INTERPOLATION BY POSITIVE HARMONIC FUNCTIONS

D. BLASI AND A. NICOLAU

Abstract. A natural interpolation problem in the cone of positive harmonic functions is considered and the corresponding interpolating sequences are geometrically described.

1. Introduction

Let $h^+ = h^+(\mathbb{D})$ be the cone of positive harmonic functions in the unit disc $\mathbb{D}$ of the complex plane. If $u \in h^+$ the classical Harnack's inequality tells that

$$\frac{1-|z|}{1+|z|} \leq \frac{u(z)}{u(0)} \leq \frac{1+|z|}{1-|z|}$$

for any $z \in \mathbb{D}$. Recall that the hyperbolic distance $\beta(z,w)$ between two points $z, w \in \mathbb{D}$ is

$$\beta(z,w) = \log_2 \frac{1+\left|\frac{z-w}{1-\bar{w}z}\right|}{1-\left|\frac{z-w}{1-\bar{w}z}\right|}$$

Hence, estimates above can be read as $|\log_2 u(z) - \log_2 u(0)| \leq \beta(z,0)$. Since these notions are preserved by automorphisms of the disc, we deduce

$$|\log_2 u(z) - \log_2 u(w)| \leq \beta(z,w) \tag{1.1}$$

for any $z, w \in \mathbb{D}$. So for any function $u \in h^+$, a sequence of points $\{z_n\} \subset \mathbb{D}$ and the corresponding sequence of values $w_n = u(z_n)$, $n = 1, 2, \cdots$ are linked by $|\log_2 w_n - \log_2 w_m| \leq \beta(z_n, z_m)$, $n, m = 1, 2, \cdots$ However, given a sequence of points $\{z_n\} \subset \mathbb{D}$, one can not expect to interpolate by a function in h^+ any sequence of positive values $\{w_n\}$ satisfying the above compatibility condition unless the sequence $\{z_n\}$ reduces to two points. Actually it is well known that having equality in (1.1) for two distinct points $z, w \in \mathbb{D}$ forces the function u to be a Poisson kernel and hence one can not expect to interpolate further

Both authors are supported in part by grants MTM2005-00544 and 2001SGR00431
The first author is also supported by DURSI and Fons Social Europeu under the grant 2003FI00116 .

values. In other words, the natural trace space given by Harnack's Lemma (1.1) is too large and one may consider the following notion.

A sequence of points $\{z_n\}$ in the unit disc will be called an interpolating sequence for h^+ if there exists a constant $\varepsilon = \varepsilon(\{z_n\}) > 0$, such that for any sequence of positive values $\{w_n\}$ satisfying

$$|\log_2 w_n - \log_2 w_m| \le \varepsilon\beta(z_n, z_m), \quad n, m = 1, 2, \cdots \tag{1.2}$$

there exists a function $u \in h^+$ with $u(z_n) = w_n$, $n = 1, 2, \cdots$

Observe that this is a conformally invariant notion, that is, if $\{z_n\}$ is an interpolating sequence for h^+, so is $\{\tau(z_n)\}$, for any automorphism τ of the unit disc. Moreover the corresponding constants satisfy $\varepsilon(\{\tau(z_n)\}) = \varepsilon(\{z_n\})$. Recall that a sequence of points $\{z_n\}$ in the unit disc is called separated if $\inf_{n\neq m} \beta(z_n, z_m) > 0$. The main result of this paper is the following.

Theorem 1. *A separated sequence $\{z_n\}$ of points in the unit disc is interpolating for h^+ if and only if there exist constants $M > 0$ and $0 < \alpha < 1$ such that*

$$\#\{z_j \colon \beta(z_j, z_n) \le l\} \le M2^{\alpha l} \tag{1.3}$$

for any $n, l = 1, 2, \cdots$

We have restricted attention to separated sequences because we want to consider an interpolation problem by positive harmonic functions and not by their derivatives. However it is worth mentioning that any interpolating sequence for h^+ is the union of at most three separated sequences. Let us now discuss condition (1.3). As it is usual in this kind of problems, the geometrical description of interpolating sequences is given in terms of a density condition which tells, in the appropriate sense, that interpolating sequences are not too dense. The number 2 shows up in (1.3) because of the normalization of the hyperbolic distance. We have chosen this normalization because it fits perfectly well with dyadic decompositions. As we will show in Section 4, there are a number of conditions which are equivalent to (1.3). For instance, a sequence $\{z_n\}$ satisfies (1.3) if and only if there exist constants $M_1 > 0$ and $0 < \alpha < 1$ such that

$$\#\left\{z_j \colon \left|\frac{z_j - z_n}{1 - \bar{z}_n z_j}\right| \le r\right\} \le M_1(1-r)^{-\alpha}$$

for any $n = 1, 2, \cdots$ and $0 < r < 1$. One can also write an equivalent condition in terms of Carleson measures. It will be shown in Section 4 that a sequence $\{z_n\} \subset \mathbb{D}$ satisfies (1.3) if and only if there exist constants $M_2 > 0$ and $0 < \alpha < 1$ such that

$$\sum_j (1-|z_j|)^\alpha \le M_2(1-|z_n|)^\alpha, \quad n = 1, 2, \cdots,$$

where the sum is taken over all points $z_j \in \{z_k\}$ such that $|z_j - z_n| \le 2(1-|z_n|)$. This resembles the usual Carleson condition with an exponent $\alpha < 1$ for the Carleson squares which contain a point of the sequence in its top part. Let us now discuss the geometrical meaning of condition (1.3). It tells that, when viewed from a point of the sequence, sequences satisfying (1.3) are —at large scales— exponentially more sparse than merely separated sequences. Actually, a sequence of points $\{z_n\} \subset \mathbb{D}$ is a finite union of separated sequences if and only if (1.3) holds with $\alpha = 1$. It should also be mentioned that in condition (1.3) one counts points in the sequence which are at hyperbolic distance less than l from a given point z_n in the sequence, instead of taking as a base point any $z \in \mathbb{D}$ as in [BN]. See also [S, p. 63–77]. This last condition is stronger. Actually it will be shown in Section 4 that there exist two separated interpolating sequences Z_1, Z_2 for h^+ with $\inf\{\beta(z,\xi)\colon z \in Z_1, \xi \in Z_2\} > 0$ such that $Z_1 \cup Z_2$ is not an interpolating sequence for h^+.

Let us now explain the main ideas of the proof. Let E^* denote the radial projection of a set $E \subset \mathbb{D}$, that is, $E^* = \{\xi \in \partial\mathbb{D}\colon r\xi \in E$ for some $0 \le r < 1\}$. An application of Hall's Lemma yields that there exists a universal constant $C > 0$ such that for any $u \in h^+$ one has

$$\left| \left\{ z \in \mathbb{D} : \frac{u(z)}{u(0)} > \lambda \right\}^* \right| \le \frac{C}{\lambda}, \quad \lambda > 0.$$

The necessity of condition (1.3) follows easily from this estimate. The proof of the sufficiency is harder. Given a sequence of points $\{z_n\} \subset \mathbb{D}$ satisfying (1.3) and a sequence of positive values $\{w_n\}$ satisfying the compatibility condition (1.2), one has to find a function $u \in h^+$ such that $u(z_n) = w_n$, $n = 1, 2, \cdots$ The construction of the function $u \in h^+$ may be splitted into three steps.

1. We will apply a classical result in Convex Analysis called Farkas Lemma which may be understood as an analogue for Cones of the Hahn-Banach Theorem. Instead of constructing directly the function $u \in h^+$ which performs the interpolation, Farkas Lemma will tell that it suffices to prove the following statement. Given any partition of the sequence $\{z_n\}$ into two disjoint subsequences, $\{z_n\} = T \cup S$, there

exists a function $u = u(T, S) \in h^+$ such that

$$u(z_n) \geq w_n, \quad \text{if } z_n \in T,$$
$$u(z_n) \leq w_n, \quad \text{if } z_n \in S.$$

2. Let $\omega(z, G)$ denote the harmonic measure in $\mathbb{D}$ of the set $G \subset \partial\mathbb{D}$ from the point $z \in \mathbb{D}$, that is,

$$\omega(z, G) = \frac{1}{2\pi} \int_G \frac{1 - |z|^2}{|\xi - z|^2} \, |d\xi|.$$

For each point z_n of the sequence $\{z_n\}$ we will construct a set $G_n \subset \partial\mathbb{D}$ and we will show that condition (1.3) provides some sort of independence of harmonic measures $\{\omega(z_n, \cdot)\colon n = 1, 2, \cdots\}$. Actually, given $0 < \delta < 1$, there exists $N > 0$ and a collection of pairwise disjoint subsets $\{G_n\}$ of $\partial\mathbb{D}$ such that

$$\omega(z_n, \cup_{k \in A(n)} G_k) \geq 1 - \delta,$$
$$\sum_{k \notin A(n)} 2^{\eta\beta(z_k, z_n)} \omega(z_n, G_k) \leq \delta.$$

Here $A(n)$ denotes the set of indexes k so that $\beta(z_k, z_n) \leq N$. The number $\eta = \eta(\delta, M, \alpha) > 0$ is a constant depending on $\delta > 0$ and on the constants $M > 0$ and $\alpha < 1$ of (1.3). The construction of the sets $\{G_n\}$ uses a certain stopping time argument and constitutes the most technical part of the proof.

3. L. Carleson ad J. Garnett found a description of the interpolating sequences for the space h^∞ of bounded harmonic functions in the unit disc (see [CG], [G1] or [G2, p. 313]). Using their result it is easy to show that a separated sequence verifying (1.3) is interpolating for h^∞. Hence there exists $\gamma > 0$ and a harmonic function h, with $\sup\{|h(z)|\colon z \in \mathbb{D}\} < 1$ such that $h(z_n) = \gamma$ if $z_n \in T$, while $h(z_n) = -\gamma$ if $z_n \in S$. Then, fixed $\varepsilon > 0$ and $\delta > 0$ sufficiently small, using the compatibility condition (1.2) and the estimates in step 2, one can show that the function

$$u(z) = \sum_{z_n \in T} w_n \int_{G_n} \frac{1 - |z|^2}{|\xi - z|^2} \, (1 + h(\xi)) \frac{|d\xi|}{2\pi}, \quad z \in \mathbb{D},$$

verifies $u(z_n) \geq w_n$ if $z_n \in T$ and $u(z_n) \leq w_n$ if $z_n \in S$.

One may consider a similar problem in higher dimensions. Let $h^+(\mathbb{R}^{d+1}_+)$ denote the cone of positive harmonic functions in the upper half space $\mathbb{R}^{d+1}_+ = \{(x, y)\colon x \in \mathbb{R}^d, y > 0\}$. A sequence of points $\{z_n\} \subset \mathbb{R}^{d+1}_+$ will be called an interpolating sequence for $h^+(\mathbb{R}^{d+1}_+)$ if there exists a constant $\varepsilon = \varepsilon(\{z_n\}) > 0$ such that for any sequence of positive values $\{w_n\}$ verifying

$$|\log_2 w_n - \log_2 w_m| \leq \varepsilon\beta(z_n, z_m), \quad n, m = 1, 2, \cdots$$

there exists $u \in h^+(\mathbb{R}^{d+1}_+)$ with $u(z_n) = w_n$, $n = 1, 2, \cdots$. When $d >$ 1 we do not have a complete geometric description of interpolating sequences. In this direction the situation is analogue to the work of L. Carleson and J. Garnett [CG] on interpolating sequences for the space $h^\infty(\mathbb{R}^{d+1}_+)$ of bounded harmonic functions in $\mathbb{R}^{d+1}_+$. See section 6 for details.

The paper is organized as follows: Section 2 is devoted to the proof of the necessity of condition (1.3). Section 3 contains the proof of the sufficiency. Section 4 is devoted to the analysis of condition (1.3). In Section 5 a related interpolation problem for bounded analytic functions in the unit disc without zeros is considered. This may be compared to [DN]. In the last section the interpolation problem for positive harmonic functions in higher dimensions is discussed. The letter C will denote an absolute constant whose value may change from line to line. Also $C(M)$ will denote a constant which depends on M.

2. Necessity

Given a set $E \subset \mathbb{D}$, let $\omega(z, E, \mathbb{D} \setminus E)$ denote the harmonic measure from the point $z \in \mathbb{D} \setminus E$ of the set E in the domain $\mathbb{D} \setminus E$. The classical Hall's Lemma tells that there exists a universal constant $C > 0$ such that $\omega(0, E, \mathbb{D} \setminus E) \geq C|E^*|$ for any set $E \subset \mathbb{D}$. See [H] or [MS]. Recall that E^* denotes the radial projection of E. The main auxiliary result is the following.

Lemma 2. *There exists a constant $C > 0$ such that for any $u \in h^+$ and $\lambda > 0$ one has*

$$\left| \left\{ z \in \mathbb{D} \colon \frac{u(z)}{u(0)} > \lambda \right\}^* \right| \leq \frac{C}{\lambda}$$

Proof. One may assume that $\lambda > 1$. Fix $u \in h^+$, let $E = \{z \in \mathbb{D} \colon u(z) > \lambda u(0)\}$. The maximum principle shows that

$$u(z) \geq \lambda u(0)\omega(z, E, \mathbb{D} \setminus E), \quad z \in \mathbb{D} \setminus E.$$

Taking $z = 0$, one gets $\omega(0, E, \mathbb{D} \setminus E) \leq \lambda^{-1}$ and applying Hall's Lemma one finishes the proof. □

Proof of the necessity of condition (1.3). Assume that $\{z_k\}$ is an interpolating sequence for h^+. By conformal invariance it is sufficient to prove (1.3) when the base point z_n is the origin. So assume $z_1 = 0$ and take $w_k = 2^{\varepsilon\beta(z_k,0)}$, $k = 1, 2, \cdots$ It is clear that the compatibility condition (1.2) holds. So, there exists $u \in h^+$ with $u(z_k) = w_k$, $k = 1, 2, \cdots$ Let D_k be the hyperbolic disc centered at z_k of hyperbolic radius 1. By Harnack's Lemma

$$u(z) \geq \frac{w_k}{2}, \quad z \in D_k, \quad k = 1, 2, \cdots$$

So, if $A(j)$ denotes the set of indexes k corresponding to points z_k with $j - 1 \leq \beta(z_k, 0) \leq j$, $j = 1, 2, \cdots$, one deduces

$$u(z) \geq 2^{\varepsilon(j-1)-1}, \quad z \in D_k, \quad k \in A(j).$$

Now since $u(0) = 1$, Lemma 2 gives

$$\left|\left(\cup_{k\in A(j)} D_k\right)^*\right| \leq C_1 2^{\varepsilon(1-j)}.$$

Since the sequence $\{z_k\}$ is separated, the discs $\{D_k\}$ are quasidisjoint and one deduces

$$\sum_{k\in A(j)} 1 - |z_k| \leq C_2 2^{\varepsilon(1-j)}.$$

Since $1 - |z_k|$ is comparable to 2^{-j} for any $k \in A(j)$, one deduces

$$\#A(j) \leq C_3 2^{(1-\varepsilon)j}.$$

Adding up for $j = 1, \cdots, l$, one gets

$$\#\{z_k \colon \beta(z_k, 0) \leq l\} \leq C_4 2^{(1-\varepsilon)l}. \quad \square$$

3. Sufficiency of Condition (1.3)

By a normal families argument, one may assume the sequence $\{z_n\}$ consists of finitely many points. As explained in the introduction the proof consists of three steps.

3.1. **First Step.** Let $e_1, \cdots, e_m$ be a collection of vectors of the euclidian space $\mathbb{R}^d$. Farkas Lemma asserts that a vector $e \in \mathbb{R}^d$ is in the cone generated by $\{e_i \colon i = 1, \cdots, m\}$, that is $e = \sum \lambda_i e_i$ for some $\lambda_i \geq 0$, $i = 1, \cdots, m$, if and only if $\langle x, e\rangle \leq 0$ for any vector $x \in \mathbb{R}^d$ for which $\langle x, e_i\rangle \leq 0$, $i = 1, \cdots, m$. See [HL]. This classical result will be used in the proof of the next auxiliary result

Lemma 3. *Let $\{z_n\}$ be a sequence of distinct points in the unit disc and let $\{w_n\}$ be a sequence of positive values. Assume that for every partition of the sequence $\{z_n\} = T \cup S$, into two disjoint subsequences T and S, there exists $u = u(T, S) \in h^+$ such that $u(z_n) \geq w_n$ if $z_n \in T$ and $u(z_n) \leq w_n$ if $z_n \in S$. Then, there exists $u \in h^+$ such that $u(z_n) = w_n$, $n = 1, 2, \cdots$*

Proof of Lemma 3. By a normal families argument, one may assume that both the sequences of points $\{z_n\}$ and values $\{w_n\}$ consist of finitely many, say d, points. Consider the set of all partitions $\{z_n\} = T_k \cup S_k$, $k = 1, \ldots, m$ of the sequence $\{z_n\}$. Let $u_1, \ldots, u_m \in h^+$ be the corresponding functions such that $u_k(z_n) \geq w_n$ if $z_n \in T_k$ and $u_k(z_n) \leq w_n$ if $z_n \in S_k$, and consider the vector

$$u_i \colon = (u_i(z_1), \ldots, u_i(z_d)), \quad i = 1, \ldots, m.$$

If $x \in \mathbb{R}^d$ satisfies $\langle x, u_i \rangle \leq 0$, $i = 1, \ldots, m$, that is $\sum_{n=1}^{d} u_i(z_n) x_n \leq 0$, let $\mathcal{F} = \{z_n \colon x_n \geq 0\}$. Then $\mathcal{F} = T_k$ for some k and let $S_k = \{z_n\} \setminus \mathcal{F}$. Its corresponding function u_k satisfies $x_n w_n \leq x_n u_k(z_n)$ for all $n = 1, \cdots, d$. So,

$$\langle x, w \rangle = \sum_{n=1}^{d} w_n x_n \leq \sum_{n=1}^{d} u_k(z_n) w_n \leq 0.$$

Now, by Farkas's Lemma, $w = (w_1, \cdots, w_d)$ is in the cone generated by the vectors $\{u_i, i = 1, \ldots, m\}$. So there exist constants $\lambda_i \geq 0$, $i = 1, \cdots, m$ such that $u(z) = \sum_{i=1}^{m} \lambda_i u_i(z) \in h^+$ and $u(z_n) = w_n$, $n = 1, 2, \ldots, d$. □

3.2. **Second Step.** The second step in the proof consists on using condition (1.3) to construct a collection of disjoint subsets $\{G_n\}$ of the unit circle which provide a suitable kind of independence of the harmonic measures $\{\omega(z_n, \cdot) \colon n = 1, 2, \cdots\}$. The precise statement is given in the following result which is the main technical part of the proof. Recall that $\omega(z, G)$ denote the harmonic measure in $\mathbb{D}$ of the set $G \subset \partial\mathbb{D}$ from the point $z \in \mathbb{D}$, that is,

$$\omega(z, G) = \frac{1}{2\pi} \int_G \frac{1 - |z|^2}{|\xi - z|^2} \, |d\xi|.$$

Lemma 4. *Let $\{z_n\}$ be a sequence of distinct points in the unit disc which satisfies condition (1.3). Then for any $\delta > 0$, there exist numbers $N = N(\delta) > 0$, $\eta = \eta(\delta) > 0$ and a collection $\{G_n\}$ of pairwise disjoint subsets of the unit circle such that*

$$\omega\left(z_n, \cup_{k \in A(n)} G_k\right) \geq 1 - \delta, \quad n = 1, 2, \cdots, \tag{3.1}$$

and

$$\sum_{k \notin A(n)} 2^{\eta \beta(z_k, z_n)} \omega(z_n, G_k) < \delta, \quad n = 1, 2, \cdots. \tag{3.2}$$

Here $A(n) = A(n, N)$ denotes the collection of indexes k such that $\beta(z_k, z_n) \leq N$.

We first introduce some notation. Given a point $z \in \mathbb{D}$ and $C > 0$ we denote

$$I(z) = \{e^{i\theta}\colon\ -\pi(1-|z|) < \theta - \operatorname{Arg} z \leq \pi(1-|z|)\},$$

$$Q(z) = \left\{re^{i\theta}\colon 0 < 1-r \leq 1-|z|,\ e^{i\theta} \in I(z)\right\},$$

$$CI(z) = \left\{e^{i\theta}\colon\ -\pi\, C(1-|z|) < \theta - \operatorname{Arg} z \leq \pi\, C(1-|z|)\right\}$$

$$CQ(z) = \left\{re^{i\theta}\colon 0 < 1-r \leq C(1-|z|),\ e^{i\,\theta} \in C\, I(z)\right\}$$

Observe that if $C(1-|z|) \geq 1$, one has $CI(z) = \partial\,\mathbb{D}$ and $CQ(z) = \mathbb{D}$. When $z = z_k \in \{z_n\}$, we simply denote $I_k = I(z_k)$. We will use the following two elementary auxiliary results.

Lemma 5. *Fixed $\delta > 0$, there exists $M_0 = M_0(\delta) > 0$ such that*

$$\omega(z_k, M_0 I_k) \geq 1 - \frac{\delta}{100}, \quad k = 1, 2, \cdots$$

Proof. If $z_k = 0$ one may take $M_0 = 1$. If $z_k \neq 0$ observe that there exists an absolute constant $C_0 > 0$ such that $|e^{it} - z_k| \geq C_0|t - \operatorname{Arg} z_k|$. Since

$$\omega(z_k, \partial\mathbb{D} \setminus M_0 I_k) = \frac{1-|z_k|^2}{2\pi} \int_{\partial\mathbb{D}\setminus M_0 I_k} \frac{|d\xi|}{|\xi - z_k|^2},$$

one gets

$$\omega(z_k, \partial\mathbb{D} \setminus M_0 I_k) \leq \frac{1-|z_k|^2}{2\pi C_0^2} \int_{\pi M_0(1-|z_k|)}^{\infty} \frac{dx}{x^2}.$$

Hence

$$\omega(z_k, \partial\mathbb{D} \setminus M_0 I_k) \leq \frac{1}{\pi^2 C_0^2 M_0}$$

and taking $M_0 = 100/\pi C_0^2 \delta$ the result follows. □

Lemma 6. *Fixed $M > 0$, there exists a constant $C(M) > 0$ such that for all pair of points $z, w \in \mathbb{D}$ with $w \in 20MQ(z)$, one has*

$$\left|\beta(z,w) - \log_2\left(\frac{1-|z|}{1-|w|}\right)\right| \leq C(M).$$

Proof. One may assume that $z, w \in \mathbb{D} \setminus \{0\}$. Since

$$|1 - \bar{w}z| \geq (1 - |z||w|) \geq (1-|z|)$$

and

$$\begin{aligned} |1 - \bar{w}z| &\leq |w| \left|\frac{1}{\bar{w}} - z\right| \\ &\leq |w| \left|\frac{1}{\bar{w}} - e^{i\operatorname{Arg} w}\right| + \left|e^{i\operatorname{Arg} w} - e^{i\operatorname{Arg} z}\right| + \left|e^{i\operatorname{Arg} z} - z\right| \\ &\leq (20M + 20M\pi + 1)(1-|z|), \end{aligned}$$

we deduce

$$1-|z| \le |1-\bar{w}z| \le K(M)(1-|z|),$$

where $K(M) = 20M + 20M\pi + 1$. So,

$$\begin{aligned}\beta(z,w) &= 2\log_2\left(1+\left|\frac{z-w}{1-\bar{w}z}\right|\right) - \log_2\left(1-\left|\frac{z-w}{1-\bar{w}z}\right|^2\right)\\ &= 2\log_2\left(1+\left|\frac{z-w}{1-\bar{w}z}\right|\right) - \log_2\frac{(1-|z|^2)(1-|w|^2)}{|1-\bar{w}z|^2}\\ &= C + \log_2\left(\frac{1-|z|}{1-|w|}\right)\end{aligned}$$

where $-2 \le C \le 2 + 2\log_2 K(M)$. □

Proof of Lemma 4. The construction of the sets $\{G_n\}$ may be splitted into three steps.

i) For each $z_k \in \{z_n\}$ and $\lambda > 0$, we will construct certain points $z_n^\gamma(k) \in \mathbb{D}$ with $I(z_n) \subset I(z_n^{(\gamma)}(k))$ and

$$\sum_{\substack{z_n \in 20M_0Q(z_k)\\ \beta(z_k,z_n)\ge N}} 1-|z_n^\gamma(k)| \le \lambda(1-|z_k|) \text{ for all } z_k \in \{z_n\}. \tag{3.3}$$

Here N is a constant depending on λ, M_0 and on the constants M and α appearing in (1.3)

ii) Next, we will construct certain sets $E_k \subset \partial\mathbb{D}$ with $E_k \cap E_j = \emptyset$ if $\beta(z_k, z_j) \ge N$ such that

$$\omega(z_k, E_k) \ge 1 - \frac{\delta}{10}. \tag{3.4}$$

In the construction of the sets E_k we will use the points $z_n^\gamma(k)$ of the first step which satisfy the estimate (3.3) above for a certain fixed λ sufficiently small.

iii) Finally we will construct the pairwise disjoint sets G_n satisfying conditions (3.1) and (3.2).

i) Construction of the points $z_n^\gamma(k)$. Fix $\delta > 0$. Applying Lemma 5, there exists a constant $M_0 = M_0(\delta) > 0$ such that

$$\omega(z_k, M_0 I_k) \ge 1 - \frac{\delta}{100}, \quad k = 1, 2, \cdots \tag{3.5}$$

Fix $z_k \in \{z_n\}$. Let $\gamma = \gamma(\alpha) > 0$ be a small number to be fixed later. For any $z_n \in 20M_0Q(z_k)$ with $\beta(z_k, z_n) \ge N$ we define $z_n^\gamma(k)$ as the point in $\mathbb{D}$ satisfying the following three conditions

$$\begin{aligned} &\operatorname{Arg}(z_n) = \operatorname{Arg}(z_n^\gamma(k)), \\ &\beta(z_n^\gamma(k), z_n) = \gamma\beta(z_k, z_n), \\ &|z_n^\gamma(k)| < |z_n|. \end{aligned} \tag{3.6}$$

Here $N = N(\gamma, M_0, \lambda)$ is a large number to be fixed later. In particular $N > 0$ will be taken so large that $z_n^\gamma(k) \in 20M_0Q(z_k)$ whenever $z_n \in 20M_0Q(z_k)$ satisfies $\beta(z_n, z_k) > N$. See Figure 1.

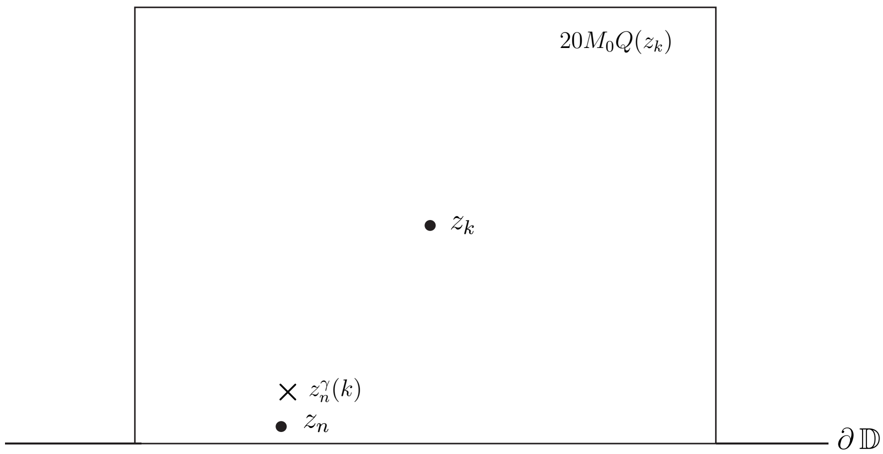

FIGURE 1

Using Lemma 6 and $\beta(z_n^\gamma(k), z_n) = \gamma\beta(z_k, z_n)$ we obtain the following inequalities:

$$\left(\frac{1-|z_k|}{1-|z_n|}\right)^{C^{-1}\gamma} \le \frac{1-|z_n^\gamma(k)|}{1-|z_n|} \le \left(\frac{1-|z_k|}{1-|z_n|}\right)^{C\gamma}, \tag{3.7}$$

where C is a constant depending on M_0. So,

$$\sum_{\substack{z_n \in 20M_0Q(z_k) \\ \beta(z_k, z_n) \ge N}} 1-|z_n^\gamma(k)| \le (1-|z_k|)^{C\gamma} \sum_{j=N}^{\infty} \sum_{\substack{z_n \in 20M_0Q(z_k) \\ j \le \beta(z_n, z_k) < j+1}} (1-|z_n|)^{1-C\gamma}.$$

Now, if $z_n \in 20M_0Q(z_k)$ and $j \le \beta(z_n, z_k) < j+1$, Lemma 6 tells that $1-|z_n| \le K(M_0)2^{-j}(1-|z_k|)$. So, using (1.3), the right hand side term is bounded by

$$K(M_0)^{1-C\gamma}(1-|z_k|) \sum_{j=N}^{\infty} M2^{\alpha j}2^{-j(1-C\gamma)}.$$

Since $\alpha < 1$, taking $\gamma > 0$ so small that $\alpha + C\gamma < 1$, the expression above may be bounded by

$$M\,K(M_0)^{1-C\gamma}\frac{2^{N(\alpha+C\gamma-1)}}{1-2^{\alpha+C\gamma-1}}\,(1-|z_k|).$$

Finally, given $\lambda > 0$ taking N sufficiently large, we obtain

$$\sum_{\substack{z_n\in 20M_0Q(z_k)\\ \beta(z_n,z_k)\geq N}} 1-|z_n^{\gamma}(k)| \leq \lambda\,(1-|z_k|) \text{ for all } z_k\in\{z_n\}.$$

ii) Construction of the sets $\{E_k\}$. For each $z_n^{\gamma}(k)$, we define $I_n^{\gamma}(k) = I(z_n^{(\gamma)}(k))$. Fixed $M_0 > 0$ and $N > 0$, we introduce the notation:

$$B(k) = \{z_n :\ |z_n| \geq |z_k|,\ \beta(z_k,z_n)\geq N,\ z_n\in 20M_0Q(z_k)\}.$$

Now we will proof that the sets $E_k = M_0I_k \setminus \bigcup_{z_n\in B(k)} I_n^{\gamma}(k)$ satisfy

$$\omega(z_k,E_k)\geq 1-\frac{\delta}{10}. \tag{3.8}$$

Using the elementary estimate of the Poisson Kernel

$$\frac{1-|z_k|^2}{|e^{it}-z_k|^2}\leq\frac{1+|z_k|}{1-|z_k|},$$

one obtains

$$\omega\big(z_k,\bigcup_{z_n\in B(k)} I_n^{\gamma}(k)\big)\leq\sum_{z_n\in B(k)}\frac{1+|z_k|}{1-|z_k|}\int_{I_n^{\gamma}(k)}\frac{dt}{2\pi}\leq\frac{2}{1-|z_k|}\sum_{z_n\in B(k)}1-|z_n^{\gamma}(k)|.$$

which by (3.3) is smaller than 2λ. Since

$$\omega(z_k,E_k)=\omega(z_k,M_0I_k)-\omega\left(z_k,\bigcup_{z_n\in B(k)} I_n^{\gamma}(k)\right),$$

the estimate (3.5) tells

$$\omega(z_k,E_k)\geq 1-\frac{\delta}{100}-\lambda.$$

If we take $\lambda > 0$ sufficiently small, we deduce (3.8). Since $M_0I_n \subset I_n^{\gamma}(k)$, it is clear from the definition that $E_k\cap E_j=\emptyset$ if $\beta(z_k,z_j)>N$.

iii) Construction of the pairwise disjoint sets G_n. We rearrange the sequence $\{z_n\}$ so that $\{1-|z_n|\}$ decreases. For each point z_n we will construct a set $G_n\subset E_n$ so that the corresponding family $\{G_n\}$ will satisfy (3.1), (3.2) and $G_n\cap G_m=\emptyset$ if $n\neq m$. The construction will proceed by induction and will ensure that the sets G_n are pairwise disjoint and verify (3.1).

Take $G_1 = E_1$. By (3.8), the estimate (3.1) is satisfied when $n = 1$. Assume that pairwise disjoint subsets $G_1, \cdots, G_{j-1}$ of the unit circle have been defined so that

$$\omega(z_n, \bigcup_{k \le n, k \in A(n)} G_k) \ge 1 - \delta, \text{ for } n = 1, 2, \cdots, j-1.$$

The set G_j will be constructed according to the following two different situations:

(1) If $\beta(z_j, \{z_1, \cdots, z_{j-1}\}) \ge N$ we define $G_j = E_j$. By (3.4) we have

$$\omega(z_j, \bigcup_{k \le j, k \in A(j)} G_k) \ge \omega(z_j, G_j) \ge 1 - \delta.$$

Now let us show that $G_k \cap G_j = \emptyset$ for any $k = 1, \cdots, j-1$. Since $G_k \subset E_k$ and $G_j \subset M_0 I_j$, it is sufficient to show that $M_0 I_j \cap E_k = \emptyset$ for $k = 1, \cdots, j-1$. Fix $k = 1, \cdots, j-1$ and consider two cases

(a) If $z_j \in 20 M_0 Q(z_k)$. Since $M_0 I_j \subset I_j^\gamma(k)$ and $E_k = M_0 I_k \setminus \bigcup I_j^\gamma(k)$, we have $E_k \cap M_0 I_j = \emptyset$

(b) If $z_j \notin 20 M_0 Q(z_k)$. Since $|z_j| > |z_k|$ we have $M_0 I_j \cap M_0 I_k = \emptyset$. Hence $E_k \cap M_0 I_j = \emptyset$.

(2) If $\beta(z_j, \{z_1, \cdots, z_{j-1}\}) \le N$, consider the set of indexes $\mathcal{F} = \mathcal{F}(j) = \{k \in [1, \cdots, j-1] \colon \beta(z_k, z_j) \le N\}$. Let us distinguish the following two cases:

(a) If $\omega(z_j, \bigcup_{k \in \mathcal{F}} G_k) \ge 1 - \delta$, define $G_j = \emptyset$. It is obvious that

$$\omega(z_j, \bigcup_{k \le j, k \in A(j)} G_k) \ge 1 - \delta.$$

(b) If $\omega(z_j, \bigcup_{k \in \mathcal{F}} G_k) < 1 - \delta$, define $G_j = E_j \setminus \bigcup_{k \in \mathcal{F}} G_k$. Arguing as in case 1 one can show that $G_k \cap G_j = \emptyset$ for any $k = 1, \cdots, j-1$. Also, applying (3.8), one gets

$$\omega(z_j, \bigcup_{k \le j, k \in A(j)} G_k) \ge \omega(z_j, E_j) \ge 1 - \delta.$$

So, by induction, a family $\{G_n\}$ of pairwise disjoint subsets of the unit circle is constructed so that condition (3.1) is satisfied. It just remains to show that the family $\{G_n\}$ verifies (3.2), that is, there exists $\eta = \eta(\delta) > 0$ such that

$$\sum_{k \colon \beta(z_k, z_n) \ge N} 2^{\eta \beta(z_k, z_n)} \omega(z_n, G_k) \le \delta, \quad n = 1, 2, \cdots$$

Fixed $n = 1, 2, \cdots$, split this sum into three parts (A), (B) and (C), corresponding to the points z_k with $\beta(z_k, z_n) \geq N$ such that:
(a) $z_k \in 20M_0Q(z_n)$ in part (A),
(b) z_k so that $2M_0I_k \cap M_0I_n = \emptyset$ in part (B) (See Figure 2)
(c) points z_k which are not in (a) or (b)

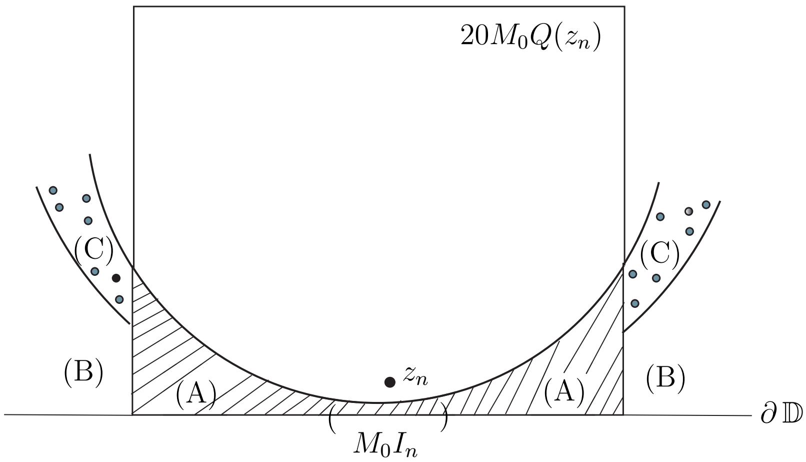

Figure 2. The sum is splitted into three parts corresponding to the location of the points z_k in the regions (A), (B) or (C)

In (A) and (B) we will use the estimate $\omega(z_n, G_k) \leq C(M_0)2^{-\beta(z_n,z_k)}$ and for (C) we will use the constant $\gamma > 0$ appearing in the construction of the sets E_k.

We first claim that there exists a constant $C = C(M_0) > 0$ such that for points z_k in part (A) or (B), that is those verifying either $z_k \in 20M_0Q(z_n)$ or $2M_0I_k \cap M_0I_n = \emptyset$, one has

$$\omega(z_n, G_k) \leq C2^{-\beta(z_k,z_n)}. \tag{3.9}$$

For the points z_k in part (A) we have $z_k \in 20M_0Q(z_n)$. Since $G_k \subseteq M_0I_k$, a trivial estimate of the Poisson kernel gives

$$\omega(z_n, G_k) \leq \int_{M_0I_k} \frac{1-|z_n|^2}{|e^{it}-z_n|^2}\frac{dt}{2\pi} \leq 2M_0\frac{1-|z_k|}{1-|z_n|}.$$

Applying Lemma 6, since $z_k \in 20M_0Q(z_n)$, one has

$$\log_2 \frac{1-|z_k|}{1-|z_n|} \leq C(M_0) - \beta(z_k, z_n).$$

Hence, if $z_k \in 20M_0Q(z_n)$ we deduce

$$\omega(z_n, G_k) \leq C2^{-\beta(z_k,z_n)}$$

with $C = 2M_0 2^{C(M_0)}$. For the points z_k in part (B) we have $2M_0 I_k \cap M_0 I_n = \emptyset$. An easy calculation shows that there exists a constant $C_1 = C_1(M_0)$ such that for any $e^{it} \in I_k$ one has

$$|e^{it} - z_n| \geq C_1 |1 - z_n \bar{z}_k|.$$

Then

$$\omega(z_n, G_k) \leq \int_{M_0 I_k} \frac{1 - |z_n|^2}{|e^{it} - z_n|^2} \frac{dt}{2\pi} \leq C_1^{-2} M_0 \frac{(1 - |z_n|^2)(1 - |z_k|^2)}{|1 - z_n \bar{z}_k|^2}.$$

It is easy to see from the estimates above that there exists a universal constant $C_2 > 0$ such that

$$\beta(z_n, z_k) \leq C_2 - \log_2 \frac{(1 - |z_n|^2)(1 - |z_k|^2)}{|1 - z_n \bar{z}_k|^2},$$

one deduces

$$\omega(z_n, G_k) \leq C 2^{-\beta(z_n, z_k)}$$

with $C = C_1^{-2} M_0 2^{C_2}$. Hence (3.9) holds for points z_k in parts (A) and (B). Therefore

$$(\mathrm{A}) + (\mathrm{B}) \leq C \sum_{k:\ \beta(z_k, z_n) \geq N} 2^{(\eta - 1)\beta(z_n, z_k)}.$$

Observe that condition (1.3) gives

$$\sum_{k:\ \beta(z_k, z_n) \leq j} 2^{(\eta - 1)\beta(z_n, z_k)} \leq M 2^{(\eta + \alpha - 1)j},$$

for any $j = 1, 2, \cdots$ Since $\alpha < 1$ one may choose $0 < \eta = \eta(\alpha) < 1 - \alpha$ so that $\alpha + \eta < 1$. So, adding up for $j \geq N$, one obtains

$$(\mathrm{A}) + (\mathrm{B}) \leq CM \frac{2^{(\eta + \alpha - 1)N}}{1 - 2^{\eta + \alpha - 1}}.$$

Hence, taking $N > 0$ sufficiently large one deduces

$$(\mathrm{A}) + (\mathrm{B}) \leq \frac{\delta}{3}.$$

The estimate of the third term (C) depends on the choice of the constant $\gamma > 0$ appearing in the construction of the sets $\{E_n\}$. Fixed z_n, consider

$$U(n) = \{z_k\ :\ \beta(z_k, z_n) \geq N,\ z_k \notin 20 M_0 Q(z_n),\ 2M_0 I_k \cap M_0 I_n \neq \emptyset\}.$$

So (C) $= \sum_{z_k \in U(n)} 2^{\eta \beta(z_k, z_n)} \omega(z_n, G_k)$.

Observe that if $z_k \in U(n)$, then $|z_k| < |z_n|$ and $z_n \in 3M_0Q(z_k)$. In particular $z_n \in 20M_0Q(z_k)$ so, by the construction of the sets $\{G_k\}$, $G_k \subset M_0I_k \setminus I_n^\gamma(k)$. Hence

$$\omega(z_n, G_k) \le \int\limits_{M_0I_k \setminus I_n^\gamma(k)} \frac{1-|z_n|^2}{|\xi - z_n|^2}\,\frac{|d\xi|}{2\pi} \le \int\limits_{\partial\mathbb{D} \setminus I_n^\gamma(k)} \frac{1-|z_n|^2}{|\xi - z_n|^2}\,\frac{|d\xi|}{2\pi}$$

and a change of variable gives an absolute constant $C_3 > 0$ such that

$$\omega(z_n, G_k) \le C_3\,(1-|z_n|) \int_{1-|z_n^\gamma(k)|}^{\infty} \frac{dx}{x^2} \le C_3\,\frac{1-|z_n|}{1-|z_n^\gamma(k)|}. \tag{3.10}$$

This estimate is worst than (3.9) which was used for (A) and (B) but it is good enough for our purposes. The key is that in (C) we sum over "few" terms corresponding to the points $z_k \in U(n)$.

Observe that if $z_k \in U(n)$, z_k belongs to the Stolz angle $\Gamma_n = \Gamma_n(M_0) = \{z \in \mathbb{D} \colon |z - e^{i\operatorname{Arg} z_n}| \le 11M_0(1-|z|)\}$ with vertex $e^{i\operatorname{Arg} z_n}$ and a certain opening depending on M_0. To see this we only need to observe that $2M_0I_k \cap M_0I_n \ne \emptyset$ implies $|\operatorname{Arg} z_k - \operatorname{Arg} z_n| \le 10M_0(1-|z_k|)$ and use this inequality to get

$$|z_k - e^{i\operatorname{Arg} z_n}| \le 11M_0(1-|z_k|).$$

Define $V(n) = \{z_k \in \Gamma_n \colon |z_k| < |z_n|, \beta(z_k, z_n) \ge N\}$ and then,

$$\text{(C)} = \sum_{z_k \in U(n)} 2^{\eta\beta(z_k,z_n)}\omega(z_n, G_k) \le \sum_{z_k \in V(n)} 2^{\eta\beta(z_k,z_n)}\omega(z_n, G_k).$$

Using inequalities (3.10) and (3.7) we obtain

$$\text{(C)} \le C_3 \sum_{z_k \in V(n)} 2^{\eta\beta(z_k,z_n)} \frac{1-|z_n|}{1-|z_n^\gamma(k)|} \le C_3 \sum_{z_k \in V(n)} 2^{\eta\beta(z_k,z_n)} \left(\frac{1-|z_n|}{1-|z_k|}\right)^{C^{-1}\gamma}.$$

Since $z_n \in 3M_0Q(z_k)$, Lemma 6 gives

$$\left|\beta(z_n, z_k) - \log_2 \frac{1-|z_k|}{1-|z_n|}\right| \le C(M_0).$$

Hence

$$\frac{1-|z_n|}{1-|z_k|} \le 2^{C(M_0) - \beta(z_n,z_k)}.$$

Therefore

$$\text{(C)} \le C_3\, 2^{C(M_0)C^{-1}\gamma} \sum_{z_k \in V(n)} 2^{(\eta - C^{-1}\gamma)\beta(z_n,z_k)}.$$

Since the sequence $\{z_n\}$ is separated, there exists $C_4 = C_4(M_0) > 0$ such that for any $j \geq 0$, the number of points $z_k \in V_n$ with $j \leq \beta(z_k, z_n) \leq j+1$ is at most C_4. Hence

$$(\mathrm{C}) \leq C_3\, C_4 2^{C(M_0)C^{-1}\gamma} \sum_{j=N}^{\infty} 2^{(\eta - C^{-1}\gamma)j}.$$

Taking $\eta > 0$ so small that $\eta - C^{-1}\gamma < 0$ and taking N sufficiently large, we deduce

$$(\mathrm{C}) \leq \frac{\delta}{3}.$$

So condition (3.2) is satisfied and the proof of Lemma 4 is finished. □

3.3. **Third Step.** On the last step given a partition $\{z_n\} = T \cup S$ the sets $\{G_n\}$ constructed on step 3.2 will be used to find a function $u = u(T, S)$ satisfying the conditions stated in Lemma 3. This will end the proof of the sufficiency of condition (1.3).

A sequence of points $\{z_n\}$ in the unit disc is called an interpolating sequence for the space h^∞ of bounded harmonic functions in the unit disc if for any bounded sequence $\{w_n\}$ of real numbers there exists $u \in h^\infty$ with $u(z_n) = w_n$, $n = 1, 2, \cdots$. L. Carleson and J. Garnett characterized interpolating sequences for h^∞ as those sequences $\{z_n\}$ satisfying $\inf_{n \neq m} \beta(z_n, z_m) > 0$ and

$$\sup \frac{1}{\ell(Q)} \sum_{z_n \in Q} 1 - |z_n| < \infty, \tag{3.11}$$

where the supremum is taken over all Carleson squares of the form

$$Q = \{re^{i\theta} \colon 0 < 1 - r < \ell(Q), \quad |\theta - \theta_0| < \ell(Q)\}$$

for some $\theta_0 \in [0, 2\pi)$. See [CG], [G1] or [G2, p. 313]. We next show that a separated sequence $\{z_n\}$ satisfying (1.3) verifies the condition above. Actually it is sufficient to show (3.11) for Carleson squares Q which contain a point of the sequence $\{z_n\}$ in its top part $T(Q) = \{re^{i\theta} \in Q \colon 1 - r > \ell(Q)/2\}$. Let Q be a Carleson square of this type. Let $z_n \in T(Q)$ and $A(j) = \{k \colon z_k \in Q, j - 1 \leq \beta(z_k, z_n) < j\}$. Lemma 6 tells that for any $k \in A(j)$ the quantity $1 - |z_k|$ is comparable to $2^{-j}\ell(Q)$. Hence condition (1.3) yields

$$\sum_{k \in A(j)} 1 - |z_k| \leq C_1 2^{-j} \ell(Q) \# A(j) \leq C_1 M 2^{(\alpha - 1)j} \ell(Q).$$

Since $\alpha < 1$, adding up over $j = 1, 2, \cdots$, one obtains (3.11). Hence $\{z_n\}$ is an interpolating sequence for h^∞. Then by the Open Mapping Theorem, there exists a constant $\gamma = \gamma(\{z_n\}) > 0$ such that for any partition of the sequence $\{z_n\} = T \cup S$ there exists $h = h(T,S) \in h^\infty$ with $\sup\{|h(z)|\colon z \in \mathbb{D}\} < 1$ and $h(z_n) > \gamma$ for $z_n \in T$ while $h(z_n) < -\gamma$ for $z_n \in S$. Let $\delta > 0$ be a small number to be fixed later and let $N = N(\delta)$, $\eta = \eta(\delta)$ be the positive constants and $\{G_n\}$ the pairwise disjoint collection of subsets of the unit circle given in Lemma 4. Let $\varepsilon = \varepsilon(\delta)$ be a small number to be fixed later which will satisfy $\varepsilon\delta^{-1} \to 0$ as δ tends to 0. Let $\{w_k\}$ be a sequence of positive numbers satisfying the compatibility condition (1.2). Given a partition $\{z_n\} = T \cup S$, consider the function $u = u(T,S) \in h^+$ defined by

$$u(z) = \sum_k w_k \int_{G_k} P_z(\xi)(1 + h(\xi))|d\xi|,$$

where $h = h(T,S)$ and

$$P_z(\xi) = \frac{1}{2\pi}\,\frac{1 - |z|^2}{|\xi - z|^2}$$

is the Poisson kernel. Our goal is to show that $u(z_n) \geq w_n$ for $z_n \in T$ and $u(z_n) \leq w_n$ for $z_n \in S$. For $n = 1, 2, \cdots$, let $A(n)$ be the set of indexes k such that $\beta(z_k, z_n) \leq N$. Write $u(z_n) = (\mathrm{I}) + (\mathrm{II})$, where

$$(\mathrm{I}) = \sum_{k \notin A(n)} \omega_k \int_{G_k} P_{z_n}(\xi)(1 + h(\xi))\,|d\xi|,$$

$$(\mathrm{II}) = \sum_{k \in A(n)} \omega_k \int_{G_k} P_{z_n}(\xi)(1 + h(\xi))\,|d\xi|.$$

We first show that

$$(\mathrm{I}) < 2\delta w_n, \quad n = 1, 2, \cdots \tag{3.12}$$

Actually if the constant $\varepsilon = \varepsilon(\delta) > 0$ is taken so that $\varepsilon < \eta$, the compatibility condition (1.2) tells that (I) can be bounded by

$$w_n \sum_{k \notin A(n)} 2^{\eta\beta(z_k, z_n)} 2\omega(z_n, G_k)$$

which, by (3.2), is bounded by $2\delta w_n$. Hence (3.12) holds.

For the other term, using that the sets $\{G_n\}$ are pairwise disjoint and the compatibility condition (1.2) we have

$$(\mathrm{II}) = \sum_{k \in A(n)} w_k \int_{G_k} P_{z_n}(\xi)(1 + h(\xi))|d\xi| \leq 2^{\varepsilon N} w_n(1 + h(z_n)).$$

Also, since $\sup\{|h(z_n)|\colon z\in\mathbb{D}\}\le 1$, the compatibility condition (1.2) and the estimate (3.1) yield

$$\begin{aligned}(\mathrm{II}) &\ge w_n 2^{-\varepsilon N}\left(1+h(z_n)-\int_{\partial\mathbb{D}\setminus\bigcup_{k\in A(n)}G_k} P_{z_n}(\xi)\,(1+h(\xi))|d\xi|\right)\\ &\ge 2^{-\varepsilon N} w_n(1+h(z_n)-2\delta).\end{aligned}$$

So

$$2^{-\varepsilon N} w_n(1+h(z_n)-2\delta)\le (\mathrm{II})\le 2^{\varepsilon N} w_n(1+h(z_n)).$$

Hence

(a) If $z_n\in T$, $h(z_n)\ge\gamma$ and then $u(z_n)\ge(\mathrm{II})\ge w_n 2^{-\varepsilon N}(1+\gamma-2\delta)$.
(b) If $z_n\in S$, $h(z_n)\le-\gamma$ and then $u(z_n)=(\mathrm{I})+(\mathrm{II})\le w_n(2\delta+2^{\varepsilon N}(1-\gamma))$.

Fixed $\gamma>0$, taking $\delta=\delta(\gamma)>0$ and $\varepsilon=\varepsilon(\delta,\eta,N)>0$ sufficiently small, we deduce that $u(z_n)\ge w_n$ if $z_n\in T$ and $u(z_n)\le w_n$ if $z_n\in S$. An application of Lemma 3 concludes the proof of the sufficiency of condition (1.3). □

4. Equivalent conditions

In this section several geometric conditions which are equivalent to (1.3) are collected.

Proposition 7. *Let $\{z_n\}$ be a sequence of distinct points in $\mathbb{D}$. Then the following are equivalent:*

(a) *Condition (1.3) holds, that is, there exist constants $M>0$ and $0<\alpha<1$ such that*

$$\#\{z_j\colon \beta(z_j,z_n)\le l\}\le M\,2^{\alpha l}$$

for any $n,l=1,2\cdots$

(b) *There exist constants $M_1>0$ and $0<\alpha<1$ such that*

$$\#\left\{z_j\colon \left|\frac{z_j-z_n}{1-\bar z_n z_j}\right|\le r\right\}\le M_1(1-r)^{-\alpha},$$

for any $0<r<1$ and any $n=1,2,\cdots$

(c) *There exist constants $M_2>0$ and $0<\alpha<1$ such that*

$$\#\{z_j\in Q(z_n)\colon 2^{-l-1}(1-|z_n|)\le 1-|z_j|\le 2^{-l}(1-|z_n|)\}\le M_2\,2^{\alpha\, l}$$

for any $n,l=1,2,\cdots$

(d) *There exist constants $M_3>0$ and $0<\alpha<1$ such that*

$$\sum_{z_j\in Q(z_n)}(1-|z_j|)^\alpha\le M_3(1-|z_n|)^\alpha,$$

for any $n=1,2,\ldots$

Proof. The equivalence between (a) and (b) follows from the following obvious observation. Let $z, w \in \mathbb{D}$, then $\beta(z,w) \leq l$ if and only if

$$\left|\frac{z-w}{1-\overline{w}z}\right| = \frac{2^{\beta(z,w)}-1}{2^{\beta(z,w)}+1} = 1 - \frac{2}{2^{\beta(z,w)}+1} \leq 1 - \frac{2}{2^l+1}$$

Assume (a) holds. Fix two positive integers n, l. Let $z_j \in Q(z_n)$ satisfying

$$2^{-l-1}(1-|z_n|) \leq 1-|z_j| \leq 2^{-l}(1-|z_n|).$$

Applying Lemma 6 one shows that there exists a universal constant $C > 0$ such that

$$|\beta(z_n, z_j) - l| \leq C.$$

Hence

$$\left\{z_j \in Q(z_n)\colon 2^{-l-1}(1-|z_n|) \leq 1-|z_j| \leq 2^{-l}(1-|z_n|)\right\} \subseteq \{z_j\colon \beta(z_j, z_n) \leq l + C\}$$

and condition (1.3) gives (c). Adding up over $l = 1, 2, \ldots$ one shows that (c) implies (d). Assume (d) holds and let us show condition (1.3). By conformal invariance one may assume $z_n = 0$. So condition (d) tells

$$\sum_{j=1}^{\infty}(1-|z_j|)^{\alpha} \leq M_3.$$

Since $\beta(z_j, 0) \leq l$ implies

$$1 - |z_j| \geq \frac{2}{2^l+1},$$

we deduce

$$\#\{z_j\colon \beta(z_j, 0) \leq l\} \leq M_3 \left(\frac{2}{2^l+1}\right)^{-\alpha}$$

which gives (1.3) □

As mentioned in the introduction, condition (1.3) tells how dense is the sequence when one looks at it from a point of the sequence. It is worth mentioning that one can not take as a base point an arbitrary point in the unit disc. This follows from the following example of two separated interpolating sequences for h^+ which will be called Z_1, Z_2 so that $\inf\{\beta(z,\xi)\colon z \in Z_1,\ \xi \in Z_2\} > 0$ but such that the union $Z_1 \cup Z_2$ is not an interpolating sequence for h^+. For instance one may take $Z_1 = \{r_k\}$ where $r_1 = 0$, $r_k \to 1$ and $\beta(r_k, r_{k+1}) \to \infty$ as $k \to \infty$. For each $k = 1, 2, \cdots$, choose points $\{z_1^{(k)}, \cdots, z_{N(k)}^{(k)}\}$, $N(k) = 2^{n_k}$, equally distributed in the hyperbolic cercle centered at r_k of hyperbolic radius n_k. Here $n_k \to \infty$ as $k \to \infty$ in such a way that $n_k < \beta(r_k, r_{k+1})/4$.

Let $Z_2 = \{z_i^{(k)}\colon i = 1, \cdots, N(k), k = 1, 2, \cdots\}$. It can be shown that Z_1 and Z_2 satisfy condition (1.3) with the exponent $\alpha = 1/2$, while $Z_1 \cup Z_2$ does not fulfill (1.3) for any $0 < \alpha < 1$ because the number of points in Z_2 at hyperbolic distance n_k from the point $r_k \in Z_1$ is 2^{n_k}.

5. An interpolation problem for bounded Analytic Functions without zeros

Let $\mathbb{H}^\infty$ denote the algebra of bounded analytic functions in the unit disc $\mathbb{D}$. Let $(\mathbb{H}^\infty)^*$ be the subalgebra of $\mathbb{H}^\infty$ which consists on the functions in $\mathbb{H}^\infty$ without zeros in $\mathbb{D}$. If $f \in (\mathbb{H}^\infty)^*$ then $\log(\|f\|_\infty/|f(z)|) \in h^+$. So if $\{z_n\}$ is a sequence in $\mathbb{D}$ and $t_n = \log(\|f\|_\infty/|f(z_n)|)$, Harnack's inequality tells that

$$|\log t_n - \log t_m| \leq \beta(z_n, z_m), \qquad n, m = 1, 2, \cdots$$

So, as before, we may consider a notion of interpolating sequence.

Definition 8. A sequence of points $\{z_n\}$ in the unit disc is called an interpolating sequence for $(\mathbb{H}^\infty)^*$ if there exist constants $\varepsilon > 0$ and $0 < C < \infty$ such that for any sequence of non-vanishing complex values $\{w_n\}$, $|w_n| < C, n = 1, 2, \cdots$, satisfying

(5.1)
$$\left|\log\left(\log\left(\frac{C}{|w_n|}\right)\right) - \log\left(\log\left(\frac{C}{|w_m|}\right)\right)\right| \leq \varepsilon\beta(z_n, z_m),\ n, m = 1, 2, \cdots$$

there exists a function $f \in (\mathbb{H}^\infty)^*$ with $f(z_n) = w_n$, $n = 1, 2, \cdots$

The characterization of the interpolating sequences for $(\mathbb{H}^\infty)^*$ is given in the following result.

Theorem 9. *A separated sequence $\{z_n\}$ of points in the unit disc is interpolating for $(\mathbb{H}^\infty)^*$ if and only if there exist constants $M > 0$ and $0 < \alpha < 1$ such that*

$$\#\{z_j\colon \beta(z_j, z_n) \leq \ell\} \leq M 2^{\alpha\ell} \text{ for any } n, \ell = 1, 2, \cdots \tag{5.2}$$

Proof of Theorem 9. Let us start by showing the neccessity of condition (5.2). Given a separated interpolating sequence $\{z_n\}$ for $(\mathbb{H}^\infty)^*$ consider the constants $\varepsilon > 0$ and $C < \infty$ given in definition 8. Define the sequence of positive values $t_n = 2^{\varepsilon\beta(0, z_n)}$, $n = 1, 2, \cdots$ It is clear that

$$|\log_2 t_n - \log_2 t_m| \leq \varepsilon\beta(z_n, z_m), \quad n, m = 1, 2, \cdots .$$

Then, if we consider a sequence of complex values $\{w_n\}$ with $t_n = \log(C/|w_n|)$, we have $\sup_n |w_n| \leq C$ and furthermore $\{w_n\}$ satisfies condition (5.1). So, there exists a function $f \in \mathbb{H}^\infty$ without zeros with $f(z_n) = w_n$, $n = 1, 2, \cdots$ The function $v(z) = \log\left(\frac{C}{|f(z)|}\right)$ is a harmonic function, $v(z) \geq \log(C/\|f\|_\infty) := -k_1$, and interpolates the values $\{t_n\}$ at the points $\{z_n\}$. So, $u(z) = v(z) + k_1 \in h^+(\mathbb{D})$ and $u(z_n) = t_n + k_1 = 2^{\varepsilon\beta(0,z_n)} + k_1$, $n = 1, 2, , \cdots$ Now, arguing as in the proof of the necessity of Theorem 1, we can conclude that there exist constants $M > 0$ and $0 < \alpha < 1$ such that

$$\#\{z_j \colon \beta(z_j, z_n) \leq \ell\} \leq M 2^{\alpha\ell} \text{ for any } n, \ell = 1, 2, \cdots$$

Let us now show the sufficiency of condition (5.2). Given a separated sequence $\{z_n\}$ satisfying (5.2) and $\{w_n\}$ satisfying (5.1) for some ε, C, consider $t_n = \log \frac{C}{|w_n|}$. We can take $C > \|w_n\|_\infty$. Then obviously $\{t_n\}$ satisfies the compatibility condition (1.2). So, there exists a function $u \in h^+(\mathbb{D})$ with $u(z_n) = \log \frac{C}{|w_n|}$, for $n = 1, 2, \cdots$ Consider $u_0(z) = u(z) - \log(C)$ and let $\widetilde{u_0}(z)$ be the harmonic conjugate function of $u_0(z)$. Then $e^{-(u_0 + i\widetilde{u_0})}$ is a bounded analytic function that interpolates de values $\{|w_n|\gamma_n\}$ at the points $\{z_n\}$, where $\gamma_n = e^{-i\widetilde{u}_0(z_n)}$, $n = 1, 2, \cdots$. The sequence $\{z_n\}$ is separated and satisfies condition (1.3), so it is an interpolating sequence for $\mathbb{H}^\infty$ (see [C1] or [G2]). So there exists a bounded analytic function $g(z)$ such that $g(z_n) = -\operatorname{Arg}(\gamma_n) + \operatorname{Arg}(w_n)$ and then the function $h(z) = e^{-u_0 - i\widetilde{u_0}} e^{ig}$ is a bounded analytic function without zeros with $h(z_n) = w_n$ for any $n = 1, 2, \cdots$ □

6. Higher Dimensions

Let $h^\infty(\mathbb{R}^{d+1}_+)$ be the space of bounded harmonic functions in the upper-half space $\mathbb{R}^{d+1}_+ = \{(x, y) \colon x \in \mathbb{R}^d, y > 0\}$. A sequence of points $\{z_n\} \subset \mathbb{R}^{d+1}_+$ is called an interpolating sequence for $h^\infty(\mathbb{R}^{d+1}_+)$ if for any bounded sequence $\{w_n\}$ of real numbers there exists $u \in h^\infty(\mathbb{R}^{d+1}_+)$ with $u(z_n) = w_n$, $n = 1, 2, \cdots$. When the dimension $d > 1$, there is no complete geometric description of the interpolating sequences for $h^\infty(\mathbb{R}^{d+1}_+)$. In [C1] and [CG], L. Carleson and J. Garnett proved the following result.

Theorem 10. [C1], [CG] *Let $\{z_n = (x_n, y_n)\}$ be a sequence of points in $\mathbb{R}^{d+1}_+$, $d > 1$.*

(a) *Assume $\{z_n\}$ is an interpolating sequence for $h^\infty(\mathbb{R}^{d+1}_+)$. Then*

$$\inf_{n \neq m} \beta(z_n, z_m) > 0 \tag{6.1}$$

and there exists a constant $C = C(\{z_n\})$ such that

$$\sum_{z_n \in Q} y_n^d \leq C \ell(Q)^d \tag{6.2}$$

for any Carleson cube Q of the form

$$Q = \{(x,y) \in \mathbb{R}^{d+1}_+ \colon |x - x_0| < \ell(Q), \quad 0 < y < \ell(Q)\},$$

where $x_o \in \mathbb{R}^d$.

(b) *Assume $\{z_n\}$ satisfies the two conditions (6.1) and (6.2) above. Then $\{z_n\}$ can be splitted into a finite number of disjoint subsequences Λ_j, $j = 1, \cdots, N$, that is,*

$$\{z_n\} = \Lambda_1 \cup \cdots \cup \Lambda_N,$$

such that $\Lambda_i \cup \Lambda_j$ is an interpolating sequence for $h^\infty(\mathbb{R}^{d+1}_+)$ for any $i, j = 1, \cdots, N$.

Here $\beta(z,w)$ denotes the hyperbolic distance between the points $z, w \in \mathbb{R}^{d+1}_+$,

$$\beta(z,w) = \log_2 \frac{1 + \rho(z,w)}{1 - \rho(z,w)},$$

where $\rho(z,w) = |z - w|/|z - \bar{w}|$ and $\bar{w} = (w_1, \cdots, w_d, -w_{d+1})$.

Moreover in [CG], the authors present several geometric conditions on the sequence $\{z_n\}$ which imply that $\{z_n\}$ is an interpolating sequence for $h^\infty(\mathbb{R}^{d+1}_+)$. However it is not known if the two necessary conditions (6.1) and (6.2) are sufficient. Related interpolation problems have been considered in [A] and [D]. The situation for interpolating sequences for the space $h^+(\mathbb{R}^{d+1}_+)$ of positive harmonic functions in $\mathbb{R}^{d+1}_+$ is quite similar. A sequence of points $\{z_n\} \subset \mathbb{R}^{d+1}_+$ will be called an interpolating sequence for $h^+(\mathbb{R}^{d+1}_+)$ if there exists a constant $\varepsilon = \varepsilon(\{z_n\}) > 0$ such that for any sequence $\{w_n\}$ of positive values satisfying

$$|\log_2 w_n - \log_2 w_m| \le \varepsilon \beta(z_n, z_m), \quad n, m = 1, 2, \cdots,$$

there exists a function $u \in h^+(\mathbb{R}^{d+1}_+)$ with $u(z_n) = w_n$, $n = 1, 2, \cdots$

As before, a sequence of points $\{z_n\} \subset \mathbb{R}^{d+1}_+$ is called separated if $\inf_{n \neq m} \beta(z_n, z_m) > 0$.

Theorem 11. *Let $\{z_n\}$ be a separated sequence of points in the upper-half space $\mathbb{R}^{d+1}_+$, $d > 1$.*

(a) *Assume that $\{z_n\}$ is an interpolating sequence for $h^+(\mathbb{R}^{d+1}_+)$. Then there exist constants $M > 0$ and $0 < \alpha < 1$ such that*

$$\#\{z_k \colon \beta(z_k, z_n) \le l\} \le M 2^{\alpha d l}, \quad l, n = 1, 2, \cdots \tag{6.3}$$

(b) *Assume that $\{z_n\}$ satisfies the condition (6.3) above. Then $\{z_n\}$ can be splitted into a finite number of disjoint subsequences Λ_i, $i = 1, \cdots, N$,*

$$\{z_n\} = \Lambda_1 \cup \cdots \cup \Lambda_n,$$

such that $\Lambda_i \cup \Lambda_j$ *is an interpolating sequence for* $h^+(\mathbb{R}^{d+1}_+)$ *for any* $i, j = 1, \cdots, N$

The proof of (a) follows the same lines of the proof of the necessity in Theorem 1. The first two steps 3.1 and 3.2 of the proof of the sufficiency in Theorem 1 can be extended to several variables. However the third step 3.3 can not be fulfilled because we have not been able to show that a separated sequence satisfying condition (6.3) is an interpolating sequence for $h^\infty(\mathbb{R}^{d+1}_+)$. Since it is clear that (6.3) implies (6.2), applying the result of L. Carleson and J. Garnett, the sequence $\{z_n\}$ can be splitted into a finite number of disjoint subsequences $\Lambda_1, \cdots, \Lambda_N$ such that $\Lambda_i \cup \Lambda_j$ is an interpolating sequence for $h^\infty(\mathbb{R}^{d+1}_+)$, $i, j = 1, \cdots, N$. Arguing as in step 3.3 of the proof of the suffiency, one can show that for any $i, j = 1, \cdots, N$, the sequence $\Lambda_i \cup \Lambda_j$ is an interpolating sequence for $h^+(\mathbb{R}^{d+1}_+)$.

It is worth mentioning that we have not been able to prove that a separated sequence verifying (6.3) is interpolating for $h^+(\mathbb{R}^{d+1}_+)$, when $d > 1$.

References

[A] E. Amar, *Suites d'interpolation harmoniques*, J. Anal. Math., **32** (1977), 197–211.

[BN] B. Boe & A. Nicolau, *Interpolation by functions in the Bloch space*, J. Anal. Math., **94** (2004), 171–194.

[C1] L. Carleson, *An interpolation problem for bounded analytic functions*, Amer. J. Math., **80** (1958), 921–930.

[C2] L. Carleson, *A moment problem and harmonic interpolation*, preprint, Institut Mittag-Leffler, 1972

[CG] L. Carleson & J. Garnett, *Interpolating sequences and separation properties*, J. Anal. Math., **28** (1975), 273–299.

[D] K. Dyakonov, *Moment problems for bounded functions*, Comm. Anal. Geom., **2(4)** (1994), 533–562.

[DN] K. Dyakonov & A. Nicolau, *Free interpolation by non-vanishing analytic functions*, to appear in Trans. Amer. Math. Soc.

[G1] J. Garnett, *Interpolating sequences for bounded harmonic functions*, Indiana Univ. Math. J., **21** (1971/1972), 187–192.

[G2] J.B. Garnett, *Bounded Analytic Functions*, Pure and Applied Mathematics, 96. Academic Press, Inc., New York-London, 1981.

[H] Hall, *Sur la mesure harmonique de certains ensembles*, Arkiv för Matematik, Astronomi och Fysik, **25A(28)** (1937).

[HL] J.B. Hiriart-Urruty & C. Lemaréchal, *Convex Analysis and Minimization Algorithms I. Fundamentals*, Grundlehren der Mathematischen Wissenschaften 305, Springer-Verlag, Berlin, Heidelberg, New York, 1993.

[MS] D.E. Marshall & C. Sundberg, *Harmonic measure and radial projection*, Trans. Amer. Math. Soc., **316(1)** (1989), 81–95. Graduate Texts in Mathematics **149** Springer-Verlag, New York, 1994.

[S] K. Seip, *Interpolating and sampling in spaces of analytic functions*, University Lectures Series, **33** American Mathematical Society, Providence, RI, 2004.

Daniel Blasi, Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Spain
URL: http://www.mat.uab.es/~dblasi/
E-mail address: dblasi@mat.uab.es

Artur Nicolau, Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Spain
URL: http://www.mat.uab.es/~artur/
E-mail address: artur@mat.uab.es