Bounded Functions in Möbius Invariant Dirichlet Spaces

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For $p \in (0, 1)$, let $Q_p(Q_p, 0)$ be the space of analytic functions $f$ on the unit disk $\Delta$ with $\sup_{w, \varphi} \| f \circ \varphi_w \|_{\mathcal{D}_p} < \infty$ ($\lim_{|w| = 1} \| f \circ \varphi_w \|_{\mathcal{D}_p} = 0$), where $\| \cdot \|_{\mathcal{D}_p}$ means the weighted Dirichlet norm and $\varphi_w$ is the Möbius map of $\Delta$ onto itself with $\varphi_w(0) = w$. In this paper, we prove the Corona theorem for the algebra $Q_p \cap H^\infty (Q_p \cap H^\infty)$; then we provide a Fefferman–Stein type decomposition for $Q_p(Q_p, 0)$, and finally we describe the interpolating sequences for $Q_p \cap H^\infty (Q_p, 0 \cap H^\infty)$.

1. INTRODUCTION

For $p \in (-1, \infty)$, let $\mathcal{D}_p$ be the space of analytic functions $f$ on the unit disk $\Delta$ with

$$\| f \|_{\mathcal{D}_p}^2 = \int_{\Delta} |f'(z)|^2 (1 - |z|^2)^p \, dm(z) < \infty,$$

where $dm(z)$ denotes the usual Lebesgue measure on $\Delta$. These are called Dirichlet type spaces because for $p = 0$ one gets the classical Dirichlet space $\mathcal{D}$ of all analytic functions on $\Delta$ whose images have finite area, counting multiplicities. Also, observe that for $p = 1$, $\mathcal{D}_p$ is just the usual Hardy space $H^2$ and for $p > 1$ is the Bergman space with weight $(1 - |z|^2)^{p-2}$.

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Here, we are mainly interested in the conformally invariant version of these spaces. More precisely, for \( p \in (-1, \infty) \), the space \( Q_p \) consists of all analytic functions \( f \) on \( \mathbb{D} \) such that

\[
\| f \|_{Q_p}^2 = \sup_{u \in \partial \mathbb{D}} \| f \circ \varphi_u \|_p^2 < \infty,
\]

where \( \varphi_u \) is the Möbius transformation of \( \mathbb{D} \) onto itself, sending the origin to \( u \). An easy computation shows that a function \( f \) in \( \mathcal{D}_p \) belongs to \( Q_p \) if and only if

\[
\sup_{u \in \partial \mathbb{D}} \int_{I} |f'(z)|^2 \left[ \log \frac{1}{|\varphi_u'(z)|} \right]^p dm(z) < \infty.
\]

Moreover, if \( f \in Q_p \) and the above integrals tend to zero as \( |w| \to 1 \) then we say \( f \in Q_p,0 \).

If \( -1 < p < 0 \), the space \( Q_p \) only contains constant functions, while \( Q_0 \) is the classical Dirichlet space.

If \( p \in (0,1) \), then \( Q_p = Q_p(\partial \mathbb{D}) \cap H^2(Q_{p,0})(\partial \mathbb{D}) \cap H^2 \), where \( \partial \mathbb{D} \) is the boundary of \( \mathbb{D} \) and \( Q_p(\partial \mathbb{D}) \) is defined as the space of all functions \( f \in L^2(\partial \mathbb{D}) \) with

\[
\| f \|_{Q_p(\partial \mathbb{D})}^2 = \sup_{I \subset \partial \mathbb{D}} \int_I \frac{1}{|I|^p} \int_{I} \left| f(e^{i\theta}) - f(e^{i\phi}) \right|^2 d\theta d\phi < \infty.
\]

As usual, the supremum is taken over all subarcs \( I \subset \partial \mathbb{D} \) and \( |I| \) denotes the normalized arc length of \( I \), while \( Q_{p,0}(\partial \mathbb{D}) \) is the set of functions \( f \in Q_p(\partial \mathbb{D}) \), for which the above integrals tend to zero when \( |I| \to 0 \) [10]. In addition, if \( 0 < p_1 < p_2 < 1 \) then \( Q_{p_1} \subset Q_{p_2} \), \( (Q_{p,0} \subset Q_{p_0}) \) [3].

If \( p = 1 \) then \( Q_1 = \text{BMOA} = \text{BMO}(\partial \mathbb{D}) \cap H^2(Q_{1,0} = \text{VMOA} = \text{VMO}(\partial \mathbb{D}) \cap H^2) \), where \( \text{BMO}(\partial \mathbb{D}) \) (\( \text{VMO}(\partial \mathbb{D}) \)) is the usual space of functions in \( L^2(\partial \mathbb{D}) \) with bounded (vanishing) mean oscillation on \( \partial \mathbb{D} \), (see [11, Chapter VI]).

If \( p \in (1,\infty) \), then \( Q_p = B(Q_{p,0} = B_0) \), where \( B \) (\( B_0 \)) is the classical Bloch space (little Bloch space), [1, 23, 24].

In this paper, we study the Banach algebra \( Q_p \cap H^\infty(Q_{p,0} \cap H^\infty) \). To be precise, we first discuss the Corona property for \( Q_p \cap H^\infty(Q_{p,0} \cap H^\infty) \); secondly consider a Fefferman–Stein type decomposition of \( Q_p(\partial \mathbb{D}) \) \( (Q_{p,0}(\partial \mathbb{D})) \) and finally investigate interpolating sequences for \( Q_p \cap H^\infty(Q_{p,0} \cap H^\infty) \). Here \( H^\infty \) stands for the space of all bounded analytic functions on \( \mathbb{D} \) and \( \| f \|_\infty = \sup \{|f(z)| : z \in \mathbb{D} \} \). For convenience, we will now state our main results.
First of all, we will prove that the Corona theorem holds for the algebra \( Q_p \cap H^\infty \) whenever \( p \in (0, 1) \), that is to say, the unit disk \( A \) is dense in the maximal ideal space of \( H^\infty \cap Q_p \) \( (H^\infty \cap Q_{p,0}), \ p \in (0, 1) \). This fact can be reformulated in the following way.

**Theorem 1.1.** Let \( p \in (0, 1) \). If \( f_1, \ldots, f_n \in Q_p \cap H^\infty \) \( (Q_{p,0} \cap H^\infty) \) with

\[
\inf_{z \in A} (|f_1(z)| + \cdots + |f_n(z)|) > 0,
\]

then there exist \( g_1, \ldots, g_n \in Q_p \cap H^\infty \) \( (Q_{p,0} \cap H^\infty) \) with

\[
f_1 g_1 + \cdots + f_n g_n \equiv 1.
\]

It is certainly well known that Theorem 1.1 holds for \( p \geq 1 \). If \( p \geq 1 \), the space \( Q_p \cap H^\infty \) is just \( H^\infty \), and the Corona Theorem holds by a celebrated result of L. Carleson ([8], [11, Chapter VIII]). Also, Theorem 1.1 was proved for the algebra \( Q_{1,0} \cap H^\infty = \text{VMO}(\partial A) \) by C. Sundberg and T. Wolff ([19]). If \( p \geq 1 \), \( Q_{p,0} \cap H^\infty = B_0 \cap H^\infty \) and the result is also known. However, since we have not found a reference in the literature, a proof is presented. The proof of Theorem 1.1 follows the usual method of solving some \( \bar{\partial} \)-equations with \( L^\infty \) and \( Q_p \) estimates simultaneously. For this, we use an explicit solution due to P. Jones [12].

As is well known, there is a close relation between \( \bar{\partial} \)-equations and the Fefferman–Stein decomposition asserting that any \( f \in \text{BMO}(\partial A) \) \( (\text{VMO}(\partial A)) \) can be decomposed into \( f = u + i v \), where \( u, v \in L^\infty(\partial A) \) \( (C(\partial A)) \) and \( i \) means the conjugate function of \( i \). So, it is not surprising that solving \( \bar{\partial} \)-equations with appropriate estimates leads to the following result.

**Theorem 1.2.** Let \( p \in (0, 1) \) and \( f \in L^\infty(\partial A) \). Then \( f \in Q_{p}(\partial A), \ Q_{p,0}(\partial A) \) if and only if \( f = u + i v \), where \( u, v \in L^\infty(\partial A) \) \( (C(\partial A)) \). When \( f \in Q_{p}(\partial A) \) and \( \int_{\partial A} f(e^{\alpha} \theta) d\theta = 0 \), the functions can be chosen so that

\[
||u||_{L^\infty} + ||\bar{v}||_{L^\infty} + ||u||_{Q_{p}(\partial A)} + ||v||_{Q_{p,0}(\partial A)} \leq C ||f||_{Q_{p}(\partial A)},
\]

where \( C > 0 \) is an absolute constant.

Finally, we will discuss the free interpolation problem in the algebra \( Q_p \cap H^\infty \) \( (Q_{p,0} \cap H^\infty) \), \( p \in (0, 1) \). A sequence \( \{z_n\} \subset A \) is called an interpolating sequence for \( Q_p \cap H^\infty \) \( (Q_{p,0} \cap H^\infty) \) if for each bounded sequence \( \{w_n\} \) of complex numbers there exists \( f \in Q_p \cap H^\infty \) \( (Q_{p,0} \cap H^\infty) \) such that \( f(z_n) = w_n \) for all \( n \). Let \( \rho \) be the pseudohyperbolic distance on \( A \), that is,

\[
\rho(z, w) = |\phi_n(z)| = \left| \frac{w - z}{1 - w \bar{z}} \right|, \quad z, w \in A.
\]
Theorem 1.3. Let \( p \in (0, 1) \). A sequence \( \{z_n\} \) of points in the unit disk is an interpolating sequence for \( Q_p \cap H^\infty \) if and only if the following conditions hold:

(a) \( \inf_{m \neq n} \rho(z_m, z_n) > 0 \)

(b) \( \sup_{n \in \mathbb{N}} \sum_{m=1}^{n} (1 - |g_n(z_m)|^2)^p < \infty \).

Theorem 1.4. Let \( p \in (0, 1) \). A sequence \( \{z_n\} \) of points in the unit disk is an interpolating sequence for \( Q_{p,0} \cap H^\infty \) if and only if the following conditions hold:

(a) \( \lim_{n \to \infty} \inf_{m \neq n} \rho(z_m, z_n) = 1 \)

(b) \( \lim_{n \to \infty} \sup_{n \in \mathbb{N}} \sum_{m=1}^{n} (1 - |g_n(z_m)|^2)^p = 0 \).

It is worth remarking that Theorems 1.3 and 1.4 still hold true for \( p = 1 \). Since \( Q_p \cap H^\infty = H^\infty \), \( p \geq 1 \), this is again a result of L. Carleson [7], while interpolating sequences for the algebra \( H^\infty \cap Q_{1,0} = H^\infty \cap \text{VMOA} \) were described in [19], as the ones satisfying (a), (b) in Theorem 1.4 with \( p = 1 \). As in [19], a sequence \( \{z_n\} \) of points in the unit disc is called \( p \)-uniformly separated (\( p \)-thin) if (a) and (b) in Theorem 1.3 ((a) and (b) in Theorem 1.4) hold. We will also show that the interpolating sequences for \( H^\infty \cap B_0 \) are the \( 1 \)-thin sequences.

The necessity in both results follows from an argument which combines the Khinchin's inequality and a reproducing formula due to R. Rochberg and Z. Wu [16]. The sufficiency for \( Q_p \cap H^\infty \) is easy and it can be derived in different ways. However the sufficiency for \( Q_{p,0} \cap H^\infty \) is more difficult as one can already see in the case of \( \text{VMOA} \cap H^\infty \) [19]. There are two main reasons. First, there are no inner functions in \( Q_{p,0} \) except for Blaschke products with a finite number of zeros and, second, it is not simple to construct non-analytic functions solving the required interpolation problems. Our proof will take some ideas from [19] but our construction of such functions is quite different from theirs and, we believe, is more elementary.

The paper is organized as follows. In Section 2 we collect some basic facts about \( Q_p \)-spaces, including their analytic and non-analytic forms. Theorems 1.1 and 1.2 will be proved in Section 3, and Section 4 is devoted to the discussion of interpolating sequences.

Throughout this paper some notations will be used repeatedly. The letters, \( C, C', C_1 \) etc will denote absolute constants, not necessarily the same at each occurrence. The notation \( a \asymp b \) means that there are absolute constants \( c_1, c_2 > 0 \) satisfying \( c_1 b \leq a \leq c_2 b \). Similarly, \( a \lesssim b \) means that the second inequality holds. Also given an arc \( I \) of the unit circle, let \( S(I) \) be the Carleson box based on \( I \), i.e.,

\[
S(I) = \{ re^{i\theta}, 1 - |I| \leq r < 1, e^{i\theta} \in I \};
\]
and \( T(S(I)) \) denotes the “top half” of \( S(I) \), i.e.,

\[
T(S(I)) = \left\{ re^{it} \in S(I) : 1 - |I| \leq r \leq 1 - \frac{|I|}{2} \right\}.
\]

For \( 0 \neq z \in A \) let \( I_z \) be the arc of the unit circle of length \((1 - |z|)/2\pi\) centered at \( z/|z| \), and further we denote by \( S(z) \) and \( T(z) \) respectively the sets \( S(I_z) \) and \( T(S(I_z)) \). If \( I \) is an arc on \( \partial A \) then \( z_I \) means the point in \( A \) such that \( I = I_z \), and for a positive integer \( n \), \( nI \) means the arc with the same center as \( I \) and length \( n|I| \).

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2. PRELIMINARY FACTS

In this section we will collect some known facts on \( Q_p \) which will be used in the following sections.

3.1. Two Characterizations of \( Q_p(\partial A) \)

Let \( p \in (0, \infty) \). A positive Borel measure \( \mu \) defined on \( A \) is called a \( p \)-Carleson measure if

\[
\|\mu\|_p = \sup_{I \subset \partial A} \mu(S(I)) / |I|^p < \infty,
\]

where the supremum is taken over all subarcs \( I \subset \partial A \). If the right hand fractions tend to zero as \( |I| \to 0 \) then \( \mu \) is said to be a \( p \)-vanishing Carleson measure. As in the case \( p = 1 \), \( p \)-Carleson measures can be characterized in conformally invariant terms ([ASX]). Actually \( \mu \) is a \( p \)-Carleson measure if and only if

\[
\sup_{z \in A} \int_{\partial A} \left( \frac{1 - |w|^2}{|1 - z\overline{w}|^2} \right)^p d\mu(w) < \infty.
\]

Actually this supremum is comparable to \( \|\mu\|_p \). Similarly, \( \mu \) is a \( p \)-vanishing Carleson measure if and only if the above integral tends to 0 when \( |z| \to 1 \).

For \( p \in (0, 1) \), this notion has been used to characterize \( Q_p(Q_{p,0}) \)-functions [2, 10]. Given \( f \in L^1(\partial A) \) let \( \hat{f} \) be its Poisson extension, that is

\[
\hat{f}(z) = \frac{1}{2\pi} \int_{0}^{2\pi} f(e^{i\theta}) \frac{1 - |z|^2}{|1 - z e^{-i\theta}|^2} d\theta.
\]
Then we will obtain the following theorem which may be viewed as an extension of Theorem 1.1 in [2], and Theorem 2.1, 5.3 and Corollary 5.2 in [10].

**Theorem 2.1.** Let $p \in (0, 1)$ and $f \in L^2(\partial \mathcal{A})$ with $f(e^{i\theta}) \sim \sum_{n=-\infty}^{\infty} a_n e^{i\theta}$. Then the following conditions are equivalent:

(i) $f \in Q_p(\partial \mathcal{A})(Q_p(\partial \mathcal{A}))$.

(ii) $|\nabla \tilde{f}(z)|^2 (1 - |z|^2)^p \, dm(z)$ is a $p$-Carleson measure (p-vanishing Carleson measure).

(iii) $\|\sigma_{n}\|_{Q_p(\partial \mathcal{A})} = O(1)(o(1))$, where

$$\sigma_{n}(e^{i\theta}) = \frac{1}{2\pi} \sum_{-n}^{n} \left( 1 - \frac{|k|}{n+1} \right) a_k e^{i\theta}.$$

**Proof.** The equivalence between (i) and (ii) is derived easily from the proof of Theorem 2.1 in [10]. Nevertheless, we provide a new proof. Indeed, for $f \in L^2(\partial \mathcal{A})$, $p \in (0, 1)$, one can check (see Lemma 2.6 of [17]), that

$$\int_{\partial \mathcal{A}} |\nabla \tilde{f}(\varphi_n)(z)|^2 (1 - |z|^2)^p \, dm(z)$$

$$\approx \int_{\partial \mathcal{A}} \int_{\partial \mathcal{A}} \frac{|f(\varphi_n(e^{i\theta})) - f(\varphi_n(e^{i\phi}))|^2}{|e^{i\theta} - e^{i\phi}|^{2-p}} \, d\theta \, d\phi$$

$$= \int_{\partial \mathcal{A}} \int_{\partial \mathcal{A}} \left( \frac{1 - |w|^2}{|1 - \overline{w} e^{i\theta}| |1 - \overline{w} e^{i\phi}|} \right)^p \, d\theta \, d\phi.$$  

The first estimate follows expressing the double integral in terms of the Fourier coefficients of $f$ and the area integral in terms of the power series of the analytic and antianalytic parts of $f$. See Lemma 2 in [20]. Let $I \subset \partial \mathcal{A}$ be any arc on $\partial \mathcal{A}$ and let $w \in \mathcal{A}$ satisfy $S(w) = S(I)$. Then

$$\frac{1}{|I|} \leq \frac{1 - |w|^2}{|1 - \overline{w} e^{i\theta}| |1 - \overline{w} e^{i\phi}|}, \quad e^{i\theta}, e^{i\phi} \in I$$

and hence (i) follows whenever (ii) is true.

Conversely, let $f \in Q_p(\partial \mathcal{A})$. For $w \in \mathcal{A}$, let $I \subset \partial \mathcal{A}$ be the arc such that $S(I) = S(w)$. Observe that if $e^{i\theta} \in 2^{k+1}I \setminus 2^k I$, $e^{i\phi} \in 2^{j+1}I \setminus 2^j I$, $k, j = 0, 1, ...$ then we have

$$\frac{1 - |w|^2}{|1 - \overline{w} e^{i\theta}| |1 - \overline{w} e^{i\phi}|} \leq \frac{1}{2^{j+k+1} |I|}.$$
Thus

\[
\int_A |\nabla (f \cdot \varphi_n)(z)|^2 \left(1 - |z|^2\right)^s \, d\mu(z)
\]

\[
\lesssim \sum_{j,k} \frac{1}{2^{j+k}} |I|^s \int_{2^j I} \int_{2^{j+1} I} \frac{|f(e^{\theta\phi}) - f(e^{\theta\psi})|^2}{|e^{\theta\phi} - e^{\theta\psi}|^{s+2}} \, d\theta \, d\psi
\]

\[
\lesssim \sum_j \sum_{k \geq 0} \frac{1}{2^{2k+j}} |I|^s \int_{2^j I} \int_{2^{j+1} I} |f(e^{\theta\phi}) - f(e^{\theta\psi})|^2 \, d\theta \, d\psi
\]

The first term is already bounded by \( \|f\|_{Q_p(\partial A)}^2 \). For the second, noting that \(|e^{\theta\phi} - e^{\theta\psi}| \geq 2^{k+1-j} |I| \) and then using a fact in [17] we get the bound

\[
\lesssim \left( \sum_j \frac{1}{2^j} \right) \|f\|_{BMO(\partial A)}^2.
\]

In the case that \( d\mu_p \) is a \( p \)-vanishing Carleson measure or \( f \in Q_p(\partial A) \), the argument only requires minor modifications and the proof is omitted here.

Now, let's turn to (i) \( \iff \) (iii), which is equivalent to saying that a complex sequence \( \{a_k\} \) is the Fourier coefficients of a function in \( Q_p(\partial A) \) (\( Q_p(\partial A) \)) if and only if \( \|a_n\|_{Q_p(\partial A)} = O(1) (o(1)) \) as \( n \to \infty \).

On the one hand, suppose that \( f \in Q_p(\partial A) \) with

\[
a_k = \frac{1}{2\pi} \int_0^{2\pi} f(e^{\theta\phi}) e^{-i\theta k} \, d\theta, \quad k = 0, \pm 1, \pm 2, \ldots
\]

So

\[
\sigma_n(e^{\theta\phi}) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i(\theta + \phi)}) K_n(e^{i\phi}) \, d\phi,
\]

where

\[
K_n(e^{i\phi}) = \sum_{k=-n}^n \left(1 - \frac{|k|}{n+1}\right) e^{i\phi} \geq 0
\]

is the Fejer kernel. The Minkowski inequality applied to \( \sigma_n \) yields

\[
\|\sigma_n\|_{Q_p(\partial A)} \leq \|f\|_{Q_p(\partial A)}.
\]
On the other hand, let \( \|\sigma_n\|_{Q_p(\partial A)} = O(1) \). Since \( Q_p(\partial A) \subset \text{BMO}(\partial A) \),
\[
\sup_n \|\sigma_n\|_{\text{BMO}(\partial A)} < \infty
\]
and hence for each \( n \), \( \sigma_n \) determines a bounded linear functional on \( H_{\mathbb{R}}^1 \), the real part of the Hardy space \( H^1 \), and so \( \{\sigma_n\} \) can be regarded as a sequence in a weakly compact subset of \( \text{BMO}(\partial A) \cong (H_{\mathbb{R}}^1)^* \). Consequently, there is a \( \sigma \in \text{BMO}(\partial A) \) which is a weak cluster point of \( \{\sigma_n\} \) and thus, if \( \epsilon > 0 \) and \( g \in H_{\mathbb{R}}^1 \), then for infinitely many \( n \), one has
\[
\left| \int_0^{2\pi} \sigma_n(e^{it}) \, g(e^{it}) \, dt - \int_0^{2\pi} \sigma(e^{it}) \, g(e^{it}) \, dt \right| < \epsilon.
\]
In particular, from \( g(e^{it}) = \cos k\theta, \sin k\theta \) respectively, it follows that
\[
\left| 2\pi \left(1 - \frac{|k|}{n+1}\right) a_k - \int_0^{2\pi} \sigma(e^{it}) \, e^{-ikt} \, dt \right| < \epsilon
\]
and then that \( \{a_k\} \) are the Fourier coefficients of \( \sigma \in \text{BMO}(\partial A) \). Next, we will further prove that \( \sigma \in Q_p(\partial A) \).

Again, using the assumption: \( \|\sigma_n\|_{Q_p(\partial A)} = O(1) \) we find a subsequence \( \{\sigma_{n_k}\} \) which converges to \( \sigma^* \) in \( L^2(\partial A) \). Moreover, applying Fatou’s Lemma to \( \|\sigma_{n_k}\|_{Q_p(\partial A)} = O(1) \) we have \( \sigma^* \in Q_p(\partial A) \). Now for any \( g \in L^2(\partial A) \), one has
\[
\left| \int_0^{2\pi} \left[ \sigma(e^{it}) - \sigma^*(e^{it}) \right] g(e^{it}) \, dt \right| \leq \left| \int_0^{2\pi} \left[ \sigma(e^{it}) - \sigma_{n_k}(e^{it}) \right] g(e^{it}) \, dt \right|
+ \left| \int_0^{2\pi} \left[ \sigma_{n_k}(e^{it}) - \sigma^*(e^{it}) \right] g(e^{it}) \, dt \right| \to 0
\]
as \( k \to \infty \),
and so \( \sigma = \sigma^* = f \) almost everywhere and then \( f \in Q_p(\partial A) \). The argument in the space \( Q_p(\partial A) \) is completely similar.

**Remark.** Let \( 0 < p < 1 \). Let \( f \) be a \( C^\infty \) function in a neighbourhood of the closed unit disk. One has
\[
\|f\|_{Q_p(\partial A)}^2 \lesssim \|\nabla f(z)\|^2 \left(1 - |z|^p \right) dm(z)\| \_p.
\]
Following the proof of (ii) ⇒ (i) in Theorem 2.1, one can show that it is sufficient to establish the following estimate

\[ \int_A |V(f \circ \varphi_w)(z)|^2 (1 - |z|^2)^p \, dm(z) \]

\[ \geq C \int_{\varphi \in A} \left( \frac{|f \circ \varphi_w(e^{i\theta}) - f \circ \varphi_w(e^{i\psi})|^2}{|e^{i\theta} - e^{i\psi}|^{2p}} \right) \, d\theta, \quad w \in A \]

(\text{where } C \text{ is an absolute constant}).

Writing \( g = f \circ \varphi_w \), it sufficies to verify

\[ \int_A |Vg(z)|^2 (1 - |z|^2)^p \, dm(z) \geq C \int_{r_1}^{r_2} \frac{1}{h^{2p}} \int_0^{2\pi} |g(e^{i(\theta + h)}) - g(e^{i\theta})|^2 \, d\theta \, dh. \]

Take \( r = 1 - h \). We have

\[ |g(e^{i(\theta + h)}) - g(e^{i\theta})| \leq |g(e^{i(\theta + h)}) - g(re^{i(\theta + h)})| + |g(re^{i(\theta + h)}) - g(e^{i\theta})| \]

\[ + |g(re^{i\theta}) - g(e^{i\theta})| \]

\[ \leq \int_{r_1}^{r_2} |Vg(re^{i(\theta + h)})| \, dt + \int_0^{h} |Vg(re^{i(\theta + s)})| \, ds \]

\[ + \int_{r_1}^{r_2} |Vg(te^{i\theta})| \, dt. \]

Apply Minkowski integral inequality \([18, \text{ p. } 271]\), to get

\[ \left( \int_{r_1}^{r_2} |g(e^{i(\theta + h)}) - g(e^{i\theta})|^2 \, d\theta \right)^{1/2} \]

\[ \leq 2 \left( \int_{r_1}^{r_2} \left( \int_0^{2\pi} |Vg(te^{i\theta})|^2 \, d\theta \right)^{1/2} \right)^{1/2} \]

\[ \times \left( \int_0^{2\pi} |Vg(re^{i\theta})|^2 \, d\theta \right)^{1/2} \]

\[ = (I) + (II). \]

Changing to planar coordinates one gets

\[ \int_{r_1}^{r_2} \frac{1}{h^{2p}} \left( \int_0^{2\pi} |Vg(re^{i\theta})|^2 \, d\theta \right) \, dh = \int_{r_1}^{r_2} \frac{1}{h^{2p}} \int_0^{2\pi} |Vg(re^{i\theta})|^2 \, d\theta \, dh \]

\[ \leq \int_A |Vg(z)|^2 (1 - |z|^2)^p \, dm(z). \]
For the term (1), put $x = 1 - t$ and apply Hardy’s inequality ([18, p. 272]) to obtain

$$\int_0^{1/2} \frac{1}{1-t} (I)^2 \, dt = 4 \int_0^{1/2} \frac{1}{1-t} \left[ \int_0^1 \left( \int_{\alpha}^{2\alpha} |Vg(te^{i\phi})|^2 \, d\phi \right)^{\frac{1}{2}} \, d\alpha \right]^2 \, dt$$

$$= 4 \int_0^{1/2} \frac{1}{1-t} \left[ \int_0^h \left( \int_{\alpha}^{2\alpha} |Vg((1-x)e^{i\phi})|^2 \, d\phi \right)^{\frac{1}{2}} \, d\alpha \right]^2 \, dh$$

$$\leq \int_0^{1/2} \frac{1}{1-t} \left[ \int_0^h \left( \int_{\alpha}^{2\alpha} |Vg((1-h)e^{i\phi})|^2 \, d\phi \right)^{\frac{1}{2}} \, d\alpha \right]^2 \, dh$$

$$\leq \int_0^1 |Vg(z)|^2 (1-|z|)^p \, dm(z).$$

This gives the Remark.

Also we remark that (i) $\Leftrightarrow$ (iii) still holds for BMO($\partial\Omega$) (VMO($\partial\Omega$)). Theorem 2.1 and the conformally invariant characterization of $p$-Carleson measures give the following result.

**Corollary 2.2.** Let $0 < p < 1$ and $f \in L^2(\partial\Omega)$. Then the quantities

$$\|f\|_{Q_p, 0, \Omega}^2 = \sup_{T \in \mathbb{A}} \frac{1}{[T]} \int_T \left[ \frac{|f(e^{i\theta}) - f(e^{i\phi})|^2}{|e^{i\theta} - e^{i\phi}|^{2-p}} \, d\theta \, d\phi \right]$$

$$A = \sup_{T \in \mathbb{A}} \frac{1}{[T]} \int_{S(T)} |\nabla f(w)|^2 (1-|w|^2)^p \, dm(w)$$

$$B = \sup_{T \in \mathbb{A}} \int_T \left( \frac{(1-|z|^2)}{|1-ar{w}z|^2} \right)^p |\nabla f(w)|^2 \, dm(w)$$

are comparable, that is, there exist constants $C_1, C_2, C_3 > 0$ independent of $f$ such that

$$\|f\|_{Q_p, 0, \Omega}^2 \leq C_1 A \leq C_2 B \leq C_3 \|f\|_{Q_p, 0, \Omega}^2.$$

Recall that VMO($\partial\Omega$) is the closure of trigonometric polynomials in BMO($\partial\Omega$). An analogous result holds for $Q_{p, 0}(\partial\Omega)$.

**Corollary 2.3.** Let $p \in (0, 1)$ and $f \in Q_p(\partial\Omega)$. Then the following conditions are equivalent:

...
(i) \( f \in Q_p(\partial \mathcal{D}) \).
(ii) \( \lim_{\theta \to 0} \| T \theta f - f \|_{Q_p(\partial \mathcal{D})} = 0 \), where \( T \theta f(\phi) = f(e^{i(\phi - \theta)}) \).
(iii) \( \lim_{\theta \to 1} \| T \theta f - f \|_{Q_p(\partial \mathcal{D})} = 0 \), where \( f(\phi) = f(\phi) \).

Proof. (i) \( \Rightarrow \) (ii). If \( f \in Q_p(\partial \mathcal{D}) \) and \( F = T \theta f - f \) then for any \( \varepsilon > 0 \) there is a \( \delta > 0 \) so that for all subarcs \( I \subset \partial \mathcal{D} \) with \( |I| < \delta \), one has

\[
\frac{1}{|I|^p} \int_I \left| \frac{f(e^{i\phi}) - f(e^{i\phi})}{|e^{i\phi} - e^{i\phi}|} \right|^2 d\phi < \varepsilon,
\]

and thus for any \( I \subset \partial \mathcal{D} \) with \( |I| < \delta \), one has

\[
\frac{1}{|I|^p} \int_I \left| \frac{F(e^{i\phi}) - F(e^{i\phi})}{|e^{i\phi} - e^{i\phi}|} \right|^2 d\phi < \varepsilon
\]

However, for any arc \( I \subset \partial \mathcal{D} \) with \( |I| \geq \delta \), applying Theorem 2.1 one gets

\[
\frac{1}{|I|^p} \int_I \left| \frac{F(e^{i\phi}) - F(e^{i\phi})}{|e^{i\phi} - e^{i\phi}|} \right|^2 d\phi
\]

\[
\leq \frac{1}{\delta^p} \int_0^{2\pi} \left| \frac{F(e^{i\phi}) - F(e^{i\phi})}{|e^{i\phi} - e^{i\phi}|} \right|^2 d\phi
\]

\[
\leq \frac{1}{\delta^p} \int I \left[ |\mathcal{F}(z)|^2 (1 - |z|^2) d|z| \right]
\]

\[
\leq \frac{1}{\delta^p} \int I \left[ |\mathcal{F}(z)|^2 (1 - |z|^2) d|z| \right] \to 0 \quad \text{as} \quad t \to 0.
\]

In other words, (ii) follows.

(ii) \( \Rightarrow \) (iii). Assuming that \( f \in Q_p(\partial \mathcal{D}) \) satisfies (ii), we use Minkowski inequality in

\[
f(e^{i\phi}) - f(e^{i\phi}) = \frac{1}{2\pi} \int_0^{2\pi} \left[ f(e^{i\phi}) - f(e^{i(\phi - \theta)}) \right] P_t(\phi) d\phi
\]
to get that for any small \( \varepsilon > 0 \),
\[
\|f - f_r\|_{Q_\rho(\partial A)} \leq \frac{1}{2\pi} \int_0^{2\pi} \|f - T_{\varphi} f\|_{Q_\rho(\partial A)} P_\lambda(\varphi) \, d\varphi \\
\leq \int_{|\varphi| < \varepsilon} \|f - T_{\varphi} f\|_{Q_\rho(\partial A)} P_\lambda(\varphi) \, d\varphi \\
+ \int_{|\varphi| \geq \varepsilon} \|f\|_{Q_\rho(\partial A)} P_\lambda(\varphi) \, d\varphi,
\]
which gives (iii)

(iii) \( \Rightarrow \) (i). This implication is obvious since \( Q_\rho(\partial A) \) is closed in \( Q_p(\partial A) \).

2.2. Inner Functions in \( Q_p \)

This paragraph is designed to discuss inner functions in \( Q_p(\partial A) \) for \( p \in (0,1) \). An inner function is a bounded analytic function on \( A \) whose radial limits have modulus 1 almost everywhere on \( \partial A \). Inner functions in \( Q_p, p \in (0,1) \) are described in [10] as the Blaschke products whose zeros \( \{z_n\} \) have the property that
\[
\sum_n (1 - |z_n|^2)^p \delta_n
\]
is a \( p \)-Carleson measure, where \( \delta_n \) is the Dirac measure at \( z_n \). Moreover, the only inner functions in \( Q_{p,0} \subset \text{VMOA}, p \in (0,1) \) are the finite Blaschke products. The later fact follows also from the equality
\[
\frac{1}{2\pi} \int_0^{2\pi} |B(e^{i\theta}) - B(z_j)|^2 P_{z_j}(\theta) \, d\theta = 1 - |B(z_j)|^2,
\]
which holds for any inner function \( B \) and any subarc \( I \subset \partial A \). Besides, all inner functions \( B \) have small mean variation on many arcs \( I \), namely, those such that \( |B(z_j)| \) is close to 1. The next property presents an analogue of this phenomenon for \( Q_p, p \in (0,1) \).

Theorem 2.4. Let \( p \in (0,1) \) and \( B \in Q_p \) be the Blaschke product with zeros \( \{z_n\} \). Then for any \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that whenever \( r > 1 - \delta \), one has
\[
\frac{1}{|I|^2} \int_{S(I)} |B'(z)|^2 (1 - |z|^2)^p \chi(z) \, dm(z) < \varepsilon,
\]
where $S(I)$ is the Carleson box based on the subarc $I \subset \partial A$ and $\chi_r$ is the characteristic function of the set $\{ z \in A : \inf_n \rho(z, z_n) \geq r \}$.

**Proof.** Let us first check the following fact.

**Claim.** Let $B$ be a Blaschke product in $Q_p$, $p \in (0, 1)$. If $\varepsilon > 0$, there exists $\delta > 0$ such that

$$ \log |B(z)|^{-2} < \varepsilon \quad \text{if} \quad \inf_n \rho(z, z_n) > 1 - \delta. $$

An elementary estimate gives that if $\inf_n \rho(z, z_n) \geq 1/2$ then there is a constant $C > 0$ such that

$$ \sum_n \frac{(1 - |z|^2)(1 - |z_n|^2)}{|1 - z_n|^2} \leq \log |B(z)|^{-2} \leq C \sum_n \frac{(1 - |z|^2)(1 - |z_n|^2)}{|1 - z_n|^2}. \quad (2.1) $$

Then the claim follows as long as we prove that

$$ \sum_n \frac{(1 - |z|^2)(1 - |z_n|^2)}{|1 - z_n|^2} \to 0 \quad \text{as} \quad r = \inf_n \rho(z, z_n) \to 1. \quad (2.2) $$

For this, it suffices to show that for any $\varepsilon > 0$ there exists $j(r) \to \infty$ (when $r \to 1$) so that

$$ \frac{1}{|2/I_r|} \sum_{z_n \in 2^j I_r} (1 - |z_n|^2) < \varepsilon, \quad j = 0, 1, \ldots, j(r). \quad (2.3) $$

Actually, since

$$ \sum_n (1 - |z_n|) \delta_{z_n} $$

is a $1$-Carleson measure, (2.3) gives (2.2) after one breaks the sum into dyadic blocks.

Observe that $z_n \in S(2/I_r)$ and $\rho(z, z_n) \geq r$, give

$$ \frac{1 - |z_n|^2}{4 \cdot 2^{2j}(1 - |z|^2)} \leq \frac{(1 - |z|^2)(1 - |z_n|^2)}{|1 - z_n|^2} \leq 1 - r^2, $$

that is,

$$ 1 - |z_n|^2 \leq 4 \cdot 2^{2j}(1 - r^2)(1 - |z|^2). $$

Now fix $j(r)$ so that $\varepsilon(r) = 4 \cdot 2^{j(r)}(1 - r^2) \to 0$ as $r \to 1$. Since $B \in Q_p$,

$$ \sum_n (1 - |z_n|^2)^r \delta_{z_n} $$
is a $p$-Carleson measure and consequently, for $0 \leq j \leq j(r)$, we have
\[
\sum_{z_n \in S(2/I_r)} (1 - |z_n|^2) \leq [2/L_r \epsilon(r)]^{1-p} \sum_{z_n \in S(2/I_r)} (1 - |z_n|^2)^p 
\leq [\epsilon(r)]^{1-p} [2/L_r] 
\]
which gives (2.3) and proves the claim.

A calculation with logarithmic derivatives gives
\[
\frac{B'(z)}{B(z)} = \sum_n \frac{1 - |z_n|^2}{(z - z_n)\overline{(1 - z_n \overline{z})}} 
\]
Hence
\[
|B'(z)| \leq \sum_n \frac{1 - |z_n|^2}{|1 - z_n \overline{z}|^2} \quad \text{ (2.4)}
\]
Using the claim we deduce that for any $\epsilon > 0$, there is an $r$ in $(0, 1)$ such that $1 - |B(z)|^2 \leq \epsilon$ if $\inf_{\rho} \rho(z, z_n) \geq r$. Hence, (2.4) gives
\[
\int_{S(I_r)} |B'(z)|^2 (1 - |z|^2)^p \chi(z) \, dm(z) 
\leq 2 \sum_n (1 - |z_n|^2) \int_{S(I)} \frac{(1 - |z|^2)^{p-1}}{|1 - z_n \overline{z}|^2} \, dm(z) 
\leq 2 \sum_n (1 - |z_n|^2) \int_{S(I)} \frac{(1 - |z|^2)^{p-1}}{|1 - z_n \overline{z}|^2} \, dm(z) 
\]
In the last line one may use the following formula. Let
\[
I(\gamma) = \int_{S(I)} \frac{(1 - |z|^2)^\gamma}{|1 - z \overline{w}|^2} \, dm(z), \quad \gamma > -1. \quad \text{ (2.5)}
\]
Then $I(\gamma) \leq 1$ if $\gamma > 0$, $I(0) \simeq \log(1 - |w|)^{-1}$, $I(\gamma) \simeq (1 - |w|)^\gamma$ if $-1 < \gamma < 0$. Since $\epsilon > 0$ is arbitrary, this completes the proof. [1]
The above Theorem 2.4 has an interesting consequence which is analogous to Lemma 3.3 in [19] and will be used to solve the interpolation problem in $Q_{p,0} \cap H^\infty$.

**Corollary 2.5.** Let $p \in (0, 1)$ and $B \in Q_p$ be a Blaschke product with zeros $\{z_n\}$. If $F \in Q_{p,0} \cap H^\infty$ satisfies $\lim_{n \to \infty} F(z_n) = 0$ then $BF \in Q_{p,0} \cap H^\infty$.

**Proof.** By the previous results, Theorems 2.1 and 2.4, we only have to show that

$$\frac{1}{|I|^p} \int_{S(I)} |F(z)|^2 |B'(z)|^2 (1 - |z|^2)^\rho [1 - \chi(z)] \ dm(z) \to 0$$

as $|I| \to 0$, where $\chi$ is still the function given in Theorem 2.4. Because $B \in Q_p$, the above integral can be bounded by

$$C \sup \left\{ |F(z)|^2 : z \in S(I), \inf_n \rho(z, z_n) \leq \rho \right\},$$

which tends to zero when $|I| \to 0$, because $\lim_{n \to \infty} F(z_n) = 0$ and $F \in Q_{p,0} \subseteq B_p$. Actually the little Bloch space consists on those analytic functions having vanishing oscillation on pseudohyperbolic disks with fixed radius [6].

**Remark.** Corollary 2.5 still holds for $p = 1$ if the sequence $\{z_n\}$ is $1$-thin. Actually, since

$$\inf_{z \in A} \left\{ |B(z)| : \inf_n \rho(z, z_n) \geq r \right\} \to 1$$

as $r \to 1$,

one has

$$\sup_{I \subseteq A} \frac{1}{|I|^p} \int_{S(I)} |B'(z)|^2 (1 - |z|^2)^\rho \chi(z) \ dm(z) \to 0$$

as $r \to 1$,

and one can mimic the last proof.

3. CORONA PROPERTY AND FEFFERMAN–STEIN DECOMPOSITION

In this section we use $p$-Carleson measures to solve $\bar{\partial}$-problems in $Q_p(\partial A) \cap L^p(\partial A)$. This leads to the Corona theorem in $Q_p \cap H^\infty$ and to the Fefferman–Stein decomposition in $Q_p(\partial A)$. 
3.1. $\overline{\partial}$-Problem in $Q_p(\partial \mathcal{D}) \cap L^\infty(\partial \mathcal{D})(Q_p,0(\partial \mathcal{D}) \cap L^\infty(\partial \mathcal{D}))$

Given a Carleson measure $\mu$ on the unit disk, it is well known (see Chapter III of [11]) that the $\overline{\partial}$-problem,

$$\overline{\partial}F = \mu,$$

has a solution $F$, in the sense of distributions, satisfying

$$\|F\|_{L^\infty(\partial \mathcal{D})} \leq C \|\mu\|_1.$$

P. Jones further found in [12] that such a solution $F$ can be given by an extremely simple and flexible formula,

$$F(z) = \int_A K \left( \frac{\mu}{\|\mu\|_1}, z, \zeta \right) d\mu(\zeta), \quad (3.1)$$

where

$$K \left( \frac{\mu}{\|\mu\|_1}, z, \zeta \right) = \frac{2i}{\pi} \frac{1 - |z|^2}{(1 - \overline{\zeta}z)(z - \zeta)} \exp \left[ \int_{|w|>|z|} \left( \frac{1 + \overline{w}\zeta}{1 - \overline{\zeta}w} - \frac{1 + w\zeta}{1 - \overline{w}z} \right) \frac{d\mu(w)}{\|\mu\|_1} \right].$$

The estimate

$$\int_A \left| K \left( \frac{\mu}{\|\mu\|_1}, e^{i\phi}, \zeta \right) \right| d\mu(\zeta) \leq C_1 \|\mu\|_1,$$

shows that $F \in L^\infty(\partial \mathcal{D})$. As a consequence, if $|g(z)| \, dm(z)$ is a 1-Carleson measure then $\overline{\partial}F = g$, has a solution $F \in L^\infty(\partial \mathcal{D})$. We want to find an analogous result for $Q_p(\partial \mathcal{D}) \cap L^\infty(\partial \mathcal{D})(Q_p,0(\partial \mathcal{D}) \cap L^\infty(\partial \mathcal{D}))$; that is to say, under which conditions on $g$ does the equation $\overline{\partial}F = g$ have a solution $F \in Q_p(\partial \mathcal{D}) \cap L^\infty(\partial \mathcal{D})(Q_p,0(\partial \mathcal{D}) \cap L^\infty(\partial \mathcal{D}))$? It is surely reasonable to observe in advance that $L^\infty(\partial \mathcal{D}) \neq Q_p(\partial \mathcal{D}) \neq Q_p(\partial \mathcal{D})$ for $p \in (0, 1)$, in fact, for example,

$$f(e^{i\theta}) = \sum_{n=0}^{\infty} 2^{n \|p\|_1/2} e^{i2^n \theta}$$

is in $C(\partial \mathcal{D}) \setminus Q_p(\partial \mathcal{D})$. See [3] where the power series with Hadamard gaps which are in $Q_p(\partial \mathcal{D})$ are characterized.

Our answer to the above question is:
Theorem 3.1. Let $p \in (0, 1)$. If $\delta \lambda(z) = |g(z)|^2 \left(1 - |z|^2\right)^p \, dm(z)$ is a $p$-Carleson measure ($p$-vanishing Carleson measure) then $\partial F = g$ has a solution $F \in Q_p(\partial A) \cap L^\infty(\partial A)$ ($Q_p,0(\partial A) \cap L^\infty(\partial A)$). Actually,

$$
\|F\|_{Q_p(\partial A)} + \|F\|_{L^\infty(\partial A)} \leq C \|\lambda\|_p^{1/2},
$$

where $C$ is an absolute constant.

Proof. We only give a proof for $Q_p(\partial A) \cap L^\infty(\partial A)$, because the proof for $Q_p,0(\partial A) \cap L^\infty(\partial A)$ is similar. Let $d\lambda(z) = |g(z)| \, dm(z)$; then $\mu$ is a $1$-Carleson measure. Actually,

$$
\mu(S(I)) = \int_{S(I)} d\mu(z) \leq \left[ \int_{S(I)} d\lambda(z) \right]^{1/2} \left[ \int_{S(I)} \left(1 - |z|^2\right)^{-p} \, dm(z) \right]^{1/2} \leq \|\lambda\|_p^{1/2} |I|.
$$

Thus the function $F$ given by (3.1) is in $L^\infty(\partial A)$ and $\partial F = g$. Next, our hope is to show that $F \in Q_p(\partial A)$. For this purpose, consider a new function $G$ on $\partial A$ which has the same boundary values on $\partial A$ as $zF$,

$$
G(z) = \frac{2\pi}{n} \int \frac{1 - |\xi|^2}{1 - \overline{z}\xi} \exp \left[ \int_{|\theta| > |z|} \left( \frac{1 + \overline{\theta}\xi}{1 - \overline{\theta}\xi} - \frac{1 + w\theta}{1 - w\theta} \right) \, dm(\theta) \right] \frac{|g(\xi)|}{\|\mu\|_1} \, dm(\xi).
$$

By the Remark after Theorem 2.1, one only has to show that $G$ satisfies

$$
\sup_{I \subset \partial A} \frac{1}{|I|^p} \int_{S(I)} |VG(z)|^2 \left(1 - |z|^2\right)^p \, dm(z) < \infty.
$$

(3.3)

Without loss of generality, let $g(z) \geq 0$ and $\|\mu\|_1 = 1$. Then,

$$
\text{Re} \left( \int_{|\theta| > |z|} \frac{1 + \overline{\theta}\xi}{1 - \overline{\theta}\xi} g(\theta) \, dm(\theta) \right) \leq 2 \int_{|\theta| > |z|} \frac{1 - |\theta|^2}{|1 - \overline{\theta}\xi|^2} g(\theta) \, dm(\theta) \leq C_2,
$$

(3.4)

where $C_2 > 0$ is a constant independent of $\xi \in A$. Moreover,

$$
\int_{|\theta| > |z|} \exp \left[ - \int_{|\theta| > |z|} \frac{1 - |\theta|^2}{|1 - \overline{\theta}\xi|^2} g(\theta) \, dm(\theta) \right] g(\xi) \, dm(\xi) \leq 1
$$

(3.5)
(see the proof of Lemma 2.1 in [12]). Using (3.4) and (3.5) one can show that
\[ |\nabla G(z)| \lesssim \int_{A} \frac{g(w)}{|1 - wz|^p} \, dm(w) \]
and hence that
\[
\int_{S(I)} |\nabla G(z)|^2 (1 - |z|^2)^p \, dm(z) \\
\lesssim \int_{S(I)} (1 - |z|^2)^p \left[ \int_{S(I)} \frac{G(w)}{|1 - wz|^2} \, dm(w) \right]^2 \, dm(z) \\
\lesssim \int_{S(I)} (1 - |z|^2)^p \left[ \int_{S(I)} \frac{g(w)}{|1 - wz|^2} \, dm(w) \right]^2 \, dm(z) \\
+ \int_{S(I)} (1 - |z|^2)^p \left[ \int_{A \setminus S(2I)} \frac{g(w)}{|1 - wz|^2} \, dm(w) \right]^2 \, dm(z) = (A) + (B). \]

For (A), we use Schur Lemma [25, p. 42]. Indeed, consider
\[ k(z, w) = \frac{(1 - |z|^2)^p (1 - |w|^2)^{-p/2}}{|1 - wz|^2} \]
and the integral operator \( T \) induced by the kernel \( k(z, w) \),
\[ (Tf)(z) = \int_{A} f(w) \, k(z, w) \, dm(w). \]
Taking \( z \in (-1, -p/2) \) and applying the formula (2.5), one gets
\[ \int_{A} k(z, w)(1 - |w|^2)^s \, dm(w) \lesssim (1 - |z|^2)^s \]
and
\[ \int_{A} k(z, w)(1 - |z|^2)^s \, dm(z) \lesssim (1 - |w|^2)^s. \]
Therefore the operator \( T \) is bounded from \( L^2(A) \) to \( L^2(A) \). Once letting
\[ f(w) = (1 - |w|^2)^{p/2} g(w) \chi_{S(2I)}(w), \]

\[ \int_{A} g(w) \chi_{S(2I)}(w) \, dm(w) \lesssim \int_{A} g(w) \chi_{S(2I)}(w) \, dm(w) \].
where $\chi_{S(2I)}$ denotes the characteristic function of $S(2I)$, we have

\[
(A) = \int_{S(2I)} \left[ \int_{\mathbb{D}} f(w) k(z, w) \, dm(w) \right]^2 \, dm(z) \lesssim \int_{\mathbb{D}} |f(z)|^2 \, dm(z)
\]

\[
= \int_{S(2I)} |g(z)|^2 (1 - |z|^2)^\nu \, dm(z) \lesssim |I|^\nu.
\]

We estimate (B) using dyadic blocks,

\[
(B) \lesssim \int_{S(I)} (1 - |z|^2)^\nu \left[ \sum_{n=1}^{\infty} \int_{S(2^n+1)I \setminus S(2^n I)} \frac{g(w)}{|1 - \overline{w} z|^2} \, dm(w) \right]^2 \, dm(z)
\]

\[
\lesssim \int_{S(I)} (1 - |z|^2)^\nu \left[ \sum_{n=1}^{\infty} \mu(S(2^n+1)I) \right] \frac{1}{(2^n |I|^2)} \, dm(z) \lesssim |I|^\nu.
\]

These estimates on (A) and (B) imply (3.3).

3.2. Corona Theorem for $Q_p \cap H^\infty(Q_{p,0} \cap H^\infty)$

The first application of Theorem 3.1 is the Corona theorem for $Q_p \cap H^\infty(Q_{p,0} \cap H^\infty)$.

Proof of Theorem 1.1. From a normal family argument, we can assume that the given functions $f_1, \ldots, f_n$ are analytic on a neighbourhood of the closed unit disk and we are forced to find functions $g_1, \ldots, g_n \in Q_p \cap H^\infty(Q_{p,0} \cap H^\infty)$ satisfying

\[
\sum_{k=1}^{n} f_k g_k \equiv 1.
\]

It is clear that

\[
h_j(z) = \frac{|f_j(z)|^2}{\sum_{k=1}^{n} |f_k(z)|^2}
\]

are nonanalytic functions making

\[
\sum_{k=1}^{n} f_k h_k \equiv 1.
\]

As in the case of $H^\infty$ (see Chapter VIII of [11]), to replace $h_k$ by functions in $Q_p \cap H^\infty$ one needs to solve the equations below,

\[
\tilde{\partial} b_{j,k} = h_j \tilde{\partial} h_k, \quad 1 \leq j, \ k \leq n
\]
in $Q_p(\partial A) \cap L^\infty(\partial A)$ ($Q_p,0(\partial A) \cap L^\infty(\partial A)$). It is sufficient to deal with an equation $\partial b = h$ where $h = h_j \partial L$. An easy calculation shows

$$|h(z)| \leq \sum_{j=1}^{n} |f_j(z)|.$$ 

So by Theorem 2.1, $|h(z)|^2 (1 - |z|^2)^p \, dm(z)$ is a $p$-Carleson measure ($p$-vanishing Carleson measure). Using Theorem 3.1 there is a solution $b \in Q_p(\partial A) \cap L^\infty(\partial A)$ ($Q_p,0(\partial A) \cap L^\infty(\partial A)$), $\partial b = h$ such that

$$\|b\|_{Q_p(\partial A)} + \|b\|_{L^\infty(\partial A)} \leq C \sum_{j=1}^{n} \|f_j\|_{Q_p}.$$ 

Remark. As mentioned in the introduction, the Corona Theorem for $Q_{1,0} \cap H^\infty = \text{VMO} \cap H^\infty$ was proved in [19]. It also holds for $Q_{p,0} \cap H^\infty$, $p > 1$, that is, for the algebra of bounded analytic functions in the little Bloch space. Actually, given $f_1, ..., f_n \in B_0 \cap H^\infty$, satisfying

$$\inf_{z \in A} (|f_1(z)| + \cdots + |f_n(z)|) = \delta > 0,$$

one can find $\varepsilon = \varepsilon(\delta) > 0$ and $\Psi_j \in C^\infty(A)$, $j = 1, ..., n$ such that

$$\Psi_j(z) = 1 \text{ if } |f_j(z)| > \delta/n,$$

$$\Psi_j(z) = 0 \text{ if } |f_j(z)| < \delta/n,$$

and $|\nabla \Psi_j(z)| \, dm(z)$ is a Carleson measure (see [11, p. 342]). Moreover since $f_j \in B_0$, one can also assume

$$(1 - |z|) \, |\nabla \Psi_j(z)| \to 0 \quad \text{as } |z| \to 1.$$

Set

$$\varphi_j = \frac{\Psi_j}{f_j \sum \Psi_j},$$

and one has

$$f_1 \varphi_1 + \cdots + f_n \varphi_n \equiv 1.$$ 

Observe that $\varphi_j$ are bounded, $|\nabla \varphi_j(z)| \, dm(z)$ is a Carleson measure and

$$(1 - |z|) \, |\nabla \varphi_j(z)| \to 0 \quad \text{as } |z| \to 1.$$
To replace \(\varphi_j\) by functions in \(H^\infty \cap B_0\) one has to solve the equations

\[
\frac{\partial b_{j,k}}{\partial z} = \varphi_j \frac{\partial \varphi_k}{\partial z}, \quad 1 \leq j \leq n
\]

\[
|b_{j,k}|_{L^1(\mathbb{D},\mu)} < +\infty
\]

$$\sup_{|z| \leq r} \{|b_{j,k}(z) - b_{j,k}(w)| : \rho(z,w) \leq 1/2\} \to 0 \quad \text{as} \quad r \to 1.$$

For this one can consider the P. Jones solution (3.1) and observe

\[
\int_{|\zeta - z| < 1/2} \frac{1 - |\zeta|^2}{|\zeta - z|} K(\zeta, z) \frac{\partial \varphi_k}{\partial z}(\zeta) \left| \frac{\partial \varphi_k}{\partial z}(\zeta) \right| dm(\zeta) \to 0 \quad \text{as} \quad |z| \to 1
\]

\[
\int_{|\zeta - z| < 1/2} \frac{1 - |\zeta|^2}{|\zeta - z|^3} \varphi_j(\zeta) \left| \frac{\partial \varphi_j}{\partial z}(\zeta) \right| dm(\zeta) \to 0 \quad \text{as} \quad |z| \to 1.
\]

3.3. Fefferman–Stein Decomposition for \(Q_p(\partial \mathbb{D})(Q_p,0(\partial \mathbb{D}))\)

The second application of Theorem 3.1 is a decomposition of \(Q_p(\partial \mathbb{D})(Q_p,0(\partial \mathbb{D}))\) similar to the Fefferman–Stein decomposition of \(\text{BMO}(\partial \mathbb{D})\) (VMO(\partial \mathbb{D})).

Proof of Theorem 1.2. Denoting by \(\bar{X}\) the conjugate space of a given space \(X\), we will show that

(a) \(Q_p(\partial \mathbb{D}) = Q_p(\partial \mathbb{D}) \cap L^\infty(\partial \mathbb{D}) + \{Q_p(\partial \mathbb{D}) \cap L^\infty(\partial \mathbb{D})\}^\perp\)

(b) \(Q_p,0(\partial \mathbb{D}) = Q_p,0(\partial \mathbb{D}) \cap C(\partial \mathbb{D}) + \{Q_p,0(\partial \mathbb{D}) \cap C(\partial \mathbb{D})\}^\perp\)

First of all, we show (a). On the one hand, if \(f = u + \bar{\nu}, u, \nu \in Q_p(\partial \mathbb{D}) \cap L^\infty(\partial \mathbb{D})\), then from Corollary 3.2 in [10] it follows that \(\bar{\nu} \in Q_p(\partial \mathbb{D})\) and hence \(f \in Q_p(\partial \mathbb{D})\).

On the other hand, suppose that \(f \in Q_p(\partial \mathbb{D})\), is real-valued and \(\hat{f}(0) = 0\). We find immediately that \(F = f + \bar{\nu} \in Q_p(\partial \mathbb{D})\) and its Poisson extension \(\tilde{F} \in Q_p\). From Theorem 2.1, one has that \(|\tilde{F}(z)|^2 (1 - |z|^2)^p dm(z)\), and then \(|f(z)|^2 (1 - |z|^2)^p dm(z)\) are \(p\)-Carleson measures. Let \(df(z) = \tilde{f}(z) dm(z)\) and let \(f(z)\) be the function given by Theorem 3.1; then \(\partial f_u = \mu\) and \(f_u \in Q_p(\partial \mathbb{D}) \cap L^\infty(\partial \mathbb{D})\). Hence \(g = f - f_u\) is analytic and \(g \in Q_p\). Put \(u = \text{Re} f_u\), then \(\bar{u} = -\text{Im} g\). So \(f = u + \bar{\nu}\), where \(u = \text{Re} f_u\) and \(\nu = -\text{Im} g\) belong to \(Q_p(\partial \mathbb{D}) \cap L^\infty(\partial \mathbb{D})\). The estimates on \(u, \nu\) in Theorem 1.2 follow from the bound on \(f_u\) in Theorem 3.1.

Next we show (b). This part may be seen as a by-product of (a) and Corollary 2.3. It suffices to check that \(Q_p,0(\partial \mathbb{D})\) is a subset of the righthand
set in (b). To this end, let $f \in Q_p(\partial A)$ with $\tilde{f}(0) = 0$. From (a) it follows that there are $g_1, g_2 \in Q_p(\partial A) \cap L^\infty(\partial A)$ satisfying $f = g_1 + \tilde{g}_2$ with

$$
\|g_j\|_{L^\infty(\partial A)} + \|g_j\|_{Q_p(\partial A)} \leq C_3 \|f\|_{Q_p(\partial A)}, \quad j = 1, 2,
$$

where $C_3 > 0$ is a constant independent of $f$ and $g_j$. Since $f \in Q_p(\partial A)$, by Corollary 2.3 it follows that there is an $r \in (0, 1)$ satisfying

$$
\|f\|_{Q_p(\partial A)} \leq \frac{\|f\|_{Q_p(\partial A)}}{2}.
$$

Also, let $f^{(1)}_1 = (g_1), (j = 1, 2)$, which obviously are in $Q_p(\partial A) \cap C(\partial A)$, and then let $f_1 = f^{(1)}_1 + f^{(2)}_2$. So

$$
\|f - (f^{(1)}_1 + f^{(2)}_2)\|_{Q_p(\partial A)} \leq \frac{\|f\|_{Q_p(\partial A)}}{2};
$$

consequently, $f_1 = f - (f^{(1)}_1 + f^{(2)}_2) = g_1 - f^{(1)}_1 + \tilde{g}_2 / f^{(2)}_2 \in Q_p(\partial A)$ with

$$
\|f_1\|_{Q_p(\partial A)} \leq \frac{\|f\|_{Q_p(\partial A)}}{2} \quad \text{and} \quad \|g_j - f^{(1)}_1\| \leq C_3 \|f\|_{Q_p(\partial A)}.
$$

Repeating the above argument with $f_1$, and iterating, we have $f = u + \tilde{v}$ where

$$
u = \sum_{k=1}^{\infty} f^{(k)}_1 \quad \text{and} \quad v = \sum_{k=1}^{\infty} f^{(k)}_2
$$

belong to $Q_p(\partial A) \cap C(\partial A)$. Hence (b) is proved.

Remarks. (i) In the above proof, we use Theorem 3.1 to deduce Theorem 1.2. Conversely, Theorem 1.2 can be used to deduce Theorem 3.1 as well. For instance, assume that

$$Q_p(\partial A) = Q_p(\partial A) \cap L^\infty(\partial A) \cap [Q_p(\partial A) \cap L^\infty(\partial A)]^{-}.$$

If $d\lambda(z) = |g(z)|^2 (1 - |z|^2)^p \ dm(z)$ is a $p$-vanishing Carleson measure then we wish to show that $\partial F = g$ has a solution $F$ in $Q_p(\partial A) \cap L^\infty(\partial A)$. For this, let

$$F_1(z) = \frac{1}{\pi} \int_A \frac{g(w)}{z - w} \ dm(w).$$

Then $\partial F_1 = g$ and $\partial F_1 = B(g)$ is the Beurling transform of $g$. We now argue as in [14]. Since $|1 - |z|^2|^p$ is an $A_p$-weight for $p \in (0, 1)$, $B(f)$ is a
bounded linear operator from $L^2(A, (1-|z|^2)^p \, dm(z))$ to itself, [11, Chapter VI], and consequently,

\[
\int_{S(2I)} |\partial F_1(z)|^2 (1-|z|^2)^p \, dm(z) \\
\leq \int_{S(2I)} |B(g_{S(2I)}(z))|^2 (1-|z|^2)^p \, dm(z) \\
+ \int_A |B(g(1-Z_{S(2I)}))|^2 (1-|z|^2)^p \, dm(z) \\
\leq \int_{S(2I)} |g(z) Z_{S(2I)}(z)|^2 (1-|z|^2)^p \, dm(z) \\
+ \int_{S(2I)} \left( \int_A |g(w)| \, dm(w) \right)^2 (1-|z|^2)^p \, dm(z) \\
\leq \int_{S(2I)} |g(z)|^2 (1-|z|^2)^p \, dm(z) \\
+ \int_{S(2I)} \left( \sum_{k=1}^N + \sum_{k=N+1}^\infty \frac{f(S(2k+1))] (2^k |1|)^2 \right)^2 (1-|z|^2)^p \, dm(z) \\
= o(|I|^p) \quad \text{as} \quad |I| \to 0,
\]

where $Z_{S(2I)}$ is the characteristic function on $S(2I)$, and $d\mu(z) = |g(z)| \, dm(z)$. Together with $\partial F_1 = g$, the estimate of $\partial F_1$ gives that $F_1 \in Q_{p,0}(\partial A)$ and also by Theorem 1.2 it yields $u \in Q_{p,0}(\partial A) \cap L^\infty(\partial A)$ such that $F_1 = u + \bar{v}$. Setting $F = F_1 + i(v + iv) = u + iv$, we obtain $\partial F = f$ with $F \in Q_{p,0}(\partial A) \cap L^\infty(\partial A)$.

(ii) Let $P$ be the Riesz projection from $L^2(\partial A)$ onto $H^2$. Then Theorem 1.2 is equivalent to $P(Q_{p,0}(\partial A) \cap L^\infty(\partial A)) = Q_{p,0}$, $P(Q_{p,0}(\partial A) \cap C(\partial A)) = Q_{p,0}$. The corresponding fact for $BMO(\partial A)$ ($VMO(\partial A)$) can be found in [25, p. 79-186].

(iii) It follows easily that an analytic function $f$ in the unit disk belongs to $Q_{p,0}(\partial A)$ if and only if $f = f_1 + if_2$, where $f_i$ are analytic functions such that $\text{Ref}_i \in Q_{p,0}(\partial A) \cap L^\infty(\partial A)$ ($Q_{p,0}(\partial A) \cap \mathcal{F}(\partial A)$) $i = 1, 2$.

4. INTERPOLATING SEQUENCES

In this section, which will occupy the rest part of this paper, we will prove Theorems 1.3 and 1.4.
4.1. Necessary Conditions

The necessity parts in both cases will follow from the same argument, which combines Khinchin’s inequality and a reproducing formula for $D_p$, $p > 0$. To be precise, given finitely many complex numbers $w_1, \ldots, w_n$, consider the $2^n$ possible sums

$$\sum_{j=1}^{n} \pm w_j$$

obtained as the plus-minus signs vary in the $2^n$ possible ways. For $q > 0$ we use

$$\mathcal{E} \left( \left| \sum_{j=1}^{n} \pm w_j \right|^q \right)$$

to denote the average value of

$$\left| \sum_{j=1}^{n} \pm w_j \right|^q$$

over the $2^n$ choices of sign. Khinchin’s inequality states an estimate on the expectation below,

$$\mathcal{E} \left( \left| \sum_{j=1}^{n} \pm w_j \right|^q \right) \leq C(q) \left( \sum_{j=1}^{n} |w_j|^2 \right)^{q/2}$$

(4.1)

(see [11, p. 302]). Actually $C_q = 1$ if $q \leq 2$. This inequality will be used in the reproducing formula for $D_p$. The reproducing formula in [16] asserts that for $f \in D_p$, one has

$$f(z) = f(0) + \int_{A} f'(w) K(z, w)(1-|w|^2)^p dm(w), \quad z \in A,$$

(4.2)

where

$$K(z, w) = \frac{(1-\overline{z}w)^{1+p} - 1}{w(1-\overline{z}w)^{1+p}}.$$

Proof of the Necessity in Theorems 1.3 and 1.4. Let $0 < p \leq 1$ and assume that $\{z_n\}$ is an interpolating sequence for $Q_p \cap H^\infty$. Then for
\[ \epsilon_{ij}^{(k)} = \pm 1, \quad j = 1, \ldots, 2^n, \quad k = 1, \ldots, n \], there are \( f_j \in Q_p \cap H^\infty \) such that

\[ \|f_j\|_{H^\infty} + \|f_j\|_{Q_p} \leq C_4, \]

where \( C_4 > 0 \) is an absolute constant. Applying (4.2) to \( f_j \cdot \varphi_n \) at \( \varphi_n(z_k) \) we get

\[ f_j(z_k) = f_j(w) + \int_d (f_j \cdot \varphi_n)'(\xi) K(\varphi_n(z_k), \xi)(1 - |\xi|^2)^{p} \, dm(\xi). \]

We have

\[ \sum_{k=1}^{n} (1 - |\varphi_n(z_k)|^2)^p \]

\[ = \sum_{k=1}^{n} \epsilon_{ij}^{(k)} f_j(z_k)(1 - |\varphi_n(z_k)|^2)^p \]

\[ = f_j(w) \sum_{k=1}^{n} \epsilon_{ij}^{(k)}(1 - |\varphi_n(z_k)|^2)^p \]

\[ + \int_d (f_j \cdot \varphi_n)'(\xi) \left[ \sum_{k=1}^{n} \epsilon_{ij}^{(k)} K(\varphi_n(z_k), \xi)(1 - |\varphi_n(z_k)|^2)^p \right] (1 - |\xi|^2)^p \, dm(\xi) \]

\[ = (A) + (B). \]

We will compute the expectation of both sides of this equality. Observe that by (4.1) with \( q = 1 \) we find

\[ \mathcal{E}(A) \leq C_4 \left[ \sum_{k=1}^{n} (1 - |\varphi_n(z_k)|^2)^2 \right]^{1/2} \leq C_4 \left[ \sum_{k=1}^{n} (1 - |\varphi_n(z_k)|^2)^p \right]^{1/2}. \] (4.4)

Also, applying Hölder's inequality and (4.1) with \( q = 2 \), we get

\[ \mathcal{E}(B) \leq \sup_j \|f_j \cdot \varphi_n\|_{H^1} \]

\[ \left[ \int_d \left| K(\varphi_n(z_k), \xi) \right|^2 \left( 1 - |\varphi_n(z_k)|^2 \right)^{2p} (1 - |\xi|^2)^p \, dm(\xi) \right]^{1/2} \leq C_5 \left[ \sum_{k=1}^{n} (1 - |\varphi_n(z_k)|^2)^2 \right]^{1/2} \]

\[ \leq C_5 \left[ \int_d \left( 1 - |\xi|^2 \right)^{2p} \, dm(\xi) \right]^{1/2} \]

\[ \leq C_5 \left[ \sum_{k=1}^{n} (1 - |\varphi_n(z_k)|^2)^p \right]^{1/2}. \] (4.5)
So, putting (4.4) and (4.5) in (4.3) we see that (b) of Theorem 1.3 holds. Since \( \{z_n\} \) is an interpolating sequence for \( H^\infty \), (a) also holds.

If \( \{z_n\} \) is an interpolating sequence for \( Q_{p,0} \cap H^\infty \subset \text{VMO} \cap H^\infty \), it is easy to show (a) in Theorem 1.4 holds (see [19] or the Remark below). For (b), we may suppose that \( \lim_{|z| \to 1} \sup_j \|f_j \circ \varphi_n\|_{\mathcal{C}_p} = 0 \); then (4.3)–(4.5) yield

\[
\sum_{k : \{z_n, z_m\} \cap \mathcal{C}_p \neq \emptyset} (1 - |\varphi_n(z_k)|^2)^p \leq C \left[ (1 - r^2)^p + \sup_j \|f_j \circ \varphi_n\|_{\mathcal{C}_p} \right]^{1/2}
\]

which implies (b).

**Remark.** For \( Q_{p,0} \cap H^\infty \), \( p \in (1, \infty) \), we can obtain a better necessary condition, that is, \( \{z_n\} \) is 1-thin. Here, we only have to check that \( \{z_n\} \) is 1-thin when \( \{z_n\} \) is an interpolating sequence for \( B_0 \cap H^\infty \). Using Bloch’s theorem we can easily show that \( f \in \mathcal{B}_0 \) if it has vanishing variation in pseudohyperbolic disks of fixed radius, that is, for fixed \( r \in (0, 1) \),

\[
\sup \{|f(z) - f(w)| : |z| = |w|, |z| > s\} \to 0 \quad \text{as} \quad s \to 1
\]

(see [6]). So if \( f \in Q_{p,0} = B_0 \) and \( r \in (0, 1) \) is fixed we have

\[
|f(z_n) - f(z_m)| \leq \varepsilon(n, m), \quad \rho(z_n, z_m) < r,
\]

where \( \varepsilon(n, m) \to 0 \) as \( n, m \to \infty \). Hence, if one wants to interpolate any bounded sequence at the points \( \{z_n\} \), it follows that

\[
\inf_{m \neq n} \rho(z_m, z_n) \to 1 \quad \text{as} \quad n \to \infty.
\]

It remains to check (b). It is not difficult to show (see [19]) that \( \{z_n\} \) is 1-thin if and only if given any integer \( N > 0 \), one has

\[
\frac{1}{(1 - |z_n|)} \sum_{z_m \in S(Nz)} (1 - |z_m|) \to 0 \quad \text{as} \quad n \to \infty.
\]

(4.7)

Let \( f = f_n \in Q_{p,0} \cap H^\infty \) with \( f(z_n) = 1, \ f(z_m) = 0, \ m \neq n \). Given \( k > 0 \), consider the arcs

\[
L_k = \{re^{i\theta} : r = |z_k|, |\theta - \arg z_k| \leq 1 - |z_k|\}.
\]

Since \( f \in B_0 \), given \( \varepsilon > 0 \) there exists \( n_0 > 0 \) such that

\[
\zeta_k = \zeta_k(n) = \sup_{z \in L_k} \frac{|f(z)|}{\|f\|_{H^\infty}} < \varepsilon \quad \text{if} \quad k \neq n, \ k > n_0.
\]
From harmonic majorization [22, p. 302] it follows that
\[
\log \left( \frac{|f(z_n)|}{\|f\|_{H^\infty}} \right) \leq \sum_{k \neq n} \omega(z_n, L_k, A \setminus L_k) \log \xi_k,
\] (4.8)
where \( \omega(z, L, A \setminus L) \) means the harmonic measure at \( z \) of \( L \) in the domain \( A \setminus L \). If (4.7) is not satisfied then there would exist \( \eta > 0 \) such that
\[
\omega \left( z_n, \bigcup_{k \neq n} L_k, A \setminus \bigcup_{k \neq n} L_k \right) \geq \eta
\]
for infinitely many \( n \). So (4.8) would imply
\[
\frac{|f(z_n)|}{\|f\|_{H^\infty}} < e^\eta,
\]
which is a contradiction.

Although we have handled the necessity for \( Q_p \cap H^\infty \) and \( Q_{p,0} \cap H^\infty \) with the same idea, we are forced to prove the sufficiency individually.

4.2. Sufficient Condition for Interpolation in \( Q_p \cap H^\infty \)

Proof of sufficiency in Theorem 1.3. Assume that \( \{z_n\} \) is \( p \)-uniformly separated and let
\[
0 < \delta = \inf_{m \neq n} \rho(z_m, z_n).
\]
Given a bounded sequence \( \{w_n\} \) of complex numbers, take a function \( \varphi \in C^\infty(A) \) with the following properties:
(i) \( \varphi(z) \equiv w_n \) if \( \rho(z, z_n) \leq \delta / 4 \);
(ii) \( \varphi(z) \equiv 0 \) if \( \inf_{n} \rho(z, z_n) \geq \delta / 2 \);
(iii) \( \sup_{z \in A} (1 - |z|^2) |\nabla \varphi(z)| < \infty \).

Let \( B \) be the Blaschke product with zeros \( \{z_n\} \). So,
\[
B \in Q_p \quad \text{and} \quad \left( |\nabla \varphi(z)|/|B(z)| \right)^2 (1 - |z|^2)^p \, dm(z)
\]
must be a \( p \)-Carleson measure. Hence from Theorem 3.1 it follows that there exists \( b \in Q_p(\partial A) \cap L^\infty(\partial A) \) solving the equation: \( \overline{\partial} b = \overline{\partial} \varphi / B \). So, \( f = \varphi - Bb \in Q_p \cap H^\infty \) and \( f(z_n) = w_n \) for \( n = 1, 2, ... \).

Remarks. Here we would like to point out that J. Earl's [9] and P. Jones' [12] constructive proofs for \( H^\infty \)-interpolation are suitable for our situation, that is to say, their solutions are in \( Q_p \cap H^\infty \) if \( \{z_n\} \) is \( p \)-uniformly separated.
In a similar way one can prove that \( \{ z_n \} \) is an interpolating sequence for \( \partial_p \cap H^\infty \) if and only if \( \{ z_n \} \) is uniformly separated and \( \sum (1 - |z_n|)^p < \infty \).

In order to give an argument for sufficiency in the case of \( Q_p \cap H^\infty \), we require some further information on \( p \)-thin sequences and non-analytic interpolation solutions.

4.3. \( p \)-Vanishing Carleson Measured and \( p \)-Thin Sequences

The following three auxiliary lemmas are similar to results given in [19] but they are proved by different methods.

**Lemma 4.1.** Let \( \mu \) be a \( p \)-vanishing Carleson measure, \( p \in (0, 1] \). Then there exists a positive function \( f \) on \( \mathbb{D} \) so that \( \lim_{|z| \to 1} f(z) = 0 \) and \( f(z) \, \mu(z) \) is a \( p \)-vanishing Carleson measure.

**Proof.** For \( z, w \in \mathbb{D} \), define

\[
F(z, w) = \left[ \int_{|\xi| > |z|} \left( \frac{1 - |w|^2}{|1 - \overline{w}\xi|^2} \right)^p \, d\mu(\xi) \right]^{-1/2}
\]

and

\[
f(z) = \inf_{w \in \mathbb{D}} F(z, w).
\]

Since

\[
\lim_{|z| \to 1} \int_{|\xi| > |z|} \left( \frac{1 - |w|^2}{|1 - \overline{w}\xi|^2} \right)^p \, d\mu(z) = 0,
\]

one has \( f(z) \to 0 \) as \( |z| \to 1 \). Using the identity

\[
\int_0^1 g(\lambda) \left[ \int_0^1 g(t) \, dt \right]^{-1/2} \, d\lambda = 2 \left[ \int_0^1 g(t) \, dt \right]^{1/2}, \quad 0 \leq g \in L^1(0, 1),
\]

one deduces

\[
\int_{|\xi| > |z|} \left( \frac{1 - |w|^2}{|1 - \overline{w}\xi|^2} \right)^p \, d\mu(z) \leq 2 \left[ \int_0^1 \left( \frac{1 - |w|^2}{|1 - \overline{w}\xi|^2} \right)^p \, d\mu(z) \right]^{1/2} \to 0 \quad \text{as} \quad |w| \to 1.
\]

Hence, Lemma 4.1 is proved.

The next result provides a characterization of \( p \)-thin sequences in terms of \( p \)-Carleson measures.

**Lemma 4.2.** Let \( p \in (0, 1] \) and \( \{ z_n \} \) be a sequence of points in the unit disc. Then the following conditions are equivalent:

1. \( \{ z_n \} \) is \( p \)-thin in \( \mathbb{D} \).
2. There exists a positive function \( f \) on \( \mathbb{D} \) so that \( \lim_{|z| \to 1} f(z) = 0 \) and \( f(z) \, \mu(z) \) is a \( p \)-vanishing Carleson measure.
(i) \( \{ z_n \} \) is \( p \)-thin;
(ii) \( \lim_{n \to \infty} \inf_{r \geq 1} \rho(z_n, z_n) = 1 \) and
\[
\lim_{r \to 1} \sup_{I \in \mathcal{D}} \frac{1}{|I|^p} \sum_{z_n \in \Sigma(I), \rho(z_n, z_I) \geq r} (1 - |z_n|^2)^p = 0.
\]

Proof. (i) \( \Rightarrow \) (ii). Let \( I \) be an arc of the unit circle and \( w = z_I \). Given \( \epsilon > 0 \), if \( 0 < 1 - r \) is sufficiently small, one has
\[
\epsilon > \sum_{n: \rho(z_n, w) \geq r} (1 - |\varphi_w(z_n)|^2)^p
\geq \sum_{z_n \in \Sigma(I)} \left( \frac{(1 - |w|^2)(1 - |z_n|^2)^p}{|1 - wz_n|^2} \right)
\geq C |I|^{-p} \sum_{z_n \in \Sigma(I)} (1 - |z_n|^2)^p.
\]

(ii) \( \Rightarrow \) (i). Given \( w \in \mathcal{D} \) let \( I \in \mathcal{D} \) be the arc such that \( w = z_I \) and \( \xi_k \) be the point associated to the arc \( 2^k I \). Observe that \( \rho(z_n, w) \geq r \) implies that there exists \( j = j(r) \to \infty \) (as \( r \to 1 \)) such that
\[
\inf_{k \leq j} \rho(z_n, \xi_k) = \rho(r) \to 1 \quad \text{as} \quad r \to 1.
\]
Since \( \sum (1 - |z_n|^2)^p \delta_{z_n} \) is a \( p \)-Carleson measure, we get
\[
\sum_{n: \rho(w, z_n) \geq r} (1 - |\varphi_w(z_n)|^2)^p \leq 2 |I|^{-p} \sum_{z_n \in \Sigma(I)} (1 - |z_n|^2)^p
+ 2 \sum_{k=1}^{\infty} (2^k |I|)^{-p} \sum_{z_n \in \Sigma(2^k I), \rho(z_n, \xi_k) \geq \rho(r)} (1 - |z_n|^2)^p
+ 2 \sum_{k=j+1}^{\infty} (2^k |I|)^{-p} \sum_{z_n \in \Sigma(2^k I)} (1 - |z_n|^2)^p \to 0
\]
as \( r \to 1 \).

So, (i) holds. \( \blacksquare \)

Remarks. (i) In the case \( p = 1 \), the conditions of Lemma 4.2 are equivalent to
\[
\sum_{m \neq n} \frac{(1 - |z_m|^2)(1 - |z_n|^2)}{|1 - z_m z_n|^2} \to 0 \quad \text{as} \quad n \to \infty \quad (4.9)
\]
(see [19]). However the corresponding condition, in the case $p \in (0, 1)$,
\[
\sum_{m \neq n} \left( \frac{|1 - |z_m|^2|1 - |z_n|^2|}{|1 - z_m z_n|^2} \right)^p \to 0 \quad \text{as } n \to \infty, \tag{4.10}
\]
is necessary but not sufficient for $\{z_n\}$ to be $p$-thin. The necessity is easily seen taking $w = z_n$ in the definition of $p$-thin sequence. To check that it is not sufficient we construct counterexamples.

**Case 1.** $p \in (0, 1/2)$. Choose integers $k_n \to \infty$ such that $2^{-n^p} k_n \to \infty$ and $2^{-2np} k_n \to 0$. Let $z_j = (1 - 2^{-n^p}) \exp(i \Theta_j)$, $\Theta_j = jk_n^{-1}$, $j = 1, \ldots, k_n$. Then
\[
\sum_{j=1}^{k_n} (1 - |z_j|^2) = 2^{-2np} k_n \to \infty
\]
while
\[
\sum_{j \neq m} \left( \frac{|1 - |z_j|^2|1 - |z_m|^2|}{|1 - z_m z_j|^2} \right)^p \leq 2^{-2np} \sum_{j=1}^{k_n} \frac{1}{(jk_n^{-1})^p} \leq 2^{-2np} k_n \to 0.
\]

**Case 2.** $p = 1/2$. Choose integers $k_n \to \infty$ such that $2^{-n^2} k_n \to \infty$ and $2^{-np} \log k_n \to 0$, and let $z_j$ be as above.

**Case 3.** $p \in (1/2, 1)$. Likewise, choose integers $k_n$ such that $2^{-n^2} k_n \to 0$ and $2^{-np} k_n \to \infty$, also take $z_j$ as in case 1, then we still find that $\{z_j\}$ satisfies (4.10) but $\{z_j\}$ is not $p$-thin.

The next result allows us to make big hyperbolic perturbations of $p$-thin sequences.

**Lemma 4.3.** Let $p \in (0, 1]$. If a sequence $\{z_n\} \subset A$ is $p$-thin, then there exists a sequence $\{\rho_n\}$ of positive numbers, with $\lim_{n \to \infty} \rho_n = 1$, such that whenever $\{\xi_n\} \subset A$ satisfies $\rho(z_n, \xi_n) \leq \rho_n$, $n = 1, 2, \ldots$, $\{\xi_n\}$ is also $p$-thin.

**Proof.** Since $\{z_n\} \subset A$ is $p$-thin, for any sequence $r_n$ tending to 1 one has
\[
\sup_{|w| \geq r_n} \sum_{m \neq n, |p(w, z_n)| \geq r_n} (1 - |\varphi_n(z_n)|^2)^p \to 0 \quad \text{as } r_n \to 1.
\]
Arguing as in Lemma 4.1, one can find a sequence of positive numbers $a_n$ tending to infinity such that
\[
\sup_{|w| \geq r_n} \sum_{m \neq n, |p(w, z_n)| \geq r_n} a_n (1 - |\varphi_n(z_n)|^2)^p \to 0 \quad \text{as } r_n \to 1.
\]
Here, we may, for example, take

$$a_n = \inf_{w \in A} \left[ \sum_{|z_m| > |z_n| \atop \rho(z_m, w) > r_n} \left(1 - |\varphi(z_m)|^2\right)^\rho \right]^{-1/2}.$$ 

Now, fix numbers $r_n$ tending to 1 satisfying

$$\frac{1 - \inf_{m=1}^n \rho(z_m, z_n)}{1 - r_n} \to 0 \quad \text{as} \quad n \to \infty. \quad (4.11)$$

Then, consider the corresponding sequence $\{a_n\}$ and pick numbers $\rho_n$ tending to 1 satisfying

$$\frac{1 - r_n}{1 - \rho_n} \to 0 \quad \text{as} \quad n \to \infty. \quad (4.12)$$

Also, by taking $\rho_n \to 1$ sufficiently slow one can assume that $\rho(z_n, \xi_n) \leq \rho_n$ implies $(1 - |z_n|^2)^\rho \leq a_d (1 - |z_n|^2)^\rho$ and that $\xi_n \in S(I)$, $\rho(\xi_n, z_I) \geq (r_n + \rho_n)/(1 + r_n \rho_n)$ implies that $z_n \in S(2I)$.

Let $\xi_n$ be a sequence of points in the unit disk, $\rho(\xi_n, z_n) \leq \rho_n$, $n = 1, 2, \ldots$. We are going to show that $\{\xi_n\}$ is $p$-thin using (ii) of Lemma 4.2. Given an arc $I$ of the unit circle let $\lambda = z_I$ and set

$$\Omega = \left\{ \xi_n \in S(I) : \rho(\xi_n, \lambda) \geq \frac{r_n + \rho_n}{1 + \rho_n r_n} \right\}.$$ 

Given $\varepsilon > 0$, if $|I|$ is small enough we have

$$\frac{1}{|I|^\rho} \sum_{\xi_n \in \Omega} (1 - |\xi_n|^2)^\rho \leq \frac{1}{|I|^\rho} \sum_{\xi_n \in S(2I), \rho(\xi_n, \lambda) \geq r_n} a_d (1 - |\xi_n|^2)^\rho$$

$$\leq \sum_{n : \rho(\xi_n, \lambda) \geq r_n} a_d (1 - |\varphi(z_n)|^2)^\rho < \varepsilon.$$

On the other hand, if $\xi_n \in S(I)$ but $\xi_n \notin \Omega$, that is,

$$\rho(\xi_n, \lambda) \leq (r_n + \rho_n)/(1 + \rho_n r_n),$$

then we deduce that

$$1 - \rho(z_n, \lambda) \geq C_n.$$
From (4.11) it follows that there is at most one \( \xi_m \in S(I) \setminus \Omega \) and hence
\[
\frac{1}{|I|^p} \sum_{\xi \in S(I)} (1 - |\xi|^2)^p \leq \frac{1}{|I|^p} \sum_{\xi \in \Omega} (1 - |\xi|^2)^p + (1 - r^2)^p.
\]
An easy calculation using the triangle inequality for the pseudohyperbolic distance [11, p. 2–6] and (4.11), (4.12) shows that
\[
\inf_{m \neq n} p(\xi_m, \xi_n) \to 1 \quad \text{as} \quad n \to \infty.
\]

4.4. Interpolation by Non-analytic Functions

As in [19], to interpolate by \( Q_{p,q} \cap H^\infty \) functions we first construct non-analytic interpolating functions.

It is clear that a Blaschke sequence \( \{z_n\} \), that is, a sequence satisfying
\[
\sum_n (1 - |z_n|^2) < \infty,
\]
can at most accumulate non-tangentially in a set of points (on \( \partial A \)) of length zero. We will refine this fact.

**Lemma 4.4.** Let \( \{z_n\} \subset A \) be a Blaschke sequence. Then there exists an increasing function \( h(t) \) on \([0, \infty)\) depending on \( \{z_n\} \) and satisfying \( h(0) = 0 \), \( \lim_{t \to \infty} t^{-1} h(t) = \infty \) and
\[
|\{ e^\alpha \in \partial A : \text{Card}(I_{\alpha}(e^\alpha) \cap \{z_n\}) = \infty \}| = 0,
\]
where Card(\( E \)) means the cardinal number of the set \( E \), and
\[
I_{\alpha}(e^\alpha) = \{ z \in A : |z - e^\alpha| < h(1 - |z|) \}.
\]

**Proof.** Assume that \( |z_n| \leq |z_{n+1}| \) for all \( n \). Let \( h_n \) be an increasing sequence, \( h_n \to \infty \) as \( n \to \infty \) so that
\[
\sum_n (1 - |z_n|^2) h_n < \infty.
\]
Define \( h(1 - |z_n|) = (1 - |z_n|) h_n \) and extend linearly between \( 1 - |z_{n+1}| \) and \( 1 - |z_n| \). It is clear that \( h(0) = 0 \) and
\[
\lim_{t \to 0} h(t)/t = \infty.
\]
In addition, the set \( E = \{ e^\alpha \in \partial A : \text{Card}(I_{\alpha}(e^\alpha) \cap \{z_n\}) = \infty \} \) can be covered by
\[
\bigcup_{n > n_0} [h(1 - |z_n|)/(1 - |z_n|)] I_{z_n}.
\]
where \( n_0 > 0 \) is any given integer. So \(|E|\) can be bounded by

\[
\sum_{n \geq n_0} h(1 - |z_n|) = \sum_{n \geq n_0} (1 - |z_n|) h_n \to 0 \quad \text{as} \quad n_0 \to \infty
\]

and hence \(|E| = 0\).

In order to construct a non-analytic interpolating function, we use the idea of \textit{Generations} in [11, p. 299–300] to split \( p \)-thin sequences. Let \( \{\xi_n\} \) be a \( p \)-thin sequence of points of the unit disk, \( p \in (0, 1) \). Given an arc \( I \subset \partial D \), consider the dyadic subarcs of \( I \), that is, for \( n = 1, 2, \ldots \)

\[
I = \bigcup_{k=1}^{2^n} I_k^{(n)},
\]

where \( I_k^{(n)} \) are \( 2^n \) disjoint subarcs of \( I \) with length \( 2^{-n} |I| \). Now, we will divide \( \{\xi_n\} \) into \textit{generations}. For \( I = \partial D \), consider the maximal dyadic subarc \( J \) of \( \partial D \) such that \( T(S(J)) \) contains some points in \( \{\xi_n\} \) and then the first generation \( G_1 \) of \( \{\xi_n\} \) consists of these points in \( \{\xi_n\} \). Since \( \{\xi_n\} \) satisfies

\[
\inf_{m \neq n} \rho(\xi_m, \xi_n) \to 1 \quad \text{as} \quad n \to \infty,
\]

\( T(S(J)) \) can contain at most one point of \( \{\xi_n\} \) whenever \( |J| \) is sufficiently small. Next, for each \( \xi_n \in G_1 \) one repeats the above selection replacing \( \partial D \) by the dyadic subarc \( J \subset \partial D \) such that \( \xi_n \in T(S(J)) \). In this way one obtains a new subcollection of the sequence which will be denoted by \( G_1(\xi_n) \), the first generation corresponding to \( \xi_n \). The second generation of \( \{\xi_n\} \) is defined as

\[
G_2 = \bigcup_{\xi_n \in G_1} G_1(\xi_n).
\]

The later generations, \( G_3, G_4, \ldots \) are defined recursively. See Figure 1.

For \( n \) large enough, if \( \xi_n \in G_j \) for some \( j \), then there exists a unique \( \xi_m \in G_{j-1} \) such that \( S(\xi_m) \subset S(\xi_n) \). Moreover,

\[
\sum_{\xi_n \in G_j, \xi_m \in S(I)} (1 - |\xi_n|^2)^p \leq C |I|^p \tag{4.13}
\]

holds for any arc \( I \subset \partial D \), where \( C > 0 \) is a constant independent of \( j \) and \( I \).

We can now state a non-analytic interpolation theorem which is the main difference between our proof and the one for \( \text{VMOA} \cap H^p \) due to C. Sundberg and T. Wolff [19].
Theorem 4.5. Let \( p \in (0, 1) \). Let \( \{z_n\} \subset \Delta \) be a \( p \)-thin sequence with \( |z_n| \leq |z_{n+1}| \) for all \( n \). Then there exists an increasing sequence \( \{K_n\} \) of positive numbers, \( K_n \to \infty \) as \( n \to \infty \), and \( c > 0 \), such that whenever \( \{w_n\} \) is a sequence of complex numbers satisfying
\[
\sup_{m, n > k} \frac{|w_n - w_m|^2}{\max \{K_n, K_m\}} \to 0 \quad \text{as} \quad k \to \infty,
\]
there exists \( \varphi \in C^\infty(\Delta) \) with \( \varphi(z) \equiv w_n \) in \( \{z: \rho(z, z_n) \leq c\} \) satisfying

(i) \( \nabla \varphi(z) \equiv 0 \) in \( \Delta \setminus \bigcup_n R_n \), where \( \{R_n\} \) are some pairwise disjoint regions on \( \Delta \) and \( z_n \in R_n \);

(ii) \( |\nabla \varphi(z)|^2 (1 - |z|^2)^p \, dm(z) \) is a \( p \)-vanishing Carleson measure;

(iii) \( |\Delta \varphi(z)| (1 - |z|^2) \leq |\nabla \varphi(z)| \) and \( (1 - |z|^2) |\nabla \varphi(z)| \leq 1 \), where \( \Delta \varphi \) means the Laplacian of \( \varphi \);

(iv) \( \sup_{z \in \Delta} |\varphi(z)| \leq 2 \sup_n |w_n|, \) if \( \sup_n |w_n| < \infty \);

(v) If \( w_n > 0 \) increases when \( (1 - |z_n|) \) decrease, one has \( 0 \leq \varphi(z) \leq 2w_n \) for \( z \in R_n \).

Proof. Observe that one only has to do the construction when \( n \) is large. Hence one may assume that \( \{z_n\} \) are very close to \( \partial \Delta \).

Let \( I_n = I_{\zeta_n} \) (i.e., \( S(\zeta_n) = S(I_n) \)) and \( \xi_n \) be the point associated to the arc \( 2\xi_n \). By Lemma 4.3, \( K_n \) can be chosen tending to \( \infty \) sufficiently slowly so that \( \{\xi_n\} \) is also \( p \)-thin.
Fig. 2. The shadowed region is the tower over \( z_n \) when \( K_n = 2 \).

For \( n = 1, 2, \ldots \), we consider a region \( T(z_n) \) which will be called the tower over \( z_n \),

\[
T(z_n) = T(S(I_n)) \cup T(S(2I_n)) \cup \cdots \cup T(S(2^K I_n)).
\]

See Figure 2. Since \( S(2^k I_n) = S(\zeta_n) \), the decomposition of \( \{ \zeta_n \} \) into generations gives a corresponding decomposition for the towers, that is,

\[
\{ T(z_n) \} = \bigcup_j \{ T(z_n): \zeta_n \in G_j \}.
\]

If \( K_n \to \infty \) sufficiently slowly, then the following properties hold:

1. \( \inf \{ \rho(T(z_n), T(z_m)) : n \neq m \text{ and } \zeta_n, \zeta_m \in G_j \} \to 1 \text{ as } n \to \infty \), where \( \rho(E, F) = \inf \{ \rho(z, w) : z \in E, w \in F \} \) stands for the pseudohyperbolic distance between \( E \subset \Delta \) and \( F \subset \Delta \).

2. For any arc \( I \subset \partial \Delta \) one has

\[
\sum_{T(z_n) \in S(I)} (1 - |\zeta_n|^2)^{\eta} \lesssim |I|^\eta.
\]

Moreover, using a slightly bigger Carleson box as a substitute for \( S(z_n) \) one can also assume...
(3) Given a tower $T(z_n)$ of the $j$-generation, that is, $z_n \in G_j$, there is a unique $z_m$ in the $(j - 1)$ generation, that is, $z_m \in G_{j-1}$, such that $T(z_n) \subset S(z_m)$.

Next, we will determine a neighborhood $R(z_n)$, the so-called extended tower over $z_n$, of $T(z_n)$. For $j = 1, \ldots, K_n$ consider the brothers of $T(S(2^j I_n))$ as

$$S^{(j)}_{-1, \ldots, -K_n, \ldots, 0, 1} \cup S^{(j)}_{0, \ldots, 0, 1} \cup S^{(j)}_{K_n, \ldots, 0, 1},$$

which are the $2(K_n - j) + 1$ adjacent (from both sides) truncated Carleson boxes to $T(S(2^j I_n))$ of the same size. So, $S^{(j)}_0 = T(S(2^j I_n))$. Also, for $j = 0$, consider

$$S^{(0)}_{-1, \ldots, -K_n, \ldots, 0, 1} \cup S^{(0)}_{0, \ldots, 0, 1} \cup S^{(0)}_{K_n, \ldots, 0, 1},$$

the $2K_n + 1$ adjacent (from both sides) Carleson boxes to $S(I_n) = S^{(0)}_0$.

Then consider

$$R(z_n) = \left( \bigcup_{j=0}^{K_n} \bigcup_{l=-j}^{K_n-j} S^{(j)}_{l} \right).$$

See Figure 3. If $K_n \rightarrow \infty$ sufficiently slowly, and again replacing $S(z_n)$ by a slightly bigger Carleson box if necessary, one can assume that properties (2) and (3) hold for $R(z_n)$ instead of $T(z_n)$.

Finally, for $l = -K_n, \ldots, K_n$, consider a point $\zeta = \zeta(l)$ in the radial projection of $S^{(0)}_l$ onto $\partial \mathcal{A}$ so that $S^{(0)}_l \cap \Gamma(\zeta)$ do not intersect any extended tower $R(z_n)$, $m \neq n$, where $h$ is the function given by Lemma 4.4. One may also assume that the usual truncated Stolz angles

$$\Gamma(\zeta(l)) = \{ z \in S^{(0)}_l : |z - \zeta(l)| < 2(1 - |z|) \}, \quad l = -K_n, \ldots, K_n$$

are pairwise disjoint. Now, define

$$R_n = \left[ \bigcup_{z \in R(z_n)} \bigcup_{l=-K_n}^{K_n} S^{(0)}_l \right] \cup \left[ \bigcup_{l=-K_n}^{K_n} (\Gamma_h(\zeta(l)) \cap S^{(0)}_l) \right].$$

It follows from properties (1) and (3) that $\{ R_n \}$ are pairwise disjoint.

We will define the function $\varphi$ as the limit of some $\varphi_j \in C^\infty(\mathcal{A})$ which will be constructed using generations. The function $\varphi_j$ will be constant outside the sets $R_n$ corresponding to points $z_n$ in generations $G_k$, $k \leq j$. The key estimate will be

$$1 - |z| \leq \max_{k < n} \left\{ \frac{|w_n - w_k|}{K_n} : z_k \in S(z_m), R(z_n) \subset S(z_m) \right\},$$

for $z \in R(z_n)$. 

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Fig. 3. The shadowed region is the extended tower over \( z_n \) when \( K_n = 3 \).

It will be clear from the construction below how to define \( \varphi_1 \). Assume we have defined \( \varphi_{j-1} \) with the following properties.

(a) \( \varphi_{j-1} \equiv w_n \) in \( \{ z : \rho(z, z_n) \leq \epsilon \} \) if \( z_n \in \bigcup_{k=1}^{j-1} G_k \).

(b) \( \text{supp}(\nabla \varphi_{j-1}) \subset \bigcup_{k=1}^{j-1} \bigcup_{z_n \in G_k} R_n \).

(c) \( (1 - |z|) |\nabla \varphi_{j-1}(z)| \leq (\max_{m \leq n} |w_n - w_m|)/K_n, \ z \in R(z_n) \) if \( R(z_n) \subset S(z_m) \).

After that, let us construct \( \varphi_j \). Given a point \( z_m \in G_{j-1} \) let \( R = R(z_n) \) be the largest extended tower of \( G_j \) contained in \( S(z_m) \), namely,

\[
1 - |z_m| = \sup \{ 1 - |z_k| : z_k \in G_j, R(z_k) \subset S(z_m) \}.
\]

We want to find a function \( \psi = \psi_R \) in \( S(z_m) \), which satisfies the analogues of (a)–(c). Let \( w_n \) be the value that one wants to interpolate at \( z_n \).

Take \( w^{(0)} = w_n, w^{(1)}, \ldots, w^{(K_n)} = w_m \) with

\[
|w^{(k)} - w^{(k+1)}| = K_n |w_n - w_m|
\]
and define
\[ \psi \equiv \psi_R = w^{(j+j_1)} \quad \text{in} \quad S_j^{(1)}, \quad l = -(K_n - j), \ldots, K_n - j, \]
j = 0, \ldots, K_n
and \( \psi = \varphi_{j-1} \) in \( D \setminus R(z_n) \). So, after a regularization we can assume that \( \psi \in C^\infty(D) \) with \( \psi \equiv w_n \) in \( \{ z \colon \rho(z, z_n) \leq \epsilon \} \), satisfies
\[
(1 - |z|) |D\psi(z)| \leq |\nabla \psi(z)|, \quad z \in D, \quad (4.15)
\]
\[
(1 - |z|) |\nabla \psi(z)| \leq 1, \quad z \in D, \quad (4.16)
\]
and
\[
(1 - |z|) |\nabla \psi(z)| \leq \frac{|w_n - w_m|}{K_n}, \quad z \in R(z_n). \quad (4.17)
\]
Furthermore, we can do the regularization in such a way that
\[
\nabla \psi(z) \equiv 0, \quad z \in S(z_m) \setminus R_n. \quad (4.18)
\]
This is just the place where the cones \( \Gamma_j(\tilde{z}(I)) \) are used. Actually, we can do the regularization in such a way that
\[
\nabla \psi(z) \equiv 0, \quad z \in \bigcup_j \left( S_j^{(0)} \setminus \Gamma(\tilde{z}(I)) \right), \quad 1 - |z| > h^{-1}(|I_n|),
\]
where \( \Gamma(\tilde{z}(I)) = \{ z \in D : |z - \tilde{z}(I)| < 2(1 - |z|) \} \) are the usual Stőlz angles and
\[
h(1 - |z|) |\nabla \psi(z)| \leq \frac{|w_n - w_m|}{K_n}, \quad z \in S_j^{(0)} \cap \Gamma_j(\tilde{z}(I)),
\]
\[
1 - |z| < \frac{1}{2} h^{-1}(|I_n|). \quad (4.19)
\]
In the sequel, we shall take care of the remaining extended towers in \( S(z_m) \) and select the maximal one, say, \( R^* = R(z_n) \) for some \( k \). If \( R^* \) does not intersect the preceding one \( R = R(z_n) \), then repeat the same construction. Otherwise, just repeat the construction with \( \varphi_{j-1} \) replaced by \( \psi \equiv w_n \) and \( S(z_m) \) by the Carleson box \( S_j^{(0)} \subset R \) which contains \( z_n \). In this way we get a function \( \psi^* \equiv w_n \) satisfying (4.15) and (4.16). Note that even if \( \varphi_{j-1} \) has been substituted by \( \psi_R \), the estimates (4.17) and (4.19) still hold if we replace the right-hand term by
\[
\max_{m \leq n} |w_n - w_m| / K_n.
\]
Continue this process. If it ends after finitely many steps, that is, if there is a finite number of points of the sequence \( \{ z_n \} \) in \( S(z_m) \), then we take \( \varphi_j \) to be the last \( \psi_k \). If there are infinitely many points of \( \{ z_n \} \) in \( S(z_m) \), then the corresponding \( \psi_k \) converges pointwise to a function which will be our \( \varphi_j \).

It is clear from (4.15)–(4.19) that \( \varphi_j \) satisfies (a)–(c). Finally, we define \( \varphi \) as the pointwise limit of \( \varphi_j \), and it is clear from the construction that \( \varphi \) satisfies (i) (iii) (iv) and (v) in Theorem 4.5. To show (ii) we will use the estimates

\[
\forall \varphi(z) = 0, \quad z \in \bigcup \{ S(z_m) \setminus \Gamma_{\xi}(l) \}, \quad 1 - |z| > h^{-1}(|I_n|),
\]

\[
(1 - |z|) |\varphi(z)| \leq \sum_{m \leq n} \left| \frac{w_n - w_m}{K_m} \right|, \quad z \in R(z_m),
\]

\[
h(1 - |z|) |\varphi(z)| \leq \sum_{m \leq n} \left| \frac{w_n - w_m}{K_m} \right|, \\
\quad z \in S(z_m) \cap \Gamma_{\xi}(l), \quad 1 - |z| < \frac{1}{2} h^{-1}(|I_n|),
\]

which follow from the similar inequalities (4.17) and (4.19) for \( \varphi_j \).

Put \( I_n = \bigcup \{ S(z_m) \setminus \Gamma_{\xi}(l) \} \cap \{ z : 1 - |z| < \frac{1}{2} h^{-1}(|I_n|) \} \), where \( S(z_m) \) are the Carleson sectors in \( R(z_m) \). Estimates (4.20) and (4.21) give respectively

\[
\left( \frac{1}{R(z)} \right) \int_{R(z)} |\varphi(z)|^2 (1 - |z|^2)^p \, dm(z) \leq \left( \sum_{m \leq n} \left| \frac{w_n - w_m}{K_m} \right| \right)^2 \sum_{j=0}^{K_n - 1} \sum_{-(K_n - j)}^{K_n - j} (2(1 - |z_n|^2))^p \]

\[
\leq \max_{m \leq n} \left| \frac{w_n - w_m}{K_m} \right|^2 \left( 2^{K_n - j} (1 - |z_n|^2) \right]^p \]

and

\[
\left( \frac{1}{S(z)} \right) \int_{S(z)} |\varphi(z)|^2 (1 - |z|^2)^p \, dm(z) \leq \left[ \max_{t \leq 1 - |z_n|} \frac{t}{h(t)} \right]^2 \left( \sum_{m \leq n} \left| \frac{w_n - w_m}{K_m} \right| \right)^2 |S(z)|^p.
\]
Now, let $S(I)$ be a Carleson box and let $n$ be such that $R_n \cap S(I) \neq \emptyset$. Then

$$\int_{R_n \cap S(I)} |\nabla \varphi(z)|^2 (1 - |z|^2)^p \, dm(z)$$

$$= \left( \int_{R_n \cap (J_n - S(I))} + \int_{R_n \cap S(I)} \right) |\nabla \varphi(z)|^2 (1 - |z|^2)^p \, dm(z)$$

$$= (A) + (B).$$

If $2^{K_n} |I_n| \leq |I|$, then as in (4.22) we have

$$(A) \leq \frac{\max_{m \leq n} |w_n - w_m|^2}{K_n} \left[ 2^{K_n} (1 - |z_n|^2) \right]^p.$$

Otherwise, that is, if $2^{K_n} |I_n| > |I|$, we get

$$(A) \leq \frac{\max_{m \leq n} |w_n - w_m|^2}{K_n} |I \cap (2^{K_n} I_n)|^p.$$

On the other hand, if $|I| \geq |I_n|$, then using (4.23) we obtain

$$(B) \leq \max_{t \leq 1 - |z_n|} t \left[ \frac{\max_{m \leq n} |w_n - w_m|}{K_n} \right]^2 \min \left\{ \frac{|I|}{|I_n|}, K_n \right\} \cdot |I|^p,$$

and if $|I| < |I_n|$, then

$$(B) \leq \max_{t \leq h(I)} t \left[ \frac{\max_{m \leq n} |w_n - w_m|}{K_n} \right] \cdot |I|^p.$$

So,

$$\int_{R_n \cap S(I)} |\nabla \varphi(z)|^2 (1 - |z|^2)^p \, dm(z)$$

$$\leq \frac{\max_{m \leq n} |w_n - w_m|^2}{K_n} \left[ (2^{K_n} I_n \cap (2I)) \right]^p$$

$$+ \left[ \max_{t \leq \max \{ |I|, |I_n| \}} t \left[ \frac{\max_{m \leq n} |w_n - w_m|^2}{K_n} \right] \right]$$

$$\left[ 1 + \min \left\{ \frac{|I|}{|I_n|}, K_n \right\} \right] \cdot \left[ \min \{ |I|, |I_n| \} \right]^p.$$
Applying the assumption (4.14) and the fact that

$$\sum_n (1 - |\xi_n|)^p \delta_{\xi_n} = \sum_n (2K_n|I_n|)^p \delta_{\xi_n}$$

is a $p$-Carleson measure we arrive at (ii). Hence, the proof is completed.

**Remarks.**

(i) In the case $p = 1$, Theorem 4.5 still holds with property (ii) in place of the weaker condition that $|V\phi(z)| \, dm(z)$ is a $1$-vanishing Carleson measure. This fact can be shown in a similar way.

(ii) Any bounded sequence $\{w_n\}$ satisfies (4.14), but one can also take unbounded sequences $\{w_n\}$ tending to infinity sufficiently slowly.

(iii) If $\phi$ satisfies (ii) and (iii) of Theorem 4.5, there exists $b \in Q_p \cap L^\infty(\mathbb{D})$ such that $\bar{\partial}b = \bar{\partial}\phi$. This follows from the fact that when $|V\phi(z)| \, dm(z)$ and $|\nabla\phi(z)|(1 - |z|) \, dm(z)$ are 1-Carleson measures and \[ \sup_{z, t \in \mathbb{D}} |\nabla\phi(z)| (1 - |z|) < \infty, \] any smooth solution $b$ of $\bar{\partial}b = \bar{\partial}\phi$ which is bounded on the unit circle is already bounded on the unit disk [21] [5, p. 53].

### 4.5. Sufficient Condition for Interpolation in $Q_{p,0} \cap H^\infty$

After this long preparation, we can give a proof for the sufficient condition for $Q_{p,0} \cap H^\infty$-interpolation.

**Proof of sufficiency in Theorem 1.4.** Without loss of generality, let $\{z_n\}$ be a $p$-thin sequence of points in the unit disk, $|z_n| \leq |z_{n+1}|$ for all $n$, and let $\{w_n\} \subset \mathbb{C}$ be a bounded sequence of complex numbers. Choose $K_\alpha \to \infty$ verifying Theorem 4.5. Also let $\phi$ be the function given by Theorem 4.5.

Since $|V\phi(z)|^2 (1 - |z|^2)^p \, dm(z)$ is a $p$-vanishing Carleson measure, Lemma 4.1 provides a function $k(z) \geq 0$ increasing to infinity (as $|z| \to 1$) such that $[k(z) \, |V\phi(z)|]^2 (1 - |z|^2)^p \, dm(z)$ is still a $p$-vanishing Carleson measure. Setting

$$k_n = \inf_{z \in R_n} k(z)^{1/2},$$

where $R_n$ are given by Theorem 4.5, and replacing $k_n$ by smaller numbers, increasing to infinity as well, one can assume that

$$\sup_{n, m \geq k} \frac{\log k_n - \log k_m}{\max\{K_n, K_m\}} \to 0 \quad \text{as} \quad k \to \infty.$$
Another application of Theorem 4.5 with the values \( \{ \log k_n \} \) gives a function \( \psi \) satisfying (i)-(v) in Theorem 4.5. So, \( \psi(z_n) \to \infty \) as \( n \to \infty \) and \( 0 \leq \psi(z) \leq 2 \log k_n \) for \( z \in R_n \). Using the third remark after Theorem 4.5, we can find a function \( a \in L^\infty(A) \cap Q_{p,0}(\partial A) \) so that \( \partial a = \partial \psi \). Hence \( F = \exp(a - \psi) \in Q_{p,0} \cap H^\infty \), \( F(z_n) \to 0 \) as \( n \to \infty \) and \( |F(z)|^{-1} \leq k(z) \) for \( z \in \bigcup_n R_n \).

Let \( B \) be the Blaschke product with zeros \( \{ z_n \} \). Then the measure

\[
\left| \frac{V\varphi(z)}{B(z) F(z)} \right|^2 (1 - |z|^2)^p \, dm(z)
\]

is a \( p \)-vanishing Carleson measure. Now, from Theorem 3.1 it follows that there is a function \( b \in Q_{p,0}(\partial A) \cap L^\infty(\partial A) \) with \( \partial b = \partial \varphi / BF \), and then that \( f = \varphi - BFb \in Q_{p,0} \cap H^\infty \) because here Corollary 2.5 is applied to \( F \in Q_{p,0} \cap H^\infty \) with \( F(z_n) \to 0 \) and so \( BF \in Q_{p,0} \cap H^\infty \). It is obvious that this function \( f \) interpolates \( w_n \) at \( z_n \) for \( n = 1, 2, \ldots \).

**Remark.** If \( \{ z_n \} \subset A \) is 1-thin, using Remark (i) after Theorem 4.5, one can assume \( |V\varphi(z)| \, dm(z) \) is a 1-vanishing Carleson measure. Repeating the same argument we also get that

\[
\left| \frac{V\varphi(z)}{B(z) F(z)} \right| \, dm(z)
\]

is a 1-vanishing Carleson measure. So, it remains to check that the equation

\[
\partial b = \frac{\partial \varphi}{BF} = g
\]

can be solved by \( b \in \text{VMO}(\partial A) \cap L^\infty(\partial A) \). This is well-known and follows, for instance, from the Fefferman-Stein decomposition. Actually the function

\[
b_0(z) = \frac{1}{\pi} \int_{\partial A} \frac{g(\zeta)}{z - \zeta} \, dm(\zeta)
\]

is in \( \text{VMO}(\partial A) \) and \( \partial b_0 = g \). Hence \( b_0 = u + \tilde{v} \), where \( u, v \in C(\partial A) \) and one can take \( b = b_0 + (v + i\tilde{v}) \in \text{VMO}(\partial A) \cap L^\infty(\partial A) \). Finally, the interpolating function \( \varphi - bBF \) is in \( \text{VMOA} \cap H^\infty \subset B_0 \cap H^\infty \).
REFERENCES