Universal localizations embedded in power-series rings

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Abstract

Let R be a ring, let F be a free group, and let X be a basis of F.

Let $\epsilon\colon RF \to R$ denote the usual augmentation map for the group ring RF, let $X\partial := \{x - 1 \mid x \in X\} \subseteq RF$, let Σ denote the set of matrices over RF that are sent to invertible matrices by ϵ, and let $(RF)\Sigma^{-1}$ denote the universal localization of RF at Σ.

A classic result of Magnus and Fox gives an embedding of RF in the power-series ring $R\langle\langle X\partial\rangle\rangle$. We show that if R is a commutative Bezout domain, then the division closure of the image of RF in $R\langle\langle X\partial\rangle\rangle$ is a universal localization of RF at Σ.

We also show that if R is a von Neumann regular ring or a commutative Bezout domain, then $(RF)\Sigma^{-1}$ is stably flat as an RF-ring, in the sense of Neeman-Ranicki.

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Dedicated to the memory of Desmond Sheiham

1 Outline

Throughout, let R be a ring (associative, with 1). Except where otherwise specified, we will work with left modules, and linear maps between left modules will be written on the right of their arguments. We denote by $\mathbb{N}$ the set of finite cardinals, $\{0, 1, 2, 3, \ldots\}$.

Let Σ be a set (or even a class) of R-linear maps between finitely generated projective R-modules. It is well-known that the universal localization of R at Σ, denoted $R\Sigma^{-1}$, need not be flat as a right or left R-module, in general. A much less demanding condition is stable flatness of $R\Sigma^{-1}$ as an R-ring, meaning that, for each positive integer n, $\mathrm{Tor}_n^R(R\Sigma^{-1}, R\Sigma^{-1}) = 0$. In the case where all the elements of Σ are injective and $R\Sigma^{-1}$ is stably flat as an R-ring, Neeman and Ranicki [18, Theorem 0.11], [19] showed that Schofield's K-theory exact sequence for universal localization [21, Theorem 4.12] can be extended to a long exact sequence involving the K-groups of R, of $R\Sigma^{-1}$, and of the exact category $\mathrm{H}(R, \Sigma)$ of projective-dimension-at-most-one Σ-torsion R-modules.

For the remainder of this section, let us fix the following notation. Let X be a set, let F denote the free group on X, let RF denote the corresponding group ring, and let $\epsilon\colon RF \to R$ denote the R-ring map which sends every element of F to 1. Let Σ denote the set of matrices over RF that are sent to invertible matrices by ϵ.

Ranicki and Sheiham [20, Theorem 5(ii), Section 0] used the above-mentioned results of [18], [19] to provide a description of the K-theory of $(RF)\Sigma^{-1}$, in the case where F is finitely generated and $(RF)\Sigma^{-1}$ is stably flat as an RF-ring. In Section 4 below, we show that if R is a von Neumann regular ring or a commutative Bezout domain, then $(RF)\Sigma^{-1}$ is stably flat as an RF-ring, and, therefore, the results of Ranicki-Sheiham [20] apply in these cases.

Let $X\partial = \{x - 1 \mid x \in X\}$, and let $R\langle\langle X\partial\rangle\rangle$ denote the corresponding power-series ring; thus $R\langle\langle X\partial\rangle\rangle$ can be viewed as the $(X\partial)$-adic completion of the free R-ring $R\langle X\partial\rangle$, $R\langle\langle X\partial\rangle\rangle = \varprojlim_{n\in\mathbb{N}} R\langle X\partial\rangle/((X\partial)^n)$.

In Section 2, we review, in ideal-theoretic language, the Magnus-Fox embedding of RF in $R\langle\langle X\partial\rangle\rangle$; we felt that there seemed to be some confusion in the literature about what had been proved by the arguments of Magnus [17] and Fox [11], and that a brief survey might be useful. The Magnus-Fox embedding was the earliest example of a non-commutative universal localization being embedded in a power-series ring.

In Section 3, we show that if R is a commutative Bezout domain, then the division closure of the image of RF in $R\langle\langle X\partial\rangle\rangle$ is a universal localization of RF at Σ. For R a commutative principal ideal domain, this was proved by Dicks-Sontag [9, Theorem 24], and, independently, by Farber-Vogel [10, Theorem 5.1]. The assertion also holds if R is a division ring; see [5, p.416].

2 Group rings of free-groups and series

In this section, we recall, in detail, the proof of the Magnus-Fox embedding theorem [17, 11] which shows that, in the notation of Section 1, the universal localization of $R\langle X\rangle$ at X is embedded in $R\langle\langle X\partial\rangle\rangle$. Of course, $R\langle X\rangle = R\langle X\partial\rangle$ and the universal localization of $R\langle X\rangle$ at X is the group ring $R\langle X\rangle X^{-1} = RF$.

2.1 Notation. Let A be a ring and let M be a free left A-module.

We let M^* denote the right A-module $M^* = \mathrm{Hom}_A({}_AM, {}_AA)$.

If $(b_x \mid x \in X)$ is a left A-basis of M, we shall let $(b_x^* \mid x \in X)$ denote its *dual*; thus, for each $x \in X$, $b_x^* \in M^*$, and, for each $m \in M$, $m = \sum_{x\in X}(m)b_x^* \cdot b_x$, and this sum is finite. □

We begin by recalling the following standard result about multiplying free ideals.

2.2 Lemma. *Let A be a ring.*

Let I be a two-sided ideal of A such that I is free as a left A-module, let $(b_x \mid x \in X)$ be a left A-basis of I, and let $(b_x^* \mid x \in X)$ denote its dual.

Let J be a left ideal of A such that J is free as a left A-module, let $(c_y \mid y \in Y)$ be a left A-basis of J, and let $(c_y^* \mid y \in Y)$ denote its dual.

Then IJ is a left ideal of A which is free as a left A-module, and

$$(b_xc_y \mid (x, y) \in X \times Y)$$

is a left A-basis of IJ; its dual $((b_xc_y)^* \mid (x, y) \in X \times Y)$ is given by $(d)(b_xc_y)^* = ((d)c_y^*)b_x^*$ for all $d \in IJ$.

Proof. We have an isomorphism

$$J \xrightarrow{\sim} \bigoplus_{y\in Y} A, \quad c \mapsto ((c)c_y^* \mid y \in Y). \tag{2.2.1}$$

Since I is a right ideal of A, left multiplying (2.2.1) by I gives an additive isomorphism

$$IJ \xrightarrow{\sim} \bigoplus_{y\in Y} I, \quad d \mapsto ((d)c_y^* \mid y \in Y).$$

Now the result follows easily. $\square$

The Magnus-Fox embedding is based on the following general fact.

2.3 Lemma. *Let R be a ring, let A be an augmented R-ring, and let I denote the augmentation ideal of A. Suppose that Y is an R-centralizing left A-basis of I. Then there exists a (unique) $R\langle Y\rangle$-ring embedding $A/(\bigcap\limits_{n\geq 0} I^n) \to R\langle\langle Y\rangle\rangle$.*

Proof. Let $\widehat{A}$ denote the I-adic completion of A, that is, $\widehat{A} = \varprojlim\limits_{n\in\mathbb{N}} A/I^n$. We will show that the natural map $R\langle Y\rangle \to A$ induces an isomorphism $R\langle\langle Y\rangle\rangle \xrightarrow{\sim} \widehat{A}$.

Suppose $n \in \mathbb{N}$. Using induction and Lemma 2.2, one can show that $Y^{\times n}$ indexes a left A-basis of I^n, namely Y^n. Now,

$$I^n/I^{n+1} \quad\simeq\quad (A/I)\otimes_A (I^n) \quad\simeq\quad R\otimes_A (\bigoplus_{w\in Y^{\times n}} A) \quad\simeq\quad \bigoplus_{w\in Y^{\times n}} R.$$

Thus $Y^{\times n}$ indexes a left R-basis of I^n/I^{n+1}, namely the image of Y^n. In particular, there exists an R-linear splitting $A/I^{n+1} \quad\simeq\quad (A/I^n)\oplus(I^n/I^{n+1})$.

Using induction, one can then show that the disjoint union $\bigvee\limits_{i=0}^{n-1}(Y^{\times i})$ indexes a left R-basis of A/I^n, namely $\bigcup\limits_{i=0}^{n-1}(Y^i)$. This means that the induced map

$$R\langle Y\rangle/((Y)^n) \quad\to\quad A/I^n$$

is an isomorphism. On taking inverse limits, we get an induced isomorphism

$$\varprojlim_{n\in\mathbb{N}} R\langle Y\rangle/((Y)^n) \quad\xrightarrow{\sim}\quad \varprojlim_{n\in\mathbb{N}} A/I^n,$$

that is, $R\langle\langle Y\rangle\rangle \xrightarrow{\sim} \widehat{A}$.

Now the natural embedding $A/(\bigcap\limits_{n\geq 0} I^n) \to \widehat{A}$ gives the desired result. $\square$

We use the following, throughout the remainder of this section.

2.4 Notation. Let R be a ring, let X be a set, let F be the free group on X, let RF denote the group ring of F with coefficients in R, and let I denote the augmentation ideal of RF.

Let $\epsilon\colon RF \to RF$, $\sum_{w\in F} r_w w \mapsto \sum_{w\in F} r_w$, be the augmentation map viewed as an endomorphism. Then ϵ is an idempotent ring endomorphism of RF, $\operatorname{im}\epsilon = R$, and $\ker\epsilon = I$.

Let $\partial\colon RF \to RF$ denote $id-\epsilon$, the idempotent R-linear endomorphism complementary to ϵ. Thus $(\sum_{w\in F} r_w w)\partial = \sum_{w\in F} r_w(w-1) = \sum_{w\in F} r_w \cdot w\partial$, $\operatorname{im}\partial = I$, and $\ker\partial = R$. For each $w\in F$, $\bar{w}\partial = -\bar{w}\cdot w\partial$, where $\bar{w}$ denotes w^{-1}.

Let Z denote the centre of R, and view ZF as a subring of RF. $\square$

2.5 Definition. Suppose that Notation 2.4 applies.

A left R-linear map $d\colon RF \to RF$ satisfying

$$(2.5.1) \qquad (fg)d \quad=\quad (fd\cdot g\epsilon) + (f\cdot gd) \quad \text{for all } f,g\in RF$$

is called a *left derivation* for RF.

In this event, d restricts to a left RF-linear map $I \to RF$.

Notice that the endomorphism ∂ is a left derivation for RF. $\square$

We will now see that a left derivation for RF is uniquely determined by its values on X, and these can be arbitrary in ZF.

2.6 Lemma. *With* Notation 2.4*, each map of sets $X \to ZF$ extends uniquely to a left derivation for RF.*

Proof. Let $d\colon X \to ZF$, $x \mapsto xd$, be a map of sets. Let $\mathrm{M}_2(RF)$ denote the ring of 2×2 matrices over RF. By universal properties, there exists a unique ring homomorphism $\phi\colon RF \to \mathrm{M}_2(RF)$ which sends each $x \in X$ to $\begin{pmatrix} 1 & 0 \\ xd & x \end{pmatrix}$, and sends each $r \in R$ to $\begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}$; moreover, ϕ can be naturally expressed in the form

$$\begin{pmatrix} \epsilon & 0 \\ d & id \end{pmatrix} : RF \to \mathrm{M}_2(RF), \quad f \mapsto \begin{pmatrix} f\epsilon & 0 \\ fd & f \end{pmatrix}.$$

The $(2,2)$-component, id, is the identity map on RF; the $(1,2)$-component, 0, sends every element of RF to 0; and the $(1,1)$-component, ϵ, is the augmentation map. The $(2,1)$-component, $d\colon RF \to RF$, is the unique left R-linear map that extends the given map $d\colon X \to ZF$ and satisfies (2.5.1). $\square$

2.7 Definitions. Suppose that Notation 2.4 applies.

Consider any $x \in X$.

By Lemma 2.6, the map

$$\delta_{x,-}\colon X \to RF, \quad y \mapsto \delta_{x,y} := \begin{cases} 1 & \text{if } y = x, \\ 0 & \text{if } y \neq x, \end{cases}$$

extends to a unique left derivation for RF; it is denoted $\frac{\partial}{x\partial}\colon RF \to RF$, and called the *left Fox derivative with respect to* x.

Now $\overline{X} := \{\bar{x} \mid x \in X\}$ is also a basis for F, and the *left Fox derivative with respect to* $\bar{x}$ is $\frac{\partial}{\bar{x}\partial} := -\frac{\partial}{x\partial}\,x$; it is the left derivation for RF which sends y to $-\delta_{x,y}x$, and $\bar{y}$ to $\delta_{x,y}$, for each $y \in X$. $\square$

We now show that restricting $(\frac{\partial}{x\partial} \mid x \in X)$ gives a dual of a basis.

2.8 Lemma. *With* Notation 2.4*, I is free as left RF-module, both $X\partial$ and $\overline{X}\partial$ are R-centralizing left RF-bases of I, and, for each $f \in I$,*

$$f = \sum_{x\in X} \tfrac{f\partial}{x\partial}\cdot x\partial = \sum_{x\in X} \tfrac{f\partial}{\bar{x}\partial}\cdot \bar{x}\partial.$$

Proof. Both $(-)\partial$ and $\sum\limits_{x\in X} \frac{(-)\partial}{x\partial}\cdot x\partial$ are left derivations for RF, and they agree on X; hence they are equal. Thus $f = f\partial = \sum\limits_{x\in X} \frac{f\partial}{x\partial}\cdot x\partial$.

Now, if $(f_x \mid x \in X)$ is some element of $\bigoplus\limits_{x\in X} RF$ such that $\sum\limits_{x\in X} f_x \cdot x\partial = 0$, then, for each $y \in X$,

$$0 = (\sum_{x\in X} f_x\cdot x\partial)\tfrac{\partial}{y\partial} = \sum_{x\in X} f_x\cdot (x\partial)\tfrac{\partial}{y\partial} = \sum_{x\in X} f_x\cdot \delta_{y,x} = f_y.$$

The same arguments apply with $\overline{X}$ in place of X. $\square$

We will next show that $\bigcap\limits_{n\in\mathbb{N}} I^n = \{0\}$, by considering word lengths.

2.9 Definitions. Suppose that Notation 2.4 applies.

Let $X^{\pm 1}$ denote $X \cup X^{-1}$.

Consider any $d \in \mathbb{N}$, and any $p = (x_1^{\eta_1}, \ldots, x_d^{\eta_d}) \in (X^{\pm 1})^{\times d}$. We say that p has *length* d and *ends with* $x_d^{\eta_d}$. We say that p is *reduced* if, for each $i \in \{1, \ldots, d-1\}$, $(x_{i+1}, \eta_{i+1}) \neq (x_i, -\eta_i)$. We say that p is *an expression for* the element $x_1^{\eta_1} \cdots x_d^{\eta_d}$ of F.

By using left Fox derivatives in $\mathbb{Z}F$, one can show that the empty product is the only reduced expression for the identity element.

Consider any $w \in F$. By the foregoing, there exists a unique reduced expression p for w. We define the *X-length of* w, $\deg(w)$, to be the length of p. We say that w *ends with* x^η if p ends with x^η.

Consider any $f = \sum\limits_{w \in F} r_w w \in RF$. We define the *support of* f and the *degree of* f to be, respectively,

$$\operatorname{supp}(f) := \{w \in F \mid r_w \neq 0\} \quad \text{and} \quad \deg(f) := \max\{\deg(w) \mid w \in \operatorname{supp}(f)\},$$

with the convention that $\deg(0) = -\infty$. The *dominant component* of f is

$$\operatorname{dom}(f) := \sum_{\{w \in F \mid \deg(w) = \deg(f)\}} r_w w. \qquad \square$$

2.10 Lemma. *Suppose that* Notation 2.4 *holds. Suppose further that* $x, y \in X$, $\eta \in \{1, -1\}$ *and* $f \in RF - \{0\}$. *Then the following hold.*

(i). $\deg(\frac{f\partial}{x^\eta \partial}) \leq \deg(f)$.

(ii). *If* $\deg(\frac{f\partial}{x^\eta \partial}) = \deg(f)$, *then* some *element of* $\operatorname{supp}(\operatorname{dom}(f))$ *ends with* $\bar{x}^\eta$, *and* all *elements of* $\operatorname{supp}(\operatorname{dom}(\frac{f\partial}{x^\eta \partial}))$ *end with* $\bar{x}^\eta$.

(iii). $\deg(\frac{f\partial^2}{x\partial \cdot \bar{y}\partial}) < \deg(f)$, *where* $\frac{f\partial^2}{x\partial \cdot \bar{y}\partial}$ *denotes* $(\frac{f\partial}{x\partial})\frac{\partial}{\bar{y}\partial}$

(iv). *For each* $n \in \mathbb{N}$, *if* $f \in I^{2n} - \{0\}$ *then* $\deg(f) \geq n$.

Proof. (i)–(iii). Let w be an element of F, and let $(x_1^{\eta_1}, \ldots, x_d^{\eta_d})$ be its unique reduced expression. Then

$$\begin{aligned}
\tfrac{w\partial}{x^\eta \partial} &= \sum_{i=1}^{d} x_1^{\eta_1} \cdots x_{i-1}^{\eta_{i-1}} \cdot \tfrac{x_i^{\eta_i}\partial}{x^\eta \partial} \\
&= \sum_{\{i \mid 1 \leq i \leq d,\ (x_i, \eta_i) = (x, \eta)\}} x_1^{\eta_1} \cdots x_{i-1}^{\eta_{i-1}} - \sum_{\{i \mid 1 \leq i \leq d,\ (x_i, \eta_i) = (x, -\eta)\}} x_1^{\eta_1} \cdots x_{i-1}^{\eta_{i-1}} x_i^{\eta_i}.
\end{aligned}$$

It is now straightforward to prove (i) and (ii), and then (iii) follows easily.

(iv). We proceed by induction on n. Clearly the implication in (iv) holds for $n = 0$. Now suppose that, for some $n \geq 1$, the implication in (iv) holds with $n-1$ in place of n.

By Lemmas 2.8 and 2.2, we have a left RF-linear isomorphism

$$I^2 \xrightarrow{\sim} \bigoplus_{(x,y) \in X^2} RF, \quad f \mapsto (\tfrac{f\partial^2}{x\partial \cdot \bar{y}\partial} \mid (x, y) \in X \times X).$$

On left multiplying by I^{2n-2}, we obtain a left RF-linear isomorphism

$$I^{2n} \xrightarrow{\sim} \bigoplus_{(x,y) \in X^2} I^{2n-2}, \quad f \mapsto (\tfrac{f\partial^2}{x\partial \cdot \bar{y}\partial} \mid (x, y) \in X \times X).$$

Now suppose that $f \in I^{2n} - \{0\}$. Then there exists $(x, y) \in X \times X$ such that $\frac{f\partial^2}{x\partial \cdot \bar{y}\partial} \in I^{2n-2} - \{0\}$. By the induction hypothesis, $\deg(\frac{f\partial^2}{x\partial \cdot \bar{y}\partial}) \geq n - 1$. By (iii), $\deg(f) \geq n$. This proves (iv). $\square$

2.11 The Magnus-Fox embedding theorem. *Let R be a ring, let X be a set, let F be the free group on X, let $X\partial$ be a set given with a bijective map $X \to X\partial$, $x \mapsto x\partial$, and let $\phi = \phi(R, X)\colon RF \longrightarrow R\langle\langle X\partial\rangle\rangle$ be the unique R-ring map such that, for each $x \in X$, $x\phi = 1 + x\partial$ and, hence,*

$$\bar{x}\phi \quad = \quad (1 + x\partial)^{-1} \quad = \quad 1 - \partial x + (\partial x)^2 - (\partial x)^3 + \cdots \quad .$$

Then ϕ is injective. □

Proof. Suppose that Notation 2.4 holds. It is clear from Lemma 2.10(iv) that $\bigcap\limits_{n\geq 0} I^n = \{0\}$. Now the result follows from Lemmas 2.8 and 2.3. □

2.12 Historical remarks. In 1935, Wilhelm Magnus [17, Satz I] proved that $\phi(\mathbb{Z}, X)$ embeds F in the group of units of $\mathbb{Z}\langle\langle \partial X\rangle\rangle$.

This embedding has many important applications. For example, F can then be ordered, since it is easy to order the group of those units of $\mathbb{Z}\langle\langle \partial X\rangle\rangle$ which have constant term 1; see, for example, the first paragraph of [1].

Luis Paris has pointed out to us that, for any prime integer p, a natural analogue of Magnus's argument shows that $\phi(\mathbb{Z}_p, X)$ embeds F in the group of units of $\mathbb{Z}_p\langle\langle X\partial\rangle\rangle$, where $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$. It then follows that if R is any nonzero ring, then $\phi(R, X)$ embeds F in the group of units of $R\langle\langle X\partial\rangle\rangle$.

In 1953, Ralph H. Fox [11, Theorem 4.3] proved that $\phi(\mathbb{Z}, X)$ is injective. In this section, we have verified the well-known fact that Fox's argument works with any ring in place of $\mathbb{Z}$. This was noted, for example, by Sheiham [24, Lemma 2.6]. □

2.13 Remarks. Suppose that Notation 2.4 holds, and let $R = \mathbb{Z}$.

Let M be any $\mathbb{Z}$-module. Let MF denote the induced left $\mathbb{Z}F$-module $\mathbb{Z}F \otimes_{\mathbb{Z}} M$.

It is straightforward to translate the arguments of this section from RF to MF and deduce that $\bigcap\limits_{n\geq 0} (I^n MF) = \{0\}$; the translation amounts to applying $- \otimes_R M$ to the maps in the proof of Lemma 2.10.

As in the work of Brian Hartley [12], it then follows easily that the wreath product

$$F \wr M = MF \rtimes F = \begin{pmatrix} 1 & 0 \\ MF & F \end{pmatrix}$$

is residually nilpotent. □

3 Rational series and universal localizations

In this section, we will consider the following situation.

3.1 Notation. Let R be a ring, and let Y be a set. We shall think of the elements of Y as R-centralizing indeterminates.

Let $R\langle Y\rangle$ be the free R-ring on the R-centralizing set Y. Let $\epsilon\colon R\langle Y\rangle \to R$ be the R-ring map which sends every element of Y to 0. Let I denote the kernel of ϵ.

Let $R\langle\langle Y\rangle\rangle$ denote the power-series R-ring on the R-centralizing set Y. We think of $R\langle\langle Y\rangle\rangle$ as the I-adic completion of $R\langle Y\rangle$, $R\langle\langle Y\rangle\rangle = \varprojlim\limits_{n\in\mathbb{N}} R\langle Y\rangle/I^n$. It is not difficult to show that the I-adic completion map $R\langle Y\rangle \to R\langle\langle Y\rangle\rangle$ is injective; we shall view $R\langle Y\rangle$ as a subring of $R\langle\langle Y\rangle\rangle$.

Let $R_{\mathrm{rat}}\langle\langle Y\rangle\rangle$ denote the *division closure* of $R\langle Y\rangle$ in $R\langle\langle Y\rangle\rangle$, that is, $R_{\mathrm{rat}}\langle\langle Y\rangle\rangle$ is the smallest subring of $R\langle\langle Y\rangle\rangle$ which contains $R\langle Y\rangle$ and is closed under taking inverses of elements that are invertible in $R\langle\langle Y\rangle\rangle$. The elements of $R_{\mathrm{rat}}\langle\langle Y\rangle\rangle$ are called *rational* power series.

A matrix A over $R\langle Y\rangle$ is said to be *sent to an invertible matrix by* ϵ if $A\epsilon$, the coordinate-wise image of A under ϵ, is invertible over R. Let Σ denote the set of matrices over $R\langle Y\rangle$ that are sent to invertible matrices by ϵ. Then Σ is also the set of matrices over $R\langle Y\rangle$ that are invertible over $R\langle\langle Y\rangle\rangle$.

Suppose that A is a matrix over $R_{\mathrm{rat}}\langle\langle Y\rangle\rangle$ that is sent to an invertible matrix by the induced augmentation map $\hat{\epsilon}\colon R\langle\langle Y\rangle\rangle \to R$. It is easy to see that A is invertible over $R\langle\langle Y\rangle\rangle$, and we claim that A is invertible over $R_{\mathrm{rat}}\langle\langle Y\rangle\rangle$. By viewing R as a subring of $R\langle\langle Y\rangle\rangle$, we can view $(A\hat{\epsilon})^{-1}$ as a matrix over $R_{\mathrm{rat}}\langle\langle Y\rangle\rangle$. By replacing A with $(A\hat{\epsilon})^{-1}A$, we may assume that (A is square and) $A\hat{\epsilon}$ is an identity matrix. Thus, the diagonal entries of A are invertible in $R\langle\langle Y\rangle\rangle$, and, hence, are invertible in $R_{\mathrm{rat}}\langle\langle Y\rangle\rangle$. By applying to A a suitable sequence of row operations consisting of adding a left multiple of some row to another row, we may further assume that A is diagonal. It follows that A is invertible over $R_{\mathrm{rat}}\langle\langle Y\rangle\rangle$, as claimed. In particular, Σ is the set of matrices over $R\langle Y\rangle$ that are invertible over $R_{\mathrm{rat}}\langle\langle Y\rangle\rangle$.

Let $\psi\colon R\langle Y\rangle \to R\langle Y\rangle\Sigma^{-1}$ denote the universal ring homomorphism such that the elements of Σ are sent to invertible matrices by ψ. We call $R\langle Y\rangle\Sigma^{-1}$ *the universal localization of* $R\langle Y\rangle$ *at* Σ.

By the universal property, there exists a unique $R\langle Y\rangle$-ring homomorphism

$$\tau = \tau(R,Y)\colon R\langle Y\rangle\Sigma^{-1} \to R_{\mathrm{rat}}\langle\langle Y\rangle\rangle.$$

It follows from [4, Proposition **7**.1.3] that the image of τ is division closed in $R_{\mathrm{rat}}\langle\langle Y\rangle\rangle$, and, hence, τ is surjective. $\square$

3.2 Historical remarks. Suppose that Notation 3.1 holds.

Desmond Sheiham [22, Proposition 1.2] showed that $\tau(R,Y)$ need not be an isomorphism, by constructing an elegant example in which the kernel of $\tau(R,Y)$ contains a nonzero element of the form $a(1-by)^{-1}c$, with a, b, $c \in R$ and $y \in Y$.

Cohn-Dicks [5, p.416] showed that if R is a division ring, then $\tau(R,Y)$ is an isomorphism.

Cohn [3, Theorem 5] conjectured that if R is a commutative integral domain, then $\tau(R,Y)$ is an isomorphism; Linnell [16, Problem 3.2] has recently resuscitated this problem.

Dicks-Sontag [9, Theorem 24] showed that if R is a commutative principal ideal domain, then $\tau(R,Y)$ is an isomorphism. We now extend this to the case where R is a commutative *Bezout* domain, that is, a commutative integral domain in which every finitely generated ideal is principal. $\square$

3.3 Theorem. *If R is a commutative Bezout domain and Y is any set, then the map $\tau(R,Y)\colon R\langle Y\rangle\Sigma^{-1} \to R_{\mathrm{rat}}\langle\langle Y\rangle\rangle$ of* Notation 3.1 *is an isomorphism.*

Proof. Let K denote the field of fractions of R. By [6, Theorem 3.3], there exists a homomorphism ϕ from $R\langle Y\rangle$ to a division ring $U(R\langle Y\rangle)$ such that ϕ preserves inner ranks of matrices. In particular, $\phi\colon R\langle Y\rangle \to U(R\langle Y\rangle)$ sends the nonzero elements of R to invertible elements. Hence, ϕ factors through the natural map $\psi\colon R\langle Y\rangle \to K\langle Y\rangle$, and, therefore, ψ, also, preserves inner ranks of matrices.

Thus, the conclusion of [9, Theorem 13] holds when R is a commutative Bezout domain. Now the proof of [9, Theorem 24] applies verbatim to show that $\tau(R,Y)$ is an isomorphism. $\square$

3.4 Remark. Let $\epsilon\colon S \twoheadrightarrow R$ be a surjective ring homomorphism.

Let Σ denote the set of all square matrices over S which are sent to invertible matrices by ϵ.

Let Σ' denote the class consisting of all S-linear maps between finitely generated projective S-modules that are sent to invertible maps by ϵ, that is, applying $R_\epsilon \otimes_S -$ gives an R-linear isomorphism. The proof of [20, Proposition 3.1] shows that every element of Σ' is also sent to an invertible map by the universal localization map $S \to S\Sigma^{-1}$. It follows thus there is a natural S-ring identification of the universal localizations, $S\Sigma^{-1} = S\Sigma'^{-1}$.

Thus, it makes no difference whether we consider square matrices, matrices, or arbitrary maps between finitely generated projective modules; the resulting universal localizations are all the same. We shall work with matrices. □

We can extend Theorem 3.3 by using the following result.

3.5 Lemma (Sheiham [23]). *Let $\phi\colon S \to R$ be a ring homomorphism, and let Σ denote the set of matrices over S that are sent to invertible matrices by ϕ.*

Let Σ_0 be any subset of Σ. Let Σ_2 denote the set of matrices over $S\Sigma_0^{-1}$ that are sent to invertible matrices by the induced homomorphism $S\Sigma_0^{-1} \to R$. Then the natural map $S\Sigma^{-1} \to (S\Sigma_0^{-1})\Sigma_2^{-1}$ is an isomorphism.

Proof. Clearly, there exist S-ring maps

$$S \to S\Sigma_0^{-1} \to S\Sigma^{-1} \to R.$$

Each element of Σ is sent to an invertible matrix by the composition

$$S \to S\Sigma_0^{-1} \to R,$$

and, hence, each element of Σ is sent to an element of Σ_2 by the map $S \to S\Sigma_0^{-1}$. This then gives us an S-ring map $S\Sigma^{-1} \to (S\Sigma_0^{-1})\Sigma_2^{-1}$. We will construct its inverse.

Now, Σ_2 is the set of matrices that are sent to invertible matrices by the composition

$$S\Sigma_0^{-1} \to S\Sigma^{-1} \to R.$$

By [22, Lemma 3.1], every matrix over $S\Sigma^{-1}$ which is sent to an invertible matrix by the map $S\Sigma^{-1} \to R$ is already invertible over $S\Sigma^{-1}$. Thus, Σ_2 is the set of matrices that are sent to invertible matrices by the first factor, $S\Sigma_0^{-1} \to S\Sigma^{-1}$. By universal properties, there exists an S-ring map $(S\Sigma_0^{-1})\Sigma_2^{-1} \to S\Sigma^{-1}$, which is easily seen to be the inverse of the above map $S\Sigma^{-1} \to (S\Sigma_0^{-1})\Sigma_2^{-1}$, as desired. □

3.6 Theorem. *Let R be a division ring or a commutative Bezout domain, let Y be a set, and let $S = R\langle Y\rangle$. Let $\epsilon\colon S \to R$ denote the R-ring map which sends every element of Y to 0.*

Let Σ_0 be any set of matrices over S that are sent to invertible matrices by ϵ. Let Σ_2 denote the set of matrices over $S\Sigma_0^{-1}$ that are sent to invertible matrices by the induced map $S\Sigma_0^{-1} \to R$. Then there exists a (unique) S-ring isomorphism $(S\Sigma_0^{-1})\Sigma_2^{-1} \xrightarrow{\sim} R_{\mathrm{rat}}\langle\langle Y\rangle\rangle$.

Proof. Let Σ denote the set of matrices over S that are sent to invertible matrices by ϵ. Thus Σ_0 is a subset of Σ.

By Lemma 3.5, there exists a (unique) S-ring isomorphism

$$S\Sigma^{-1} \xrightarrow{\sim} (S\Sigma_0^{-1})\Sigma_2^{-1}.$$

If R is a division ring, resp. a commutative Bezout domain, then, by [5, p.416], resp. Theorem 3.3, there exists a (unique) S-ring isomorphism

$$\tau(R,Y)\colon S\Sigma^{-1} \xrightarrow{\sim} R_{\mathrm{rat}}\langle\langle Y\rangle\rangle.$$

Now the result follows. □

3.7 Corollary. *Let R be a division ring or a commutative Bezout domain, and let F be a free group. Let $\epsilon\colon RF \to R$ denote the R-ring map that sends every element of F to* 1

Let Σ denote the set of matrices over RF that are sent to invertible matrices by ϵ.

Let X be a basis of F, and let $X\partial$ be a set given with a bijective map $X \to X\partial$, $x \mapsto x\partial$.

Then there exists a (unique) $R\langle X\rangle$-ring isomorphism $(RF)\Sigma^{-1} \xrightarrow{\sim} R_{\text{rat}}\langle\langle X\partial\rangle\rangle$ where each $x \in X$ corresponds to $x \in (RF)\Sigma^{-1}$ and to $1 + x\partial \in R_{\text{rat}}\langle\langle X\partial\rangle\rangle$.

Proof. Let $S = R\langle X\partial\rangle$, let $\Sigma_0 = X = \{x\partial + 1 \mid x\partial \in X\partial\}$, and let $\Sigma_2 = \Sigma$. Now $S\Sigma_0^{-1} = (R\langle X\partial\rangle)\Sigma_0^{-1} = R\langle X\rangle X^{-1} = RF$. Applying Theorem 3.6, we obtain the desired result. □

3.8 Remarks. (i). By the Magnus-Fox embedding theorem 2.11, RF itself embeds in $R_{\text{rat}}\langle\langle X\partial\rangle\rangle$, although this information is not used in the above proof of Corollary 3.7.

(iii). The case of Corollary 3.7 where R is a commutative principal ideal domain was obtained by Farber-Vogel [10, Theorem 5.1]; notice their result is equivalent to [9, Theorem 24], by Sheiham's Lemma 3.5. □

3.9 Open Questions. Let R be a commutative Bezout domain, let F be a free group, and let Y be a set. By [6, Theorem 3.3], $R\langle Y\rangle$ is a Sylvester domain. This suggests the following questions.

(1). Is RF a Sylvester domain?

(2). Is $R_{\text{rat}}\langle\langle Y\rangle\rangle$ a Sylvester domain?

(3). Is $R\langle\langle Y\rangle\rangle$ a Sylvester domain?

Even for R a commutative principal ideal domain, we do not know the answers to any of these questions. □

3.10 Digression. Although the referee of [10] did not draw the authors' attention to [5], [11], [9], or [6], said referee did mention a deep result of Jacques Lewin, see [10, Remark 5.4], and it is worth elaborating on this.

Let R be a division ring.

(i). Let G be an ordered group, and let $R\langle\langle G\rangle\rangle$ denote the Mal'cev-Neumann power-series ring of G, consisting of all functions $G \to R$ with well-ordered support, where the functions are written, multiplied and added as formal sums over the elements of G.

Both A. I. Mal'cev and B. H. Neumann showed that $R\langle\langle G\rangle\rangle$ is a division ring; see, for example, [8, Corollary 2.2].

Let $R_{\text{rat}}\langle\langle G\rangle\rangle$ denote the division closure of the group ring RG in $R\langle\langle G\rangle\rangle$; thus, $R_{\text{rat}}\langle\langle G\rangle\rangle$ is an RG-division ring.

F. W. Levi [14, p.201] showed that G is locally indicable. Ian Hughes [13, top of page 183] showed that the isomorphism class of the RG-ring $R_{\text{rat}}\langle\langle G\rangle\rangle$ does not depend on the ordering of G; the main part of Hughes' argument has been recast in [7].

(ii) Let F be a free group. Let $\epsilon\colon RF \to R$ denote the usual augmentation map.

As we saw in Remarks 2.12, the free group F can be ordered. By (i), there exists an RF-division ring $R_{\text{rat}}\langle\langle F\rangle\rangle$, unique up to RF-ring isomorphism.

P. M. Cohn showed that RF is a semifir, and, hence, that there exists a universal RF-division ring, $U(RF)$, and that $U(RF)$ has the form $(RF)\Phi^{-1}$, where Φ denotes the set of all full square matrices over RF; see [4].

Jacques Lewin [15, Theorem 2] used results of Cohn and Hughes to show that $U(RF)$ and $R_{\mathrm{rat}}\langle\langle F\rangle\rangle$ are isomorphic as RF-rings.

Let Σ denote the set of matrices over RF that are sent to invertible matrices by ϵ. By [9, Proposition 4] with $(R,\Sigma,S,T) = (RF,\Sigma,RF,RF\Phi^{-1})$, the natural map

$$(RF)\Sigma^{-1} \to (RF)\Phi^{-1}$$

is injective.

Let X be a basis of F. By the foregoing, together with Corollary 3.7, there exists a (unique) $R\langle X\rangle$-ring embedding of $R_{\mathrm{rat}}\langle\langle X\partial\rangle\rangle$ in $R_{\mathrm{rat}}\langle\langle F\rangle\rangle$, where it is understood that each $x \in X$ is mapped to $1 + x\partial \in R_{\mathrm{rat}}\langle\langle X\partial\rangle\rangle$ and to $x \in R_{\mathrm{rat}}\langle\langle F\rangle\rangle$. $\square$

4 Stable flatness

In this section we apply Corollary 3.7 to obtain new examples of stable flatness.

4.1 Definition. Let $R \to S$ be an arbitrary ring homomorphism, that is, S has the structure of an R-ring. We say that S is a *stably flat* R-ring, if, for each positive integer n, $\mathrm{Tor}_n^R(S_R,\ {}_RS) = 0$. This extends the usage, introduced by Neeman and Ranicki [19, Theorem 0.7], from universal localizations to arbitrary ring homomorphisms.

Obviously S is stably flat as an R-ring whenever R is a von Neumann regular ring, or, more generally, whenever S is flat as a left or a right R-module. $\square$

4.2 Lemma. *Let R be a ring, let F be a free group, and let Σ be a set of RF-linear maps whose domains and codomains are finitely generated projective left RF-modules.*

If $(RF)\Sigma^{-1}$ is stably flat as an R-ring, then $(RF)\Sigma^{-1}$ is also stably flat as an RF-ring.

In particular, if R is a von Neumann regular ring, or, more generally, if $(RF)\Sigma^{-1}$ is flat as a left R-module, then $(RF)\Sigma^{-1}$ is stably flat as an RF-ring.

Proof. Let $S = (RF)\Sigma^{-1}$ and let n be a positive integer. Recall that we want to show that $\mathrm{Tor}_n^{RF}(S,S) = 0$.

Bergman and Dicks [2, (95)] showed that $\mathrm{Tor}_1^{RF}(S,S) = 0$; a simpler proof was given by Dicks, Dlab, Ringel, and Schofield [21, pp. 57–58].

Thus it remains to show that $\mathrm{Tor}_{n+1}^{RF}(S,S) = 0$.

Let X be a basis of the free group F.

There exists an exact sequence of $R\langle X\rangle$-bimodules of the form

$$0 \to \bigoplus_{x\in X}(R\langle X\rangle \otimes_R R\langle X\rangle) \to R\langle X\rangle \otimes_R R\langle X\rangle \xrightarrow{\text{mult}} R\langle X\rangle \to 0;$$

see, for example, [2, (17)].

There exists an exact sequence of RF-bimodules of the form

$$0 \to \bigoplus_{x\in X}(RF \otimes_R RF) \to RF \otimes_R RF \xrightarrow{\text{mult}} RF \to 0;$$

see, for example, [2, (4)].

There exists an exact sequence of left RF-modules of the form

$$0 \to \bigoplus_{x\in X}(RF \otimes_R S) \to RF \otimes_R S \to S \to 0;$$

see, for example, [2, (62)].

The long exact sequence resulting from applying $\mathrm{Tor}_*^{RF}(S,-)$ to this short exact sequence contains

$$\text{(4.2.1)}\quad \mathrm{Tor}_{n+1}^{RF}(S, RF\otimes_R S) \quad\to\quad \mathrm{Tor}_{n+1}^{RF}(S,S) \quad\to\quad \mathrm{Tor}_n^{RF}(S, \bigoplus_{x\in X} RF\otimes_R S).$$

Now RF is free, and, hence, flat, as a right R-module; thus any flat, left R-resolution of S lifts to a flat, left RF-resolution of $RF\otimes_R S$. It follows that (4.2.1) can be rewritten as

$$\mathrm{Tor}_{n+1}^{R}(S,S) \quad\to\quad \mathrm{Tor}_{n+1}^{RF}(S,S) \quad\to\quad \bigoplus_{x\in X}\mathrm{Tor}_n^{R}(S,S).$$

By hypothesis, S is a stably flat R-ring, and, hence, $\mathrm{Tor}_n^R(S,S) = \mathrm{Tor}_{n+1}^R(S,S) = 0$. Thus, $\mathrm{Tor}_{n+1}^{RF}(S,S) = 0$, as desired. □

4.3 Review. In the proof of the next result, we shall use the well-known fact that if R is a commutative Bezout domain, and M is a torsion-free R-module, then M is a flat R-module. This fact can be proved as follows.

It suffices to show that, for each finitely generated ideal I of R, the multiplication map $I\otimes_R M\to M$ is injective. Here $I = aR$ for some $a\in R$, and we may assume that a is nonzero. Thus $I_R \simeq R_R$, and $I\otimes_R M\simeq M$, and the map $I\otimes_R M\to M$ is equivalent to the map $M\to M$ given by multiplying by a. Since M is torsion-free, the latter map is injective. □

4.4 Theorem. *Let R be a ring, let F be a free group, let $\epsilon\colon RF\to R$ be the R-ring map which sends each element of F to 1, and let Σ denote the set of matrices over RF that are sent to invertible matrices by ϵ.*

If R is a von Neumann regular ring or a commutative Bezout domain, then $(RF)\Sigma^{-1}$ is stably flat as an RF-ring.

Proof. If R is a von Neumann regular ring, then $(RF)\Sigma^{-1}$ is stably flat as an RF-ring, by Lemma 4.2.

Now, let R be a commutative Bezout domain. Let X be a basis of F. By Corollary 3.7, there exists an isomorphism of $R\langle X\rangle$-rings $(RF)\Sigma^{-1} \xrightarrow{\sim} R_{\mathrm{rat}}\langle\langle X\partial\rangle\rangle$, and, as an R-module, $R_{\mathrm{rat}}\langle\langle X\partial\rangle\rangle$ is obviously torsion-free. Thus, as an R-module, $(RF)\Sigma^{-1}$ is torsion-free. Hence, by Review 4.3, $(RF)\Sigma^{-1}$ is flat as left R-module. By Lemma 4.2, $(RF)\Sigma^{-1}$ is a stably flat RF-ring. □

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