# APPROXIMATING THE MODULUS OF AN INNER FUNCTION

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ABSTRACT. We show that the modulus of an inner function can be uniformly approximated in the unit disk by the modulus of an interpolating Blaschke product.

## 1. INTRODUCTION

Let $H^\infty$ be the algebra of bounded analytic functions in the unit disk $\mathbb{D}$. A function in $H^\infty$ is called inner if it has radial limit of modulus one at almost every point of the unit circle. A Blaschke product is an inner function of the form

$$B(z) = z^m \prod_{n=1}^{\infty} \frac{\bar{z}_n}{|z_n|} \frac{z_n - z}{1 - \bar{z}_n z},$$

where $m$ is a non-negative integer and $\{z_n\}$ is a sequence of points in $\mathbb{D} \setminus \{0\}$ satisfying the Blaschke condition $\sum_n (1 - |z_n|) < \infty$. A classical result of O. Frostman tells that for any inner function $f$, there exists an exceptional set $E = E(f) \subset \mathbb{D}$ of logarithmic capacity zero such that the Möbius shift

$$\frac{f - \alpha}{1 - \bar{\alpha} f}$$

is a Blaschke product for any $\alpha \in \mathbb{D} \setminus E$. See [3] or [4, p. 79]. Hence any inner function can be uniformly approximated by a Blaschke product.

A Blaschke product $B$ is called an interpolating Blaschke product if its zero set $\{z_n\}$ form an interpolating sequence, that is, for any bounded sequence of complex numbers $\{w_n\}$, there exists a function $f \in H^\infty$ such that $f(z_n) = w_n$, $n = 1, 2, \ldots$. A celebrated result by L. Carleson tells that this holds precisely when the following two conditions are satisfied:

(1) $\inf_{n \neq m} \left| \frac{z_n - z_m}{1 - \bar{z}_m z_n} \right| > 0$,

(2) there exists a constant $C$ such that $\sum_{z_n \in Q} (1 - |z_n|) < C\ell(Q)$ for any Carleson square $Q$ of the form

$$Q = \left\{ re^{i\theta} : 0 < 1 - r < \ell(Q),\ |\theta - \theta_0| < \pi \ell(Q) \right\} \tag{1}$$

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where $\theta_0 \in [0,2\pi)$ and $0 < \ell(Q) < 1$.

See [1] or [4, p. 287]. Although the interpolating Blaschke products comprise a small subset of all Blaschke products, they play a central role in the theory of the algebra $H^\infty$. See the last three chapters of [4].

In [9] D. Marshall proved that any function $f \in H^\infty$ can be uniformly approximated by finite linear combinations of Blaschke products. That is, for any $\varepsilon > 0$ there are constants $c_1,\dots,c_N$ and Blaschke products $B_1,\dots,B_N$ such that

$$\Big\|f - \sum_{i=1}^{N} c_i B_i\Big\|_\infty < \varepsilon.$$

Here the $\infty$-norm is given by $\|g\|_\infty = \sup\{|g(z)| : z \in \mathbb{D}\}$. This result was improved in [5] by showing that one can take each of $B_1,\dots,B_N$ to be an interpolating Blaschke product. However the following problem remains open.

(1) For any inner function $B$ and $\varepsilon > 0$, is there an interpolating Blaschke product $I$ such that $\|B - I\|_\infty < \varepsilon$?

This question was posed in [4, p. 430], [6, pp. 268–269], [7] and [13, p. 202]. The purpose of this note is to provide a positive answer if one restricts attention to the modulus.

**Theorem 1.** *Let $B$ be an inner function and $\varepsilon > 0$. Then there exists an interpolating Blaschke product $I$ such that*

$$\big||B(z)| - |I(z)|\big| < \varepsilon$$

*for all $z \in \mathbb{D}$.*

The proof may be described as follows. The first step consists of constructing a system $\Gamma = \bigcup_i \Gamma_i$ of disjoint closed curves $\Gamma_i \subset \mathbb{D}$ such that arclength of $\Gamma$ is a Carleson measure, and verifying that

(a) $|B(z)|$ is uniformly small on hyperbolic disks of fixed radius centered at points of $\Gamma$,

(b) in any hyperbolic disk of fixed radius centered at a point outside the union of the interiors of $\Gamma_i$, $\bigcup_i \operatorname{int}\Gamma_i$, there is a point $z$ where $|B(z)|$ is not small.

Write $B = B_1 \cdot B_2$ where $B_1$ is the Blaschke product formed with the zeros of $B$ which are in $\bigcup_i \operatorname{int}\Gamma_i$. Statement (b) gives that $B_2$ is a finite product of interpolating Blaschke products. Since D. Marshall and A. Stray proved in [10] that any finite product of interpolating Blaschke products may be approximated by a single interpolating Blaschke product, the relevant zeros of $B$ lie in $\bigcup_i \operatorname{int}\Gamma_i$, that is, are those of $B_1$. The construction of $\Gamma$ is a variation of the original Corona construction introduced by L. Carleson. See [2] or [4, pp. 342–347].

Next, for each $i = 1,2,\dots$, let $\mu_i$ be the sum of harmonic measures in $\operatorname{int}\Gamma_i$ from the zeros of $B_1$ contained in $\operatorname{int}\Gamma_i$. Then the mass $\mu_i(\Gamma_i)$ is the total number of zeros of $B_1$ contained in $\operatorname{int}\Gamma_i$. The second step consists of splitting $\Gamma_i = \bigcup_k \Gamma_{i,k}$, into pieces $\Gamma_{i,k}$ with $\mu_i(\Gamma_{i,k}) = 1$, $k = 1,2,\dots$ and choosing points $\xi_{i,k} \in \Gamma_{i,k}$ which match a certain moment of the measure $\mu_i$ on $\Gamma_{i,k}$. This choice may be compared with [8] where a related discretization argument is performed in a different context.

Let $I_1$ be the Blaschke product with zeros $\xi_{i,k}$, $i,k = 1,2,\ldots$. Finally the last step of the proof is to use $(b)$ above to show that $I_1$ is an interpolating Blaschke product and to use the location of $\{\xi_{i,k}\}$, as well as (a) above, to show that $|I_1(z)\cdot B_2(z)|$ approximates $|B(z)|$.

Besides the individual problem mentioned above, some questions concerning approximation by arguments of interpolating Blaschke products remain open. Let $B$ be an inner function.

(2) Given $\varepsilon > 0$, is there an interpolating Blaschke product $I$ such that

$$\|\operatorname{Arg} B - \operatorname{Arg} I\|_{\mathrm{BMO}(\partial\mathbb{D})} < \varepsilon?$$

(3) Is there an interpolating Blaschke product $I$ such that $\operatorname{Arg} B - \operatorname{Arg} I = \widetilde{v}$ where $v \in L^\infty(\partial\mathbb{D})$?

(4) Is there an interpolating Blaschke product $I$ such that $\operatorname{Arg} B - \operatorname{Arg} I = u + \tilde{v}$ where $u, v \in L^\infty(\partial\mathbb{D})$ and $\|u\|_\infty < \frac{\pi}{2}$?

It is clear that a positive answer to any of these problems would lead to a positive answer to the next one. Moreover a positive answer to Problem 2 would imply the main result of this note. Problem 4 was posed by N. K. Nikol′skiĭ in [6] and [13] in connection to Toeplitz operators and complete interpolating sequences in model spaces. Problem 3 and Problem 4 have been discussed in the nice monograph by K. Seip [14, p. 92].

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## 2. CONSTRUCTION OF THE CONTOUR

The hyperbolic distance between two points $z, w \in \mathbb{D}$ is

$$\beta(z,w) = \tfrac{1}{2}\log\tfrac{1+\rho(z,w)}{1-\rho(z,w)}$$

where $\rho(z,w)$ is the pseudohyperbolic distance,

$$\rho(z,w) = \left|\frac{z-w}{1-\bar{w}z}\right|.$$

Recall that a positive measure $\mu$ in the unit disk is called a Carleson measure if there exists a constant $M = M(\mu) > 0$ such that $\mu(Q) \le M\ell(Q)$ for any Carleson square of the form (1). The infimum of the constants $M$ verifying the inequality above is called the Carleson norm of the measure $\mu$ and it is denoted by $\|\mu\|_C$.

The main result of this section is a variant of the classical construction of the Carleson contour introduced by L. Carleson in his original proof of the Corona Theorem. See [2] or [4, pp. 342–347].

**Lemma 2.** *Let $B \in H^\infty$ with $\|B\|_\infty = 1$. Let $0 < \varepsilon < 1$ and $K > 0$ be fixed constants. Then, there exists a constant $\delta = \delta(\varepsilon, K) > 0$ and a system $\Gamma = \bigcup \Gamma_i$ of disjoint closed curves $\Gamma_i$ contained in $\mathbb{D}$ such that*

(a) *if $\inf_i \beta(z, \operatorname{int}\Gamma_i) \le K$, one has $|B(z)| \le \varepsilon$,*

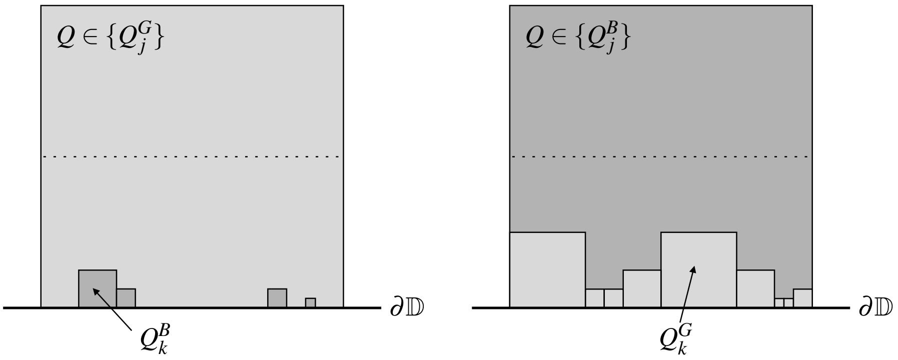

FIGURE 1. Choosing good and bad squares for constructing the contour

(b) *if* $z \notin \bigcup \operatorname{int}\Gamma_i$, *one has*

$$\sup\{|B(w)|\colon \beta(w,z) \le K+14\} > \delta,$$

(c) *arclength on* $\Gamma$, $\mathrm{d}s_{|\Gamma}$, *is a Carleson measure and* $\|\mathrm{d}s_{|\Gamma}\|_C \le 68$.

*Proof.* The proof is essentially contained in the recent paper [12], but we sketch it for the convenience of the reader. Given a set $E \subset \mathbb{D}$, let $\Omega_K(E)$ denote the set of points that are at most at hyperbolic distance $K$ from the set $E$, that is,

$$\Omega_K(E) = \big\{z\colon \inf_{w\in E} \beta(z,w) \le K\big\}.$$

Consider dyadic Carleson squares of the form

$$Q_{n,j} = \big\{re^{i\theta}\colon 1-2^{-n} < r < 1,\ 2\pi j 2^{-n} < \theta < 2\pi(j+1)2^{-n}\big\},$$

for $j = 0,1,\dots,2^n-1$ and $n = 1,2,\dots$, and their top halves $T(Q_{n,j}) = \{re^{i\theta} \in Q_{n,j}\colon r < 1-2^{-n-1}\}$. Let $0 < \delta < \varepsilon$ be a constant to be fixed later. A dyadic Carleson square $Q$ will be called good if

$$\sup\big\{|B(z)|\colon z \in \Omega_K\big(T(Q)\big)\big\} > \varepsilon.$$

The collection of good dyadic Carleson squares will be denoted by $\{Q_j^G\colon j = 1,2,\dots\}$. A dyadic Carleson square $Q$ will be called bad if

$$\sup\big\{|B(z)|\colon z \in \Omega_K\big(T(Q)\big)\big\} < \delta.$$

We denote the collection of bad dyadic Carleson squares by $\{Q_j^B\colon j = 1,2,\dots\}$. The construction goes as follows.

1. For each good dyadic Carleson square $Q = Q_j^G$, we choose the maximal bad dyadic Carleson squares $Q_k^B$ contained in $Q$. The main estimate in the construction is

$$\sum_{Q_k^B \subset Q} \ell(Q_k^B) \le \frac{1}{2}\ell(Q). \tag{2}$$

Since $|B(z)| < \delta$ if $z \in T(Q_k^B)$, while $|B(z)| > \varepsilon$ for some $z \in \Omega_K(T(Q))$, taking $\delta = \delta(\varepsilon,K)$ sufficiently small, standard arguments lead to (2). See Lemma 2.1 of [12] for details.

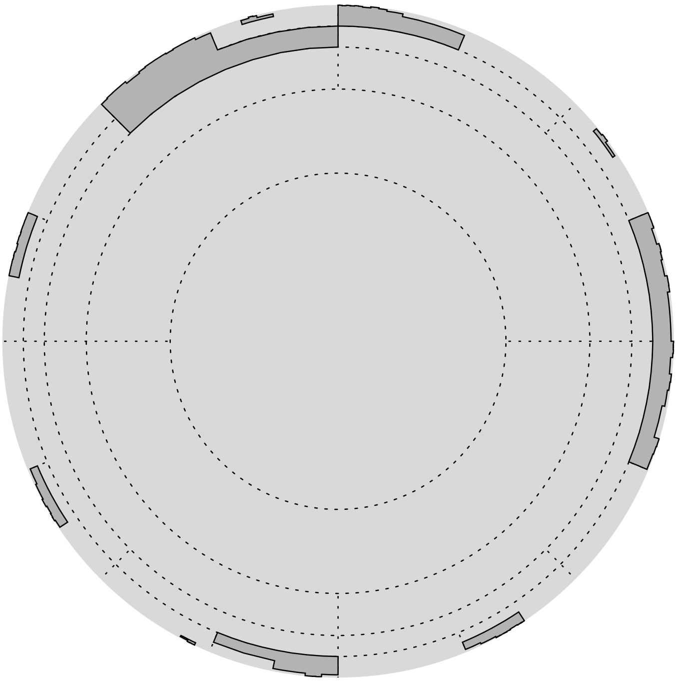

FIGURE 2. The unit disk, some dyadic Carleson contours and an example of a contour.

2. For each bad dyadic Carleson square $Q = Q_j^B$, we choose the maximal good dyadic Carleson squares $Q_k^G$ contained in $Q$. This family is denoted by $G(Q) = \{Q_k^G \colon k = 1,2,\dots\}$.

So, from each good dyadic Carleson square we move to bad ones fulfilling the estimate (2) and from each bad one we again move to good ones. See Figure 1. Now for each bad square $Q = Q_j^B$, let $R(Q)$ be the region

$$R(Q) = Q \setminus \overline{\bigcup_{G(Q)} Q_k^G}$$

and let $R$ be the open set

$$R = \bigcup_j R(Q_j^B).$$

Finally, decompose $R$ into its connected components $R_i$ and denote $\Gamma_i = \partial R_i$, $i = 1,2,\dots$. Observe that each $\Gamma_i$ consists of pieces of boundaries of dyadic Carleson squares. See Figure 2. By construction if $z \in R$ we have

$$\sup\big\{|B(w)| \colon \beta(w,z) \le K\big\} \le \varepsilon$$

and hence part $(a)$ in the statement follows. Similarly, if $z \notin R$, the point $z$ is not in the top part of a bad dyadic Carleson square. As the hyperbolic diameter of a top part of a Carleson square is uniformly bounded, say by 14, we deduce that there exists $w \in \mathbb{D}$ with $\beta(z,w) \le K + 14$ such that $|B(w)| > \delta$. Hence part $(b)$ in the statement follows. Since the length of $\partial R(Q)$ is bounded by $17\ell(Q)$, the scaling (2) shows that for any bad dyadic square $Q$, one has

$$\sum_{Q_j^B \subsetneq Q} |\partial R(Q_j^B)| \leq 17\ell(Q).$$

Then easy geometric considerations show that arclength on $\bigcup \Gamma_i$ is a Carleson measure and its Carleson norm is smaller than 68. □

## 3. CONSTRUCTION OF THE INTERPOLATING BLASCHKE PRODUCT

We now use Lemma 2 to construct a contour $\Gamma$. Note that by Frostman's Theorem we can assume that $B$ is a Blaschke product. Given $\varepsilon > 0$, let $N$ be a big constant dependent on $\varepsilon$ to be fixed later. Apply Lemma 2 with $\frac{\varepsilon}{2}$ and $2N$ instead of $\varepsilon$ and $K$ to obtain $\Gamma$ and $\delta > 0$ such that

(a) $|B(z)| < \frac{\varepsilon}{2}$ if $\beta(z, \operatorname{int}\Gamma) \leq 2N$,

(b) $\sup\{|B(w)| \colon \beta(w,z) \leq 2N + 14\} > \delta$ if $z \notin \operatorname{int}\Gamma$,

(c) arclength on $\Gamma$ is a Carleson measure with Carleson norm $\|\mathrm{d}s_{|\Gamma}\|_C \leq 68$.

With the contour $\Gamma$ in place, we want to construct the interpolating Blaschke product $I$. Split $B$ into two Blaschke products $B_1$ and $B_2$. That is $B = B_1 \cdot B_2$, where $B_1$ is formed with the zeros $\{z_j\}$ of $B$ which are inside $\operatorname{int}\Gamma$ and at hyperbolic distance more than 1 from the contour $\Gamma$. Now for each zero $z$ of $B_2$, part (b) provides a point $w \in \mathbb{D}$, $\beta(w,z) \leq 2N + 15$ such that $|B_2(w)| \geq |B(w)| > \delta$. This implies that $B_2$ is a finite product of interpolating Blaschke products (see Theorem 2.2 of [11]).

Hence the dangerous part of $B$ will be $B_1$ which has all its zeros contained deeply inside the contour $\Gamma$. We want to mimic the behavior of $|B_1|$ by constructing a Blaschke product $I_1$ with zeros on $\Gamma$. To this end, for each component $\Gamma_i$ of the contour we consider the measure

$$\mathrm{d}\mu_i(\xi) = \sum_{\substack{z_j \in \operatorname{int}\Gamma_i \\ \beta(z_j,\Gamma_i) > 1}} \omega(z_j, \xi; \operatorname{int}\Gamma_i)$$

defined for $\xi \in \Gamma_i$. Here $\omega(z, \xi; \Omega)$ denotes the harmonic measure from the point $z \in \Omega$ in the domain $\Omega \subseteq \mathbb{D}$. Clearly $\mu_i(\Gamma_i)$ will be equal to the number of zeros $z_j$ of $B_1$ inside $\Gamma_i$. Next we split $\Gamma_i$ into disjoint arcs $\Gamma_{i,k}$ such that $\mu_i(\Gamma_{i,k}) = 1$ for each $k$. This is illustrated in Figure 3. On each such arc we locate one zero $\xi_{i,k}$ of $I_1$ such that

$$1 - |\xi_{i,k}|^2 = \int_{\Gamma_{i,k}} \left(1 - |\xi|^2\right) \mathrm{d}\mu_i(\xi). \tag{3}$$

This will in general not determine the points $\xi_{i,k}$ uniquely. However, there seems to be a lot of freedom for placing the zeros of $I_1$ in this construction, and the condition (3) will be sufficient for our purposes.

Let $I_1$ be the Blaschke product with the zeros $\xi_{i,k}$, and factor $I_1 = I_1^o \cdot I_1^e$ where $I_1^o$ is the Blaschke product with zeros $\xi_{i,k}$ with $k$ odd, while $I_1^e$ is the Blaschke product with zeros $\xi_{i,k}$ with $k$ even. In Figure 3, $I_1^o$ has its zeros placed in the dark arcs, while the zeros of $I_1^e$ are placed in the light arcs. We claim that both $I_1^o$ and $I_1^e$ are interpolating Blaschke products, and hence $I_1$ can be approximated by an interpolating Blaschke product [10]. To show this claim we will observe that

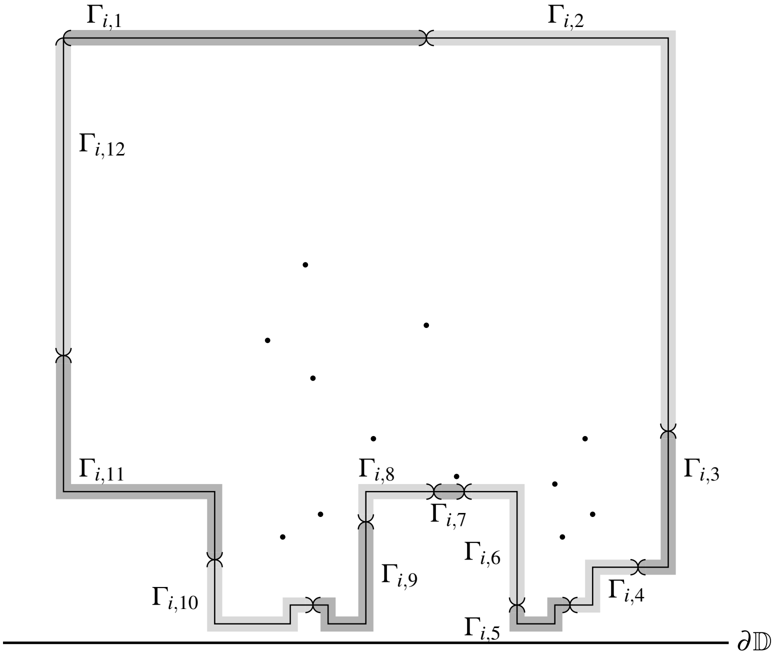

FIGURE 3. Each component $\Gamma_i$ of the contour is split into arcs $\Gamma_{i,k}$ such that the $\mu$-measure of each arc is 1.

their zero sets satisfy the two conditions of Carleson's theorem [1], stated in the introduction.

In this case, property (2) follows from the fact that arclength is a Carleson measure on $\Gamma$, while the first property follows from the following lemma and the geometry of the contour.

**Lemma 3.** *The hyperbolic length, $\ell_\beta(\Gamma_{i,k})$, of $\Gamma_{i,k}$ is bounded from below,*

$$\ell_\beta(\Gamma_{i,k}) \ge \delta^{2^{2N+14}}.$$

*Proof.* We first note that for any point $w \in \Gamma$, $|B_1(w)| \ge \delta^{2^{2N+14}}$. To see this, recall that there is a point $\zeta$ such that $\beta(\zeta,w) \le 2N+14$ and $|B_1(\zeta)| \ge |B(\zeta)| > \delta$. As the zeros of $B_1$ are located inside $\Gamma$ the function $\log|B_1(z)|^{-1}$ is positive and harmonic on $\mathbb{D} \setminus \operatorname{int}\Gamma$. By applying Harnack's principle we get the bound stated above.

Intuitively, this lower bound for the values of $|B_1|$ should imply that the arcs $\Gamma_{i,k}$ can not be too short hyperbolically. To make this observation rigorous we argue as follows. Using that the harmonic measure $\omega$ is positive and harmonic, we have that for any $z \in \operatorname{int}\Gamma_i$,

$$\omega(z,\Gamma_{i,k};\operatorname{int}\Gamma_i) \le \omega(z,\Gamma_{i,k};\mathbb{D}\setminus\Gamma_{i,k}) \le \frac{\displaystyle\int_{\Gamma_{i,k}} \log\left|\frac{z-w}{1-\bar{w}z}\right|^{-1} \frac{|\mathrm{d}w|}{1-|w|^2}}{\displaystyle\min_{z\in\Gamma_{i,k}} \int_{\Gamma_{i,k}} \log\left|\frac{z-w}{1-\bar{w}z}\right|^{-1} \frac{|\mathrm{d}w|}{1-|w|^2}}$$

and

$$\begin{aligned} 1 = \mu_i(\Gamma_{i,k}) &= \sum_{z_j \in \operatorname{int}\Gamma_i} \omega(z_j, \Gamma_{i,k}; \operatorname{int}\Gamma_i) \\ &\leq \frac{1}{C_{i,k}} \int_{\Gamma_{i,k}} \log\left( \prod_{z_j \in \operatorname{int}\Gamma_i} \left| \frac{z_j - w}{1 - \bar{w} z_j} \right|^{-1} \right) \frac{|dw|}{1 - |w|^2} \end{aligned}$$

where $C_{i,k} = \min_{z \in \Gamma_{i,k}} \int_{\Gamma_{i,k}} \log\left|\frac{z-w}{1-\bar{w}z}\right|^{-1} \frac{|dw|}{1-|w|^2}$ is a constant dependent on $\Gamma_{i,k}$. Let $B_{1,i}$ denote the Blaschke product with the zeros of $B_1$ that fall inside the component $\Gamma_i$. Then for $w \in \Gamma_i$,

$$\log\left( \prod_{z_j \in \operatorname{int}\Gamma_i} \left| \frac{z_j - w}{1 - \bar{w} z_j} \right|^{-1} \right) = \log |B_{1,i}(w)|^{-1} \leq \log |B_1(w)|^{-1} \leq 2^{2N+14} \log \delta^{-1}.$$

Thus

$$1 \leq \frac{1}{C_{i,k}} 2^{2N+14} \log \delta^{-1} \int_{\Gamma_{i,k}} \frac{|dw|}{1 - |w|^2} = \frac{1}{C_{i,k}} 2^{2N+14} \log \delta^{-1} \ell_\beta(\Gamma_{i,k})$$

such that

$$\ell_\beta(\Gamma_{i,k}) \geq \frac{C_{i,k}}{2^{2N+14} \log \delta^{-1}}.$$

To estimate $C_{i,k}$ we use the substitution $\xi = \varphi_z(w) = \frac{z-w}{1-\bar{w}z}$ and the conformal invariance of the hyperbolic metric. A calculation then gives that

$$C_{i,k} \geq \log\big(\tanh(\ell_\beta(\Gamma_{i,k}))\big) \ell_\beta(\Gamma_{i,k}),$$

which implies the desired bound, $\ell_\beta(\Gamma_{i,k}) \geq \delta^{2^{2N+14}}$. □

## 4. PROOF OF THE APPROXIMATION

In this section we will show that the constructed function, $I = I_1 \cdot B_2$, approximates the given Blaschke product uniformly in modulus. We first claim that it suffices to prove Theorem 1 for points $z \in \mathbb{D}$ far away from the contour. Indeed, assume that we can prove that

$$\big| |B_1(z)| - |I_1(z)| \big| < \tfrac{\varepsilon}{2} \tag{4}$$

for all $z$ such that $\beta(z, \operatorname{int}\Gamma) \geq 2N$, where $N$ is as in the construction of the contour. Then for points $z$ with $\beta(z, \operatorname{int}\Gamma) = 2N$

$$\begin{aligned} |I(z)| &= \big( |I_1(z)| - |B_1(z)| + |B_1(z)| \big) |B_2(z)| \\ &\leq \big| |B_1(z)| - |I_1(z)| \big| + |B(z)| < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

By the maximum principle $|I(z)| < \varepsilon$ for all $z \in \Omega_{2N}(\operatorname{int}\Gamma)$ as well. Hence

$$\big| |B(z)| - |I(z)| \big| = \big| |B_1(z)| - |I_1(z)| \big| |B_2(z)| < \begin{cases} \frac{\varepsilon}{2} & \text{if } \beta(z, \operatorname{int}\Gamma) \geq 2N, \\ \varepsilon & \text{if } \beta(z, \operatorname{int}\Gamma) < 2N. \end{cases}$$

So Theorem 1 follows from (4).

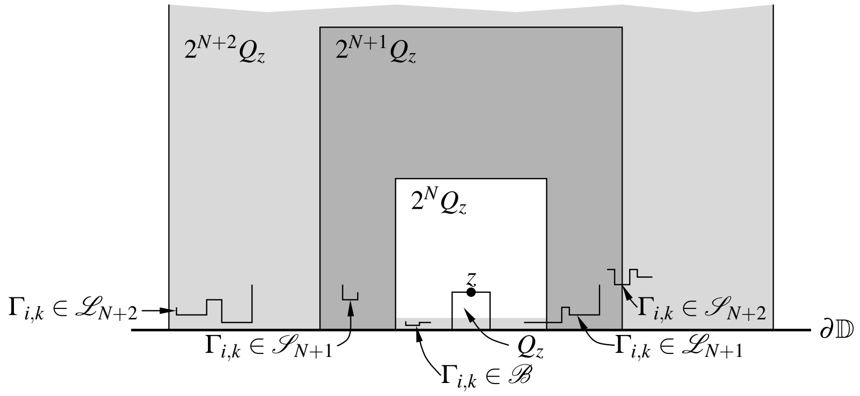

FIGURE 4. We divide the arcs $\Gamma_{i,k}$ into classes denoted $\mathscr{B}$, $\mathscr{S}_n$ and $\mathscr{L}_n$

The rest of the paper will be dedicated to prove that (4) holds. Fix a point $z$ such that $\beta(z, \operatorname{int}\Gamma) \geq 2N$. We will consider the logarithm of $|B_1|$. As all the zeros of $B_1$ lie inside the contour $\Gamma$, $\log\left|\frac{z-z_j}{1-\bar{z}_j z}\right|$ is harmonic inside $\Gamma$ as a function of $z_j$. Hence

$$\log|B_1(z)| = \sum_j \log\left|\frac{z-z_j}{1-\bar{z}_j z}\right| = \int_\Gamma \log\left|\frac{z-\xi}{1-\bar{\xi}z}\right| \mathrm{d}\mu(\xi),$$

where $\mathrm{d}\mu = \sum_i \mathrm{d}\mu_i$. As the $\mu$-measure of each arc $\Gamma_{i,k}$ is 1, we have

$$\begin{aligned}\log|B_1(z)| - \log|I_1(z)| &= \int_\Gamma \log\left|\frac{z-\xi}{1-\bar{\xi}z}\right| \mathrm{d}\mu(\xi) - \sum_{i,k} \log\left|\frac{z-\xi_{i,k}}{1-\bar{\xi}_{i,k}z}\right| \\ &= \sum_{i,k} \int_{\Gamma_{i,k}} \left(\log\left|\frac{z-\xi}{1-\bar{\xi}z}\right| - \log\left|\frac{z-\xi_{i,k}}{1-\bar{\xi}_{i,k}z}\right|\right) \mathrm{d}\mu(\xi) \\ &= \sum_{i,k} \int_{\Gamma_{i,k}} \log\frac{\rho(z,\xi)}{\rho(z,\xi_{i,k})} \mathrm{d}\mu(\xi) \stackrel{\text{def}}{=} \sum_{i,k} H_{i,k}(z). \qquad (5)\end{aligned}$$

To estimate this sum we consider different types of arcs. By $Q_z$ we denote the Carleson square with $z$ as the midpoint on the top-side. We say that an arc $\Gamma_{i,k}$ is in the class $\mathscr{B}$ if $\Gamma_{i,k} \subset 2^N Q_z$. Note that since $\beta(z, \operatorname{int}\Gamma) \geq 2N$, this implies that such an arc lies very close to the boundary. The rest of the arcs we split into short and long arcs. For $n \geq N+1$ define

$$\mathscr{S}_n = \left\{\Gamma_{i,k} \colon \ell_\beta(\Gamma_{i,k}) < 1,\ \Gamma_{i,k} \subset 2^n Q_z\right\} \setminus \left(\mathscr{B} \cup \bigcup_{i<n} \mathscr{S}_i\right)$$

and

$$\mathscr{L}_n = \left\{\Gamma_{i,k} \colon \ell_\beta(\Gamma_{i,k}) \geq 1,\ \Gamma_{i,k} \subset 2^n Q_z\right\} \setminus \left(\mathscr{B} \cup \bigcup_{i<n} \mathscr{L}_i\right).$$

Consult Figure 4 for some examples of this classification. This partition is such that each arc $\Gamma_{i,k}$ belongs to one and only one of the classes $\mathscr{B}$, $\mathscr{S}_n$ and $\mathscr{L}_n$, $n \geq N+1$. Hence we may decompose the sum (5) as follows

$$\sum_{i,k} H_{i,k}(z) = \sum_{\Gamma_{i,k} \in \mathscr{B}} H_{i,k}(z) + \sum_{n=N+1}^{\infty} \left(\sum_{\Gamma_{i,k} \in \mathscr{S}_n} H_{i,k}(z) + \sum_{\Gamma_{i,k} \in \mathscr{L}_n} H_{i,k}(z)\right).$$

Our goal is to show that the absolute value of the left hand side is small. To accomplish this we will show that each of the terms

$$\Big|\sum_{\Gamma_{i,k}\in\mathscr{B}} H_{i,k}(z)\Big|,\quad \Big|\sum_{n=N+1}^{\infty}\sum_{\Gamma_{i,k}\in\mathscr{S}_n} H_{i,k}(z)\Big|\quad\text{and}\quad \Big|\sum_{n=N+1}^{\infty}\sum_{\Gamma_{i,k}\in\mathscr{L}_n} H_{i,k}(z)\Big|$$

are small.

Let us begin with the boundary arcs $\Gamma_{i,k}\in\mathscr{B}$. Using that $\log(1-t) = -t+\mathscr{O}(t^2)$ we get

$$\begin{aligned}\sum_{\Gamma_{i,k}\in\mathscr{B}}&\int_{\Gamma_{i,k}}\log\frac{\rho(z,\xi)}{\rho(z,\xi_{i,k})}\,\mathrm{d}\mu(\xi)\\ &= -\frac{1}{2}\sum_{\Gamma_{i,k}\in\mathscr{B}}\int_{\Gamma_{i,k}}\left(1-\frac{\rho(z,\xi)^2}{\rho(z,\xi_{i,k})^2}+\mathscr{O}\left((1-\frac{\rho(z,\xi)^2}{\rho(z,\xi_{i,k})^2})^2\right)\right)\mathrm{d}\mu(\xi).\end{aligned}$$

Taking absolute values,

$$\begin{aligned}\Big|\sum_{\Gamma_{i,k}\in\mathscr{B}}\int_{\Gamma_{i,k}}\log\frac{\rho(z,\xi)}{\rho(z,\xi_{i,k})}\,\mathrm{d}\mu(\xi)\Big| &\le \frac{1}{2}\Big|\sum_{\Gamma_{i,k}\in\mathscr{B}}\int_{\Gamma_{i,k}}1-\frac{\rho(z,\xi)^2}{\rho(z,\xi_{i,k})^2}\,\mathrm{d}\mu(\xi)\Big|\\ &+\frac{1}{2}\Big|\sum_{\Gamma_{i,k}\in\mathscr{B}}\int_{\Gamma_{i,k}}\mathscr{O}\left((1-\frac{\rho(z,\xi)^2}{\rho(z,\xi_{i,k})^2})^2\right)\mathrm{d}\mu(\xi)\Big| \overset{\text{def}}{=} E_{\mathscr{B},1}+E_{\mathscr{B},2},\end{aligned}\tag{6}$$

where we define $E_{\mathscr{B},1}$ and $E_{\mathscr{B},2}$ for convenience. At first we focus on the first term, $E_{\mathscr{B},1}$, of this expression. Note that as $z$ is far away from $\xi_{i,k}\in\Gamma_i$, $\rho(z,\xi_{i,k})^{-2}$ is bounded, say $\rho(z,\xi_{i,k})^{-2}\le 2$. By expanding $1-\rho(z,\xi)^2$ and $1-\rho(z,\xi_{i,k})^2$, we can write

$$\begin{aligned}E_{\mathscr{B},1} &\le \sum_{\Gamma_{i,k}\in\mathscr{B}}\Big|\int_{\Gamma_{i,k}}(1-|z|^2)\left(\frac{1-|\xi|^2}{|1-\bar{\xi}z|^2}-\frac{1-|\xi_{i,k}|^2}{|1-\bar{\xi}_{i,k}z|^2}\right)\mathrm{d}\mu(\xi)\Big| \qquad (7)\\ &= \sum_{\Gamma_{i,k}\in\mathscr{B}}\Big|\int_{\Gamma_{i,k}}(1-|z|^2)\left(\frac{1-|\xi|^2}{|1-\bar{\xi}z|^2}-\frac{1-|\xi|^2}{|1-\bar{\xi}_{i,k}z|^2}+\frac{|\xi_{i,k}|^2-|\xi|^2}{|1-\bar{\xi}_{i,k}z|^2}\right)\mathrm{d}\mu(\xi)\Big|.\end{aligned}$$

By the placement, (3), of the zeros $\xi_{i,k}$, the integral of the last term is zero. We now move the modulus under the integral to get

$$E_{\mathscr{B},1}\le(1-|z|^2)\sum_{\Gamma_{i,k}\in\mathscr{B}}\int_{\Gamma_{i,k}}(1-|\xi|^2)\left|\frac{1}{|1-\bar{\xi}z|^2}-\frac{1}{|1-\bar{\xi}_{i,k}z|^2}\right|\mathrm{d}\mu(\xi).\tag{8}$$

Because $\xi$ and $\xi_{i,k}$ should be close to each other in some sense, compared to $z$, we suspect some cancellation. Therefore we use the estimate

$$\left|\frac{1}{|1-\bar{\xi}z|^2}-\frac{1}{|1-\bar{\xi}_{i,k}z|^2}\right|\le\frac{2|\xi-\xi_{i,k}|}{\left(1-|z|\right)^3}\tag{9}$$

and the more trivial inequalities $|\xi-\xi_{i,k}|\le\ell(\Gamma_{i,k})$ and $1-|z|^2\le 2(1-|z|)$ to obtain

$$E_{\mathscr{B},1}\le 2^3\cdot\left(1-|z|\right)^{-2}\sum_{\Gamma_{i,k}\in\mathscr{B}}\ell(\Gamma_{i,k})\int_{\Gamma_{i,k}}\left(1-|\xi|\right)\mathrm{d}\mu(\xi).$$

All the arcs $\Gamma_{i,k} \in \mathscr{B}$ are contained in a rectangle at the boundary with height $2^{-2N}(1-|z|)$ and width $2^{N}(1-|z|)$. Using that $1-|\xi| \leq 2^{-2N}(1-|z|)$ and that the arclength $\mathrm{d}s_{|\Gamma}$ is a Carleson measure, we then get

$$E_{\mathscr{B},1} \leq 2^3 \cdot \|\mathrm{d}s_{|\Gamma}\|_C \cdot 2^{-N}$$

where $\|\mathrm{d}s_{|\Gamma}\|_C$ is the Carleson norm of arclength on $\Gamma$.

Next we focus our attention on the higher order terms, and give the estimate for $E_{\mathscr{B},2}$. From (6) and (7) and the inequality $(a+b)^2 \leq 2(a^2+b^2)$ we see that $E_{\mathscr{B},2}$ is bounded by a fixed multiple of

$$\begin{aligned}\left(1-|z|^2\right)^2 \sum_{\Gamma_{i,k}\in\mathscr{B}} \int_{\Gamma_{i,k}} \left(1-|\xi|^2\right)^2 \left|\frac{1}{|1-\bar{\xi}z|^2} - \frac{1}{|1-\bar{\xi}_{i,k}z|^2}\right|^2 \mathrm{d}\mu(\xi)& \\ + \left(1-|z|^2\right)^2 \sum_{\Gamma_{i,k}\in\mathscr{B}} \int_{\Gamma_{i,k}} \frac{(|\xi_{i,k}|^2-|\xi|^2)^2}{|1-\bar{\xi}_{i,k}z|^4}\,\mathrm{d}\mu(\xi).&\end{aligned}$$

For the first term, we use as above the estimate (9) as well as $1-|\xi| \leq 2^{-2N}(1-|z|)$ and $|\xi-\xi_{i,k}| \leq 2\cdot 2^{N}(1-|z|)$. Then we find

$$\begin{aligned}&\left(1-|z|^2\right)^2 \sum_{\Gamma_{i,k}\in\mathscr{B}} \int_{\Gamma_{i,k}} \left(1-|\xi|^2\right)^2 \left|\frac{1}{|1-\bar{\xi}z|^2} - \frac{1}{|1-\bar{\xi}_{i,k}z|^2}\right|^2 \mathrm{d}\mu(\xi) \\ &\quad\leq 2^4 \cdot 2^{-N} \cdot \left(1-|z|^2\right) \sum_{\Gamma_{i,k}\in\mathscr{B}} \int_{\Gamma_{i,k}} \left(1-|\xi|^2\right) \left|\frac{1}{|1-\bar{\xi}z|^2} - \frac{1}{|1-\bar{\xi}_{i,k}z|^2}\right| \mathrm{d}\mu(\xi).\end{aligned}$$

Observe that the last sum is just (8) and by the earlier argument the last expression is bounded by $2^7 \cdot \|\mathrm{d}s_{|\Gamma}\|_C \cdot 2^{-2N}$.

For the second term we use that $|1-\bar{\xi}_{i,k}z| \geq 1-|z|$, $||\xi_{i,k}|-|\xi|| \leq 2^{-2N}(1-|z|)$ and $||\xi_{i,k}|-|\xi|| \leq \ell(\Gamma_{i,k})$ to arrive at

$$\begin{aligned}&\left(1-|z|^2\right)^2 \sum_{\Gamma_{i,k}\in\mathscr{B}} \int_{\Gamma_{i,k}} \frac{(|\xi_{i,k}|^2-|\xi|^2)^2}{|1-\bar{\xi}_{i,k}z|^4}\,\mathrm{d}\mu(\xi) \\ &\qquad\leq 2^4 \cdot \left(1-|z|\right)^{-2} \sum_{\Gamma_{i,k}\in\mathscr{B}} \int_{\Gamma_{i,k}} \left||\xi_{i,k}|-|\xi|\right|^2 \mathrm{d}\mu(\xi) \\ &\qquad\leq 2^4 \cdot 2^{-2N} \cdot \left(1-|z|\right)^{-1} \sum_{\Gamma_{i,k}\in\mathscr{B}} \ell(\Gamma_{i,k}) \leq 2^4 \cdot \|\mathrm{d}s_{|\Gamma}\|_C \cdot 2^{-N}.\end{aligned}$$

Thus we get that $E_{\mathscr{B},2} \leq C\cdot(2^4+1)\cdot\|\mathrm{d}s_{|\Gamma}\|_C \cdot 2^{-N}$ for big $N$.

For the short arcs $\Gamma_{i,k} \in \mathscr{S}_n$, $n \geq N+1$ we will use similar estimates as above, however we do not need to be as delicate. For these arcs, we can use that $|\log x| \leq |1-x^2|$ to obtain

$$\begin{aligned}E_{\mathscr{S}} &\overset{\mathrm{def}}{=} \left|\sum_{n=N+1}^{\infty} \sum_{\Gamma_{i,k}\in\mathscr{S}_n} \int_{\Gamma_{i,k}} \log\frac{\rho(z,\xi)}{\rho(z,\xi_{i,k})}\,\mathrm{d}\mu(\xi)\right| \\ &\leq \sum_{n=N+1}^{\infty} \sum_{\Gamma_{i,k}\in\mathscr{S}_n} \int_{\Gamma_{i,k}} \left|1-\frac{\rho(z,\xi)^2}{\rho(z,\xi_{i,k})^2}\right| \mathrm{d}\mu(\xi).\end{aligned}$$

The same calculations that gave (7) show that

$$\left|1-\frac{\rho(z,\xi)^2}{\rho(z,\xi_{i,k})^2}\right| \le 2\big(1-|z|^2\big)\left(\left|\frac{1-|\xi|^2}{|1-\bar{\xi}z|^2}-\frac{1-|\xi|^2}{|1-\bar{\xi}_{i,k}z|^2}\right|+\frac{\big||\xi_{i,k}|^2-|\xi|^2\big|}{|1-\bar{\xi}_{i,k}z|^2}\right).$$

For $\xi \in \Gamma_{i,k} \in \mathscr{S}_n$, using $|1-\bar{\xi}z| \ge 2^{n-3}(1-|z|)$ we get

$$\big(1-|\xi|^2\big)\left|\frac{1}{|1-\bar{\xi}z|^2}-\frac{1}{|1-\bar{\xi}_{i,k}z|^2}\right| \le 2^{11}\frac{\big(1-|\xi|\big)|\xi-\xi_{i,k}|}{2^{3n}\big(1-|z|\big)^3} \le 2^{11}\frac{|\xi-\xi_{i,k}|}{2^{2n}\big(1-|z|\big)^2}.$$

Similarly

$$\frac{\big||\xi_{i,k}|^2-|\xi|^2\big|}{|1-\bar{\xi}_{i,k}z|^2} \le 2^7\frac{|\xi-\xi_{i,k}|}{2^{2n}\big(1-|z|\big)^2}.$$

Adding up, we obtain

$$\left|1-\frac{\rho(z,\xi)^2}{\rho(z,\xi_{i,k})^2}\right| \le 2^{14}\frac{|\xi-\xi_{i,k}|}{2^{2n}\big(1-|z|\big)}.$$

Hence

$$E_{\mathscr{S}} \le 2^{14}\sum_{n=N+1}^{\infty}\frac{1}{2^{2n}\big(1-|z|\big)}\sum_{\Gamma_{i,k}\in\mathscr{S}_n}\ell(\Gamma_{i,k}) \le 2^{14}\cdot\|\mathrm{d}s_{|\Gamma}\|_C\cdot 2^{-N}.$$

Finally, we estimate the long arcs $\Gamma_{i,k} \in \mathscr{L}_n$, $n \ge N+1$. As the zeros on these arcs are well separated, one can expect only a small contribution from these arcs. We will use an auxiliary interpolating Blaschke product to find a bound for the $\mathscr{L}_n$-terms of (5). By the same reasoning that led to (7) and the triangle inequality,

$$\begin{aligned}
E_{\mathscr{L}} &\overset{\text{def}}{=} \left|\sum_{n=N+1}^{\infty}\sum_{\Gamma_{i,k}\in\mathscr{L}_n}\int_{\Gamma_{i,k}}\log\frac{\rho(z,\xi)}{\rho(z,\xi_{i,k})}\,\mathrm{d}\mu(\xi)\right| \\
&\le 2\sum_{n=N+1}^{\infty}\sum_{\Gamma_{i,k}\in\mathscr{L}_n}\int_{\Gamma_{i,k}}\big(1-|z|^2\big)\left(\frac{1-|\xi|^2}{|1-\bar{\xi}z|^2}+\frac{1-|\xi_{i,k}|^2}{|1-\bar{\xi}_{i,k}z|^2}\right)\mathrm{d}\mu(\xi) \\
&\le 2^2\sum_{n=N+1}^{\infty}\sum_{\Gamma_{i,k}\in\mathscr{L}_n}\max_{\xi\in\bar{\Gamma}_{i,k}}\frac{\big(1-|z|^2\big)\big(1-|\xi|^2\big)}{|1-\bar{\xi}z|^2}.
\end{aligned}$$

For each $\Gamma_{i,k} \in \mathscr{L}_n$, let $\zeta_{i,k} \in \Gamma_{i,k}$ be such that

$$\frac{1-|\zeta_{i,k}|^2}{|1-\bar{\zeta}_{i,k}z|^2} = \max_{\xi\in\bar{\Gamma}_{i,k}}\frac{1-|\xi|^2}{|1-\bar{\xi}z|^2},$$

and define $B_\zeta$ to be the Blaschke product with $\{\zeta_{i,k}\}$ as zeros. Now we reorder the summation, and sum with respect to the placement of the $\zeta_{i,k}$ instead. Then

$$E_{\mathscr{L}} \le 2^3\cdot\big(1-|z|\big)\sum_{n=0}^{\infty}\sum_{\zeta_{i,k}\in U_n}\frac{1-|\zeta_{i,k}|^2}{|1-\bar{\zeta}_{i,k}z|^2}$$

where $U_0 = Q_z$ and $U_n = 2^nQ_z \setminus 2^{n-1}Q_z$ for $n \ge 1$. The scaling property (2) implies that at most four of the points $\zeta_{i,k}$ are contained in $2^{N-1}Q_z$. These must be close to the boundary, so that

$$2^3\cdot\big(1-|z|\big)\sum_{n=0}^{N-1}\sum_{\zeta_{i,k}\in U_n}\frac{1-|\zeta_{i,k}|^2}{|1-\bar{\zeta}_{i,k}z|^2} \le 4\cdot 2^4\cdot 2^{-2N}.$$

For the rest of the terms, we then get

$$2^3 \cdot (1-|z|) \sum_{n=N}^{\infty} \sum_{\zeta_{i,k}\in U_n} \frac{1-|\zeta_{i,k}|^2}{|1-\bar{\zeta}_{i,k}z|^2} \leq 2^8 \sum_{n=N}^{\infty} \frac{1}{2^n} \sum_{\zeta_{i,k}\in U_n} \frac{1-|\zeta_{i,k}|}{2^n(1-|z|)} \leq 2^9 \cdot C_\zeta \cdot 2^{-N},$$

where $C_\zeta$ is the Carleson norm of the measure $\sum(1-|\zeta_{i,k}|)\delta_{\zeta_{i,k}}$, which is bounded by a fixed multiple of $\|\mathrm{d}s_{|\Gamma}\|_C$. Thus $E_{\mathscr{L}} \leq 2^9 \cdot (C_\zeta + 1) \cdot 2^{-N}$.

We have now estimated the contribution from all the arcs $\Gamma_{i,k}$, and we have found that for some constant $C$,

$$\big|\log|B_1(z)| - \log|I_1(z)|\big| \leq C \cdot 2^{-N}.$$

This means that given $\varepsilon > 0$, taking $N$ so that $C \cdot 2^{-N} < \frac{\varepsilon}{2}$, we obtain

$$\big||B_1(z)| - |I_1(z)|\big| < \tfrac{\varepsilon}{2},$$

which was what we needed.

## REFERENCES

[1] Lennart Carleson, *An interpolation problem for bounded analytic functions*, Amer. J. Math. **80** (1958), 921–930. MR 0117349 (22:8129)

[2] ———, *Interpolations by bounded analytic functions and the corona problem*, Ann. of Math. (2) **76** (1962), 547–559. MR 0141789 (25:5186)

[3] Otto Frostman, *Potentiel d'équilibre et capacité des ensembles avec quelques applications á la théorie des fonctions*, Medd. Lund. Univ. Math. Sem. **3** (1935), no. 3, 1–118.

[4] John B. Garnett, *Bounded analytic functions*, Academic Press, 1981. MR 0628971 (83g:30037)

[5] John B. Garnett and Artur Nicolau, *Interpolating Blaschke products generate $H^\infty$*, Pacific J. Math. **173** (1996), no. 2, 501–510. MR 1394402 (97f:30050)

[6] Victor P. Havin and Nikolai K. Nikol′skiĭ (eds.), *Linear and complex analysis. Problem book 3. Part II*, Lecture Notes in Mathematics, vol. 1574, Springer, Berlin, 1994. MR 1334345 (96c:00001a)

[7] Peter W. Jones, *Ratios of interpolating Blaschke products*, Pacific J. Math. **95** (1981), no. 2, 311–321. MR 0632189 (82m:30032)

[8] Yurii Lyubarskii and Eugenia Malinnikova, *On approximation of subharmonic functions*, J. Anal. Math. **83** (2001), 121–149. MR 1828489 (2002b:30043)

[9] Donald E. Marshall, *Blaschke products generate $H^\infty$*, Bull. Amer. Math. Soc. **82** (1976), 494–496. MR 0402054 (53:5877)

[10] Donald E. Marshall and Arne Stray, *Interpolating Blaschke products*, Pacific J. Math. **173** (1996), no. 2, 491–499. MR 1394401 (97c:30042)

[11] Raymond Mortini and Artur Nicolau, *Frostman shifts of inner functions*, J. Anal. Math. **92** (2004), 285–326. MR 2072750 (2005e:30088)

[12] Artur Nicolau and Daniel Suárez, *Approximation by invertible functions of $H^\infty$*, To appear in Math. Scandinavica.

[13] Nikolai K. Nikol′skiĭ, *Treatise on the shift operator*, Springer, Berlin, 1986. MR 0827223 (87i:47042)

[14] Kristian Seip, *Interpolation and sampling in spaces of analytic functions*, American Mathematical Society, Providence, RI, 2004, University Lecture Series 33. MR 2040080 (2005c:30038)

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