Vertex coloring of a graph

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1 Basic definitions (from the wikipedia)

Vertex coloring When used without any qualification, a coloring of a graph means (almost always) a proper vertex coloring, namely a labeling of the graph's vertices with colors such that no two vertices sharing the same edge have the same color. Since a vertex with a loop (i.e. a connection directly back to itself) could never be properly colored, it is understood that graphs in this context are loopless.

The terminology of using colors for vertex labels goes back to map coloring. Labels like red and blue are only used when the number of colors is small, and normally it is understood that the labels are drawn from the integers $\{1, 2, 3, \ldots\}$.

A coloring using at most k colors is called a *(proper) k-coloring.* The smallest number of colors needed to color a graph G is called its *chromatic number*, and is often denoted $\chi(G)$. Sometimes $\gamma(G)$ is used, since $\chi(G)$ is also used to denote the *Euler characteristic* of a graph. A graph that can be assigned a (proper) k-coloring is *k-colorable*, and it is *k-chromatic* if its chromatic number is exactly k. A subset of vertices assigned to the same color is called a *color class.* A k-coloring is the same as a partition of the vertex set into k independent sets.

Bounds on the chromatic number Assigning distinct colors to distinct vertices always yields a proper coloring, so

$$1 \leq \chi(G) \leq n.$$

The only graphs that can be 1-colored are edgeless graphs. A complete graph K_n of n vertices requires $\chi(K_n) = n$ colors. In an optimal coloring there must be at least one of the graph's m edges between every pair of color classes, so

$$\chi(G)(\chi(G) - 1) \leq 2m.$$

A *clique* in an undirected graph $G = (V, E)$ is a subset of the vertex set $C \subset V$, such that for every two vertices in C, there exists an edge connecting the two. This is equivalent to saying that the subgraph induced by C is complete (in some cases, the term clique may also refer to the subgraph).

A *maximum clique* is a clique of the largest possible size in a given graph. The clique number $\omega(G)$ of a graph G is the number of vertices in a maximum clique in G.

If G contains a clique of size k, then at least k colors are needed to color that clique; in other words, the chromatic number is at least the clique number:

$$\chi(G) \geq \omega(G).$$

The 2-colorable graphs are exactly the bipartite graphs, including trees and forests. By the four color theorem, every planar graph can be 4-colored.

A greedy coloring shows that every graph can be colored with one more color than the maximum vertex degree $\Delta(G)$:

$$\chi(G) \leq \Delta(G) + 1.$$

2 The assignment

By using a genetic algorithm find a vertex coloring of the following graphs with the number of colors as small as possible. If it is possible determine also the chromatic number. Discuss also how appropriate are the genetic algorithms for this problem.

Do the computation for the (undirected) graphs `dsjc250.5`, `r250.5` and `flat300_28_0` from the DIMACS page `http://www.info.univ-angers.fr/pub/porumbel/graphs/`

The graph files are formatted by using three types of lines:

- Lines starting with `c`. These are comment lines.
- A unique line of the form `p edge 250 31336` which is the first "non-comment" line. It specifies the number of vertices `p` (in the example 250) and edges `edge` in the graph (in the example 31336).
- Lines of the form `e 2 1`. They specify an (unordered) edge between the two vertices specified after the symbol `e`.