0,n} \), we define the shadow of \( w \) in \( \mathcal{E} \) to be

\[ (w \downarrow) \:= \{ a_{1[n]} \in \mathcal{E} \mid \Pi a_{1[n]} = w \} \].
Thus, for example, \((1 \bullet) = \mathcal{E}\).

We now give \(\mathcal{E}\) an ordering, \(<\), as follows. For each \(w \in \Sigma_{0,1,n}\), we assign an ordering, \(<\), to a partition of \((w \bullet)\) into \(2n\) or \(2n-1\) subsets, depending as \(w = 1\) or \(w \neq 1\), as follows. We set

\[
(t_1 \bullet) < (\tilde{t}_1 \bullet) < (t_2 \bullet) < (\tilde{t}_2 \bullet) < \cdots < (t_n \bullet) < (\tilde{t}_n \bullet).
\]

If \(i \in [1 \uparrow n]\) and \(w \in (\star t_i)\), then we set

\[
(w \tilde{t}_i \bullet) < (w t_{i+1} \bullet) < (w \tilde{t}_{i+1} \bullet) < (w t_{i+2} \bullet) < \cdots
\]

\[
\cdots < (w t_n \bullet) < (w \tilde{t}_n \bullet) < (w t_{n+1} \bullet) < (w \tilde{t}_{n+1} \bullet) < (w t_2 \bullet) < \cdots
\]

\[
\cdots < (w t_{i-1} \bullet) < (w \tilde{t}_{i-1} \bullet) < (w t_i \bullet).
\]

If \(i \in [1 \uparrow n]\) and \(w \in (\star t_i)\), then we set

\[
(w t_{i+1} \bullet) < (w \tilde{t}_{i+1} \bullet) < (w t_{i+2} \bullet) < (w \tilde{t}_{i+2} \bullet) < \cdots
\]

\[
\cdots < (w t_n \bullet) < (w \tilde{t}_n \bullet) < (w t_{n-1} \bullet) < (w \tilde{t}_{n-1} \bullet) < (w t_1 \bullet) < \cdots
\]

\[
\cdots < (w t_{i-1} \bullet) < (w \tilde{t}_{i-1} \bullet) < (w t_i \bullet).
\]

Hence, for each \(w \in \Sigma_{0,1,n}\), we have an ordering \(<\) of a partition of \((w \bullet)\) into \(2n\) or \(2n-1\) subsets.

If \(a_{[1 \uparrow \infty]}\) and \(b_{[1 \uparrow \infty]}\) are two different ends, then there exists \(i \in \mathbb{N}\) such that \(a_{[1 \uparrow i]} = b_{[1 \uparrow i]}\) and \(a_{i+1} \neq b_{i+1}\). Let \(w = \Pi a_{[1 \uparrow i]} = \Pi b_{[1 \uparrow i]} \in \Sigma_{0,1,n}\). Then \(a_{[1 \uparrow \infty]}\) and \(b_{[1 \uparrow \infty]}\) lie in \((w \bullet)\), but lie in different elements of the partition of \((w \bullet)\) into \(2n\) or \(2n-1\) subsets. We then order \(a_{[1 \uparrow \infty]}\) and \(b_{[1 \uparrow \infty]}\) using the order of the elements of the partition of \((w \bullet)\) that they belong to. This completes the definition of the ordering \(<\) of \(\mathcal{E}\).

We remark that the smallest element of \(\mathcal{E}\) is \(\bar{z}_1^{\infty} = (\Pi t_{[1 \uparrow n]})^{\infty}\) and the largest element of \(\mathcal{E}\) is \(\bar{z}_1^{\infty} = (\Pi t_{[n \uparrow 1]})^{\infty}\).

\[\square\]

### 7.2 Review

By work of Nielsen-Thurston [9], [29], there is an order-preserving action of \(\mathcal{B}_n\) on \((\mathcal{E}(\Sigma_{0,1,n}), \leq)\); we shall give an elementary version of this result.

We assume that \(n \geq 2\), and we first define the action of \(\sigma_1\) on \(\mathcal{E}\).

Consider any \(\epsilon \in \mathcal{E}\). There is then a unique factorization \(\epsilon = \Pi w_{[1 \uparrow i]}\) or \(\epsilon = \Pi w_{[1 \uparrow \infty]}\), where, in the former case, \(w_{[1 \uparrow (i-1)]}\) is a finite sequence of non-trivial group elements, and \(w_i\) is an end, and, in the latter case, \(w_{[1 \uparrow \infty]}\) is an infinite sequence of non-trivial group elements, and in both cases, the \(w_i\) alternate between elements of \(\langle t_{[1 \uparrow 2]} \rangle \cup \mathcal{E}(\langle t_{[1 \uparrow 2]} \rangle)\), and elements of \(\langle t_{[3 \uparrow n]} \rangle \cup \mathcal{E}(\langle t_{[3 \uparrow n]} \rangle)\). We shall express this factorization as \(\epsilon = [w_1][w_2] \cdots\).

Recall, from Notation 6.5, that we have the factorization \(\sigma_1 = \sigma_1'\sigma_1''\). On \(\langle t_{[1 \uparrow 2]} \rangle \cup \mathcal{E}(\langle t_{[1 \uparrow 2]} \rangle)\), \(\sigma_1'\) acts as conjugation by \(t_1\), while \(\sigma_1''\) interchanges the two free generators. On \(\langle t_{[3 \uparrow n]} \rangle \cup \mathcal{E}(\langle t_{[3 \uparrow n]} \rangle)\), \(\sigma_1'\) and \(\sigma_1''\) act as the identity map. This completes the description of the action of \(\sigma_1', \sigma_1''\) and \(\sigma_1\) on \(\mathcal{E}\).
It is not difficult to show that, for any ends $a_{[1,\infty]}$ and $b_{[1,\infty]}$, if $(a_{[1,\infty]})^{\sigma_1} = b_{[1,\infty]}$, then for all $i, j \in \mathbb{N}$, if $j \geq 2i$, then $(\Pi a_{[1,j]})^{\sigma_1} \in (\Pi b_{[1,i]})^{\sigma_1}$. Thus, $(a_{[1,\infty]})^{\sigma_1} = \lim_{j \to \infty} ((\Pi a_{[1,j]})^{\sigma_1})$

It is clear that $\sigma_1, \sigma_2$ and, hence, $\sigma_1$ act bijectively on $\mathfrak{C}$. Hence we have the action of $\mathfrak{S}_1$ on $\mathfrak{C}$. It is then not difficult to verify that we have an action of $\mathfrak{B}_n$ on $\mathfrak{C}$.

We next show that $\sigma_1$ respects the ordering of $\mathfrak{C}$. We do this by considering all the ways that two ends can be compared, and the resulting effect of $\sigma_1'$ and $\sigma_1$. We represent the information in tables. In all of the following, we understand that $t_1a, \bar{t}_1b, t_2c,$ and $\bar{t}_2d$ are reduced expressions for elements of $(t_1\cup \mathfrak{C}(t_1\cup t_2))$, and $b \neq 1$. Since $a$ does not begin with $\bar{t}_1$, $a\sigma_1t_2$ begins with $t_1$ or $\bar{t}_1$ or $t_2$. We make the convention that $\Sigma_{0,1,n}$ acts trivially on the right on $\mathfrak{C}$.

Here, the case $w = 1$ does not present any problems.

Here, $w$ does not end with $t_1$, and, hence, $(\bar{t}_2\sigma_1^\prime)$ ends with $t_1$, $\bar{t}_1$ or $t_2$.

Here, $w$ does not end with $t_2$, and, hence, $(\bar{t}_2\sigma_1^\prime)$ ends with $t_2$, $\bar{t}_1$ or $\bar{t}_2$.
\[\sigma_{\max(\cdot)}(\cdot) \quad \sigma_1(\cdot) \quad \sigma_{\max(\cdot)}(\cdot)\]

\[
\begin{array}{c|c|c}
\cdots t_3 t_4|\tilde{t}_n \cdots & \cdots t_3 t_4|\tilde{t}_n \cdots & \cdots t_3 t_4|\tilde{t}_n \cdots \\
\cdots t_3[(a_1) \vdots] \cdots & \cdots t_3[(a_1 \sigma_1) [t_2] \vdots] \cdots & \cdots t_3[t_2] [t_3] \tilde{t}_n \cdots \\
\cdots t_3[t_1 b] \cdots & \cdots t_3[t_1 (b_1) \vdots] \cdots & \cdots t_3[t_2 t_3 (b^{\sigma_1} t_2) \vdots] \\
\cdots t_3[t_2 c] \cdots & \cdots t_3[t_1 t_2 (c_1) \vdots] \cdots & \cdots t_3[t_1 c_2 (c_2) \vdots] \\
\cdots t_3[t_3 d] \cdots & \cdots t_3[t_1 d (d_1) \vdots] \cdots & \cdots t_3[t_2 t_3 (d^{\sigma_1} t_2) \vdots] \\
\cdots t_3 \cdots & \cdots t_3 \cdots & \cdots t_3 \cdots \\
\end{array}
\]

The remaining tables are clearly of the same form as the last one. Thus, we have proved that the action of \(\sigma_1\) respects the ordering of \(\mathcal{E}\). It follows that the action of \(\sigma_1\) respects the ordering of \(\mathcal{E}\). Similarly, the action of \(\sigma_{[2\lfloor n-1\rfloor]} \vee \mathcal{E}_{[2\lfloor n-1\rfloor]}\) respects the ordering of \(\mathcal{E}\). Hence \(\mathcal{B}_n\) acts on \((\mathcal{E}, \leq)\).

### 7.3 Remarks (Thurston [29])

The (right) action of \(\mathcal{B}_n\) on \((\mathcal{E}, \leq)\) gives rise to many right orderings of \(\mathcal{B}_n\).

Let us use the left-to-right lexicographic ordering on \((\mathcal{E}^n, \leq)\), and consider the \(\mathcal{B}_n\)-orbit of \(t_{[1]}^{\infty} := (t_{i}^{\infty})_{i \in [1,n]}\). It is not difficult to show that the \(\mathcal{B}_n\)-stabilizer of \(t_{[1]}^{\infty}\) is trivial. Thus we have an injective map

\[
\mathcal{B}_n \to \mathcal{E}^n, \quad \phi \mapsto t_{[1]}^{\infty} := ((t_{i}^{\infty})_{i \in [1,n]}).
\]

Let \(\leq\) denote the ordering of \(\mathcal{B}_n\) induced by pullback from \(\mathcal{E}^n\). Clearly \(\leq\) is a right-ordering of \(\mathcal{B}_n\).

If \(n \geq 2\) and \(\phi \in \mathcal{B}_n\) is \(\sigma_1\)-negative, then, as in the proof of Theorem 6.6(b2)\(\Rightarrow\)(c2), we have \((t_{i}^{\phi}) \leq (t_{i}^{\phi})\). Since \(\max(t_{i}^{\phi}) = t_{i}^{\infty}\) and \(\phi\) respects the ordering, we see that \(t_{i}^{\infty} \leq t_{i}^{\infty}\). Hence \(\phi < 1\) and \(1 < \tilde{\phi}\). Similar arguments with \((t_{i}^{\phi})\), \(i \in [2\lfloor n\rfloor]\), show that, if \(\phi \in \mathcal{B}_n\) is \(\sigma\)-positive (resp. \(\sigma\)-negative), then \(1 < \phi\) (resp. \(1 > \phi\)). Hence the right-ordering of \(\mathcal{B}_n\) obtained from \((t_{[1]}^{\infty})_{\mathcal{B}_n} \subseteq (\mathcal{E}^n, \leq)\) coincides with the Dehornoy right-ordering.

The following will be useful in the study of squarefreeness.

### 7.4 Lemma

Let \(n \geq 1\), let \(i \in [1,n]\), and let \(w \in \Sigma_{0,1,n} - (\ast i) - (\ast \tilde{t}_i)\). Then, in \((\mathcal{E}(\Sigma_{0,1,n}), \leq)\), the following hold:

(i) \(wt_i \overline{w}(\Pi \xi_{[1]}^{\infty}) \leq wt_i(\Pi \xi_{[1]}^{\infty} \Pi | i = 1-1\Pi) = \min(w t_i, \tilde{t}_i)\);

(ii) \(\min(w t_i, \tilde{t}_i) < \max(w t_i, \tilde{t}_i)\);

(iii) \(\max(w t_i, \tilde{t}_i) = \overline{w} t_i(\Pi \xi_{[1]}^{\infty} \Pi | i = 1+1\Pi) \leq \overline{w} t_i \overline{w}(\Pi \Pi | i = 1\Pi)\);

(iv) \((wt_i, \tilde{t}_i) \cup (\overline{w} t_i, \tilde{t}_i) \subseteq [(wt_i \overline{w}(\Pi \xi_{[1]}^{\infty}) \Pi | \overline{w} t_i \overline{w}(\Pi \Pi | i = 1\Pi))]\).

(v) If \(n \geq 3\), then one of the following holds:

(a) \(t_1(\Pi \Pi | i = 1\Pi) < \overline{w} t_i \overline{w}(\Pi \Pi | i = 1\Pi)\);

(b) \(t_1(\Pi \Pi | i = 1\Pi) > \overline{w} t_i \overline{w}(\Pi \Pi | i = 1\Pi)\);
Recall that:

Case 1.

Case 2.

Case 3.

Case 4.

Proof. Recall that:

\[(t_1 \triangledown) < (t_1 \triangledown_1) < (t_2 \triangledown) < \cdots < (t_n \triangledown) < (\bar{t}_n \triangledown),\]

\[(t_i t_{i+1} \triangledown) < (t_i \bar{t}_{i+1} \triangledown) < \cdots < (t_i \bar{t}_n \triangledown) < (t_i \bar{t}_1 \triangledown) < \cdots < (t_i \bar{t}_{i-1} \triangledown) < (t_i \bar{t}_i \triangledown),\]

\[(\bar{t}_i \bar{t}_i \triangledown) < (\bar{t}_i \bar{t}_{i+1} \triangledown) < \cdots < (\bar{t}_i \bar{t}_n \triangledown) < (\bar{t}_i \bar{t}_1 \triangledown) < \cdots < (\bar{t}_i \bar{t}_{i-1} \triangledown) < (\bar{t}_i \bar{t}_i \triangledown).\]

(i). It is straightforward to see that \(w \bar{t}_i((\Pi \bar{t}_{[n+1]})) = \min(w t_i \triangledown).\)

Let \(x\) denote the element of \(t_{[1:n]} \cup \bar{t}_{[1:n]}\) such that \((w \bar{t}_i((\Pi t_{[1:n]})) \in (x \triangledown)\); notice that \(x \neq \bar{t}_i\).

If \(x \neq t_i\), then \((w t_i \triangledown) < (w t_i \triangledown)\), and we have

\[w \bar{t}_i((\Pi t_{[1:n]})) \in (w t_i \triangledown) < (w t_i \triangledown) \ni \min(w t_i \triangledown).\]

If \(x = t_i\), then \(w \bar{t}_i((\Pi t_{[1:n]})) = w \bar{t}_i((\Pi t_{[1:n]})) = \min(w t_i \triangledown).\)

Thus, (i) holds.

(ii) is clear.

(iii). It is straightforward to see that \(w \bar{t}_i((\Pi \bar{t}_{[n+1]})) = \max(w \bar{t}_i \bar{t}_i \triangledown).\)

Let \(x\) denote the element of \(t_{[1:n]} \cup \bar{t}_{[1:n]}\) such that \((w \bar{t}_i((\Pi \bar{t}_{[n+1]})) \in (x \triangledown)\); notice that \(x \neq \bar{t}_i\).

If \(x \neq \bar{t}_i\), then \((w \bar{t}_i \bar{t}_i \triangledown) < (w \bar{t}_i \bar{t}_i \triangledown)\), and we have

\[\max(w \bar{t}_i \bar{t}_i \triangledown) \in (w \bar{t}_i \bar{t}_i \triangledown) < (w \bar{t}_i \bar{t}_i \triangledown) \ni w \bar{t}_i((\Pi \bar{t}_{[n+1]})).\]

If \(x = \bar{t}_i\), then \(w \bar{t}_i((\Pi \bar{t}_{[n+1]})) = w \bar{t}_i((\Pi \bar{t}_{[n+1]})) = \max(w \bar{t}_i \bar{t}_i \triangledown).\)

Thus, (iii) holds.

(iv) follows from (i)-(iii).

(v). It is not difficult to see that

\[w t_i((\Pi t_{[1:n]})) \in (w t_i \triangledown) \quad \text{and} \quad w (\bar{t}_i \triangledown) \in (w \bar{t}_i \triangledown).\]

Case 1. \(w = 1\).

Here, \(t_1((\Pi t_{[1:n]})) \in (t_1 \bar{t}_n \triangledown) \ni (t_1 t_1 \triangledown) = t_1((\Pi t_{[1:n]})),\) and (a) holds.

Case 2. \(w \notin (t_1 \star) \cup \{1\}\).

Here, \(t_1((\Pi t_{[1:n]})) \in (t_1 \bar{t}_n \triangledown) \ni w t_i((\Pi t_{[1:n]}))\), and (a) holds.

Case 3. \(w \in (t_1 t_1 \star)\).

Here, \(t_1((\Pi t_{[1:n]})) \in (t_1 \bar{t}_n \triangledown) \ni w t_i((\Pi t_{[1:n]}))\), and (a) holds.

Case 4. \(w \in (t_1 \star) - (t_1 t_1 \star)\).

Here, \(w \bar{t}_i((\Pi \bar{t}_{[n+1]})) \in (w \bar{t}_i \triangledown) = (t_1 \bar{t}_n \triangledown) = \max(t_1 \bar{t}_n \triangledown) = t_1((\Pi \bar{t}_{[n+1]})).\)
To prove that (b) holds, it remains to show that
\[ w\bar{t}_1\bar{w}((\Pi t_{[n;1]}))^\infty \neq t_1((\Pi t_{[n;1]}))^\infty, \]
that is, \( t_1w\bar{t}_1\bar{w}((\Pi t_{[n;1]}))^\infty \neq ((\Pi t_{[n;1]}))^\infty \), that is, \( t_1w\bar{t}_1\bar{w} \notin (\Pi t_{[n;1]}) \). We can write \( w = t_1u \) where \( u \notin (\bar{t}_1\star) \). Then \( t_1w\bar{t}_1\bar{w} = u\bar{t}_1\bar{u}\bar{w}_1 \), in normal form. Thus it suffices to show that \( u\bar{t}_1\bar{u}\bar{w}_1 \notin (\Pi t_{[n;1]}) \).

If \( u = 1 \), then \( u\bar{t}_1\bar{u}\bar{w}_1 = \bar{t}_1\bar{t}_1 \notin (\Pi t_{[n;1]}) \), since \( n \geq 3 \).

If \( u \neq 1 \), then \( u\bar{t}_1\bar{u}\bar{w}_1 \notin (\Pi t_{[n;1]}) \), since \( u\bar{t}_1\bar{u}\bar{w}_1 \) does not lie in the submonoid \( \Sigma_{0,1,n} \) generated by \( t_{[1;1]} \), nor in the submonoid generated by \( \bar{t}_{[1;1]} \).

In all subcases, (b) holds.

In all four cases, (v) holds. \( \square \)

The following appeared as [5, Lema 2.2.17].

7.5 Theorem. If \( n \geq 1 \) then, for each \( \phi \in B_n \), \( t_1^\phi((\Pi t_{[n;1]}))^\infty \) is a squarefree end.

Proof. This is clear if \( n = 1 \).

For \( n = 2 \), \( B_2 = \{\sigma_1\} \), and
\[ t_1^{B_2} = \{t_1^{t_1^{2m}}, t_1^{t_1^{1+2m}} \mid m \in \mathbb{Z}\} = \{t_1^{(t_1t_2)^m}, t_2^{(t_1t_2)^m} \mid m \in \mathbb{Z}\}. \]
Thus, every element of \( t_1^{B_2} \) is squarefree and does not end in \( \bar{t}_2 \). Hence, every end in \( t_1^{B_2}((\Pi t_{[n;1]})^\infty) \) is squarefree.

Thus, we may assume that \( n \geq 3 \).

Recall that \( z_1 = \Pi t_{[1;1]} \) and \( z_1 = \Pi t_{[1;1]} \). Let \( \cup[t_{[1;1]}] \) denote \( \bigcup_{i \in [1;1]} \{t_i\} \). By Lemma 7.4(v), \( t_1(z_1^\infty) \) does not lie in
\[ \bigcup_{x \in \cup[t_{[1;1]}]} [(x(z_1^\infty), t(z_1^\infty)) \cup (t(x(z_1^\infty)), z_1^\infty))] = \bigcup_{i=1}^{n} w \in \Sigma_{0,1,n}(\star t_i) \cup (w\bar{t}_1\bar{w}(z_1^\infty))). \]
Notice that \( \phi \) permutes the elements of each of the following sets:
\[ \cup[t_{[1;1]}]; \{z_1^\infty\}; \{z_1^\infty\}; \text{ and } \bigcup_{x \in \cup[t_{[1;1]}]} [x(z_1^\infty), z_1^\infty]. \]
Hence \( t_1(z_1^\infty)^\phi \) does not lie in \( \bigcup_{x \in \cup[t_{[1;1]}]} [x(z_1^\infty), z_1^\infty] \). By Lemma 7.4(iv),
\[ \bigcup_{x \in \cup[t_{[1;1]}]} [x(z_1^\infty), z_1^\infty] \supset \bigcup_{i=1}^{n} w \in \Sigma_{0,1,n}(\star t_i) \cup (t_1\bar{t}_1\bar{w}(z_1^\infty))). \]
Hence, \( t_1(z_1^\infty)^\phi \) does not lie in the latter set either, and, hence, \( t_1(z_1^\infty)^\phi \) is a squarefree end. Since \( t_1(z_1^\infty)^\phi = t_1^\phi(z_1^\infty) \), the desired result holds. \( \square \)
Throughout this section, we assume that

Recall from Notation 3.1 that we write $w$ a positive integer

8 Actions on free products of cyclic groups

7.6 Corollary. Let $n \geq 1$, let $\phi \in \mathbb{B}_n$, and let $k \in [1|n]$.

(i). $t_1^\phi$ is squarefree.

(ii). $t_1^\phi \notin (\Pi^t_{[n|k+1]})t_1^{\phi}$.

(iii). $t_1^\phi \notin (\Pi^t_{[1|k-1]})t_1^{\phi}$.

Proof. Recall from Notation 3.1 that we write $t_1^\phi = t_1^{w_1(\phi)}$. Let $\pi = \pi(\phi)$ and $w_1 = w_1(\phi)$.

It is not difficult to see that

$$t_1^\phi(z_1^n) = w_1(t_1^\phi(w_1((\Pi^t_{[1|1]})^\infty)) \in (w_1 \prec).$$

By Theorem 7.5, $t_1^\phi(z_1^n)$ is a squarefree end. Hence, $w_1$ is squarefree, and $w_1 \notin (\Pi^t_{t_k|t_{(k+1)}|n})$.

Since $w_1$ is squarefree, $t_1^\phi$ is also squarefree. Hence (i) holds.

Also, $w_1 \notin (\Pi^t_{t_k|t_{(k+1)}|n})$ implies that $w_1 \notin ((\Pi^t_{[n|k+1]})t_1^{\phi}$ and, hence,

$$t_1^\phi \notin (\Pi^t_{[n|k+1]})t_1^{\phi} \text{ and, also, } t_1^\phi \notin ((\Pi^t_{[n|1]})t_1^{\phi}).$$

In particular, (ii) holds.

Let $\xi$ be the automorphism of $\Sigma_{0,1,n}$ determined by

$$j \in [1|n] \quad (t_j)^\xi = (\tau_{n+1-j}).$$

$\xi^2 = 1$ and $\xi \in \text{Out}_0 \subset \text{Out}_{0,1,n} - \text{Out}_{0,1,n}^+$. Also,

$$t_1^\phi = t_1^{\phi,\xi} = t_1^\phi \notin (\Pi^t_{[n|k+1]})t_1^{\phi} = (\Pi^t_{[1|n-1]}t_1^{n+1-k}).$$

It follows that $t_1^\phi \cap (\Pi^t_{[1|n-k]})t_1^{n+1-k} = \emptyset$. Since $\mathbb{B}_n = \mathbb{B}_{n}$ and $t_1^{\phi,\xi} = t_1^{\phi,n}$, we see that $t_1^\phi \notin ((\Pi^t_{[1|n-k]})t_1^{n+1-k})$. Now replacing $k$ with $n + 1 - k$ gives (iii).

In Remark IV.3, we shall give a second proof of Corollary 7.6 using Larué-Whitehead diagrams.

8 Actions on free products of cyclic groups

8.1 Notation. Throughout this section, we assume that $n \geq 1$ and we fix a positive integer $N$.

Let $p_{[1|N]}$ be a partition of $n$, that is, $p_{[1|N]}$ is a sequence in $[1|\infty]$ such that

$\sum_{i=1}^{p} p_i = n$.

Let $m = [1|\infty]$ be a sequence in $\mathbb{N} - \{1\}$.

We let $\Sigma_{0,1,p_{[1|N]}} = \Sigma_{0,1,p_{[1|N]}}$ denote the group with presentation

$$\langle z, \pi_{[1|n]} \mid z\Pi\pi_{[1|n]}, \{t_j^{\pi_{[1|n]}} \mid j \in [1|n], j \in [1|n] \rangle.$$
Thus, $\Sigma_{0,1,p_1^{(m_1)} \perp p_2^{(m_2)} \perp \cdots \perp p_N^{(m_N)}}$ is isomorphic to a free product of cyclic groups, $C_{m_1}^{\ast p_1} \ast C_{m_2}^{\ast p_2} \ast \cdots \ast C_{m_N}^{\ast p_N}$, where $C_0$ is interpreted as $C_{\infty}$, and $p_i^{(0)}$ is also written $p_i$.

We let $\text{Out}_0^{+} p_1^{(m_1)} \perp p_2^{(m_2)} \perp \cdots \perp p_N^{(m_N)}$ denote the group of all automorphisms of $\Sigma_{0,1,p_1^{(m_1)} \perp p_2^{(m_2)} \perp \cdots \perp p_N^{(m_N)}}$ which respect the $\tau_1$ and $\{[\tau_1]_i \in \{(p_1+\cdots+p_{i-1}+1)!/(p_1+\cdots+p_j)\}$ for each $j \in [1 \mid N]$.

We let $\text{Out}_0 p_1^{(m_1)} \perp p_2^{(m_2)} \perp \cdots \perp p_N^{(m_N)}$ denote the group of all automorphisms of $\Sigma_{0,1,p_1^{(m_1)} \perp p_2^{(m_2)} \perp \cdots \perp p_N^{(m_N)}}$ which respect the $z$ and $\{[\tau_1] \cup [\tau_2]_{i \in \{(p_1+\cdots+p_{i-1}+1)!/(p_1+\cdots+p_j)\}}$ for each $j \in [1 \mid N]$.

In the case where all the $m_i$ are 0, we get groups denoted $\text{Out}_0 1 \perp p_1 \perp p_2 \perp \cdots \perp p_N$ and $\text{Out}_0^{+} 1 \perp p_1 \perp p_2 \perp \cdots \perp p_N$. Notice that $\text{Out}_0 1 \perp p_1 \perp p_2 \perp \cdots \per p_N$ is the subgroup of $\text{Out}_0 1 \perp p_1 \perp p_2 \perp \cdots \per p_N$ consisting of those elements such that the permutation in $\text{Sym}_n$, arising from the permutation of $\{(t_i \cup [t_i])_{i \in \{1 \mid n\}} \}$, lies in the natural image of $\text{Sym}_{p_1} \times \text{Sym}_{p_2} \times \cdots \times \text{Sym}_{p_N}$ in $\text{Sym}_n$.

There are natural maps

\begin{align}
(8.1.1) & \quad \text{Out}_0 1 \perp p_1 \perp p_2 \perp \cdots \per p_N \rightarrow \text{Out}_0 p_1^{(m_1)} \perp p_2^{(m_2)} \perp \cdots \per p_N^{(m_N)}, \\
(8.1.2) & \quad \text{Out}_0^{+} 1 \perp p_1 \perp p_2 \perp \cdots \per p_N \rightarrow \text{Out}_0^{+} p_1^{(m_1)} \perp p_2^{(m_2)} \perp \cdots \per p_N^{(m_N)}.
\end{align}

Since (8.1.2) is of index two in (8.1.1), we see that (8.1.1) is injective, surjective or bijective, if and only if (8.1.2) has the same property.

For topological reasons, we suspect that (8.1.1) and (8.1.2) are isomorphisms. In this section, we shall prove that this holds in the case where all the $m_i$ are equal, which includes the case $N = 1$. We begin by proving that (8.1.1) and (8.1.2) are injective, which seems to be new.

**8.2 Theorem.** With Notation 8.1, the maps

\begin{align}
(8.1.1) & \quad \text{Out}_0 1 \perp p_1 \perp p_2 \perp \cdots \per p_N \rightarrow \text{Out}_0 p_1^{(m_1)} \perp p_2^{(m_2)} \perp \cdots \per p_N^{(m_N)}, \\
(8.1.2) & \quad \text{Out}_0^{+} 1 \perp p_1 \perp p_2 \perp \cdots \per p_N \rightarrow \text{Out}_0^{+} p_1^{(m_1)} \perp p_2^{(m_2)} \perp \cdots \per p_N^{(m_N)}
\end{align}

are injective.

**Proof.** Suppose that $\phi$ is an element of the kernel of (8.1.1) or (8.1.2). Clearly, $\phi \in \text{Out}_0 1 \perp p_1 \perp p_2 \perp \cdots \per p_N$. Also $t^{\phi}_{[1 \mid n]}$ and $t_{[1 \mid n]}$ both have the same image in $\Sigma_{0,1,p_1^{(m_1)} \perp p_2^{(m_2)} \perp \cdots \per p_N^{(m_N)}}$. By Theorem 7.5, $t^{\phi}_{[1 \mid n]}$ is a sequence of squarefree elements of $\Sigma_{0,1,n}$, and, hence, they have the same normal form in $\Sigma_{0,1,n}$ and in $\Sigma_{0,1,p_1^{(m_1)} \perp p_2^{(m_2)} \perp \cdots \per p_N^{(m_N)}}$. Hence $t^{\phi}_{[1 \mid n]} = t_{[1 \mid n]}$, as sequences in $\Sigma_{0,1,n}$. Thus $\phi = 1$, and the result is proved.
8.3 Historical Remarks. Let us now restrict to the classic case where \( N = 1 \).
Here, for an integer \( m \geq 2 \), we are considering the action of \( \text{Out}_{0,1,n} \) on \( C_m^{\ast n} \),
and it induces maps

\[
\begin{align*}
(8.3.1) \quad \text{Out}_{0,1,n} & \to \text{Out}_{0,1,n}^{(m)}, \\
(8.3.2) \quad \text{Out}_{0,1,n}^+ & \to \text{Out}_{0,1,n}^{+ (m)}.
\end{align*}
\]

Theorem 8.2 shows that these maps are injective. Birman-Hilden \([6, \text{Theorem 7}]\) gave a topological proof that (8.3.2) is injective, thus answering a question of Magnus. Crisp-Paris \([11]\) gave an elegant algebraic proof of the injectivity of (8.3.2) using the \( \sigma_1 \)-trichotomy and the technique of Larue \([22]\) and Shpilrain \([30]\). Here is the essence of their proof.

Suppose that \( \phi \) is a non-trivial element of \( \mathcal{B}_n = \text{Out}_{0,1,n}^+ \). We will show that \( \phi \) acts non-trivially on \( \Sigma_{0,1,n}^{(m)} \).

We may assume that \( n \geq 3 \). By Theorem 6.7, we may replace \( \phi \) with \( \phi \) if necessary, and assume that \( \phi \) is \( \sigma \)-negative. Thus there exists some \( i \in [1 \uparrow (n - 1)] \) such that \( \phi \) has a monoid expression in \( \sigma_{[i+1 \downarrow (n-1)]} \lor \sigma_{[i \uparrow (n-1)]} \), and \( \sigma_i \) appears at least once in the expression.

Let \( (\tau_i^{\ast \ast}) \) denote the set of elements of \( \Sigma_{0,1,n}^{(m)} \) whose free-product normal form begins with an element of \( \langle \tau_i \rangle - \{1\} \). With Notation 6.5,

\[
(\tau_i^{\ast \ast})^\phi = (\tau_i^{\ast \ast})^{\sigma_i^\phi} = (\tau_{i+1}^{\ast \ast})^{\sigma_i^\phi} \subseteq \tau_i((\tau_{i+1}^{\ast \ast})^{\sigma_i^\phi})^{(n \geq 2)} \subset (\tau_i^{\ast \ast}).
\]

Because the elements of \( \sigma_{[i+1 \downarrow (n-1)]} \lor \sigma_{[i \uparrow (n-1)]} \) act as injective self-maps on \( (\tau_i^{\ast \ast}) \), it follows that \( (\tau_i^{\ast \ast})^\phi \subset (\tau_i^{\ast \ast}) \), and, hence, \( \phi \) acts non-trivially on \( \Sigma_{0,1,n}^{(m)} \), as desired. \(\square\)

Let us now verify the surjectivity of the maps (8.3.1) and (8.3.2). The case where \( m = 2 \) is due to Stephen Humphries \([2, \text{Lemma 2.1.7}]\). \(\square\)

8.4 Notation. Let \( m, n \in \mathbb{N} \) with \( n \geq 1 \) and \( m \geq 2 \). Let \( \lfloor \frac{m}{2} \rfloor \) denote the greatest integer not exceeding \( \frac{m}{2} \). Then \( [0 \lfloor \frac{m}{2} \rfloor ] \lor (-1) \downarrow (- \lfloor \frac{m-1}{2} \rfloor ) \) is a sequence of representatives for the integers modulo \( m \). For \( \tau^k \in \langle \tau \mid \tau^m = 1 \rangle \), we define \( |\tau^k| \) by

\[
\begin{array}{c|c|c|}
\left( |\tau^k| \right) & k \in [0 \lfloor \frac{m}{2} \rfloor ] & k \in (-1) \downarrow (- \lfloor \frac{m-1}{2} \rfloor ) \\
\left( \right) & (2k) & (-2k - 1)
\end{array}
\]

and we then extend \( | \cdot | \) to all of \( \Sigma_{0,1,n}^{(m)} \) additively on normal forms for the free product \( C_m^{\ast n} \).

Let \( \phi \in \text{Out}_{0,1,n}^+ \). There exists a unique permutation \( \pi \in \text{Sym}_n \), and a
unique sequence \( w_{[0(l+1)]} \) in \( \Sigma_{0,1,n}^{(m)} \) such that \( w_0 = 1 \) and \( w_{n+1} = 1 \), and, for
each \( i \in [1 \uparrow n] \), \( w_i \not\in (\tau_i^{\ast \ast}) \) and \( \tau_i^\phi = \tau_i^{\omega_i} \). For each \( i \in [0 \uparrow n] \), let \( u_i = w_i \omega_i \).
We define \( \pi(\phi) := \pi, \ w_i(\phi) := w_i, \ i \in [0 \uparrow (n + 1)] \), and \( u_i(\phi) := u_i, \ i \in [0 \uparrow n] \). We write \(|\phi| := n + 2 \sum_{i \in [1 \uparrow n]} |w_i(\phi)| \). \(\square\)
The following is similar to Artin’s Lemma 3.2.

**8.5 Lemma.** Let $n \geq 1$, $m \geq 2$ and let $\phi \in \text{Out}_{0,1,n(m)}$. Let $\pi = \pi(\phi)$. For each $i \in [0|n]$, let $u_i = u_i(\phi)$. For each $i \in [1|n]$, let $a_i$, $b_i$ denote the elements of $[0|(m - 1)]$ determined by the following:

- there exists some $u'_i \in \Sigma_{0,1,n(m)} - (\star \tau_i^n)$ such that $u_{i-1} = u'_i \tau_i^n$;
- there exists some $u''_i \in \Sigma_{0,1,n(m)} - (\tau_i^n \ast)$ such that $u_i = \tau_i^n u''_i$.

In particular, $a_1 = b_n = 0$.

(i). Let $i \in [2|n]$. If $a_i \in [\mid \frac{m}{2} \mid |(m - 1)|$, then $\|\sigma_{i-1}\phi\| < \|\phi\|$.

(ii). Let $i \in [1|(n - 1)]$. If $b_i \in [\mid \frac{m-1}{2} \mid |(m - 1)|$, then $\|\sigma_i\phi\| < \|\phi\|$.

(iii). If $\phi \neq 1$, there exists some $\sigma_i \in \sigma_{[1|(n-1)]} \cup \sigma_{[1|(n-1)]}$ such that $\|\sigma_i\phi\| < \|\phi\|$.

**Proof.** (i). Let $a = a_i$. There exists some $v \in \Sigma_{0,1,n(m)} - (\star \tau_i^n)$ such that $u_{i-1} = v \tau_i^n$. Since $w_{i-1}(\phi) = u_{i-1} w_i(\phi)$, we have

\[(8.5.1) \quad w_{i-1}(\phi) = v \tau_i^n w_i(\phi);\]

since $w_i(\phi) \not\in (\tau_i^n \ast)$ and $v \not\in (\ast \tau_i^n)$, $v \tau_i^n w_i(\phi)$ is a free-product normal form for $w_{i-1}(\phi)$.

**Claim.** $|\tau_i^{a+1}| < |\tau_i^a|$.

**Proof of claim.** If $a' \in [\mid\frac{m}{2}\mid + 1| |(m - 1)|$, then $a' - m \in [\mid\frac{m-1}{2}\mid |(-1)|]$, and, hence,

$$\mid\tau_i^a\mid = \mid\tau_i^{a-m}\mid = -2(a' - m) - 1 = 2m - 2a' - 1.$$  

Therefore, if $a \in [\mid\frac{m}{2}\mid |(m - 2)|$, $\mid\tau_i^{a+1}\mid = 2m - 2(a + 1) - 1 = 2m - 2a - 3$. Thus, $\mid\tau_i^{a+1}\mid < \mid\tau_i^a\mid$ if $a \in [\mid\frac{m}{2}\mid + 1| |(m - 2)|]$. $\Box$

For $a = m - 1$, $\mid\tau_i^a\mid = 1$ and $\mid\tau_i^{a+1}\mid = 0$. This proves the claim.

Thus, $\mid w_{i-1}(\phi)\mid = \mid v\mid + \mid \tau_i^a\mid + \mid w_i(\phi)\mid > \mid v\mid + \mid \tau_i^{a+1}\mid + \mid w_i(\phi)\mid$.  

By (8.5.1), $w_{i-1}(\phi) \tau_i^n = v \tau_i^{a+1}$; hence

$$\tau_{\sigma_{i-1}} = (\tau_i \tau_{i-1})^\phi = (\tau_{w_{i-1}(\phi)} \tau_{w_i(\phi)})^{\tau_{w_i(\phi)}} = \tau_{w_{i-1}(\phi)} \tau_{w_i(\phi)}.$$  

Hence, $\mid w_i(\sigma_{i-1}\phi)\mid = \mid v \tau_i^{a+1} w_i(\phi)\mid \leq \mid v\mid + \mid \tau_i^{a+1}\mid + \mid w_i(\phi)\mid < \mid w_{i-1}(\phi)\mid$.

For each $j \in [1|2(i - 2)| \cup [2(i + 1)|n]$, $\tau_j^{a_{j-1}} = \tau_j^{\phi}$, and, hence, $\mid w_j(\sigma_{i-1}\phi)\mid = \mid w_j(\phi)\mid$.

Also, $\tau_i^{a_{i-1}} = \tau_i^{\phi}$; in particular, $\mid w_{i-1}(\sigma_{i-1}\phi)\mid = \mid w_i(\phi)\mid$.

It now follows that $\|\sigma_{i-1}\phi\| < \|\phi\|$.
(ii). Let \( b = b_i \). There exists some \( v \in \Sigma_{0,1,n(m)} - (\tau^*_i \ast) \) such that \( u_i = \tau_i^b v \). Since \( w_{i+1}(\phi) = \overline{v}w_i(\phi) \), we have

\[
(8.5.2) \quad w_{i+1}(\phi) = \overline{v} \tau_i^b w_i(\phi).
\]

Since \( w_i(\phi) \not\in (\langle \tau_i \rangle \ast) \) and \( \overline{v} \not\in (\langle \tau_i \rangle \ast) \), \( \overline{v} \tau_i^b w_i(\phi) \) is a free-product normal form for \( w_{i+1}(\phi) \). Hence, \( |w_{i+1}(\phi)| = |\overline{v}| + |\tau_i^b| + |w_i(\phi)| \).

**Claim.** \(|\tau_i^{b+1}| < |\tau_i^b|\).

**Proof of claim.** Suppose that \( b' \in \left[ \left[ \frac{m+1}{2} \right] \uparrow m \right] \). Then \( m - b' \in \left[ \left[ \frac{m}{2} \right] \downarrow 0 \right] \), and, hence,

\[
|\tau_i^{b'}| = |\tau_i^{m-b'}| = 2(m - b) = 2m - 2b'.
\]

Since \( b \in \left[ \left[ \frac{m+1}{2} \right] \uparrow (m - 1) \right] \),

\[
|\tau_i^{b+1}| = 2m - 2(b + 1) = 2m - 2b - 2 < |\tau_i^b|.
\]

This proves the claim.

Hence \( |w_{i+1}(\phi)| > |\overline{v}| + |\tau_i^{b+1}| + |w_i(\phi)| \).

For all \( j \in [1 \uparrow (i - 1)] \setminus \{i\} \), \( \tau_i^{\overline{v}} = \tau_i^\phi \); hence, \( |w_j(\overline{\sigma}_i \phi)| = |w_j(\phi)| \).

Since \( \tau_i^{\overline{v}} = \tau_i^\phi \), we see that \( |w_i(\overline{\sigma}_i \phi)| = |w_i(\phi)| \).

By (8.5.2), \( w_{i+1}(\phi)\overline{w}_i(\phi)\overline{\tau}_i = \overline{v} \tau_i^{b+1} \); hence,

\[
\tau_i^{\overline{v}} = (\tau_i^{w_i(\phi)})(\tau_i^{w_i(\phi)}) = \tau_i^{\overline{v}} \tau_i^{b+1} w_i(\phi).
\]

Hence, \( |w_i(\overline{\sigma}_i \phi)| = |\overline{v} \tau_i^{b+1} w_i(\phi)| \leq |\overline{v}| + |\tau_i^{b+1}| + |w_i(\phi)| < |w_{i+1}(\phi)| \).

It now follows that \( \|\overline{\sigma}_i \phi\| < \|\phi\| \), and (ii) is proved.

(iii). If \( \phi \neq 1 \), we choose a distinguished element of \([1 \uparrow n] \) as follows.

If, for some \( i \in [1 \uparrow n] \), \( \tau_i^{a_i+1+b_i} = 1 \), we take any such \( i \) to be our distinguished element of \([1 \uparrow n] \).

Consider then the case where, for all \( i \in [1 \uparrow n] \), \( \tau_i^{a_i+1+b_i} \neq 1 \). Thus, there is no further cancellation in \( \Pi \tau_i^{\phi} \). Since \( \phi \) fixes \( \Pi \tau_i^{\phi} \), it is not difficult to see that, for all \( i \in [1 \uparrow n] \), \( \tau_i^{a_i+1+b_i} = \tau_i \). Since \( \phi \neq 1 \), it is then not difficult to show that there exists some \( i \in [1 \uparrow n] \) such that \( (a_i, b_i) \neq (0, 0) \). We take any such \( i \) to be our distinguished element of \([1 \uparrow n] \).

In each case, let \( i \) denote our distinguished element of \([1 \uparrow n] \).

Notice that \( (a_i, b_i) \neq (0, 0) \) and that \( \tau_i^{a_i+1+b_i} \in \{1, \tau_i \} \).

Hence, \( a_i + 1 + b_i \in \{m, m + 1\} \), and, hence, \( b_i \in \{m - a_i - 1, m - a_i\} \).

**Case 1.** \( a_i \in \left[ \left[ \frac{m}{2} \right] \uparrow (m - 1) \right] \).

Here, \( i \in [2 \uparrow n] \) and, by (i), \( \|\sigma_{i-1} \phi\| < \|\phi\| \).

**Case 2.** \( a_i \in \left[ \left[ \frac{m-2}{2} \right] \right] \).

Here, \( m - a_i - 1 \in \left[ \left[ (m - 1) \downarrow \frac{m+1}{2} \right] \right] \), and, hence, \( b_i \in \left[ \left[ \frac{m+1}{2} \right] \uparrow (m - 1) \right] \). Here, \( i \in [1 \uparrow (n - 1)] \) and, by (ii), \( \|\sigma_i \phi\| < \|\phi\| \).
8.6 Theorem. Let \( n \geq 1, \ m \geq 2 \). The natural map \( \text{Out}_{0,1,n}^+ \to \text{Out}_{0,1,n(m)}^+ \) is an isomorphism, and, hence, the natural map \( \text{Out}_{0,1,n}^+ \to \text{Out}_{0,1,n(m)}^+ \) is an isomorphism.

With Notation 8.1, the maps \( \text{Out}_{0,1,p_1 \perp p_2 \perp \ldots \perp p_N}^+ \to \text{Out}_{0,1,p_1 \perp p_2 \perp \ldots \perp p_N}^+ \) and \( \text{Out}_{0,1,p_1 \perp p_2 \perp \ldots \perp p_N}^+ \to \text{Out}_{0,1,p_1 \perp p_2 \perp \ldots \perp p_N}^+ \) are isomorphisms. \( \square \)

The following is essentially an algebraic translation of a part of a topological argument in [27, Section 3].

8.7 Proposition. With Notation 8.1, in \( \Sigma_{0,1,p_1 \perp p_2 \perp \ldots \perp p_N}^+ \) let \( H \) be any subgroup of finite index, and now in \( \text{Out}_{0,1,p_1 \perp p_2 \perp \ldots \perp p_N}^+ \) let \( A \) be any subgroup consisting of automorphisms which map \( H \) to itself. Then, either the induced map \( A \to \text{Aut} H \) is injective or \( (n,N,m_1) = (2,1,2) \).

Proof. Suppose that \( \phi \in \text{Out}_{0,1,p_1 \perp p_2 \perp \ldots \perp p_N}^+ \), and that \( \phi \) acts as the identity on \( H \). We shall show that \( \phi = 1 \) or \( (n,N,m_1) = (2,1,2) \).

Let \( G = \Sigma_{0,1,p_1 \perp p_2 \perp \ldots \perp p_N}^+ \). For any \( g \in G \), right multiplication by \( g \) permutes the elements of the finite set \( H \backslash G \), so there exists some positive integer \( k \) such that \( g^k \) acts trivially on \( H \backslash G \). In particular, \( Hg^k = H \) and, hence, \( g^k \in H \).

Hence, there exists some positive integer \( k \) such that \( (\Pi \tau_{[1\n]}^k) \in H \). Now \( (\Pi \tau_{[1\n]}^k) = (\Pi \tau_{[1\n]}^\epsilon)^k \) for some \( \epsilon \in \{1,-1\} \), and, hence,

\[
(\Pi \tau_{[1\n]}^k) = (\Pi \tau_{[1\n]}^\epsilon)^k = (\Pi \tau_{[1\n]}^\epsilon)^k = (\Pi \tau_{[1\n]}^\epsilon)^k = (\Pi \tau_{[1\n]}^\epsilon)^k.
\]

Since \( \Pi \tau_{[1\n]}^\epsilon \) has infinite order in \( G \), we see that \( \epsilon = 1 \). Thus \( \phi \) fixes \( \Pi \tau_{[1\n]}^\epsilon \).

Consider any \( i \in [1\n] \). Since \( (\Pi \tau_{[1\n]}^\epsilon)^k \in G \), there exists some positive integer \( k \) such that \( (\Pi \tau_{[1\n]}^\epsilon)^k = H. \) Hence,

\[
(\Pi \tau_{[1\n]}^\epsilon)^k = (\Pi \tau_{[1\n]}^\epsilon)^k = (\Pi \tau_{[1\n]}^\epsilon)^k = (\Pi \tau_{[1\n]}^\epsilon)^k = (\Pi \tau_{[1\n]}^\epsilon)^k.
\]

Hence \( \tau_i^\epsilon \tau_i \) commutes with \( (\Pi \tau_{[1\n]}^\epsilon)^k \). A straightforward normal-form argument shows that \( \tau_i^\epsilon \tau_i \in (\Pi \tau_{[1\n]}^\epsilon)^k \).

Hence there exists an integer \( j \) such that \( \tau_i^\epsilon = (\Pi \tau_{[1\n]}^\epsilon)^j \tau_i \). Since \( \tau_i^\epsilon \) is a conjugate of \( \tau_i \), the cyclically-reduced form of \( (\Pi \tau_{[1\n]}^\epsilon)^j \tau_i \) is \( \tau_i \). Either \( j = 0 \), or there must be cyclic cancellation, and a straightforward analysis then shows that \( (n,N,m_1) = (2,1,2) \). Since \( i \) was arbitrary, this completes the proof. \( \square \)

9 The \( \mathcal{B}_{n+1} \)-group \( \Phi_n \)

9.1 Notation. Recall that \( \Sigma_{0,1,(n+1)(2)} = C_2^{(n+1)} = \langle \tau_{[1\n]}^2 \mid \tau_{[1\n]}^2 = 1 \rangle \).

By Theorem 8.6, \( \mathcal{B}_{n+1} = \text{Out}_{0,1,n+1}^+ = \text{Out}_{0,1,(n+1)(2)}^+ \). We define \( \Phi_n \) to be the
subgroup of $\Sigma_{0,1,(n+1)(2)}$ consisting of the elements which have even exponent sum in the $\tau_i$. It is not difficult to see that $\Phi_n$ is a free group of rank $n$, and that there is induced a map from $\text{Out}_{0,1,(n+1)(2)}$ to $\text{Aut} \Phi_n$. Hence $\Phi_n$ has a $\mathcal{B}_{n+1}$-action; we say that $\Phi_n$ is a $\mathcal{B}_{n+1}$-group, and that $\Phi_n$ is a $\mathcal{B}_{n+1}$-subgroup of $\Sigma_{0,1,(n+1)(2)}$.

Proposition 8.7 shows that, if $n \neq 1$, then the map from $\text{Out}_{0,1,(n+1)(2)}$ to $\text{Aut} \Phi_n$ is injective, and we say that the $\mathcal{B}_{n+1}$-action is faithful, and that $\Phi_n$ is a faithful $\mathcal{B}_{n+1}$-group.

Over the course of this section, we shall choose various free generating sets of $\Phi_n$ to obtain interesting actions. In the next two examples, we identify $\Sigma_{g,1,0}$ with $\Phi_{2g}$ and identify $\Sigma_{g,2,0}$ with $\Phi_{2g+1}$.

9.2 Example. Now that algebraic proofs of the requisite theorems are known to us, let us review [18, Example 15.6] which was an algebraic approximation of results in [27, Section 3].

Let $g \in \mathbb{N}$. Let

$$\Sigma_{g,1,0} := \langle x_{[1|g]}, y_{[1|g]}, z_1 \mid (\Pi_i \in [1|g] [x_i, y_i]) z_1 = 1 \rangle,$$

where the commutator $[x, y]$ of group elements $x, y$ is $xyx^{-1}yx$. Let $\text{Out}^+_{g,1,0}$ denote the group of all automorphisms of $\Sigma_{g,1,0}$ which fix $z_1$. Then $\Sigma_{g,1,0}$ is free of rank $2g$, freely generated by $x_{[1|g]} \cup y_{[1|g]}$, and $\text{Out}^+_{g,1,0}$ is the group of all automorphisms of $\Sigma_{g,1,0}$ which fix $\Pi_i \in [1|g] [x_i, y_i]$.

We now recall some Dehn-twist elements of $\text{Out}^+_{g,1,0}$ from Definitions 3.10 and Remarks 5.1 of [18].

For each $i \in [1|g]$, we define $\alpha_i, \beta_i \in \text{Out}^+_{g,1,0}$ by

$$k \in [1|(i-1)] \hspace{1cm} \hspace{1cm} k \in [1|(i+1)|g] \hspace{1cm} \hspace{1cm} k \in [1|2(i-1)] \hspace{1cm} \hspace{1cm} k \in [1|(i+1)|g]$$

$$(x_k, y_k, x_i, y_i, x_k, y_k)^{\alpha_i}, \hspace{1cm} \hspace{1cm} (x_k, y_k, x_i, y_i, x_k, y_k)^{\beta_i} \hspace{1cm} \hspace{1cm} (x_k, y_k, x_i, y_i, x_k, y_k)^{\gamma_i}$$

$$(x_k, y_k, \bar{y}_i, x_i, y_i, x_k, y_k).$$

For each $i \in [1|(g-1)]$, write $f_i = y_i \bar{x}_{i+1} y_{i+1} x_{i+1}$ and define $\gamma_i \in \text{Out}^+_{g,1,0}$ by

$$k \in [1|(i-1)] \hspace{1cm} \hspace{1cm} k \in [1|(i+2)|g]$$

$$(x_k, y_k, x_i, y_i, x_{i+1}, y_{i+1}, x_k, y_k)^{\gamma_i} \hspace{1cm} \hspace{1cm} (x_k, y_k, \bar{y}_{i+1}, x_{i+1}, f_i, y_i, x_k, y_k).$$

Let us identify $\Sigma_{g,1,0}$ with $\Phi_{2g}$ via

$$k \in [1|g]$$

$$(x_k, y_k, z_1) \Sigma_{g,1,0} \tilde{\Phi}_{2g}. \hspace{1cm} \hspace{1cm} (\Pi_{\tau_{[1|1]}(2k+1)}) (\Pi_{\tau_{[1|1]}(2k)}) (\Pi_{\tau_{[1|1]}(2k+1)}) (\Pi_{\tau_{[1|1]}(2k+1)}) (\Pi_{\tau_{[1|1]}(2k+1)})$$

Notice that $[x_k, y_k] = \bar{x}_k \bar{y}_k x_k y_k$ is then identified with

$$(\Pi_{\tau_{[1|1]}(2k+1)}) (\Pi_{\tau_{[1|1]}(2k)}) (\Pi_{\tau_{[1|1]}(2k+1)})$$
which equals \((\Pi_{\tau(2k-1)})_{1}\)(\(\Pi_{\tau(2k+1)})_{1}\)(\(\Pi_{\tau(2k+1)})_{1}\)). It follows that \(\Pi_{k\in[1|g]}\{x_{k}, y_{k}\}\) is identified with \((\Pi_{\tau(1|2g+1)})_{1}\)^2.

This corresponds to the surface of genus \(g\) with one boundary component arising as a two-sheeted branched cover of a sphere with one boundary component and \(2g + 1\) double points. Then  \(\mathcal{B}_{2g+1} = \text{Out}_{0,1,2g+1}^{+} = \text{Out}_{0,1,(2g+1)2}^{+}\) becomes embedded in \(\text{Out}_{g,1,0}^{+}\) via the homomorphism represented as

\[
\left(\begin{array}{cccccc}
\sigma_{1} & \sigma_{2} & \sigma_{3} & \sigma_{4} & \sigma_{5} & \cdots & \sigma_{2g-2} & \sigma_{2g-1} & \sigma_{2g} \\
\alpha_{1} & \beta_{1} & \gamma_{1} & \beta_{2} & \gamma_{2} & \cdots & \beta_{g-1} & \gamma_{g-1} & \beta_{g}
\end{array}\right).
\]

Clearly, in the preceding example, the subgroup \(\mathcal{B}_{2g}\) of \(\mathcal{B}_{2g+1}\) is also embedded in \(\text{Out}_{g,1,0}\), but it is more natural to remove from the surface a handle containing the boundary component (a sphere with three boundary components, a ‘pair of pants’), and embed \(\mathcal{B}_{2g}\) in \(\text{Out}_{g-1,2,0}\), as follows.

**9.3 Example.** Now that algebraic proofs of the requisite theorems are known to us, let us review [18, Example 15.7] which was an algebraic approximation of results in [27, Section 3].

Let \(g \in \mathbb{N}\). Let

\[\Sigma_{g,2,0} := \langle x_{[1|g]}, y_{[1|g]}, z_{[12]} | \prod_{i\in[1|g]} [x_{i}, y_{i}] \Pi z_{[12]} = 1 \rangle.\]

Recall that \([x, y] := x\bar{y}yx\). Then \(\Sigma_{g,2,0}\) is free of rank \(2g + 1\) with free generating sequence \(x_{[1|g]} \vee y_{[1|g]} \vee (z_{1})\) and distinguished element \(z_{2}\) such that \(\Sigma_{2} = (\prod_{i\in[1|g]} [x_{i}, y_{i}]) z_{1}\). Let \(\text{Out}_{g,1,1,0}^{+}\) denote the group of all automorphisms of \(\Sigma_{g,2,0} \ast \langle e_{1} | \rangle\) which map \(\Sigma_{g,2,0}\) to itself, and fix \(z_{1}^{e_{1}}\) and \(z_{2}\). It can be shown that \(\text{Out}_{g,1,1,0}^{+}\) acts faithfully on the subset \(\Sigma_{g,2,0} \cup \Sigma_{g,2,0} e_{1}\) of \(\Sigma_{g,2,0} \ast \langle e_{1} | \rangle\).

Here, \(e_{1}\) represents an arc from the basepoint of one boundary component, to the basepoint of the other boundary component. Karen Vogtmann calls such an arc a ‘tether joining the basepoint to the second boundary component’. For any surface-with-boundaries, A’Campo [1, Section 4, Remarque 6], [27, p.232] identifies basepoints of all the boundary components, which makes tethers into loops, to obtain a topological quotient space whose (free) fundamental group is (faithfully) acted on by the mapping-class group of the surface-with-boundaries.

We now recall some Dehn-twist elements of \(\text{Out}_{g,1,1,0}^{+}\) from Definitions 3.10 and Remarks 5.1 of [18].

For each \(i \in [1|g]\), we define \(\alpha_{i}, \beta_{i} \in \text{Out}_{g,1,1,0}^{+}\) by

\[
\begin{align*}
\begin{array}{c|cc|c|cc}
& x_{k} & y_{k} & x_{i} & y_{i} & x_{k} & y_{k} & z_{1} & e_{1} \\
\hline
k \in [1(i-1)] & \alpha_{i} & & & & & & & \\
k \in [(i+1)|g] & & & & & & & & \beta_{i} \end{array}
\end{align*}
\]

\[
\begin{align*}
& (x_{k} y_{k} x_{i} y_{i} x_{k} y_{k} z_{1} e_{1})^{\alpha_{i}} \\
& (x_{k} y_{k} y_{i} x_{i} y_{k} y_{k} z_{1} e_{1}),
\end{align*}
\]
Lemma. Let \( n \geq 2 \). Then, \( \text{Artin}(\langle D_n \rangle) \) has a unique automorphism \( \nu \) of order two which fixes \( d_{11(n-2)} \) and interchanges \( d_{n-1} \) and \( d_n \). The semidirect product \( \text{Artin}(\langle D_n \rangle) \rtimes \langle \nu \rangle \) has presentation
\[
\text{Artin}(\langle d_1 d_2 \cdots d_{n-3} d_{n-2} d_{n-1} = \nu \mid \nu^2 = 1 \rangle).
\]
Proof. Notice that

$$\langle d_{n-1}, d_n, v \mid v^2 = 1, d_{n-1}d_n = d_n d_{n-1} \rangle = \langle d_{n-1}, d_n, d_{n-1}d_n = d_n d_{n-1} \rangle = \text{Artin}(d_{n-1} \equiv d_n \equiv v \mid v^2 = 1).$$

The result now follows easily. \[\square\]

Part of the following appears in [27] and [10].

9.5 Theorem (Perron-Vannier [27]). Let $n \geq 2$.

(i). $B_n \ltimes \Phi_{n-1} = \text{Artin}(\sigma_1 \sigma_2 \cdots \sigma_{n-3} \sigma_{n-2} \sigma_{n-1})$

(ii). $B_n \ltimes \Phi_{n-1}$ has a unique automorphism $\upsilon$ of order two which fixes $\sigma_{[1] \uparrow (n-2)}$ and interchanges $\sigma_{n-1}$ and $\sigma_{n-1} \tau_n \tau_{n-1}$.

(iii). $(B_n \ltimes \Phi_{n-1}) \ltimes \langle \upsilon \rangle = \text{Artin}(\sigma_1 \tau_n \tau_{n-1} \sigma_1 \sigma_2 \cdots \sigma_{n-3} \sigma_{n-2} \sigma_{n-1} \equiv v \mid v^2 = 1)$.

Proof. By Corollary 5.5, we have a presentation

$$B_n \ltimes \Sigma_{0,1,n} = \text{Artin}(\sigma_1 \cdots \sigma_{n-1} \equiv \tau_n).$$

If we impose the relation $t_n^2 = 1$, we transform $B_n \ltimes \Sigma_{0,1,n}$ into $B_n \ltimes \Sigma_{0,1,n(2)}$, and we have

$$B_n \ltimes \Sigma_{0,1,n(2)} = \text{Artin}(\sigma_1 \cdots \sigma_{n-1} \equiv \tau_n \mid t_n^2 = 1)$$

by Lemma 9.4. This group has a retraction to $\langle \tau_n \mid t_n^2 = 1 \rangle$ with kernel the normal subgroup generated by $\sigma_{[1] \uparrow (n-1)}$. This normal subgroup contains $\sigma_i^{\tau_{i+1}} = \sigma_i \sigma_{i+1} \tau_i$ for all $i \in [1 \uparrow (n-1)]$, and we see that this normal subgroup is

$$B_n \ltimes \Phi_{n-1} = \text{Artin}(\sigma_1 \cdots \sigma_{n-3} \sigma_{n-2} \sigma_{n-1});$$

this agrees with the desired presentation. \[\square\]
9.6 Remarks. Corollary 5.5 says that, for $n \geq 1$, we can go down by index $n + 1$ from $\text{Artin}(A_n)$ by squaring the last generator, and arrive at $\text{Artin}(B_n) \cong \text{Artin}(A_{n-1}) \ltimes \Sigma_{0,1,n}$.

Theorem 9.5 says that, for $n \geq 2$, we can kill the square of the new last generator, go down by index 2, and arrive at $\text{Artin}(D_n) \cong \text{Artin}(A_{n-1}) \ltimes \Phi_{n-1}$.

We now review some other free generating sets of $\Phi_n$ which appear in the literature.

9.7 Examples. Recall Notation 9.1. In particular, the $\mathcal{B}_{n+1}$-action on $\Phi_n$ is faithful if $n \neq 1$.

(1). For each $k \in [1 \, n]$, set $x_k = \tau_k \tau_{k+1}$ in $\Phi_n$. Then $x_{[1\, n]}$ is a free generating set for $\Phi_n$, and, for each $i \in [1 \, n]$, the action of $\sigma_i$ on $\Phi_n$ is determined by

\[
\begin{array}{cccc}
& k \in [1\,(i-2)] & & k \in [(i+2)\, n] \\
(x_k & x_{i-1} & x_i & x_{i+1} & x_k)^{\sigma_i} = (x_k & x_{i-1} x_i & x_i & \tau_i x_{i+1} & x_k),
\end{array}
\]

interpreted appropriately for $i = 1$ and $i = n$.

(2). For each $k \in [1 \, n]$, set $x_k = \tau_{n+1} \tau_k$ in $\Phi_n$. Then $x_{[1\, n]}$ is a free generating set for $\Phi_n$, and, for each $i \in [1 \, (n-1)]$, $\sigma_i$ acts on $x_{[1\, n]}$ as follows.

\[
\begin{array}{cccc}
k \in [1\,(i-1)] & k \in [(i+2)\, n] & k \in [1\,(n-1)] \\
(x_k & x_i & x_{i+1} & x_k)^{\sigma_i} = (x_k & x_{i+1} & \tau_i x_{i+1} & x_k), & (x_k & x_n)^{\sigma_n} = (x_{n-1} x_k & x_n).
\end{array}
\]

(3). We next consider a free generating set indicated by the proof of [11, Proposition A.1(2)].

For each $k \in [1 \, n]$, set $x_k = (\prod_{\tau_{n+1}}^{\tau_k})^{\prod_{\tau_k}^{\tau_{n+1}}}$ in $\Phi_n$. Then $x_{[1\, n]}$ is a free generating set for $\Phi_n$, and, for each $i \in [1 \, (n-1)]$, $\sigma_i$ acts on $x_{[1\, n]}$ as follows.

\[
\begin{array}{cccc}
k \in [1\,(i-1)] & k \in [(i+2)\, n] \\
(x_k & x_i & x_{i+1} & x_k)^{\sigma_i} = (x_k & x_i \Pi x_{(i-1)\, i} & \Pi x_{(i+1)\, i}) x_{i+1} & x_k).
\end{array}
\]

Let $w = (\Pi x_{[1\, (n-1)]})^2 x_n$; then $\sigma_n$ acts as follows.

\[
\begin{array}{c}
k \in [1\,(n-1)] \\
(x_k & x_n)^{\sigma_n} = (w^{(-1)^k} \Pi x_{[i+1\,(n-1)]} x_k & x_n).
\end{array}
\]

9.8 Historical Remarks. Let us view $\mathcal{B}_n$ as a subgroup of $\mathcal{B}_{n+1}$ by suppressing $\sigma_n$. Then the $\mathcal{B}_{n+1}$-group $\Phi_n$ becomes a faithful $\mathcal{B}_n$-group, even if $n = 1$. 

Wada [31] defined various left actions of \( \mathcal{B}_n \) on a free group of rank \( n \). All but four of the actions are obviously non-faithful, and two of the remaining four are obviously equivalent up to changing the free generating set, leaving three actions to be studied for faithfulness. Shpilrain [30] ingeniously used the \( \sigma_1 \)-trichotomy to prove that these three are all faithful. Crisp-Paris [11, Proposition A.1(2)] showed that the second and third of these three faithful actions are equivalent up to changing the free generating set. In fact, they correspond to Examples 9.7(2), (3), above, with \( \sigma_n \) suppressed, where our actions on the right are the inversions of their actions on the left. Thus, the second and third of the faithful Wada actions of \( \mathcal{B}_n \) are both obtained by choosing suitable free generating sets of the Perron-Vannier \( \mathcal{B}_{n+1} \)-group \( \Phi_n \) and suppressing \( \sigma_n \). Hence, Shpilrain [30] had given the first algebraic proof that \( \mathcal{B}_n \) acts faithfully on \( \Phi_n \); this includes the information that \( \mathcal{B}_n \) acts faithfully on the overgroup \( \Sigma_{0,1,n^2}^{(2)} \), and on the free factor thereof \( \Sigma_{0,1,n^2}^{(2)} \).

Sakuma [28] observed that the third Wada action of \( \mathcal{B}_n \) on \( \langle x_{[1,n]} \mid \rangle \) induces an action of \( \mathcal{B}_n \) on \( \langle x_{[1,n]} \mid x_{[1,n]}^2 \rangle \) which, when pre-composed with the inversion-of-the-generators automorphism, agrees with the Artin action of \( \mathcal{B}_n \) on \( \Sigma_{0,1,n^2}^{(2)} \). Since the latter is faithful by the Birman-Hilden Theorem [6, Theorem 7], the third Wada action is faithful.

Shpilrain [30], unaware of Sakuma’s article, repeats the observation that the third Wada action of \( \mathcal{B}_n \) on \( \langle x_{[1,n]} \mid \rangle \) induces an action of \( \mathcal{B}_n \) on \( \langle x_{[1,n]} \mid x_{[1,n]}^2 \rangle \) and notes that it does not agree with the Artin action of \( \mathcal{B}_n \) on \( \Sigma_{0,1,n^2}^{(2)} \). It seems to be tacitly understood in his discussion that the second Wada action of \( \mathcal{B}_n \) on \( \langle x_{[1,n]} \mid \rangle \) induces an action of \( \mathcal{B}_n \) on \( \langle x_{[1,n]} \mid x_{[1,n]}^2 \rangle \) which clearly agrees with the Artin action of \( \mathcal{B}_n \) on \( \Sigma_{0,1,n^2}^{(2)} \), and then, by the Birman-Hilden Theorem, the second Wada action is faithful.

The first faithful Wada action is constructed by choosing a non-zero integer \( m \), and, for each \( i \in [1 \uparrow (n-1)] \), letting \( \sigma_i \) act on \( \langle x_{[1,n]} \mid \rangle \) by

\[
\begin{align*}
&\quad \begin{cases}
\text{for } k \in [1 \uparrow (i-1)] & \quad \begin{cases}
(x_k & x_i & x_{i+1} & x_{k})^\sigma_i \\
(x_k & x_{i+1} & x_{i} & x_{k})
\end{cases}
\end{cases}
\end{align*}
\]

Edward Formanek has pointed out that \( x_{[1,n]}^m \) freely generates a faithful \( \mathcal{B}_n \)-subgroup of \( \langle x_{[1,n]} \mid \rangle \), where faithfulness can be seen from the fact that the \( \mathcal{B}_n \)-action is the standard Artin action with respect to this free generating set. This argument gives a transparent proof that this action is faithful. □

**Appendix. Larue-Whitehead diagrams**

In this appendix, we rework ideas from Chapter 2 and Appendix A of Larue’s thesis [21], using combinatorial arguments to obtain a description of the \( \mathcal{B}_n \)-orbit of \( t_1 \). A topological treatment of similar ideas was given by
Fenn-Greene-Rolfsen-Rourke-Wiest [19], and it was arrived at independently of Larue’s work; see [16, Chapters 5, 6].

I Self-homeomorphisms

This section is purely motivational. We shall briefly indicate the mapping-class viewpoint of the braid group, and the Jordan-curve nature of the Whitehead graphs of the elements in the $\mathcal{B}_n$-orbit of $t_1$ if $n \geq 1$.

Let $\mathbb{C}$ denote the complex plane, and $\hat{\mathbb{C}}$ the Riemann sphere, or projective complex line, $\mathbb{C} \cup \{\infty\}$. For each $z \in \mathbb{C}$ and each non-negative real number $r$, let $D(z, r)$ denote the closed disc in $\mathbb{C}$ with centre $z$ and radius $r$, and let $D^\circ(z, r)$ denote the interior of $D(z, r)$.

Let $S_{0,1,n}$ denote the surface formed by deleting from a sphere one open disc and $n$ points. We shall think of the discs and points as being distinguished rather than deleted; for example, it is then meaningful to speak of the self-homeomorphisms of $S_{0,1,n}$ as permuting the points. We take as our model of $S_{0,1,n}$ the sphere $\hat{\mathbb{C}}$ having $[1 \uparrow n]$ as its set of distinguished points, and $D^\circ(0, \frac{1}{2})$ as its distinguished open disc. We are particularly interested in the set $[0 \uparrow n]$, and, in our diagrams, we shall indicate these points by drawing small discs around them.

For each $k \in [0 \uparrow n]$, we have a distinguished oriented tether, or arc, 

$$\{k + ri \mid r \text{ is } \infty \text{ or real, with } r \text{ decreasing from } \infty \text{ to } 0\},$$

joining $\infty$ to $k$. We label the right flank of this oriented arc $t_k$, and label the left flank $\bar{t}_k$; we then cut $\hat{\mathbb{C}}$ open along these arcs and obtain a $(2n + 2)$-gon, with clockwise boundary label $\Pi_{k \in [0 \uparrow n]} (t_k \bar{t}_k)$; see Fig. I.1.4. We shall use $t_0$ and $z_1$ interchangeably in this section. Performing the boundary identifications then gives back $\hat{\mathbb{C}}$.

The self-homeomorphism $\lambda$ of $D(0,1)$ given by $\lambda(re^{i\theta}) = re^{i(\theta - 2\pi r)}$ fixes the boundary of $D(0, 1)$ and interchanges $\frac{1}{2}$ and $-\frac{1}{2}$; see Fig. I.1.1. For each $i \in [1 \uparrow (n - 1)]$, let $\phi_i$ denote the self-homeomorphism of $\hat{\mathbb{C}}$ which acts

![Diagram](image-url)
as the identity map on $\hat{\mathbb{C}} - \mathbf{D}(i + \frac{1}{2}, 1)$.

and by $z \mapsto \lambda(z - i - \frac{1}{2}) + i + \frac{1}{2}$ on $\mathbf{D}(i + \frac{1}{2}, 1)$.

Then $\phi_{[1](n-1)}$ generates a group $\langle \phi_{[1](n-1)} \rangle$ of self-homeomorphisms of $\hat{\mathbb{C}}$ which sheds light on the fundamental group of $S_{0,1,n}$, we first give $\hat{\mathbb{C}}$ a CW-structure by specifying a graph $S_{0,1,n}$ embedded in $\hat{\mathbb{C}}$.

For each $k \in [(−1)[n]]$, we have vertices $w_k := k + \frac{1}{2} - i$ and $v_k := k + \frac{1}{2} + i$, and (in $\mathbb{C}$) an oriented straight edge $f_k$ joining $w_k$ to $v_k$. For each $k \in [0\cdots n]$, we have an oriented straight edge $e_k$ joining $w_{k-1}$ to $w_k$, and an oriented straight edge $d_k$ joining $v_{k-1}$ to $v_k$. This completes the description of the graph $S_{0,1,n}$.

Each distinguished point $k \in [0\cdots n]$ is the midpoint of the rectangle in $\mathbb{C}$ cut out by the path $f_{k-1}d_k\bar{f}_k\bar{e}_k$. For $n = 3$, $S_{0,1,3}$ can be seen in Fig. I.1.2.

![Diagram](image)

Figure I.1.2: $S_{0,1,3}$.

Let $\langle S_{0,1,n}^{(1)} \ | \ \rangle$ denote the (free) fundamental groupoid of $S_{0,1,n}^{(1)}$, and let $\langle S_{0,1,n}^{(1)} \ | \ f(-1)[n] \rangle$ denote the (free) fundamental group of $S_{0,1,n}^{(1)}$ at $w_{-1}$. The subgraph of $S_{0,1,n}^{(1)}$ spanned by $e_{[0\cdots n]} \lor f_{(-1)[n]}$ is a maximal subtree of $S_{0,1,n}^{(1)}$, and $d_{[0\cdots n]}$ then determines a free generating set $t_{[0\cdots n]}$ of $\langle S_{0,1,n}^{(1)} \ | \ (w_{-1}, w_{-1}) \rangle$ explicitly, for each $k \in [0\cdots n]$, $t_k = \Pi e_{[0\cdots (k-1)]} f_{k-1} d_k \bar{f}_k \Pi \bar{e}_{[k \cdots n]}$.

The path $f_{-1} \Pi d_{[0\cdots n]} \bar{f}_n \Pi \bar{e}_{[n \cdots 0]}$ cuts out a rectangle in $\mathbb{C}$; the complementary region in $\hat{\mathbb{C}}$ together with the graph $S_{0,1,n}^{(1)}$ is then a retract of $\hat{\mathbb{C}} - [0\cdots n]$. Let $\sim$ denote homotopy for closed paths at $w_{-1}$ in $\hat{\mathbb{C}} - [0\cdots n]$. We can identify the fundamental groupoid of $S_{0,1,n}$ with $\langle S_{0,1,n}^{(1)} \ | \ f_{-1} \Pi d_{[0\cdots n]} \bar{f}_n \Pi \bar{e}_{[n \cdots 0]} \sim w_{-1} \rangle$. We then identify $\Sigma_{0,1,n}$ with the fundamental group of $S_{0,1,n}$ at $w_{-1}$,

$$
\Sigma_{0,1,n} = \langle S_{0,1,n}^{(1)} \ | \ f_{-1} \Pi d_{[0\cdots n]} \bar{f}_n \Pi \bar{e}_{[n \cdots 0]} \sim w_{-1} \rangle (w_{-1}, w_{-1})
= \langle t_{[0\cdots n]} \ | \ \Pi t_{[0\cdots n]} = 1 \rangle.
$$

Consider the action of $\phi_1$ on the graph $S_{0,1,n}^{(1)}$. For $n = 3$, the result can be
seen in Fig. I.1.3. The crucial point is that \( f_1^{\phi_1} \sim e_2 f_2 d_2 f_1 e_1 f_0 d_1 \), and all the other elements of \( S_{0,1,3}^{(1)} \) are fixed by \( \phi_1 \); this makes the action quite simple algebraically. Then, \( f_1^{\phi_1} \sim d_1 f_0 e_1 f_1 d_2 f_2 e_2 \), and, for the free generator \( t_1 = e_0 f_0 d_1 \), we have

\[
t_1^{\phi_1} \sim e_0 f_0 d_1 (d_1 f_0 e_1 f_1 d_2 f_2 e_2) \Pi \sigma_1 \sim \Pi e_0 f_1 d_2 f_2 \Pi \sigma_2 = t_2.
\]

Similarly, for this element, \( t_2 \), we have

\[
t_2^{\phi_1} \sim \Pi e_0 f_2 d_2 (e_2 f_2 d_2 f_1 e_1 f_0 d_1) d_2 f_2 \Pi \sigma_2 \sim \Pi e_0 f_2 d_2 f_1 e_0 d_1 \Pi \sigma_0 = t_2 t_1 t_2,
\]

where the latter homotopy can be seen directly by collapsing the elements of \( e_0 f_2 \) and \( f_1 d_2 f_2 e_2 \), which lie in the maximal subtree. Thus, we see that \( \phi_1 \) acts on \( \Sigma_{0,1,n} \) as the automorphism \( \sigma_1 \). It follows that the action of any given element of \( B_n \) on \( \Sigma_{0,1,n} \) is induced by some self-homeomorphism \( \phi \in \langle \phi_{[1]}^{(n+1)} \rangle \). The interesting feature now is that \( \phi \) carries the oriented Jordan curve \( f_{-1} \Pi d_0 f_1 \Pi \sigma_{[1]} \) (\( \sim t_0 t_1 \)) to an oriented Jordan curve \( f_{-1} \Pi d_0 f_1 \Pi \sigma_{[1]} \phi \) (\( \sim (t_0 t_1)^{\phi} \sim t_0 t_1^{\phi} \)).
Actions of the braid group

Figure I.1.4: Jordan curves for $z_1\bar{t}_1$ and a Whitehead graph for $t_1^{2n} = t_1t_2\bar{t}_1$.

Recall that $\tilde{C}$ is obtained by edge identification from the $(2n + 2)$-gon with clockwise boundary label $\prod_{i \in [0,n]} (t_i\bar{t}_i)$. The Jordan curve $f_{-1}\Pi d_{[0|1]} \bar{f}_1 \Pi \pi_{[1|0]}$ has as its preimage, in the $(2n + 2)$-gon, the union of a family of disjoint oriented arcs. These arcs can be used to reconstruct $t_1^0$, since the Jordan curve cyclically reads off $t_0t_1^0$ from its meetings with the labelled oriented tethers; notice that the set of tethers is now dual to the set of generators $t_{[0|1]}$. The purpose of this appendix is to define and study a combinatorial representation of the family of arcs, and recover Larue’s characterization of the elements of $t_1^{B_n}$.

Although it will not be used in our arguments, let us mention the fact that, on collapsing the interior of each labelled edge of the $(2n + 2)$-gon to a labelled vertex, each oriented arc in the family becomes an oriented edge, and we recover the (directed, multi-edge, non-cyclic) Whitehead graph of $t_1^0$; see Fig. I.1.4.

II Nested sets

We now introduce some formal definitions that will allow us to associate a combinatorial Jordan curve to each element of $t_1^{B_n}$.

II.1 Definitions. Let $(A, \leq)$ be a finite ordered set, and let $m \in \mathbb{N}$.

Let $N$ denote the number of elements of $A$. Then $A$ is order-isomorphic to $[1 \uparrow N]$ in a unique way, and we assign to $A$ the induced metric, denoted $d_A$. Thus $d_A(a_1, a_2) = 1$ if and only if $a_1 \neq a_2$ and no element of $A$ lies strictly between $a_1$ and $a_2$. 

Let $a_1, a_2, b_1, b_2$ be elements of $A$. We say that $\{a_1, b_1\}$ is nested with $\{a_2, b_2\}$ (for $(A, \leq)$) if $a_1, a_2, b_1, b_2$ are distinct elements of $A$, and either both of, or neither of, $a_2$ and $b_2$ lie between $a_1$ and $b_1$ in $(A, \leq)$. It is not difficult to see that, in this event, $\{a_2, b_2\}$ is nested with $\{a_1, b_1\}$.

Let $a_{[1]^m}$ and $b_{[1]^m}$ be sequences in $A$.

We say that $a_{[1]^m}$ is a sequence without repetitions if $a_i \neq a_j$ for all $i \neq j$ in $[1]^m$.

We say that $a_{[1]^m}$ is an ascending sequence (in $(A, \leq)$) if $a_1 \leq a_2 \leq \cdots \leq a_m$ in $(A, \leq)$.

We say that $\{a_i, b_i\}_{i \in [1]^m}$ is nested (for $(A, \leq)$) if, for all $i \neq j$ in $[1]^m$, $\{a_i, b_i\}$ is nested with $\{a_j, b_j\}$ for $(A, \leq)$.

We let $\text{Sym}_m$ act on $A^m$, on the left, by $\pi(a_{[1]^m}) := a_{[1]^m}^{\pi}$. For example, $(1,2,3)(a_1, a_2, a_3) = (a_3, a_1, a_2)$, and, hence, $(1,2,3)(a, b, c) = (c, a, b)$. The ascending rearrangement of $a_{[1]^m}$ is the unique ascending sequence in $(A, \leq)$ that lies in the $\text{Sym}_m$-orbit of $a_{[1]^m}$.

Let $a_{[1]^2(m)}$ be a sequence in $A$.

A permutation $\pi \in \text{Sym}_{2m}$ is said to embed $a_{[1]^2(2m)}$ in a plane if $\pi a_{[1]^2(2m)}$ is ascending for $(A, \leq)$, and both $\{(2i-1, 2i)^\pi\}_{i \in [1]^m}$ and $\{(2i, 2i+1)^\pi\}_{i \in [1](m-1)}$ are nested in $(\mathbb{N}, \leq)$. We call $\{(2i-1, 2i)^\pi\}_{i \in [1]^m}$ the odd-even pairing, and call $\{(2i, 2i+1)^\pi\}_{i \in [1](m-1)}$ the even-odd pairing.

We say that $a_{[1]^2(2m)}$ is a planar sequence (in $(A, \leq)$) if there exists some $\pi \in \text{Sym}_{2m}$ which embeds $a_{[1]^2(2m)}$ in a plane. (If no two consecutive terms of $a_{[1]^2(2m)}$ are equal, $\pi$ is then unique, but we shall not need this fact.) There is then an associated diagram in $\mathbb{C}$ formed as follows. We assign, to each point $i \in [1]^2(m) \subset \mathbb{C}$ the label $a_i^{\pi}$; notice that this means that the label of $i^{\pi}$ is $a_i$.

For each $i \in [1]^m$, we join $(2i-1)^\pi$ (labelled $a_{2i-1}$) to $(2i)^\pi$ (labelled $a_{2i}$) by an oriented semi-circle in the upper half-plane. For each $i \in [1](m-1)$, we join $(2i)^\pi$ (labelled $a_{2i}$) to $(2i+1)^\pi$ (labelled $a_{2i+1}$) by an oriented semi-circle in the lower half-plane. These oriented semi-circles link up to form an oriented arc which traces out the sequence $a_{[1]^2(2m)}$, and the nesting property means that the arc has no self-crossings.

\[\square\]

**II.2 Example.** Suppose that $a_{[1]^8} = (\pi_1, t_1, \bar{t}_1, t_2, \bar{t}_2, t_1, t_1, z_1)$ is a sequence in some ordered set $(A, \leq)$, and that the ascending rearrangement of $a_{[1]^8}$ is $(\pi_1, t_1, t_1, \bar{t}_1, t_2, \bar{t}_2, t_1, t_1, z_1)$.

The permutation \[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 2 & 5 & 6 & 7 & 4 & 3 & 8
\end{pmatrix}
\]
embeds $a_{[1]^8}$ in a plane since both $\{\{1, 2\}, \{5, 6\}, \{7, 4\}, \{3, 8\}\}$ and $\{\{2, 5\}, \{6, 7\}, \{4, 3\}\}$ are nested in $(\mathbb{N}, \leq)$, and $^{(3,5,7)(4,6)}(\pi_1, t_1, \bar{t}_1, t_2, \bar{t}_2, t_1, t_1, z_1) = (\pi_1, t_1, t_1, \bar{t}_1, t_2, \bar{t}_2, t_1, t_1, z_1)$. The associated diagram can be seen in Fig. II.2.1.
Let us record two results which will be useful later.

**II.3 Lemma.** Let \((A, \leq)\) be an ordered set, let \(m \in \mathbb{N}\), and let \(a_{[1\{2m\}]}\) be a sequence in \(A\).

Then \(a_{[1\{2m\}]}\) is planar for \((A, \leq)\) if and only if there exists an ordered set \((B, \leq)\) with \(|B| = 2m\), and a sequence \(b_{[1\{2m\}]}\) in \(B\), without repetitions, and an ordered-set map \(B \to A, b \mapsto \text{label}(b)\), such that \(\text{label}(b_{[1\{2m\}]})) = a_{[1\{2m\}]}\), and \(\{\{b_{2i}, b_{2i+1}\}\}_{i \in [1/m-1]}\) and \(\{\{b_{2i-1}, b_{2i}\}\}_{i \in [1/m]}\) are nested for \((B, \leq)\).

**Proof.** Suppose first that \(a_{[1\{2m\}]}\) is planar for \((A, \leq)\), and let \(\pi\) be an element of \(\text{Sym}_2m\) that embeds \(a_{[1\{2m\}]}\) in a plane. We take \(B\) to be \([1\{2m\}]\) with the usual ordering. For each \(i \in [1\{2m\}])\), let \(\text{label}(i) = a_{i\pi}\) and let \(b_i = i_\pi\); thus, \(\text{label}(b_i) = \text{label}(i_\pi) = a_i\). All the conditions are satisfied.

Conversely, if \(B\) exists, we can identify \(B\) with \([1\{2m\}]]\) with the usual ordering, in a unique way. Then the map \(i \mapsto b_i\) is an element \(\pi\) of \(\text{Sym}_2m\) that embeds \(a_{[1\{2m\}]}\) in a plane. \(\square\)

**II.4 Lemma.** Let \((A, \leq)\) be an ordered set, and let \(m\) be a positive integer. Let \(c_{[1\{m\}]}\) and \(\overline{c}_{[1\{m\}]}\) be sequences without repetitions in \((A, \leq)\) such that \(\{\{c_i, \overline{c}_i\}\}_{i \in [1/m]}\) is nested, and \(\max(c_{[1\{m\}]})) < \min(\overline{c}_{[1\{m\}]}))\). If \(c_{[1\{m\}]}\) is ascending in \((A, \leq)\), then \(\overline{c}_{[m\{1\}]}\) is also ascending in \((A, \leq)\).

**Proof.** We argue by induction on \(m\). If \(m = 1\), the conclusion is trivial. Now, assume that \(m \geq 2\) and that the implication holds with \(m - 1\) in place of \(m\). We see that \(c_1 < c_2 \leq \max(c_{[1\{m\}]})) < \min(\overline{c}_{[1\{m\}]}))\). Since \(\{c_1, \overline{c}_1\}\) is nested with \(\{c_2, \overline{c}_2\}\), we also see that \(c_1 < \overline{c}_2 < \overline{c}_1\). By the induction hypothesis, \(\overline{c}_{[m\{2\}]}\) is ascending, and hence \(\overline{c}_{[m\{1\}]}\) is ascending. The result is proved. \(\square\)

### III Planar elements of \(\Sigma_{0,1,n}\)

#### III.1 Definitions.

Let \(A\) denote \((z_1, \overline{z}_1) \lor t_{[1\{n\}]} \lor \overline{t}_{[1\{n\}]}\), the usual monoid-generating sequence of \(\Sigma_{0,1,n}\). We form the ordered set \((A, \leq)\) with

\[
\overline{z}_1 < t_1 < \overline{t}_1 < \cdots < t_n < \overline{t}_n < z_1.
\]
We remark that, for \( n \neq 1 \), the ordering on \( A \) is reminiscent of the ordering of the ends of \( \Sigma_{0,1,n} \) in Section 7. We emphasize that, even if \( n = 1 \), \( z_1 \neq t_1 \) in \( A \).

Let \( m \in \mathbb{N} \). Consider a sequence \( a_{[1;m]} \) in \( t_{[1;m]} \lor \bar{t}_{[1;m]} \), and let \( w = \Pi a_{[1;m]} \in \Sigma_{0,1,n} \); thus \( a_{[1;m]} \) is an expression for \( w \). We define the Whitehead expansion of \( a_{[1;m]} \) to be the sequence

\[
(\bar{z}_1) \lor (\lor_{i \in [1;m]}(a_i, \bar{a}_i)) \lor (z_1) = (\bar{z}_1, a_1, \bar{a}_1, a_2, \bar{a}_2, \ldots, a_m, \bar{a}_m, z_1)
\]
in \( A \). We say that \( a_{[1;m]} \) is a planar expression for \( w \) if the Whitehead expansion of \( a_{[1;m]} \) is planar for \( (A, \leq) \). If the unique reduced expression for \( w \) is a planar expression for \( w \), then we say that \( w \) is a planar element of \( \Sigma_{0,1,n} \).

III.2 Examples. (i). \( t_1 \bar{t}_2 t_1 \) is planar, since the Whitehead expansion of the reduced expression is \((\bar{z}_1, t_1, t_1, t_1, t_1, t_1, t_1, z_1)\), which is planar for \((A, \leq)\), by Example II.2; in a sense, Fig. II.2.1 reflects Fig. I.1.4. We call Fig. II.2.1 the Larue-Whitehead diagram of \( t_1 \bar{t}_2 t_1 \).

(ii). \( t_1 \bar{t}_2 \) is not planar; there is only one permutation to consider.

(iii). \( t_1^2 \) is not planar; there are four permutations to consider.

(iv). \( t_3 \bar{t}_3 \bar{t}_1 \) is planar, while \( t_3^2 \bar{t}_3 \) is not planar, and these two group elements have the same Whitehead graph.

III.3 Proposition. Let \( w \in \Sigma_{0,1,n} \). If there exists some planar expression for \( w \), then (the reduced expression for) \( w \) is planar.

Proof. Suppose that \( a_{[1;m]} \) is a planar expression for \( w \), as in Definitions III.1.

By Lemma II.3, there exists an ordered set \((B, \leq)\) with \(|B| = 2m + 2\), and a planar sequence \( b_{[1;2m+2]} \) in \((B, \leq)\), without repetitions, and an order-preserving bijection \( B \to A, b \mapsto \text{label}(b) \), such that \( \text{label}(b_{[1;2m+2]}) \) is the Whitehead expansion of \( a_{[1;m]} \).

Suppose that the given planar expression \( a_{[1;m]} \) is not reduced. We shall find a shorter planar expression for \( w \).

There exists some \( j \in [1; (m - 1)] \) such that \( a_{j+1} = \bar{a}_j \) in \( t_{[1;n]} \lor \bar{t}_{[1;n]} \), and we may suppose that we have chosen this \( j \) in such a way that \( d_B(b_{2j+1}, b_{2j+2}) \) has the minimum possible value. Notice that \( \text{label}(b_{[2j;2j+3]}) = (a_j, \bar{a}_j, a_j, a_j). \)

Clearly, \( w = \Pi a_{[1;(j-1)]} \Pi a_{[j+1;m]} \), and

\[
\text{label}(b_{[1;2j-1]} \lor (2j+4) \lor (2m+2))) = (\bar{z}_1) \lor (\lor_{i \in [1;(j-1)]} (a_i, \bar{a}_i)) \lor (\lor_{i \in ([j+2];m]} (a_i, \bar{a}_i)) \lor (z_1).
\]

It suffices to show that \( b_{[1;2j-1]} \lor (2j+4) \lor (2m+2] \) is planar for \((B, \leq)\).

Claim. \( d_B(b_{2j}, b_{2j+3}) = 1 \).

Proof of claim. Consider any \( k \in [1; (2m - 1)] \) such that \( b_k \) lies between \( b_{2j} \) and \( b_{2j+3} \).

Let \( \eta \) denote \((-1)^k \).
Since \( \text{label}(b_{2j}) = \text{label}(b_{2j+3}) = a_j \), we see that \( \text{label}(b_k) = a_j \). Hence \( \text{label}(b_{k+\eta}) = \bar{a}_j = \text{label}(b_{2j+1}) = \text{label}(b_{2j+2}) \).

Either \( a_j < \bar{a}_j \) or \( a_j > \bar{a}_j \) in \( (A, \leq) \). Hence,

either \( \max\{b_{2j}, b_k, b_{2j+3}\} < \min\{b_{2j+1}, b_{k+\eta}, b_{2j+2}\} \) in \( (B, \leq) \),

or \( \min\{b_{2j}, b_k, b_{2j+3}\} > \max\{b_{2j+1}, b_{k+\eta}, b_{2j+2}\} \) in \( (B, \leq) \), respectively.

Since \( \{b_{2j}, b_{2j+1}\}, \{b_{2j+2}, b_{2j+3}\}, \{b_k, b_{k+\eta}\} \) is nested, and \( b_k \) lies between \( b_{2j} \) and \( b_{2j+3} \), we see, from Lemma II.4, that \( b_{k+\eta} \) lies between \( b_{2j+1} \) and \( b_{2j+2} \).

Since \( \{b_{2j+1}, b_{2j+2}\} \) is nested with \( \{b_{k+\eta}, b_{k+2\eta}\} \) and \( b_{k+\eta} \) lies between \( b_{2j+1} \) and \( b_{2j+2} \), we see that \( b_{k+2\eta} \) lies between \( b_{2j+1} \) and \( b_{2j+2} \). Hence,

\[
d_B(b_{k+2\eta}, b_{k+\eta}) \leq d_B(b_{2j+1}, b_{2j+2}),
\]

with equality holding only if \( \{b_{k+2\eta}, b_{k+\eta}\} = \{b_{2j+1}, b_{2j+2}\} \). Also, \( \text{label}(b_{k+2\eta}) = \bar{a}_j \), and, hence, \( \text{label}(b_{k+3\eta}) = a_j \). Thus

\[
\text{label}(b_k, b_{k+\eta}, b_{k+2\eta}, b_{k+3\eta}) = (a_j, \bar{a}_j, \bar{a}_j, a_j).
\]

By the minimality of \( d_B(b_{2j+1}, b_{2j+2}) \), we see that \( k = 2j \) or \( k = 2j + 3 \). This proves the claim.

Now consider the passage from \( b_{[1 \mid (2m+2)]} \) to \( b_{[1 \mid (2j-1)]} \lor b_{[2j+1 \mid (2m+2)]} \).

For the odd-even pairing, we pass from \( \{\{b_{2i-1}, b_{2i}\}\}_{i \in [1 \mid (j-1)]} \lor \{\{b_{2j-1}, b_{2j+4}\}\} \) to

\[
\{\{b_{2i-1}, b_{2i}\}\}_{i \in [1 \mid (j-1)]} \lor \{\{b_{2j-1}, b_{2j+4}\}\}.
\]

Thus, we remove \( \{\{b_{2j-1}, b_{2j}\}, \{b_{2j+1}, b_{2j+2}\}, \{b_{2j+3}, b_{2j+4}\}\} \), and we add only \( \{\{b_{2j-1}, b_{2j+4}\}\} \). To see that, for all \( k \in [1 \mid (j-1)] \lor [(j+3) \mid (m+1)] \), \( b_{2k-1}, b_{2k} \) is nested with \( \{b_{2j-1}, b_{2j+4}\} \), we note the following:

\( (b_{2j-1} \text{ lies between } b_{2k-1} \text{ and } b_{2k}) \)

\( \iff (b_{2j} \text{ lies between } b_{2k-1} \text{ and } b_{2k}) \)

\( \text{since } \{b_{2j-1}, b_{2j}\} \text{ is nested with } \{b_{2k-1}, b_{2k}\} \)

\( \iff (b_{2j+3} \text{ lies between } b_{2k-1} \text{ and } b_{2k}) \)

\( \text{since } d_B(b_{2j}, b_{2j+3}) = 1 \)

\( \iff (b_{2j+4} \text{ lies between } b_{2k-1} \text{ and } b_{2k}) \)

\( \text{since } \{b_{2j+3}, b_{2j+4}\} \text{ is nested with } \{b_{2k-1}, b_{2k}\} \).

For the even-odd pairing, we pass from \( \{\{b_{2i}, b_{2i+1}\}\}_{i \in [1 \mid m]} \) to

\[
\{\{b_{2i}, b_{2i+1}\}\}_{i \in [1 \mid (j-1)]} \lor [(j+2) \mid m].
\]

Thus, we remove \( \{\{b_{2j}, b_{2j+1}\}, \{b_{2j+2}, b_{2j+3}\}\} \), and we add nothing. Hence this remains nested.

This completes the proof. \( \square \)
At the end of the next section, we shall see that the following generalizes Corollary 7.6.

**III.4 Proposition.** Let \( w \) be a planar element of \( \Sigma_{0,1,n} \), and let \( k \in [1\uparrow n] \).

(i). \( w \) is squarefree.

(ii). \( w \notin ((\Pi_{[1\uparrow (k+1)]})t_k) \). Let \( t_k \in ((\Pi_{[1\uparrow (k+1)]})t_k) \).

(iii). \( w \notin ((\Pi_{[1\uparrow (k-1)]})t_k) \).

**Proof.** Suppose that \( a_{[1\uparrow m]} \) is the reduced planar expression for \( w \), as in Definitions III.1. By Lemma II.3, there exists an ordered set \((B, \leq)\) with \(|B| = 2m + 2\), and a planar sequence \( b_{[1\uparrow (2m+2)]} \) in \((B, \leq)\), without repetitions, and an order-respecting labelling \( B \rightarrow A, b \mapsto \text{label}(b) \), such that

\[
\text{label}(b_{[1\uparrow (2m+2)]}) = (\overline{z_1}) \cup \bigwedge_{i\in[1\uparrow m]} (a_i, \overline{a_i}) \cup (z_1).
\]

(i). Suppose that \( w \) is not squarefree; hence, for some \( i \in [1\uparrow m] \) and some \( \ell \in [1\uparrow (m-1)] \), \( a_{[\ell\uparrow (\ell+1)]} \) is \((t_i, t_i)\) or \((\overline{t}_i, \overline{t}_i)\). Hence \( \text{label}(b_{[2\uparrow (2\ell+3)]}) = (t_i, \overline{t}_i, t_i, \overline{t}_i) \) or \((\overline{t}_i, t_i, \overline{t}_i, t_i)\).

Let \( m_i \) be the number of elements of \( B \) with label \( t_i \). Let \( c_{[1\uparrow m]} \) be the ascending sequence in \((B, \leq)\) which is the interval of elements labelled \( t_i \). For each \( k \in [1\uparrow m_i] \), let \( \overline{c}_k \) denote the element of \( B \) such that \( \{c_k, \overline{c}_k\} \) is an element of the even-to-odd pairing for \( b_{[1\uparrow (2m+2)]} \). By the definition of the Whitehead expansion, the label of \( \overline{c}_k \) is \( \overline{t}_i \). By Lemma II.4, \( \overline{c}_{[m_i]} \) is the ascending sequence in \((B, \leq)\) which is the interval of elements of \( B \) labelled \( \overline{t}_i \).

By hypothesis, there exists \( \ell \in [1\uparrow (m-1)] \) such that \( \{b_{2\ell+1}, b_{2\ell+2}\} = \{\overline{c}_j, c_k\} \) for some \( j, k \in [1\uparrow m_i] \). Let us choose \( \ell \) so that \( j + k \) is as large as possible. We claim that \( k = m_i \). Suppose not; then \( c_k < c_{k+1} < \overline{c}_j \). Consider the \( d \in B \) such that \( \{c_{k+1}, d\} \) lies in the odd-even pairing for \( b_{[1\uparrow (2m+2)]} \). Then \( d \in [c_k \setminus \overline{c}_j \setminus \overline{c}_j \setminus \overline{c}_j \setminus \overline{c}_j] \) such that \( \text{label}(d) = t_i \). Since \( a_{[1\uparrow m]} \) is reduced, \( \text{label}(d) \neq t_i \). Hence \( \text{label}(d) = \overline{t}_i \). Thus, \( d = \overline{c}_{j'} \) for some \( j' \in [m_i \setminus (j + 1)] \). This contradicts the maximality of \( k + j \). Hence \( k = m_i \), as claimed. Similarly, \( j = m_i \). Thus \( \{c_{m_i}, \overline{c}_{m_i}\} \) lies in both the even-odd pairing and the odd-even pairing. This gives a closed-curve component within an arc which joins \( z_1 \) to \( z_1 \). Hence, we have a contradiction.

(ii). Suppose that \( w \in ((\Pi_{[1\uparrow (k+1)]})t_k^*) \).

Then \( a_{[1\uparrow (n-k+1)]} = \overline{t}_{[1\uparrow (n-k+1)]} \cup t_k \) and

\[
\text{label}(b_{[1\uparrow (2n-2k+3)]}) = (\overline{z}_1, \overline{t}_n, t_n, \overline{t}_{n-1}, t_{n-1}, \ldots, \overline{t}_{k+1}, t_{k+1}, t_k, \overline{t}_k).
\]

Since \( \text{label}(b_{[2n-2k+3]} \cup (2n-2k+3)] = (t_{k+1}, t_k, \overline{t}_k) \), we see that, in \((B, \leq)\),

\[
b_{2n-2k+2} < b_{2n-2k+3} < b_{2n-2k+1} \text{ with labels } t_k, \overline{t}_k, t_{k+1}.
\]

Since \( \{b_{2n-2k+1}, b_{2n-2k+2}\} \) and \( \{b_{2n-2k+3}, b_{2n-2k+4}\} \) are nested in \((B, \leq)\), we see that \( b_{2n-2k+4} \equiv b_{2n-2k+2} \). In particular, \( \text{label}(b_{2n-2k+4}) \in \{t_k, \overline{t}_k, t_{k+1}\} \).
and label($b_{2n-2k+4}$) = $a_{n-k+2}$. Since $a_{[1]\cap m}$ is reduced, $a_{n-k+2} \neq \overline{t}_k$. By (i), $a_{n-k+2} \neq t_k$. Hence $a_{n-k+2} = t_{k+1}$ and the nesting is

$$b_{2n-2k+2} < b_{2n-2k+3} < b_{2n-2k+4} < b_{2n-2k+1}$$

with labels $t_k, \overline{t}_k, t_{k+1}, t_{k+1}$.

Using the last inequality and Lemma II.4, we see that

$$b_{2n-2k+4} < b_{2n-2k+1} < b_{2n-2k} < b_{2n-2k+5}$$

with labels $t_{k+1}, t_{k+1}, \overline{t}_{k+1}, \overline{t}_{k+1}$.

Now label($b_{[1]\cap (2n-k+7)}$) is

$$(\overline{z}_1, \overline{t}_n, t_n, \overline{t}_{n-1}, \overline{t}_{n-2}, \overline{t}_{k+2}, t_{k+2}, \overline{t}_{k+1}, t_{k+1}, t_k, \overline{t}_k, t_k, t_{k+1}, \overline{t}_{k+1}, a_{n-k+3}, \overline{a}_{n-k+3})$$

Notice that

$$b_{2n-2k} < b_{2n-2k+5} < b_{2n-2k-1}$$

with labels $t_{k+1}, \overline{t}_{k+1}, t_{k+2}$.

Also $\{b_{2n-2k+5}, b_{2n-2k+6}\}$ is nested with $\{b_{2n-2k}, b_{2n-2k-1}\}$. Hence label($b_{2n-2k+6}$) lies in $\{t_{k+1}, \overline{t}_{k+1}, t_{k+2}\}$, and label($b_{2n-2k+6}$) = $a_{n-k+3}$. Since $a_{[1]\cap m}$ is reduced, $a_{n-k+3} \neq \overline{t}_{k+1}$. By (i), $a_{n-k+3} \neq t_{k+1}$. Hence $a_{n-k+3} = t_{k+2}$ and the nesting is

$$b_{2n-2k} < b_{2n-2k+5} < b_{2n-2k+6} < b_{2n-2k-1}$$

with labels $t_{k+1}, \overline{t}_{k+1}, t_{k+2}, t_{k+2}$.

Using the last inequality and Lemma II.4, we see that

$$b_{2n-2k+6} < b_{2n-2k-1} < b_{2n-2k-2} < b_{2n-2k+7}$$

with labels $t_{k+2}, t_{k+2}, \overline{t}_{k+2}, \overline{t}_{k+2}$.

By repeating the argument in the last paragraph, we eventually find that

$$w = t_k^{\Pi t_{[k+1]\cap n}}.$$

(iii). Suppose that $w \in (\Pi t_{[1]\cap k-1}) \overline{t}_k^*$. Then $a_{[1]\cap k} = t_{[1]\cap (k-1)} \vee (\overline{t}_k)$.

$$\text{label}(b_{[1]\cap (2k+1)}) = (z_1, t_1, \overline{t}_1, t_2, \overline{t}_2, \ldots, t_{k-1}, \overline{t}_{k-1}, \overline{t}_k, t_k),$$

and by an argument similar to that given in (ii), we find that this is impossible.

\[ \square \]

IV \quad $\mathcal{B}_n$ permutes the planar elements of $\Sigma_{0,1,n}$

**IV.1 Proposition.** Let $w \in \Sigma_{0,1,n}$ and let $i \in [1\uparrow (n-1)]$. If $w$ is planar, then $w^{\sigma_i}$ is planar.

**Proof.** Suppose that $r_{[1]\cap m}$ is any planar expression for $w$, as in Definitions III.1. In applying $\sigma_i$ to $(\overline{z}_1) \vee (\bigvee_{i \in [1]\cap m} (r_i, \overline{t}_i)) \vee (z_1)$ we replace each $(t_i, \overline{t}_i)$ with $(t_{i+1}, \overline{t}_{i+1})$,

each $(\overline{t}_i, t_i)$ with $(\overline{t}_{i+1}, t_{i+1})$,

each $(t_{i+1}, \overline{t}_{i+1})$ with $(t_{i+1}, t_{i+1}, t_i, \overline{t}_i, t_{i+1}, \overline{t}_{i+1})$.
each \((\bar{t}_{i+1}, t_{i+1})\) with \((\bar{t}_{i+1}, t_{i+1}, \bar{t}_i, t_i, t_{i+1}, \bar{t}_{i+1})\).

We will not perform any cancellations in the resulting sequence.

By Lemma II.3, there exists an ordered set \((B, \leq)\) with \(|B| = 2m + 2\), and a planar sequence \(p_{[1](2m+2)}\) in \((B, \leq)\), without repetitions, and an order-respecting labelling \(B \to A, b \mapsto \text{label}(b)\), such that

\[
\text{label}(p_{[1](2m+2)}) = (\bar{z}_1) \lor (\lor_{i \in [1,m]} (r_i, \bar{r}_i)) \lor (z_1).
\]

Let \(m_i\) denote the number of elements of \(B\) with label \(t_i\), and let \(m_{i+1}\) denote the number of elements of \(B\) with label \(t_{i+1}\). To begin, we have to add \(4m_{i+1}\) elements to \(B\), and we have to specify the ordering on the expanded set.

Let \(c_{[1|m_i]}\) denote the ascending sequence of those elements of \(B\) which have label \(t_i\). Let \(\bar{c}_{[m_i|1]}\) denote the ascending sequence of those elements of \(B\) which have label \(\bar{t}_i\). Let \(d_{[1|m_{i+1}]}\) denote the ascending sequence of those elements of \(B\) which have label \(t_{i+1}\). Let \(\bar{d}_{[m_{i+1}|1]}\) denote the ascending sequence of those elements of \(B\) which have label \(\bar{t}_{i+1}\). We then have an interval in \(B\)

\[
[c_1 \bar{d}_1] = c_{[1|m_i]} \lor \bar{c}_{[m_i|1]} \lor d_{[1|m_{i+1}]} \lor \bar{d}_{[m_{i+1}|1]}.
\]

We create an interval of \(4m_{i+1}\) new elements

\[
[a_1 \bar{b}_1] = a_{[1|m_{i+1}]} \lor \bar{a}_{[m_{i+1}|1]} \lor b_{[1|m_{i+1}]} \lor \bar{b}_{[m_{i+1}|1]}
\]

and expand \(B\) by inserting this interval \([a_1 \bar{b}_1]\) just before the interval \([c_1 \bar{d}_1]\). We then have a new ordered set \(B'\) with \(2m + 2 + 4m_{i+1}\) elements.

We now specify the labelling \(B' \to A\). On \(c_{[1|m_i]}\), we change the labels from \(t_i\) to \(t_{i+1}\). On \(\bar{c}_{[m_i|1]}\), we change the labels from \(\bar{t}_i\) to \(\bar{t}_{i+1}\). On \(d_{[1|m_{i+1}]}\), we change the labels from \(t_{i+1}\) to \(\bar{t}_{i+1}\). On \(\bar{d}_{[m_{i+1}|1]}\), we keep the same labels. On \(B - [c_1 \bar{d}_1]\), we keep the same labels. We give all the elements of \(a_{[1|m_{i+1}]}\) the label \(t_i\); all the elements of \(\bar{a}_{[m_{i+1}|1]}\) the label \(\bar{t}_i\); and all the elements of...
Let \( p_{2k}, p_{2k+1} \) be the sequence in \( B' \) obtained from \( p_{1\{2m+2\}} \) as follows. For each \( j \in [1 \ldots m], \) there exists a unique \( k \in [1 \ldots m] \) such that \( \{p_{2k}, p_{2k+1}\} = \{d_j, \bar{d}_j\}. \) If \( (p_{2k}, p_{2k+1}) = (d_j, \bar{d}_j) \) in \( p_{1\{2m+2\}}, \) then it is to be expanded to \( (d_j, \bar{b}_j, a_j, b_j, d_j) \) in \( q_{1\{2m+4m+1\}}. \) If \( (p_{2k}, p_{2k+1}) = (d_j, \bar{d}_j) \) in \( p_{1\{2m+2\}}, \) then it is to be expanded to \( (d_j, b_j, \bar{a}_j, a_j, \bar{b}_j, d_j) \) in \( q_{1\{2m+4m+1\}}. \) This completes the definition of \( q_{1\{2m+4m+1\}}. \)

In passing from \( \{p_{2k-1}, p_{2k}\} \) to \( \{q_{2k-1}, q_{2k}\} \), we add \( \{\bar{a}_j, b_j\}, \{a_j, \bar{b}_j\}\). In \( B' \), for each \( j \in [1 \ldots m+1], \)

\[
[a_j, b_j] = \{a_{1\{j\}} \cup b_{1\{j\}}\}
\]

has induced odd-even pairing \( \{\bar{a}_k, b_k\}\).

\[
[a_j, \bar{b}_j] = \{a_{1\{j\}} \cup \bar{a}_{1\{j\}} \cup b_{1\{j\}+1} \cup \bar{b}_{1\{j\}+1}\}
\]

has induced odd-even pairing \( \{\bar{a}_k, b_k\}, \{a_k, \bar{b}_k\}\).

Both types of intervals are closed under the odd-even pairing; this shows that \( \{q_{2k-1}, q_{2k}\} k \in [1\{m+2m+1\}] \) is nested.

**Figure IV.1.2:** \( a_{1\{2\}} \cup \bar{a}_{1\{1\}} \cup b_{1\{2\}} \cup \bar{b}_{1\{2\}} \cup c_{1\{1\}} \cup \bar{c}_{1\{1\}} \cup d_{1\{2\}} \cup \bar{d}_{1\{2\}}. \)
In passing from \( \{\{p_{2k}, p_{2k+1}\}\}_{k \in [1\parallel m]} \) to \( \{\{q_{2k}, q_{2k+1}\}\}_{k \in [1\parallel (m+2m_{i+1}−1)]} \), we delete \( \{\{d_j, \overline{d}_j\}\}_{j \in [1\parallel m_{i+1}]} \), and add \( \{\{d_j, \overline{d}_j\}, \{a_j, \overline{a}_j\}, \{b_j, \overline{b}_j\}\}_{j \in [1\parallel m_{i+1}]} \). In \( B' \), for each \( j \in [1\parallel m_{i+1}] \),

\[
[a_j \uparrow \overline{a}_j] \cup [b_j \uparrow \overline{b}_j] \upsilon [c_j \uparrow \overline{c}_j] \upsilon [d_j \uparrow \overline{d}_j] \text{ has induced even-odd pairing} \ 
\{\{a_k, \overline{a}_k\}\}_{k \in [1\parallel j]}, \\
\{\{b_k, \overline{b}_k\}\}_{k \in [1\parallel m_{i+1}]} \cup \{\{c_i, \overline{c}_i\}\}_{i \in [1\parallel m_{i}]} , \\
\{\{b_j, \overline{b}_j\}\}_{j \in [1\parallel m_{i+1}]} \cup \{\{c_i, \overline{c}_i\}\}_{i \in [1\parallel m_{i}]} \cup \{\{b_k, \overline{b}_k\}\}_{k \in [1\parallel m_{i+1}]} \cup \{\{c_i, \overline{c}_i\}\}_{i \in [1\parallel m_{i}]}.
\]

All three types of intervals are closed under the even-odd pairing; this shows that \( \{\{q_{2k}, q_{2k+1}\}\}_{k \in [1\parallel (m+2m_{i+1}−1)]} \) is nested.

A similar argument shows that \( \Phi_i \) carries planar elements to planar elements.

**IV.2 Theorem.** The group \( B_n \) acts on the set of planar elements of \( \Sigma_{0,1,n} \), and, hence, if \( n \geq 1 \), every element of \( t^B_n \) is planar.

**IV.3 Remark.** By combining Theorem IV.2 and Proposition III.4, we get another proof of Corollary 7.6.

**V The \( B_n \)-orbits of the planar elements of \( \Sigma_{0,1,n} \)**

In this section we rework \([21, Lemma 2.3.12]\) and in this case our argument is longer than Larue’s. The object is to show that the number of \( B_n \)-orbits in the set of all planar elements of \( \Sigma_{0,1,n} \) is \( n+1 \), and that \( \{\Pi_l[1\parallel n]\}_{k \in [0\parallel n]} \) is a complete set of representatives.

**V.1 Lemma.** Let \( i, j \) be elements of \( [1\parallel n] \) such that \( j \leq i−1 \), let \( \phi = \Pi \sigma_{[j\parallel (i−1)]} \), and let \( w \) be a planar element of \( \Sigma_{0,1,n} \).

(i) If \( w \in ((\Pi_l[1\parallel n])t_{j∗}) \), then \( |w^\phi| < |w| \).

(ii) If \( w \in ((\Pi_l[1\parallel n])t_{j∗}) \), then \( |w^\phi| < |w| \).

**Proof.** It is straightforward to show that \( \phi \) acts as

\[
\begin{array}{ccc}
\{k \in [1\parallel (j−1)]\} & \{k \in [(j+1)\parallel i]\} & \{k \in [(i+1)\parallel n]\} \\
(t_{k} & t_j & \overline{t}_k) & (t_{k} \overline{t}_k & t_{k}^\phi) & (t_{k}^\phi & t_j & t_k^\phi).
\end{array}
\]

(i) Suppose that \( w \in ((\Pi_l[1\parallel n])t_{j∗}) \).
Actions of the braid group

Figure V.1.1: $w \in ((\Pi t_{[1 \uparrow i]} t_j) \ast), j \leq i - 1.$

Since $t_it_j$ is a subword of $w$ and $w$ is planar, every letter occurring in $w$ that belongs to $t_{[j \uparrow i]} \vee t_{[j \uparrow i]}$ belongs to a (reduced) subword of $w$ of the form $avb$, where $a, b \in \{\bar{t}_i, t_j\}$ and $v \in \langle t_{[j \uparrow i]} \rangle$. Since, moreover, $w$ begins with $\Pi t_{[1 \uparrow i]}$, it can be shown that it is not possible to have $a = \bar{t}_i$ or $b = \bar{t}_i$. Thus $a = b = t_j$.

Here, $|avb| - 2$.

We factor $w$ into syllables consisting of all such subwords together with the individual remaining letters, all of which lie in $t_{[1 \uparrow j]} \vee t_{[1 \uparrow j]}$, and all of which are mapped to single letters by $\phi$.

Since $t_j$ occurs in $w$, we see that $|w| < |w|$.

(ii). Suppose that $w \in ((\Pi t_{[1 \uparrow i]} \bar{t}_j) \ast)$.

Figure V.1.2: $w \in ((\Pi t_{[1 \uparrow i]} \bar{t}_j) \ast), j \leq i - 1.$

Since $t_it_j$ is a subword of $w$ and $w$ is planar, every letter occurring in $w$ that belongs to $t_{[j \uparrow i]} \vee t_{[j \uparrow i]}$ belongs to a (reduced) subword of $w$ of the form $avb$, where $a, b \in \{t_j, \bar{t}_i\}$ and $v \in \langle t_{[j \uparrow i]} \rangle$. Since, moreover, $w$ begins with $\Pi t_{[1 \uparrow i]}$, it can be shown that it is not possible to have $a = \bar{t}_i$ or $b = \bar{t}_i$. Thus $a = b = t_j$.

Here, $|avb| - 2$.

We factor $w$ into syllables consisting of all such subwords together with the individual remaining letters, all of which lie in $t_{[1 \uparrow j]} \vee t_{[1 \uparrow j]}$, and all of which are mapped to single letters by $\phi$.

Since $t_i$ occurs in $w$, it is then clear that $|w| - 2$.  

**V.2 Lemma.** Let $i, j$ be elements of $[1 \uparrow n]$ such that $j \geq i + 2$, let $\phi = \Pi \sigma_{[j-1],[i+1]}$, and let $w$ be a planar element of $\Sigma_{0,1,n}$. 

\[ \text{\Box} \]
(i) If \( w \in (\Pi t_{[1|t]}t_j^\star) \), then \( |w^\phi| \leq |w| \), and, moreover, if \( |w^\phi| = |w| \) then \( w^\phi \in (\Pi t_{[1|t+1]}^\star) \).

(ii) If \( w \in ((\Pi t_{[1|t]}t_j^\star)) \), then \( |w^\phi| < |w| \).

**Proof.** It is straightforward to show that \( \phi \) acts as

\[
\begin{array}{cccc}
( t_k & t_k & t_j & t_k^\phi \\
& & & \\
( t_k & t_{k+1} & t_{i+1} & t_k )
\end{array}
\]

(i). Suppose that \( w \in ((\Pi t_{[1|t]}t_j^\star)) \).

Since \( t_i t_j \) is a subword of \( w \) and \( w \) is planar, every letter occurring in \( w \) that belongs to \( t_{[(i+1)\downarrow (j-1)]} \) belongs to a (reduced) subword of \( w \) of the form \( avb \), where \( a, b \in \{ t_i, t_j \} \) and \( v \in (t_{[(i+1)\downarrow (j-1)]}) \). Since, moreover, \( w \) begins with \( \Pi t_{[1|t]}^\star \), it can be shown that it is not possible to have \( a = t_i \) or \( b = t_i \). Thus \( a = b = t_j \). Here, \( |(avb)^\phi| = |avb| - 2 \).

We factor \( w \) into syllables consisting of all such subwords together with the individual remaining letters, all of which lie in \( t_{[1|t]}^\star \), and all of which are mapped to single letters by \( \phi \).

It is then clear that \( |w^\phi| \leq |w| \).

Moreover, if \( |w^\phi| = |w| \), then \( w \in (t_{[1|t]}^\star \cup t_{[1|\n]}^\star) \), and \( w^\phi \in (\Pi t_{[1|t+1]}^\star) \).

(ii). Suppose that \( w \in ((\Pi t_{[1|t]}\overline{t}_j^\star)) \).

Since \( t_i \overline{t}_j \) is a subword of \( w \) and \( w \) is planar, every letter occurring in \( w \) that belongs to \( t_{[(i+1)\downarrow (j)]} \) belongs to a (reduced) subword of \( w \) of the form \( avb \), where \( a, b \in \{ t_i, \overline{t}_j \} \) and \( v \in (t_{[(i+1)\downarrow (j)]}) \). Since, moreover, \( w \) begins with
\( \Pi t_{[1|k]} \), it can be shown that it is not possible to have \( a = t_i \) or \( b = t_i \). Thus \( a = b = 7_j \). Here, \(|\langle avb\rangle^\phi| = |avb| - 2\).

We factor \( w \) into syllables consisting of all such subwords together with the individual remaining letters, all of which lie in \( t_{[1|k]} \cup ([j+1]|n] \cup \tilde{t}_{[1|k]} \cup ([j+1]|n] \), and all of which are mapped to single letters by \( \phi \).

Since \( \tilde{t}_j \) occurs in \( w \), it is then clear that \(|w^\phi| \leq |w| - 2\). \( \square \)

V.3 Theorem (Larue). The set \( \{\Pi t_{[1|k]}\}_{k \in [0|n]} \) is a complete set of representatives of the \( B_n \)-orbits in the set of all planar elements of \( \Sigma_{0,1,n} \).

Proof. Let \( w \) be a planar element of \( \Sigma_{0,1,n} \). We wish to show that there exists some \( k \in [0|n] \) such that \( \Pi t_{[1|k]} \in B_n \).

Let \( i \) be the largest integer such that \( w \in (\Pi t_{[1|i]} \ast) \).

We may assume that, for all \( v \in B_n \), \(|v| \geq |w| \), and if \(|v| = |w| \), then \( v \notin (\Pi t_{[1|i+1]} \ast) \).

By Lemma V.1, for all \( j \in [i(i - 1)] \), \( w \notin ((\Pi t_{[1|i]} t_j) \ast) \cup ((\Pi t_{[1|i]} \tilde{t}_j) \ast) \).

By Proposition III.4(i), \( w \notin ((\Pi t_{[1|i]} t_i) \ast) \).

By the maximality of \( i \), \( w \notin ((\Pi t_{[1|i]} t_{i+1}) \ast) \).

By Proposition III.4(iii), \( w \notin ((\Pi t_{[1|i]} \tilde{t}_{i+1}) \ast) \).

By Lemma V.2, for all \( j \in [(i + 2)|n] \), \( w \notin ((\Pi t_{[1|i]} t_j) \ast) \cup ((\Pi t_{[1|i]} \tilde{t}_j) \ast) \).

Hence, \( w = \Pi t_{[1|i]} \), as desired. \( \square \)

V.4 Remarks. (i). Let \( w \) be a planar element of \( \Sigma_{0,1,n} \).

Lemmas V.1 and V.2 give an effective procedure for finding \( \phi \in B_n \) first to minimize \(|w^\phi| \), and then to obtain the form \( w^\phi = \Pi t_{[1|k]} \) for some \( k \in [0|n] \).

(ii). Let \( n \geq 1 \) and let \( w \in \Sigma_{0,1,n} \).

Theorem V.3 shows that \( w \) lies in the \( B_n \)-orbit of \( t_1 \) if and only if the cyclically-reduced form of \( w \) lies in \( t_{[1|n]} \) and \( w \) is planar. Moreover, in this event, Lemmas V.1 and V.2 effectively produce a \( \phi \in B_n \) such that \( w^\phi = t_1 \).

(iii). There is an algorithm which, for any \( k \in [1|n] \), and any sequence \( w_{[1|k]} \) in \( \Sigma_{0,1,n} \), decides if there exists some \( \phi \in B_n \) such that \( w_{[1|k]}^\phi = t_{[1|k]} \), and effectively finds such a \( \phi \), by using (ii) to convert \( w_1 \) to \( t_1 \) if possible, and then restricting to \( \{\sigma_{2(1|n)}\} \).

This algorithm for \( B_n \) is simpler than the Whitehead algorithm for the larger group \( \text{Aut} \Sigma_{0,1,n} \), essentially because the information carried by planarity is more detailed than the information carried by the Whitehead graph used in the Whitehead algorithm. Enric Ventura has pointed out to us that Whitehead’s algorithm has the power to decide whether any pair of conjugacy classes in \( \Sigma_{0,1,n} \) lie in the same \( B_n \)-orbit or not; see, for example, [23, Proposition I.4.21]. \( \square \)

Let us conclude by emphasizing (ii).

V.5 Theorem (Larue). Let \( n \geq 1 \) and let \( w \in \Sigma_{0,1,n} \). Then \( w \) lies in the \( B_n \)-orbit of \( t_1 \) if and only if the cyclically-reduced form of \( w \) lies in \( t_{[1|n]} \) and \( w \) is planar. \( \square \)
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References

Errata at: http://mat.uab.cat/~dicks/DDerr.html
Errata and addenda at: http://mat.uab.cat/~dicks/Boundaries.html
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