Coexistence of uncountably many attracting sets for skew products on the cylinder

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Introduction—Motivation

We want to show that the existence of attracting sets for quasiperiodically forced systems can be extended to appropriate skew-products on the cylinder, homotopic to the identity, in such a way that the general system will have (at least) one attracting set corresponding to every irrational rotation number ϱ in the rotation interval of the base map. This attracting set is a copy of the attracting set of the system quasiperiodically forced by a (rigid) rotation of angle ϱ. This shows the co-existence of uncountably many attracting sets, one for each irrational in the rotation interval of the basis.

To fix ideas we will start by studying several examples of systems quasiperiodically forced by a (rigid) rotation (monotone and non-monotone). We also study these examples because this is the kind of models and dynamics that we want to extend simultaneously for uncountably many rotation numbers.

The Keller model

It is a skew product where the function in the second component has separated variables:

$$(1) \qquad \begin{cases} \theta_{n+1} & = R(\theta_n) = \theta_n + \omega \pmod 1, \\ x_{n+1} & = f(x_n) g(\theta_n) \end{cases}$$

where $x \in \mathbb{R}^+, \theta \in \mathbb{S}^1$, $\omega \in \mathbb{R} \setminus \mathbb{Q}$ and

1. $f \colon [0, \infty) \longrightarrow [0, \infty)$ is $\mathcal{C}^1$, bounded, strictly increasing, strictly concave and verifies $f(0) = 0$ (to fix ideas take $f(x) = \tanh(x)$ as in the [GOPY] model).
   Thus, $x = 0$ will be invariant.
2. $g \colon \mathbb{S}^1 \longrightarrow [0, \infty)$ is bounded and continuous (to fix ideas take $g(\theta) = 2\sigma |\cos(2\pi\theta)|$ with $\sigma > 0$ in a similar way to the [GOPY] model – except for the absolute value).

Pinching

There are big differences between the cases when g takes the value 0 at some point: the *pinched* case and the case when g is strictly positive.

Remark

In the pinched case any T–invariant set has to be 0 on a point and hence on a dense set because the circle $x \equiv 0$ is invariant and the θ-projection of every invariant object must be invariant under R.

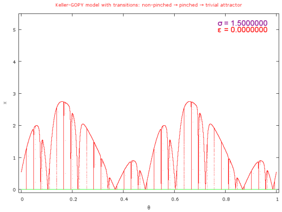
A particular example (based in the [GOPY] model)

$$(2) \qquad \begin{cases} \theta_{n+1} & = \theta_n + \omega \pmod 1, \\ x_{n+1} & = 2\sigma \tanh(x_n)(\varepsilon + |\cos(2\pi\theta_n)|) \end{cases}$$

where $x \in \mathbb{R}, \theta \in \mathbb{S}^1$, $\omega = \frac{\sqrt{5}+1}{2}$. $\sigma > 0$ and $\varepsilon \geq 0$.

Remark

The attractor of the above system (if it exists) will be pinched if and only if $\varepsilon = 0$.



▸ Movie: A family of Keller Attractors

The following theorem due to Keller [Kel] makes the above informal ideas rigorous. Before stating it we need to introduce the constant σ:

Since the line $x = 0$ is invariant, by using Birkhoff Ergodic Theorem, it turns out that

$$\sigma := f'(0) \exp\left(\int_{\mathbb{S}^1} \log g(\theta) d\theta\right) < \infty.$$

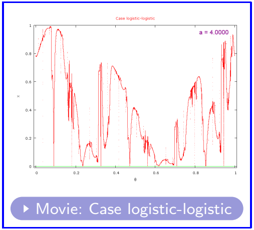
is the vertical Lyapunov exponent on the circle $x = 0$.

Keller Theorem

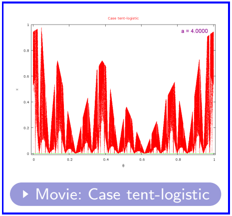
There exists an upper semicontinuous map $\phi\colon \mathbb{S}^1 \longrightarrow [0, \infty)$ whose graph is invariant under the Model (2). Moreover,

1. The Lebesgue measure on the circle, lifted to the graph of ϕ is a Sinai-Ruelle-Bowen measure,
2. if $\sigma \leq 1$ then $\phi \equiv 0$,
3. if $\sigma > 1$ then $\phi(\theta) > 0$ for almost every θ,
4. if $\sigma > 1$ and $g(\theta_0) = 0$ for some θ_0 then the set $\{\theta\colon \phi(\theta) > 0\}$ is meager and ϕ is almost everywhere discontinuous,
5. if $\sigma > 1$ and $g > 0$ then ϕ is positive and continuous; if g is $\mathcal{C}^1$ then so is ϕ,
6. if $\sigma \neq 1$ then $|x_n - \phi(\theta_n)| \to 0$ exponentially fast for almost every θ and every $x > 0$.

We consider two basic situations. One with strict concavity and another one with non-strict concavity. In any case we take the golden mean as a rotation number ($\omega = \frac{\sqrt{5}+1}{2}$ and $g(\theta) = a\theta(1-\theta)$ (thus we are in the pinched case).



$$f(x) = 4x(1-x)$$
$$a \in [1.8, 4]$$



$$f(x) = 1 - |2x-1|$$
$$a \in [3.68, 4]$$

The model we study

We denote the cylinder by $\mathbb{S}^1 \times \mathcal{K}$ (where $\mathcal{K}$ is either a closed interval of $\mathbb{R}$ containing zero, $\mathbb{R}^+$ or $\mathbb{R}$ itself). We consider the class of skew-products on $\mathbb{S}^1 \times \mathcal{K}$ of the form:

$$(3) \quad \begin{pmatrix} \theta_{n+1} \\ x_{n+1} \end{pmatrix} = T \begin{pmatrix} \theta_n \\ x_n \end{pmatrix} \qquad \text{where} \qquad T \begin{pmatrix} \theta \\ x \end{pmatrix} = \begin{pmatrix} f(\theta) \\ p(x)q(\theta) \end{pmatrix},$$

f is a continuous circle map of degree one with a lift F such that $\operatorname{Rot}(F)$ is non-empty (non-degenerate), q is a continuous map from $\mathbb{S}^1$ to $\mathcal{K}$ and p is a continuous map from $\mathcal{K}$ to itself.

The notion of attracting set

Let μ be an ergodic measure of f and let $\mathscr{U}$ be a measurable f-invariant set such that $\mu(\mathscr{U}) = 1$. Let $\varphi\colon \mathscr{U} \longrightarrow \mathcal{K}$ be a correspondence whose graph is T-invariant on $\mathscr{U}$ (i.e. $T(\text{graph}(\varphi)) = \text{graph}(\varphi)$). The closure of graph($\varphi$) will be called an *attracting set with support $\mathscr{U}$ and generated by φ* whenever

$$\lim_{n\to\infty} \| T^n(\theta, x) - T^n(\theta, z(x)) \| = 0$$

for every $\theta \in \mathscr{U}$ and x in a subset of $\mathcal{K}$ of positive Lebesgue measure, and some $z(x) \in \varphi(\theta)$ (in particular, $\omega_T(\theta, x) \subset \omega_T(\theta, \varphi(\theta))$).

A survey on rotation theory in the circle and water functions

We are only interested in continuous degree one circle maps. These are continuous maps such that $F(x+1) = F(x) + 1$ for every $x \in \mathbb{R}$ and every lifting F. We denote by $\mathfrak{L}_1$ the set of all liftings of continuous circle maps of degree one.

Definition

For each $F \in \mathfrak{L}_1$ and $x \in \mathbb{R}$ we define the *F-rotation number of x* as

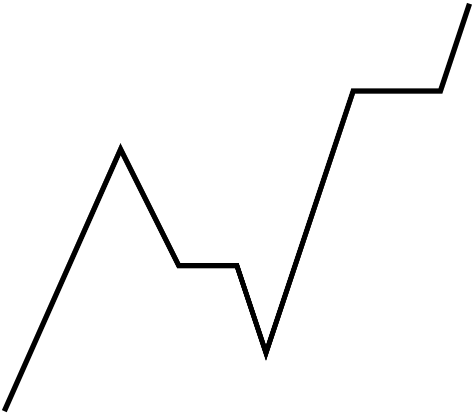
$$\rho_F(x) := \limsup_{n\to\infty} \frac{F^n(x) - x}{n},$$

and the *rotation set of F* as:

$$\operatorname{Rot}(F) := \{\rho_F(x) : x \in \mathbb{R}\}.$$

From [Ito] we know that $\operatorname{Rot}(F)$ is a closed interval of $\mathbb{R}$.

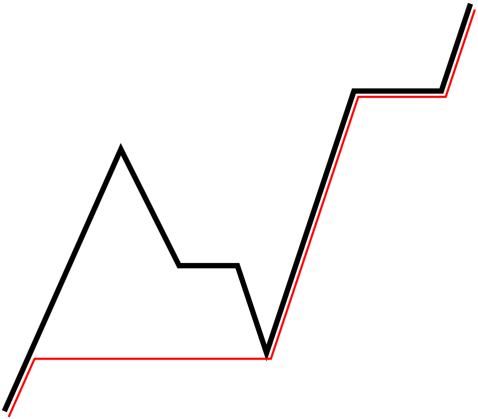
Given $F \in \mathfrak{L}_1$ we define the *lower* and *upper* liftings as follows:



Upper and lower functions

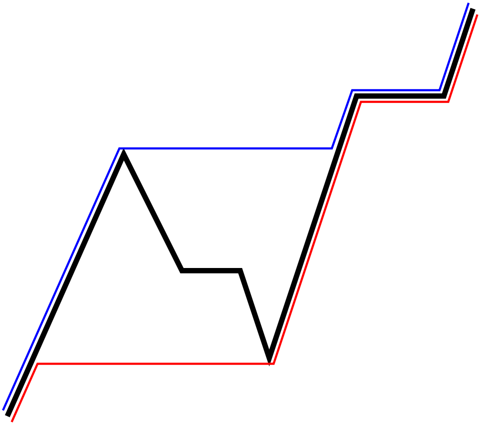
Given $F \in \mathfrak{L}_1$ we define the *lower* and *upper* liftings as follows:

$$F_l(x) = \min\{F(y) : y \geq x\}$$



Given $F \in \mathfrak{L}_1$ we define the *lower* and *upper* liftings as follows:

$$F_l(x) = \min\{F(y) : y \geq x\}$$
$$F_u(x) = \max\{F(y) : y \leq x\}$$



Properties of upper and lower functions

Clearly, F_l and F_u are non-decreasing functions from $\mathfrak{L}_1$, $F_l \leq F \leq F_u$ and, if F is non-decreasing then $F = F_l = F_u$. Moreover, if $F, G \in \mathfrak{L}_1$ are such that $F \leq G$, then $F_l \leq G_l$ and $F_u \leq G_u$.

If $F \in \mathfrak{L}_1$ and it is non-decreasing,

$$\rho_F(x) = \lim_{n\to\infty} \frac{F^n(x) - x}{n}$$

and it is independent on $x \in \mathbb{R}$. This number is denoted as $\rho(F)$ and called the *rotation number of F*.

Theorem

For every $F \in \mathfrak{L}_1$, $\operatorname{Rot}(F) = [\rho(F_l), \rho(F_u)]$.

Water functions—a homotopy family from F_l to F_u

Definition

Given $F \in \mathfrak{L}_1$ and $0 \leq \alpha \leq \|F - F_l\|_\infty$, we define the *water function of level α* as

$$F_\alpha = (\min\{F, F_l + \alpha\})_u$$

Water Functions movie

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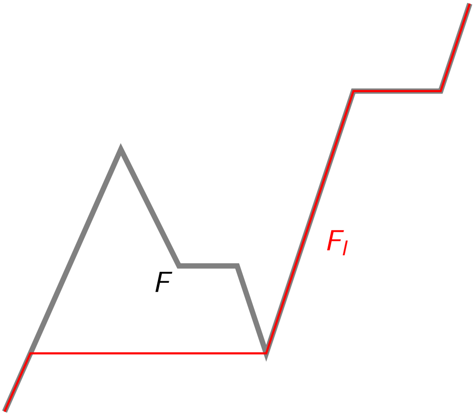

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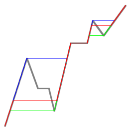
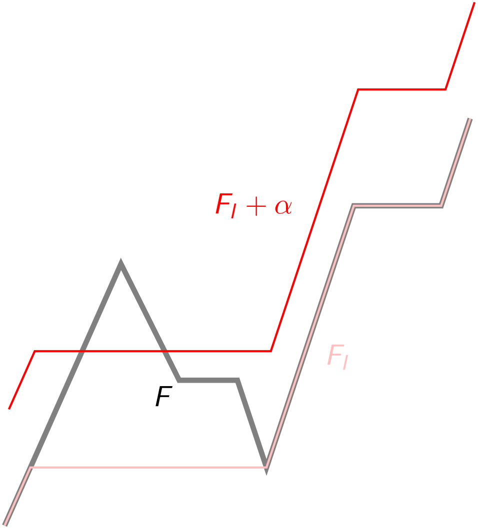


▸ Water Functions movie



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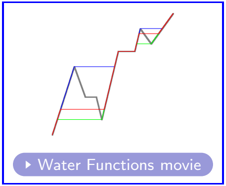

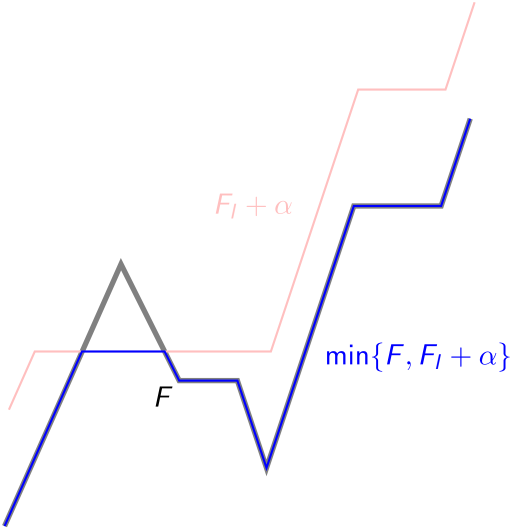

Observe that, $\min\{F, F_l + \alpha\} \in \mathfrak{L}_1$

Water functions—a homotopy family from F_l to F_u

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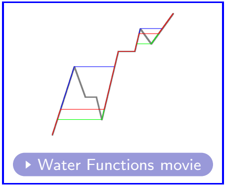

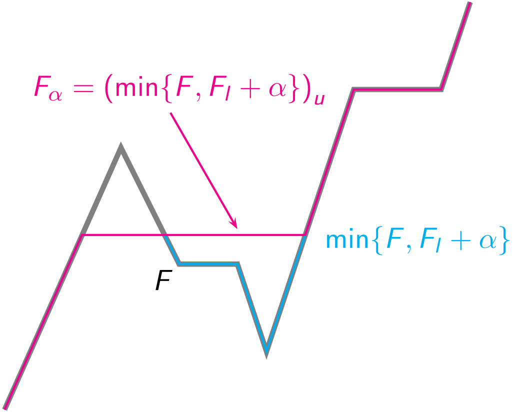

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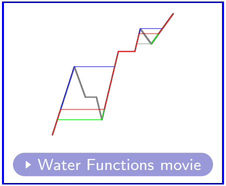
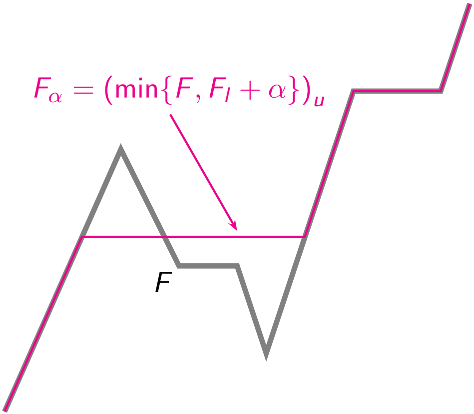




Remarks

- F_α is a non-decreasing functions from $\mathfrak{L}_1$,
- $F_0 = F_l$ and $F_{\|F-F_l\|_\infty} = F_u$
- $F_\alpha \leq F_{\alpha'}$ whenever $\alpha \leq \alpha'$
- For every α, F_α coincides with F in the complement of $\text{Const}(F_\alpha) \supset \text{Const}(F)$.

The map $\alpha \longmapsto \rho(F_\alpha)$ from $[0, \|F - F_l\|_\infty]$ to $\text{Rot}(F)$ is continuous, onto and non-decreasing.

So, for every $\varrho \in \text{Rot}(F)$, there exists an $\alpha_\varrho \in [0, \|F - F_l\|_\infty]$ such that $\rho(F_{\alpha_\varrho}) = \varrho$.

A consequence of Auslander-Katznelson Theorem

Definition

For every $\varrho \in \mathbb{R}$ we denote the rotation by angle ϱ by $\Phi_\varrho(\theta) := \theta + \varrho \pmod 1$.

Let $F \in \mathfrak{L}_1$ be non-decreasing and such that $\rho(F)$ is irrational. From [G. R. Hall] we know that f is semiconjugate to the irrational rotation $\Phi_{\rho(F)}$ by a *non-decreasing* map h: $h \circ f = \Phi_{\rho(F)} \circ h$.

Proposition

Let $F \in \mathfrak{L}_1$ be non-decreasing and such that $\rho(F)$ is irrational. Then f, the projection of F to the circle, has a measurable invariant set $\mathscr{U} \subset \mathbb{S}^1$ and a unique ergodic measure μ such that $\mu(\mathscr{U}) = 1$, $\mathrm{Cl}(\mathscr{U})$ is disjoint from $\mathrm{Const}(f)$, $h\big|_{\mathscr{U}}$ is a homeomorphism and $h(\mathscr{U})$ is a dense $\Phi_{\rho(F)}$-invariant set. If f is not a homeomorphism, then $\mathscr{U}$ is nowhere dense in $\mathbb{S}^1$.

The main result — Theorem

Consider a system of the form (3). To every $\varrho \in \operatorname{Rot}(F) \setminus \mathbb{Q}$ we can associate a measurable f-invariant set $\mathscr{U}_\varrho \subset \mathbb{S}^1$, a continuous non-decreasing circle map of degree one h_ϱ, and an f-ergodic measure μ_ϱ such that $\mu_\varrho(\mathscr{U}_\varrho) = 1$, $\rho_F(e^{-1}(\operatorname{Cl}(\mathscr{U}_\varrho))) = \varrho$, $h_\varrho\big|_{\mathscr{U}_\varrho} \colon \mathscr{U}_\varrho \longrightarrow h_\varrho(\mathscr{U}_\varrho)$ is a homeomorphism and $h_\varrho(\mathscr{U}_\varrho)$ is a dense Φ_ϱ-invariant set. Additionally, the sets $\operatorname{Cl}(\mathscr{U}_\varrho)$ are pairwise disjoint.

Assume that, for every $\varrho \in \operatorname{Rot}(F) \setminus \mathbb{Q}$, the system

$$(4) \qquad \begin{pmatrix} \theta_{n+1} \\ x_{n+1} \end{pmatrix} = S_\varrho \begin{pmatrix} \theta_n \\ x_n \end{pmatrix} \text{ where } S_\varrho \begin{pmatrix} \theta \\ x \end{pmatrix} = \begin{pmatrix} \Phi_\varrho(\theta) \\ p(x) q(h_\varrho^{-1}(\theta)) \end{pmatrix},$$

has an attracting set with support $h_\varrho(\mathscr{U}_\varrho)$ which is the closure of the graph of a correspondence $\varphi_\varrho \colon h_\varrho(\mathscr{U}_\varrho) \longrightarrow \mathcal{K}$. Then, the closure of the graph of $\varphi_\varrho \circ h_\varrho$ is an attracting set of T with support $\mathscr{U}_\varrho$. Thus, whenever $\operatorname{Rot}(F)$ is non-degenerate, T has uncountably many attracting sets coexisting dynamically.

We start with system (3) and consider $\varrho \in \operatorname{Rot}(F) \setminus \mathbb{Q}$.

Then we take the water function F_{α_ϱ} with $\alpha_\varrho \in [0, \|F - F_l\|_\infty]$ such that $\rho(F_{\alpha_\varrho}) = \varrho$.

We denote respectively by $\mathscr{U}_\varrho$, h_ϱ and μ_ϱ the set $\mathscr{U}$, the map h and the measure μ given by the above Proposition for the map f_{α_ϱ}. This already gives that $\mathscr{U}_\varrho$ is measurable and f_{α_ϱ}-invariant, $\mu_\varrho(\mathscr{U}_\varrho) = 1$, $h_\varrho\big|_{\mathscr{U}_\varrho} \colon \mathscr{U}_\varrho \longrightarrow h_\varrho(\mathscr{U}_\varrho)$ is a homeomorphism and $h_\varrho(\mathscr{U}_\varrho)$ is a dense Φ_ϱ-invariant set.

Since f_{α_ϱ} coincides with f in the complement of $\operatorname{Const}(f_{\alpha_\varrho})$ and $\operatorname{Cl}(\mathscr{U}_\varrho)$ is disjoint from $\operatorname{Const}(f_{\alpha_\varrho})$,

$$f\big|_{\operatorname{Cl}(\mathscr{U}_\varrho)} = f_{\alpha_\varrho}\big|_{\operatorname{Cl}(\mathscr{U}_\varrho)}. \tag{5}$$

Hence, $f(\mathscr{U}_\varrho) = \mathscr{U}_\varrho$ because $\mathscr{U}_\varrho$ is f_{α_ϱ}-invariant.

Idea of the proof (II)

Since F_{α_ϱ} is non-decreasing,

$$\rho_F(e^{-1}(\mathrm{Cl}(\mathscr{U}_\varrho))) = \rho_{F_{\alpha_\varrho}}(e^{-1}(\mathrm{Cl}(\mathscr{U}_\varrho))) = \varrho(F_{\alpha_\varrho}) = \varrho.$$

Consequently, if $\varrho, \varrho' \in \mathrm{Rot}_I(F)$ and $\varrho \neq \varrho'$, then $\mathrm{Cl}(\mathscr{U}_\varrho) \cap \mathrm{Cl}(\mathscr{U}_{\varrho'}) = \emptyset$.

Also, it can be proved that μ_ϱ is an ergodic measure of f.

Now we assume that, for every $\varrho \in \mathrm{Rot}(F) \setminus \mathbb{Q}$, the system (4):

$$\begin{pmatrix}\theta_{n+1}\\ x_{n+1}\end{pmatrix} = S_\varrho \begin{pmatrix}\theta_n\\ x_n\end{pmatrix} \text{ where } S_\varrho \begin{pmatrix}\theta\\ x\end{pmatrix} = \begin{pmatrix}\Phi_\varrho(\theta)\\ p(x)q(h_\varrho^{-1}(\theta))\end{pmatrix},$$

has an attracting set with support $h_\varrho(\mathscr{U}_\varrho)$ which is the closure of the graph of a correspondence $\varphi_\varrho \colon h_\varrho(\mathscr{U}_\varrho) \longrightarrow \mathcal{K}$.

Idea of the proof (III)

We have to prove that the closure of the graph of $\varphi_\varrho \circ h_\varrho$ is an attracting set of T with support $\mathscr{U}_\varrho$.

Since h_ϱ is a semiconjugacy between f_{α_ϱ} and Φ_ϱ, from (5) we get

$$h_\varrho(f(\theta)) = h_\varrho(f_{\alpha_\varrho}(\theta)) = \Phi_{\rho(F)}(h_\varrho(\theta)) \tag{6}$$

for every $\theta \in \mathscr{U}_\varrho$.

Set $H := (h_\varrho, \mathrm{Id})$ which is a homeomorphism from $\mathscr{U}_\varrho \times \mathcal{K}$ to $h_\varrho(\mathscr{U}_\varrho) \times \mathcal{K}$. We have

$$\begin{aligned} H(T(\theta, z)) &= (h_\varrho(f(\theta)), p(z)q(\theta)) = (\Phi_\varrho(h_\varrho(\theta)), p(z)q(h_\varrho^{-1}(h_\varrho(\theta)))) \\ &= S_\varrho(H(\theta, z)), \end{aligned}$$

for every $\theta \in \mathscr{U}_\varrho$ and $z \in \mathcal{K}$.

Idea of the proof (and IV)

That is, Systems (3) and (4) are conjugate by H on the set $\mathscr{U}_\varrho \times \mathcal{K}$.

From this fact it is not difficult to prove the T- invariance of the graph of $\varphi_\varrho \circ h_\varrho$. Also it follows that the closure of this graph is an attracting set of T with support $\mathscr{U}_\varrho$.

Comments to the main theorem

Remark

It is precisely the need of the above conjugacy what forces us to replace the map q by $q \circ h_\varrho^{-1}$ on System (4).

Remark

The map $q \circ h_\varrho^{-1}$ is continuous in $h_\varrho(\mathscr{U}_\varrho)$ which is dense in $\mathbb{S}^1$. Hence, if $q \circ h_\varrho^{-1}$ is discontinuous, it has only jump discontinuities in the complement of $h_\varrho(\mathscr{U}_\varrho)$. Therefore, since we want to reuse attractors from systems with a (rigid irrational) rotation in the base we are forced to consider System (4) on $h_\varrho(\mathscr{U}_\varrho) \times \mathcal{K}$ and not on $\mathbb{S}^1 \times \mathcal{K}$.

On the number of pieces of an attracting set

Remark

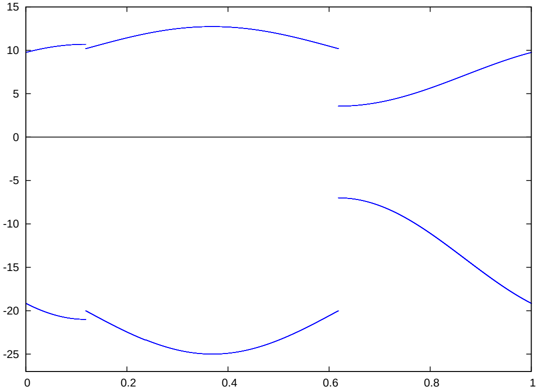
The measure μ_ϱ lifted to the closure of the graph of $\varphi_\varrho \circ h_\varrho$ is an invariant measure of T. Moreover, this measure is ergodic if and only if the closure of the graph of $\varphi_\varrho \circ h_\varrho$ is a minimal set of T. In particular this is a criterion to decide the undecomposability of the attracting set into several smaller attracting sets.

When f is defined in the whole real line or in the interval $[-1, 1]$ so that $\text{sign}(x) = \text{sign}(f(x))$, there are two possibilities for such an attracting set: either its closure is a minimal attractor or it splits into two different minimal attractors and each of these attractors is the closure of the graph of a map from $\mathscr{U}$ to the fibres.

2-periodic function graphs as attracting set—q negative

$f = \Phi_{\frac{\sqrt{5}-1}{2}}$, $\mathcal{K} = \mathbb{R}$.

When q is negative, the orbits keep alternating between $\mathbb{R}^+$ and $\mathbb{R}^-$. Thus, typically, there will be an attracting set which is a 2-periodic orbit of function graphs.

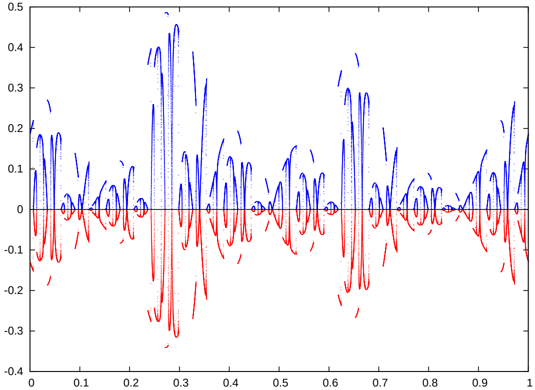


$$p(x) = \begin{cases} \tanh(x) & \text{if } x \geq 0 \\ \frac{\tanh(x-2)+\tanh(2)}{1-\tanh(2)^2} & \text{if } x \leq 0 \end{cases} \quad q(\theta) = \begin{cases} 7(\cos(2\pi\theta) - 2) & \text{if } \theta \in [0, \frac{1}{2}) \\ 5(\cos(2\pi\theta) - 4) & \text{otherwise.} \end{cases}$$

Attracting set with two minimal components—q positive

$f = \Phi_{\frac{\sqrt{5}-1}{2}}$,
$\mathcal{K} = [-1, 1]$.

When q is positive, System (3) can be split into two (one restricted to $\mathbb{R}^+$ and the other one restricted to $\mathbb{R}^-$). Consequently we get two attracting sets intersecting at $x \equiv 0$.



$$p(x) = \begin{cases} x(1-x) & \text{if } x \geq 0 \\ \frac{1}{2}x(x+1)(x+2) & \text{if } x \leq 0 \end{cases} \quad q(\theta) = \begin{cases} 2.1|\cos(2\pi\theta)| & \text{if } \theta \in [0, \frac{1}{2}) \\ 2.5|\sin(2\pi\theta)| & \text{otherwise.} \end{cases}$$