A strongly invariant pinched core strip that does not contain any arc of curve.

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   - Pseudo-curves
2. The inductive construction of a pseudo-curve
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Motivation

In the paper

[FJJK] R. Fabbri, T. Jäger, R. Johnson and G. Keller, *A Sharkovskii-type theorem for minimally forced interval maps*, Topological Methods in Nonlinear Analysis, Journal of the Juliusz Shauder Center, **26** (2005), 163–188.

the Sharkovskiĭ theorem was extended to a class of systems that are essentially quasi-periodically forced interval maps.

This is a first step towards the understanding of the *quasi-periodically forced Combinatorial dynamics*.

Sharkovskiĭ's Theorem for quasi-periodically forced interval maps

In what follows we consider the cylinder $\mathbb{S}^1 \times I$ and the following family of *skew products* on it:

$$\begin{pmatrix} \theta_{n+1} \\ x_{n+1} \end{pmatrix} = T \begin{pmatrix} \theta_n \\ x_n \end{pmatrix} = \begin{pmatrix} R(\theta_n) \\ f(\theta_n, x_n) \end{pmatrix}$$

where $R(\theta_n) = \theta_n + \omega \pmod 1$ with $\omega \in \mathbb{R} \setminus \mathbb{Q}$ and $f\colon \mathbb{S}^1 \times I \longrightarrow I$ is continuous in both variables.

Observation

In fact, in [JFFK] they consider a slightly more general situation. Indeed, instead of taking the cylinder they consider the product of a compact metric space Θ with I. Then, $R\colon \Theta \longrightarrow \Theta$ is a minimal homeomorphism with the property that R^ℓ is minimal for every ℓ. We work in the cylinder case for simplicity and clarity.

The Sharkovskiĭ Ordering ${}_{\text{Sh}}\!\geq$

It is the ordering

$$
\begin{array}{l}
3 \;{}_{\text{Sh}}\!> 5 \;{}_{\text{Sh}}\!> 7 \;{}_{\text{Sh}}\!> \cdots \;{}_{\text{Sh}}\!> \\
2\cdot 3 \;{}_{\text{Sh}}\!> 2\cdot 5 \;{}_{\text{Sh}}\!> 2\cdot 7 \;{}_{\text{Sh}}\!> \cdots \;{}_{\text{Sh}}\!> \\
4\cdot 3 \;{}_{\text{Sh}}\!> 4\cdot 5 \;{}_{\text{Sh}}\!> 4\cdot 7 \;{}_{\text{Sh}}\!> \cdots \;{}_{\text{Sh}}\!> \\
\qquad\qquad\vdots \\
2^n\cdot 3 \;{}_{\text{Sh}}\!> 2^n\cdot 5 \;{}_{\text{Sh}}\!> 2^n\cdot 7 \;{}_{\text{Sh}}\!> \cdots \;{}_{\text{Sh}}\!> \\
\qquad\qquad\vdots \\
2^\infty \;{}_{\text{Sh}}\!> \cdots \;{}_{\text{Sh}}\!> 2^n \;{}_{\text{Sh}}\!> \cdots \;{}_{\text{Sh}}\!> 16 \;{}_{\text{Sh}}\!> 8 \;{}_{\text{Sh}}\!> 4 \;{}_{\text{Sh}}\!> 2 \;{}_{\text{Sh}}\!> 1.
\end{array}
$$

defined on the set $\mathbb{N}_{\text{Sh}} = \mathbb{N} \cup \{2^\infty\}$ (we have to include the symbol 2^∞ to assure the existence of supremum for certain sets).

In the ordering ${}_{\text{Sh}}\!\geq$ the least element is 1 and the largest is 3. The supremum of the set $\{1, 2, 4, \dots, 2^n, \dots\}$ is 2^∞.

Theorem (Fabbri, Jäger, Johnson and Keller)

Suppose that $T \colon \mathbb{S}^1 \times I \longrightarrow \mathbb{S}^1 \times I$ of the above form admits a q-periodic strip and let $p \in \mathbb{N}$ be such that $p \leq_{\mathsf{Sh}} q$. Then T admits a p-periodic core strip.

Theorem (Fabbri, Jäger, Johnson and Keller)

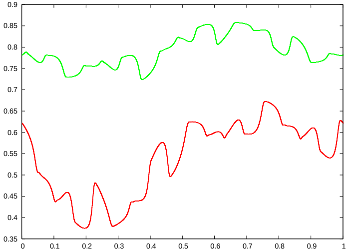
Suppose that $T\colon \mathbb{S}^1 \times I \longrightarrow \mathbb{S}^1 \times I$ *of the above form admits a* q*-periodic strip and let* $p \in \mathbb{N}$ *be such that* $p \leq_{\mathrm{Sh}} q$*. Then* T *admits a* p*-periodic core strip.*

Remark

In the trivial case when f does not depend on θ then the periodic strips are sets of circles in the cylinder which are obtained as a product of periodic orbits P (or periodic orbits of intervals) of f by the circle $\mathbb{S}^1$: $\mathbb{S}^1 \times P$.

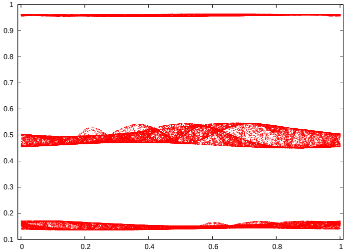
Examples of periodic (core) strips

In both cases, $\omega = \frac{\sqrt{5}-1}{2}$ and the map $f(\theta, x)$ is specified below the figure in each case.



$3.28x(1-x) + \frac{4}{100}\cos(2\pi\theta)$

A two periodic orbit of periodic curves.



$3.85x(1-x)(1+\frac{111}{10^5}\cos(2\pi\theta))$

A numerical three periodic orbit of periodic solid strips (needs analytical proof of its existence).
They correspond to the three periodic orbit of transitive intervals exhibited by the map $\mu x(1-x)$ with $\mu = 3.85\ldots$

The notation in the theorem

We will not define the [FJJK] notion of *core*. Rather we will directly define the notion of a strip and the two possible kinds of *core strips*.

Definition (Strip)

A *strip* is a closed subset A of the cylinder such that

$$\{\theta \in \mathbb{S}^1 \colon A \cap (\{\theta\} \times I) \text{ is an interval}\}$$

is a residual set on $\mathbb{S}^1$.

Remember

that $G \subseteq \mathbb{S}^1$ is *residual* if it contains the intersection of a countable family of open dense subsets of $\mathbb{S}^1$.

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that $G \subseteq \mathbb{S}^1$ is *residual* if it contains the intersection of a countable family of open dense subsets of $\mathbb{S}^1$.

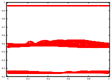
As it has been said, there are two kinds of core strips: *solid* or *pinched*.

Core strips

Definition (solid strip)

A strip A is *solid* if for each $\theta \in \mathbb{S}^1$, $A \cap (\{\theta\} \times I)$ is an interval and

$$\inf_{\theta \in \mathbb{S}^1} |A \cap (\{\theta\} \times I)| > 0.$$

An example is the picture shown before: 

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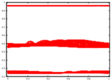
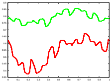
An example is the picture shown before: 

Definition (pinched strip)

A strip A is *pinched* if $A \cap (\{\theta\} \times I)$ is a point for a dense set of $\theta \in \mathbb{S}^1$.

An example is the picture shown before: 

Pseudo-curves

The pinched core strips are the *pseudo-curves* according to the following definition.

Definition (Pseudo-curve)

A subset of the cylinder is a *pseudo-curve* if it is the closure of the graph of a continuous function from a residual set of $\mathbb{S}^1$ into I.

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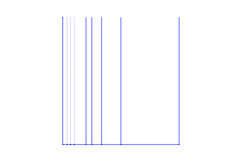
A subset of the cylinder is a *pseudo-curve* if it is the closure of the graph of a continuous function from a residual set of $\mathbb{S}^1$ into I.

Observe that a pseudo-curve is a pinched strip by definition but not conversely.

Example: the *harmonic comb* (a pinched non-core strip)



Properties of pseudo-curves

Remark

A *curve* (that is, the graph of a continuous function from $\mathbb{S}^1$ to I) is a pseudo-curve.

Properties of pseudo-curves

1. If a pseudo-curve contains a curve then it is a curve.
2. Any strongly T-invariant pseudo-curve is a minimal set.
3. If a strongly T-invariant pseudo-curve contains an arc of a curve, then it is also a curve (since the base map is an irrational rotation).

A subset A of the cylinder is *strongly T-invariant* if $T(A) = A$.
An *arc of a curve* is the graph of a continuous function from an arc of $\mathbb{S}^1$ to I.

Motivation

In this context, a natural question is whether the [FJJK] theorem is valid restricted to curves. That is:

Question 1

is it true that if T has a q-periodic curve and $p \leq_{\text{Sh}} q$ then all p-periodic strips of T are curves?

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A counterexample to Question 1 would be given by the *positive* answer to:

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Can a pseudo-curve which is not a curve occur as the unique strongly invariant object forced by a 2-periodic orbit of curves?

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Question 2

Can a pseudo-curve which is not a curve occur as the unique strongly invariant object forced by a 2-periodic orbit of curves?

The aim of this talk is to construct the example required in Question 2.

Motivation II

More precisely, we will construct an example of a skew product T on the cylinder which will have a 2-periodic orbit of curves and a strongly T-invariant pseudo-curve that does not contain any arc of a curve. Moreover, our example is monotone (decreasing) on the fibres and the pinched set has Lebesgue measure one. However, it is not a continuous curve.

Motivation II

More precisely, we will construct an example of a skew product T on the cylinder which will have a 2-periodic orbit of curves and a strongly T-invariant pseudo-curve that does not contain any arc of a curve. Moreover, our example is monotone (decreasing) on the fibres and the pinched set has Lebesgue measure one. However, it is not a continuous curve.

The construction is done in two steps:

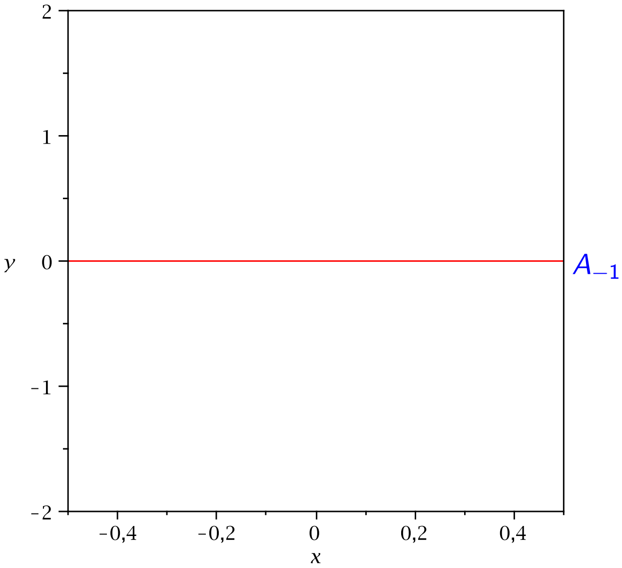
1. First we topologically construct a pseudo-curve as a *limit* of sets A_i defined inductively.
2. Second we construct a quasi-periodically forced skew product T on the cylinder which has a 2-periodic orbit of curves (the upper and lower circles) and the pseudo-curve as a totally invariant object.

The inductive construction of a pseudo-curve

Our cylinder is $\Omega = \mathbb{S}^1 \times [-2, 2]$.

The pseudo-curve is constructed as a limit of sets A_i defined inductively.

A rough idea of the construction is given by the following first three elements of the sequence:



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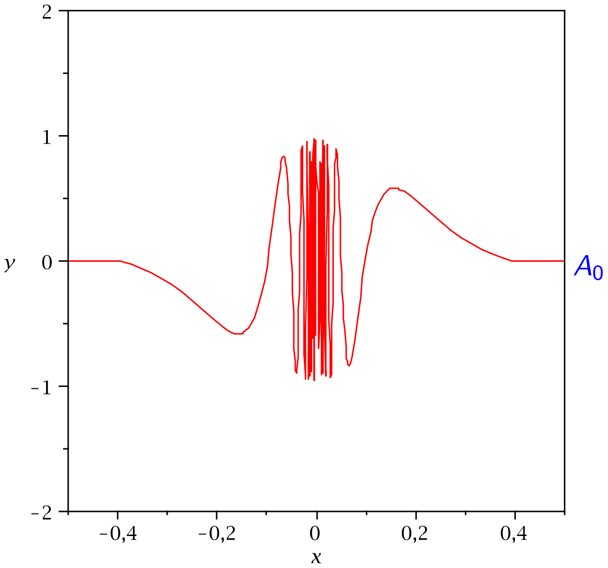

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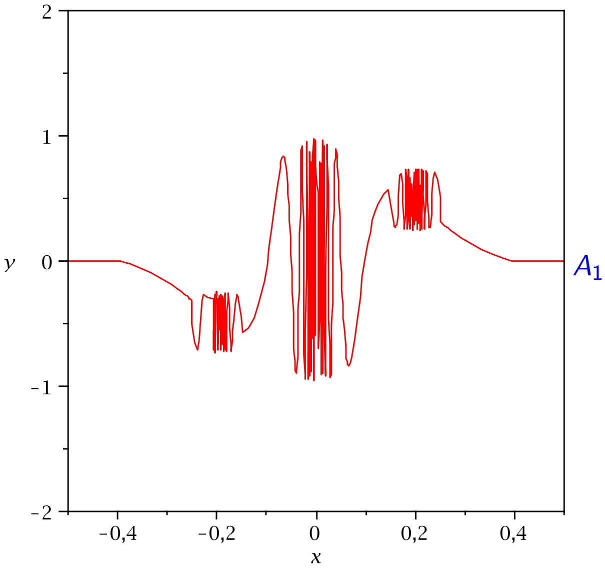


Notation to construct the sets A_i

Notation

For every $\ell \in \mathbb{Z}$ we denote

$$\ell^* := R^\ell(0) = \ell\omega \pmod 1 \text{ and}$$
$$\mathrm{Orb}_R(0) := \{\ell^* \colon \ell \in \mathbb{Z}\}.$$

Now we start with $A_{-1} := \mathbb{S}^1 \times \{0\}$ and construct iteratively compact sets $A_0, A_1, \ldots$ such that each A_n is the closure of the graph of a continuous function

$$\mathbb{S}^1 \setminus \{\ell^* \colon |\ell| \le n\} \longrightarrow [-2, 2].$$

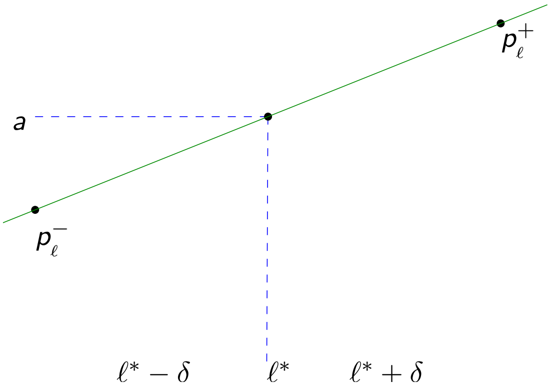
The construction is done by "perturbing" the set A_{n-1} in a neighbourhood of the the points $(\{\ell^*\} \times [-2,2]) \cap A_{n-1}$ with $\ell \in \{n, -n\}$ so that $(\{\ell^*\} \times [-2,2]) \cap A_n$ will now be an interval for $\ell \in \{n, -n\}$.

The scalable "bricks" of our construction

For $\ell \in \{-n, n\}$, the box $\mathcal{R}(\ell^*, n, \alpha, \delta, p_\ell, p_\ell^+, p_\ell^-)$ around the point $p_\ell = (\ell^*, a)$ which is the unique point of $(\{\ell^*\} \times [-2, 2]) \cap A_{n-1}$.

Note

The green line is the set A_{n-1}.



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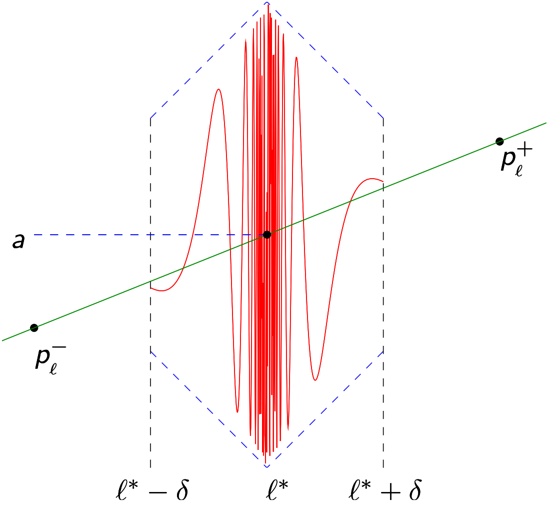

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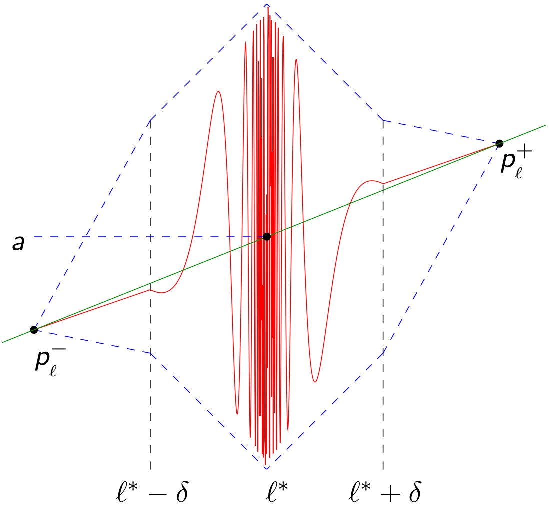

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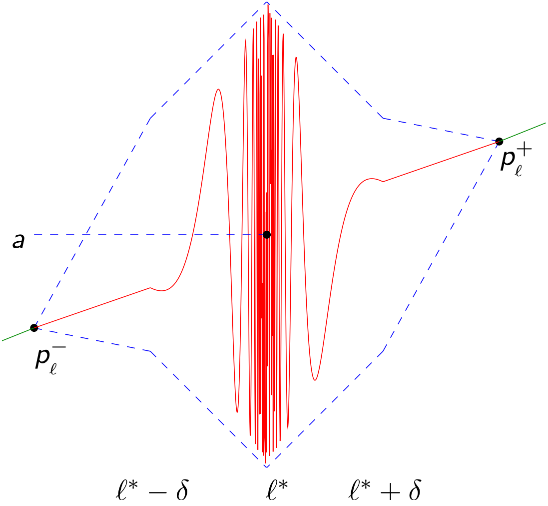

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The above boxes satisfy the following important condition (among many other which are highly technical):

Let $l, z \in \mathbb{Z}$ be such that $|l| \geq |z|$. Then either $\mathcal{R}(l^*) \cap \mathcal{R}(z^*) = \emptyset$ or $|l| > |z|$ and $\mathcal{R}(l^*)$ is contained in one of the two connected components of the interior of $\mathcal{R}(z^*) \setminus \big(A_{|z|} \cap (\{z^*\} \times [-2, 2])\big)$.

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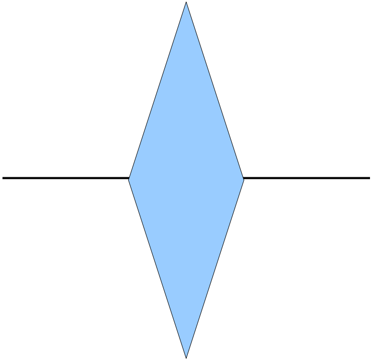

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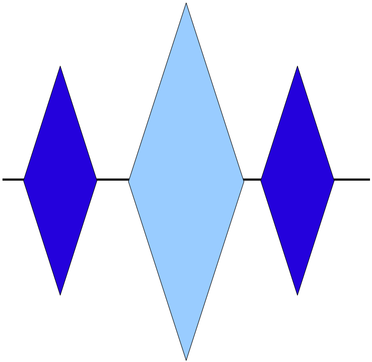

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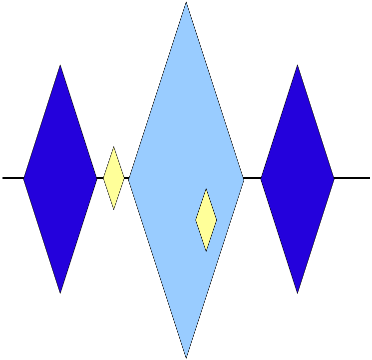

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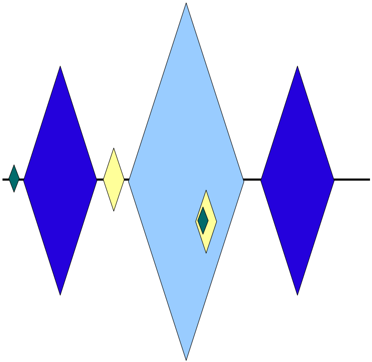

Reinterpreting the sets A_i

Now observe that each set A_n is the closure of the graph of a continuous function

$$\varphi_n \colon \mathbb{S}^1 \setminus \mathrm{Orb}_R(0) \longrightarrow [-2, 2];$$

and $\mathbb{S}^1 \setminus \mathrm{Orb}_R(0)$ is residual in $\mathbb{S}^1$.

On the other hand, the space of continuous functions from a residual set of $\mathbb{S}^1$ into $[-2, 2]$ can be endowed with the supremum pseudo-metric. Then it is a complete metric space.

Remark

The supremum pseudo-metric in the space of pseudo-curves is equivalent to the Hausdorff distance between the corresponding pseudo-curves.

It is not difficult to prove that the sequence φ_n is a Cauchy sequence in this space.
With this in mind we can define:

Passing to the limit to obtain the pseudo-curve

Definition

We denote by A the closure of the graph of the function

$$\varphi := \lim_{n\to\infty} \varphi_n.$$

Thus, A is a pseudo-curve (and hence compact) which can be shown to have the following properties:

- $A \cap (\{\ell^*\} \times [-2,2]) = A_\ell \cap (\{\ell^*\} \times [-2,2])$ is a non-degenerate interval for each $\ell^* \in \mathrm{Orb}_R(0)$.
- $A \cap (\{\theta\} \times [-2,2]) = \{\varphi(\theta)\}$ is a point for each $\theta \notin \mathrm{Orb}_R(0)$.

Clearly, since A is a pseudo-curve, it is not a curve.

The dynamics on the cylinder: making A invariant

Our next goal is to define a continuous map

$$T\colon \Omega \longrightarrow \Omega \qquad T(\theta, x) = (R(\theta), f(\theta, x))$$

such that $T(A) = A$ and, for each $\theta \in \mathbb{S}^1$, $T(\theta, 2) = (R(\theta), -2)$ and $T(\theta, -2) = (R(\theta), 2)$.

Thus, A is a T-strongly invariant pseudo-curve (hence it does not contain any arc of a curve) which coexists with a 2-periodic orbit of curves.

A sequence of maps

This map is obtained as limit of a Cauchy sequence of continuous skew products

$$T_n \colon \Omega \longrightarrow \Omega \qquad T_n(\theta, x) = (R(\theta), f_n(\theta, x))$$

such that

- $T_n(\theta, 2) = (R(\theta), -2)$ and $T_n(\theta, -2) = (R(\theta), 2)$ (that is $f_n(\theta, -2) = 2$ and $f_n(\theta, 2) = -2$) for each $\theta \in \mathbb{S}^1$.
- For each θ the function $f_n(\theta, \cdot)$ is defined piecewise linear and monotonically decreasing in such a way that T_n is globally continuous.

The sequence is constructed inductively in the following way:

The functions F_n

We define L_i as the set of all $\ell \in \mathbb{Z}$ such that ℓ^* is contained in exactly i boxes $\mathcal{R}$.

Then we set $B_i = \cup_{z \in L_i} \pi(\mathcal{R}(z^*))$ where $\pi\colon \Omega \longrightarrow \mathbb{S}^1$ denotes the projection with respect to the first component. It follows that each B_i is a dense set of $\mathbb{S}^1$ and that $B_i \supsetneq B_{i+1}$.

We also set $A^\theta := A \cap \{\theta\} \times I$.

The basic idea of the construction of the maps T_n is that, for every $m \in \mathbb{N}$ and $k \geq m$, T_k sends each vertical segment A^{ℓ^*} to $A^{(\ell+1)^*}$ in reversing order for every $\ell \in L_m$.

This will imply that $F(A^{\ell^*}) = A^{(\ell+1)^*}$ for every $\ell \in \mathbb{N}$ and, by the density of $\cup_{\ell \in \mathbb{N}} A^{\ell^*}$ in A, $F(A) = A$.

The function $f_1(\theta, \cdot)$

- For every $\theta \notin B_1$ we define $f_1(\theta, \cdot)$ as piecewise linear in two pieces such that $f_1(\theta, \varphi(\theta)) = \varphi(R(\theta))$ (notice that if $\theta \notin B_1$ then $\theta \notin \mathrm{Orb}_R(0)$ and so $R(\theta) \notin \mathrm{Orb}_R(0)$).

When $\theta \in B_1$ there exists $\ell \in L_1$ such that $\theta \in \pi(\mathcal{R}(\ell^*))$.

- If $\theta \in [\ell^* - \delta(\ell), \ell^* + \delta(\ell)] \subset \pi(\mathcal{R}(\ell^*))$ then the map $f_1(\theta, \cdot)$ is piecewise linear with three pieces so that $\mathcal{R}(\ell^*) \cap \{\theta\} \times [2, 2]$ is mapped (reversing orientation) to $\mathcal{R}((\ell + 1)^*) \cap \{R(\theta)\} \times [2, 2]$.
- The fibres corresponding to $\theta \in \pi(\mathcal{R}(\ell^*)) \setminus [\ell^* - \delta(\ell), \ell^* + \delta(\ell)]$ leave room for connecting homotopically the maps $f_1(\theta, \cdot)$ already defined.

The functions $f_n(\theta, \cdot)$

We define $f_n(\theta, \cdot)$ from $f_{n-1}(\theta, \cdot)$ as follows:

- If $\theta \notin B_n$ we set $f_n(\theta, x) = f_{n-1}(\theta, x)$ for every $x \in I$.

If $\theta \in B_n$ then there exist $\ell \in L_n$ and $m \in L_{n-1}$ such that $\mathcal{R}(\ell^*) \subset \mathcal{R}(m^*)$ and $\theta \in \pi(\mathcal{R}(\ell^*))$.

For $\theta \in [\ell^* - \delta(\ell), \ell^* + \delta(\ell)] \subset \pi(\mathcal{R}(\ell^*))$ we set:

- $f_n(\theta, x) = f_{n-1}(\theta, x)$ for every $x \notin \mathcal{R}(m^*) \cap \{\theta\} \times [2, 2]$.
- The map $f_n(\theta, \cdot)$ maps $\mathcal{R}(\ell^*) \cap \{\theta\} \times [2, 2]$ reversing orientation to $\mathcal{R}((\ell + 1)^*) \cap \{R(\theta)\} \times [2, 2]$.
- $f_n(\theta, \cdot)$ is continuous and piecewise affine in the two intervals $(\mathcal{R}(m^*) \setminus \mathcal{R}(\ell^*)) \cap \{\theta\} \times [2, 2]$.
- In the fibres corresponding to $\theta \in \pi(\mathcal{R}(\ell^*)) \setminus [\ell^* - \delta(\ell), \ell^* + \delta(\ell)]$ we define the maps $f_n(\theta, \cdot)$ to be a homotopy (with respect to θ) between the maps already defined.

The skew product T

In this way we obtain

Theorem

The function $T := \lim_{n\to\infty} T_n$ is a continuous skew product of the form

$$T(\theta, x) = (R(\theta), f(\theta, x))$$

with $T(\theta, 2) = (R(\theta), -2)$ and $T(\theta, -2) = (R(\theta), 2)$ for each $\theta \in \mathbb{S}^1$. Moreover, $T(A) = A$ and this is the only invariant object of T. In particular T has no invariant curves.

Conclusions

We have constructed a skew product map having a strongly invariant pseudo-curve which is not a curve and the pseudo curve is forced by a 2-periodic orbit of curves.

This answers the two questions that we have raised and clarifies the [FJJK] theorem in the sense that these kind of complicate objects must be taken into account.