Transitivity without (relative) specification in dendrites

Vladimír Špitalský

Matej Bel University, Banská Bystrica, Slovakia

10th AIMS Conference on Dynamical Systems, Differential Equations and Applications

July 7–11, 2014
Madrid, Spain

Contents

Transitivity on trees

Theorem (Blokh 1987)

If X is a tree and $f : X \to X$ is transitive, then

- *f has the relative specification property*

Transitivity on trees

Theorem (Blokh 1987)

If X is a tree and $f : X \to X$ is transitive, then

- *f has the relative specification property*

$f : X \to X$ is transitive if

- $\forall U, V$ – nonempty open $\exists n \in \mathbb{N} : \quad f^n(U) \cap V \neq \emptyset$

Transitivity on trees

Theorem (Blokh 1987)

If X is a tree and $f : X \to X$ is transitive, then

- *f has the relative specification property*

$f : X \to X$ has the specification property if [Bowen 1971]

- $\forall \varepsilon > 0 \quad \exists m \quad \forall k \geq 2 \quad \forall x_1, \dots, x_k \in X$
  $\forall a_1 \leq b_1 < \cdots < a_k \leq b_k$ with $a_i - b_{i-1} \geq m$ $(i = 2, \dots, k)$
  and $\forall p \geq m + b_k - a_1$,
  there is a point $x \in X$ with $f^p(x) = x$ and

$$d(f^n(x), f^n(x_i)) \leq \varepsilon \qquad \text{for} \quad a_i \leq n \leq b_i,\ 1 \leq i \leq k.$$

Transitivity on trees

Theorem (Blokh 1987)

If X is a tree and $f : X \to X$ is transitive, then

- *f has the relative specification property*

$f : X \to X$ has the relative property $\mathcal{P}$ if [Banks 1997]

- there exist regular closed sets $D_0, \ldots, D_{m-1}$ covering X such that, for every $0 \leq i < j < m$, $D_i \cap D_j$ is nowhere dense,

$$f(D_i) \subseteq D_{(i+1) \bmod m}$$

and

$$f^m|_{D_i} : D_i \to D_i \text{ has the property } \mathcal{P}.$$

Transitivity on trees

Theorem (Blokh 1987)

If X is a tree and $f : X \to X$ is transitive, then

- *f has the relative specification property*

Consequently, every transitive tree map

- is relatively mixing
- has positive entropy
- has dense periodic points
- ...

Transitivity on dendrites

Dendrites

Dendrite

- a locally connected metric continuum which contains no circle

A point x of a dendrite X is

- end point if $X \setminus \{x\}$ is connected
- cut point if $X \setminus \{x\}$ is not connected
  - branch point if $X \setminus \{x\}$ has at least 3 components

$E(X)$ and $B(X)$

- the sets of all end points and branch points

Tree

- a dendrite with finitely many end points

Dendrites

An arc $A = [a, b]$ in a dendrite X is called free if

- $A \setminus \{a, b\}$ is open in X

For a dendrite X the following are equivalent

- X does not contain a free arc
- branch points of X are dense in X
- end points of X are dense (i.e. residual) in X

Transitivity on dendrites: Positive results

Theorem (Alsedà-Kolyada-Llibre-Snoha 1999; Kwietniak 2011; Harańczyk-Kwietniak-Oprocha 2011; Dirbák-Snoha-Š. 2012)

If X is a dendrite containing a free arc and $f : X \to X$ is transitive, then

- *f is relatively mixing*
- *f has positive entropy*
- *f has dense periodic points*

Transitivity on dendrites: Positive results

Theorem (Alsedà-Kolyada-Llibre-Snoha 1999; Kwietniak 2011; Harańczyk-Kwietniak-Oprocha 2011; Dirbák-Snoha-Š. 2012)

If X is a dendrite containing a free arc and $f : X \to X$ is transitive, then

- *f is relatively mixing*
- *f has positive entropy*
- *f has dense periodic points*

Theorem (Acosta-Hernández-Naghmouchi-Oprocha 2013)

If X is a dendrite and $f : X \to X$ has a transitive cut point, then

- *f is relatively weakly mixing*
- *f has dense periodic points*

Transitivity on dendrites: Negative results

Theorem (Hoehn-Mouron 2013)

There is a dendrite X (with dense $B(X)$) admitting a map $f : X \to X$ which is

- *weakly mixing but*
- *not mixing*

Moreover, [Acosta-Hernández-Naghmouchi-Oprocha 2013]

- *f is proximal, and thus*
- *it has a unique periodic ($=$ fixed) point*

Transitivity on dendrites: Negative results

Theorem (Š.)

There is a dendrite X (with dense $B(X)$) admitting a map $f : X \to X$ such that

- *f is transitive*
- *f has infinite decomposition ideal (that is, f is not relatively totally transitive)*
- *f has a unique periodic (= fixed) point*

Transitivity on dendrites: The main theorem

$C(X)$

- the space of all subcontinua (= subdendrites) of X equipped with the Hausdorff metric

$N_f(U, V)$

- the return time set $\{n \in \mathbb{N} : \; f^n(U) \cap V \neq \emptyset\}$

Transitivity on dendrites: The main theorem

Theorem (Š.)

Let $\sigma : \Sigma \to \Sigma$ be a subshift. Then there are a dendrite X (with dense $B(X)$) and maps $f = f_\sigma : X \to X$ and $D : \Sigma \to C(X)$ s.t.

- $f \circ D = D \circ \sigma$; *i.e.* $f(D(\gamma)) = D(\sigma(\gamma))$ *for every* $\gamma \in \Sigma$
- *for every cylinders* $[\alpha], [\beta]$ *in* Σ *and every non-empty open sets* $U \subseteq D[\alpha]$, $V \subseteq D[\beta]$ *in* X *there is* $n_0 \in \mathbb{N}$ *such that, for every* $n \geq n_0$,

  $$n \in N_\sigma([\alpha], [\beta]) \qquad \Longleftrightarrow \qquad n \in N_f(U, V)$$

  consequently, f is transitive (totally transitive, weakly mixing, mixing) if and only if σ is
- *if σ is aperiodic then f has a unique periodic ($=$ fixed) point*

Transitivity on dendrites: The main theorem

Corollary

There is a dendrite X and maps $f, g, h : X \to X$ such that

- *f is transitive and has infinite decomposition ideal*
- *g is weakly mixing but not mixing*
- *h is mixing but has not dense periodic* points

3. Proof of the main result

Structure of the proof

Theorem

For a subshift $\sigma : \Sigma \to \Sigma$ there is a dendrite X, a continuous map $f : X \to X$ and a map $D : \Sigma \to C(X)$ such that

- $f \circ D = D \circ \sigma$
- $N_f(U, V) \approx N_\sigma([\alpha], [\beta])$ *for every ...*
- *if σ is aperiodic then $\mathrm{Per}(f) = \mathrm{Fix}(f)$ is a singleton*

Main steps of the proof.

1. construct the dendrite X
2. define $D : \Sigma \to C(X)$
3. construct the map $f : X \to X$
4. prove the properties of f

Step 1: The dendrite X

The dendrite X: the universal dendrite of order 3

- branch points are dense
- every branch point has order 3

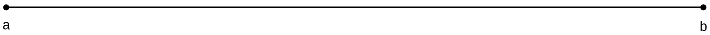
We can write

$$X = \bigcup_{m=0}^{\infty} X_m \;\cup\; X_\infty$$

- $X_0 = [a, b]$ is a segment
- $X_1 = X_0 \cup \bigcup_{r\in Q}[a_r, b_r]$
- $X_2 = X_1 \cup \bigcup_{rs\in Q^2}[a_{rs}, b_{rs}]$
- ...
- $X_\infty = \{b_{\mathbf{r}} : \mathbf{r} \in Q^\infty\}$ totally disconnected dense G_δ set

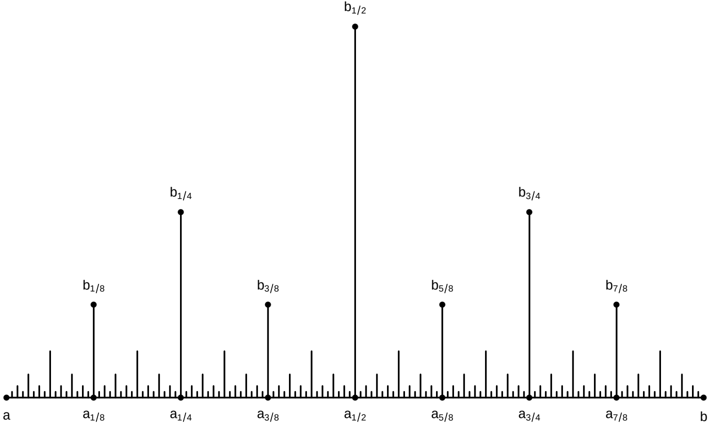
where Q is the set of all dyadic rationals in $(0, 1)$

Step 1: The dendrite X



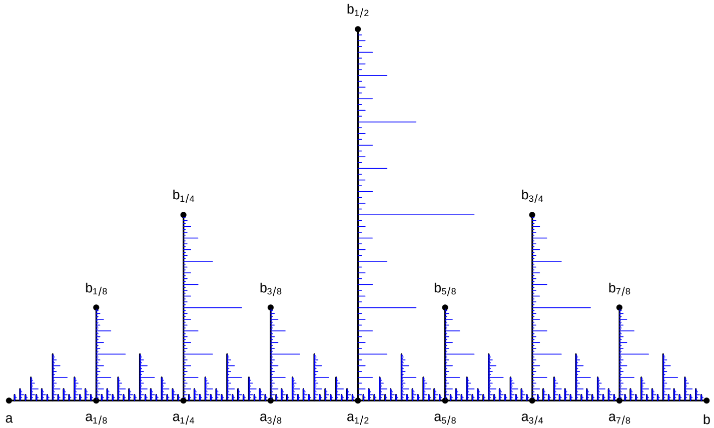
The set X_0

Step 1: The dendrite X



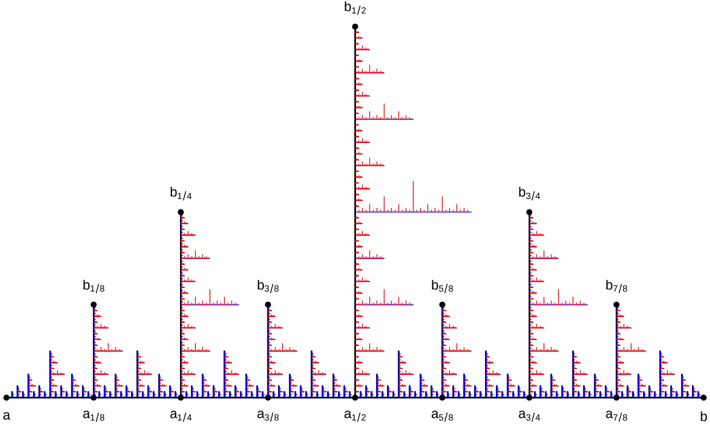
The set X_1

Step 1: The dendrite X



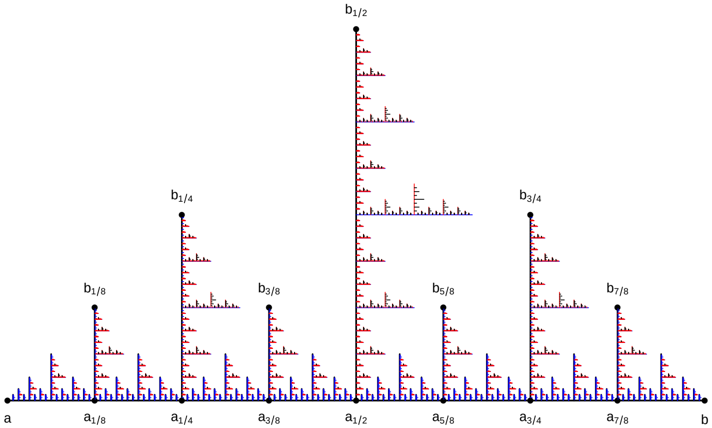
The set X_2

Step 1: The dendrite X



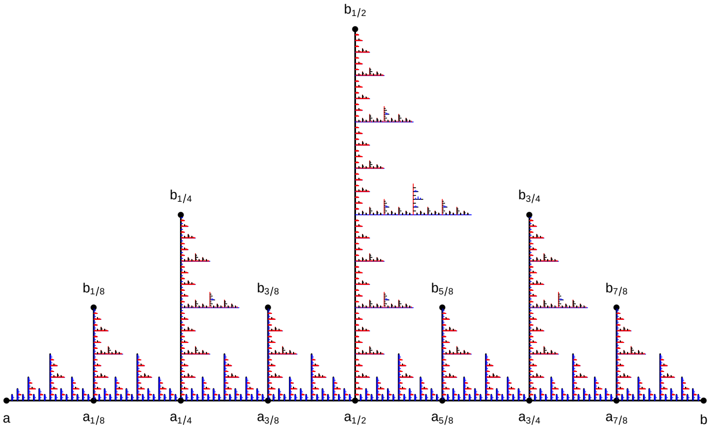
The set X_3

Step 1: The dendrite X



The set X_4

Step 1: The dendrite X



The set X_5

Step 1: The dendrite X

We can write

$$X = \text{closure}\left(\bigcup_{m=0}^{\infty} X_m\right) = \bigcup_{m=0}^{\infty} X_m \ \cup \ X_\infty$$

- $X_0 = [a, b]$ is a segment
- $X_1 = X_0 \cup \bigcup_{r \in Q} [a_r, b_r]$
- $X_2 = X_1 \cup \bigcup_{rs \in Q^2} [a_{rs}, b_{rs}]$
- ...
- $X_\infty = \{b_\mathbf{r} : \mathbf{r} \in Q^\infty\}$ totally disconnected dense G_δ set

where Q is the set of all dyadic rationals in $(0, 1)$

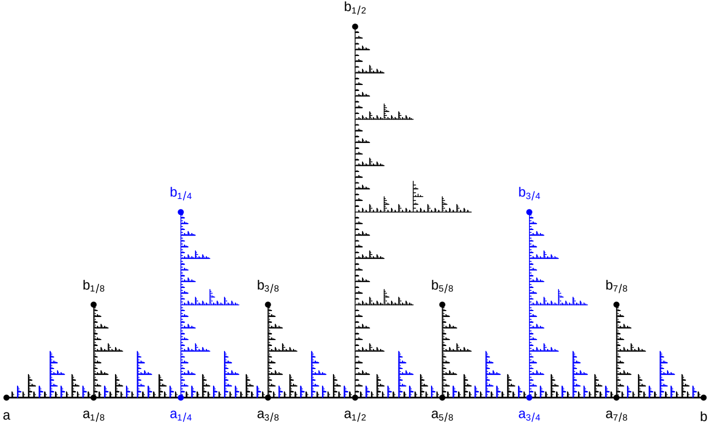
Step 2: The subdendrites $D(\gamma)$

Assume that $\Sigma = \{0,1\}^{\mathbb{N}}$ and $\sigma : \Sigma \to \Sigma$ is the full shift

D_0, D_1

- D_0, D_1 are regular closed subdendrites of X
- $D_0 \cup D_1 = X$
- $D_0 \cap D_1 = X_0$ is nowhere dense in X

Step 2: The subdendrites $D(\gamma)$



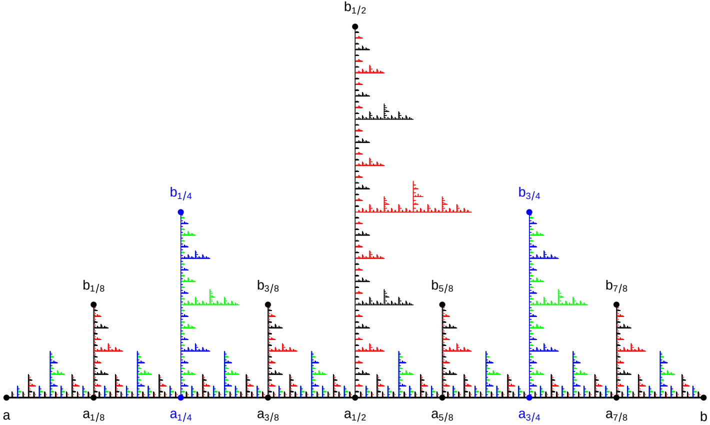
D_0, D_1

Step 2: The subdendrites $D(\gamma)$

$D_{00}, D_{01}, D_{10}, D_{11}$

- $D_{00}, \dots, D_{11}$ are regular closed subdendrites of X
- $D_0 = D_{00} \cup D_{01}$, $D_1 = D_{10} \cup D_{11}$
- $\bigcup_{i_0 i_1 \neq j_0 j_1} (D_{i_0 i_1} \cap D_{j_0 j_1}) = X_1$ is nowhere dense in X

Step 2: The subdendrites $D(\gamma)$



$$D_{00}, D_{01}, D_{10}, D_{11}$$

Step 2: The subdendrites $D(\gamma)$

For $\gamma = \gamma_0\gamma_1 \cdots \in \Sigma$:

- $D(\gamma) = D_{\gamma_0} \cap D_{\gamma_0\gamma_1} \cap D_{\gamma_0\gamma_1\gamma_2} \cap \dots$

Step 3: The map $f : X \to X$

$f_0 : X_0 \to X_0$ $X_0 = [a, b]$

- a surjective map
- "agrees with" the shift σ
- $f_0(x) < x$ for $x \in (a, b)$
- $\lim_n f_0^n(x) = a$ for every $x \neq b$

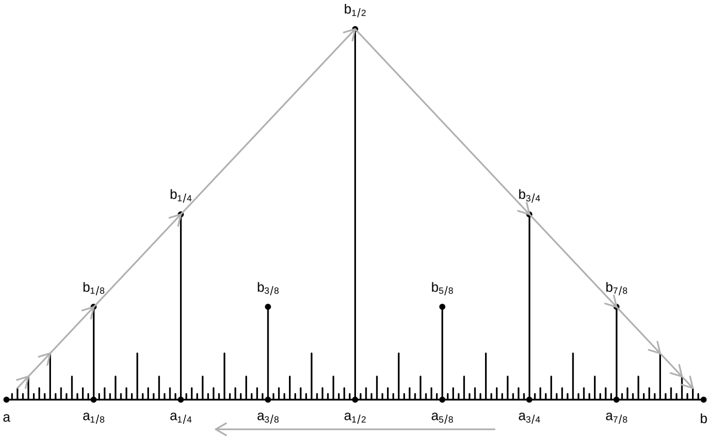
Step 3: The map $f : X \to X$

$f_1 : X_1 \to X_1$ $\qquad$ $X_1 = X_0 \cup \bigcup_{r\in Q}[a_r, b_r]$

- $f_1(a_r) = f_0(a_r)$
- maps every end point b_r onto an end point $b_{\varrho(r)}$ in such a way that

$$\lim_{n\to\infty} f_1^n(b_r) = b \qquad \text{for every } r \in Q$$

Step 3: The map $f : X \to X$



The map $f_1 : X_1 \to X_1$

Step 3: The map $f : X \to X$

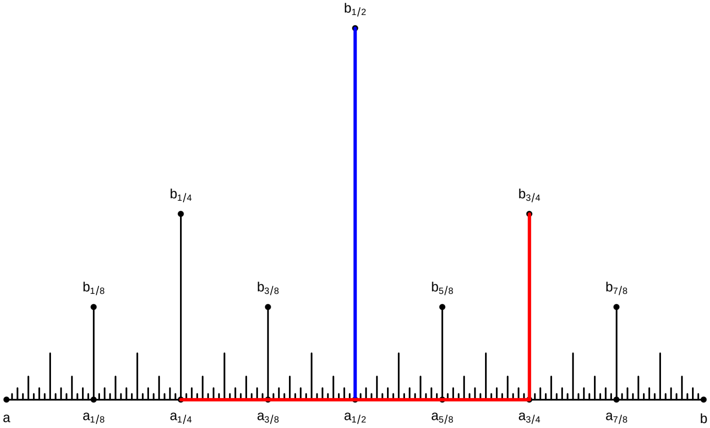
$f_1 : X_1 \to X_1$ $\qquad$ $X_1 = X_0 \cup \bigcup_{r \in Q} [a_r, b_r]$

- continuous surjective extension of f_0
- "agrees with" the shift σ
- maps every $[a_r, b_r]$ onto

$$[f_0(a_r), b_{\varrho(r)}] = [f_0(a_r), a_{\varrho(r)}] \cup [a_{\varrho(r)}, b_{\varrho(r)}]$$

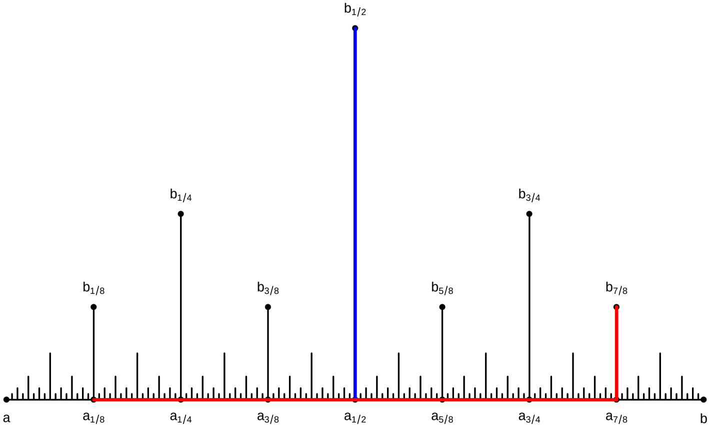
- $\lim_n f_1^n(x) = a$ for every $x \neq b, b_r \quad (r \in Q)$

Step 3: The map $f : X \to X$



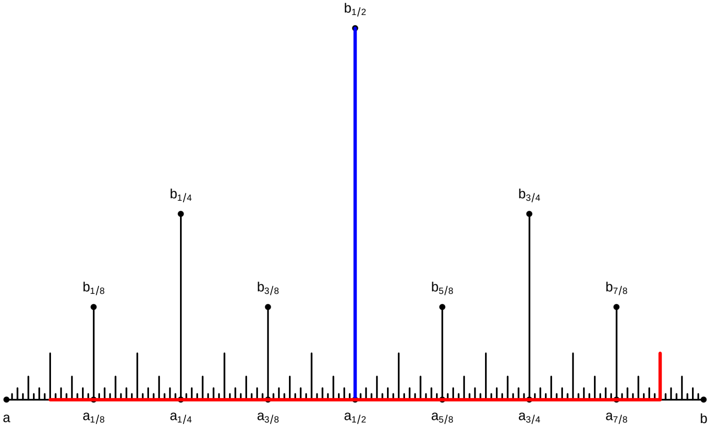
The image of $[a_{1/2}, b_{1/2}]$

Step 3: The map $f : X \to X$



The second image of $[a_{1/2}, b_{1/2}]$

Step 3: The map $f : X \to X$



The third image of $[a_{1/2}, b_{1/2}]$

Step 3: The map $f : X \to X$

$f_m : X_m \to X_m$ $(m \geq 2)$ $\qquad X_m = X_{m-1} \cup \bigcup_{\mathbf{r} \in Q^m} [a_{\mathbf{r}}, b_{\mathbf{r}}]$

- continuous surjective extension of f_{m-1}
- "agrees with" the shift σ
- maps every $[a_{\mathbf{r}}, b_{\mathbf{r}}]$ onto $[a_{\varrho(\mathbf{r})}, b_{\varrho(\mathbf{r})}]$
  - $\varrho : Q^m \to (Q^m \cup Q^{m-1})$ is such that every $\mathbf{r} \in Q^m$ eventually falls into Q^{m-1}

- $\lim_n f_m^n(b_{\mathbf{r}}) = b$ for every $\mathbf{r} \in Q^m$
- $\lim_n f_m^n(x) = a$ for every $x \neq b, b_{\mathbf{r}}$ $\quad (\mathbf{r} \in Q^1 \cup \cdots \cup Q^m)$

Step 3: The map $f : X \to X$

$f : X \to X$ $\qquad X = \bigcup_m X_m \cup X_\infty,\ X_\infty = \{b_{\mathbf{r}} : \mathbf{r} \in Q^\infty\}$

$$f(x) = \begin{cases} f_m(x) & \text{if } x \in X_m,\ m \geq 0 \\ b_{\varrho(\mathbf{r})} & \text{if } x = b_{\mathbf{r}},\ \mathbf{r} \in Q^\infty \end{cases}$$

- $\varrho : Q^\infty \to Q^\infty$ is determined by $\varrho|_{Q^m}$ $(m \geq 1)$
- X_∞ is an f-invariant (not closed) set

Step 4: Properties of $f : X \to X$

f is a continuous surjection

- every X_m is a closed invariant set with "trivial" dynamics
- X_∞ is an invariant set with "shift-like" dynamics

Step 4: Properties of $f : X \to X$

f is a continuous surjection

- every X_m is a closed invariant set with "trivial" dynamics
- X_∞ is an invariant set with "shift-like" dynamics

f "agrees" with the shift σ

- f maps $D(\gamma)$ onto $D(\sigma(\gamma))$

Step 4: Properties of $f : X \to X$

f is a continuous surjection

- every X_m is a closed invariant set with "trivial" dynamics
- X_∞ is an invariant set with "shift-like" dynamics

f "agrees" with the shift σ

- f maps $D(\gamma)$ onto $D(\sigma(\gamma))$

f has the "same" return time sets as σ

- ...

Step 4: Properties of $f : X \to X$

Subshifts $\tilde{\sigma} : \tilde{\Sigma} \to \tilde{\Sigma}$ $\qquad (\tilde{\Sigma} \subseteq \Sigma)$

- correspond to subsystems

$$\tilde{f} = f|_{D(\tilde{\Sigma})} : D(\tilde{\Sigma}) \to D(\tilde{\Sigma})$$

Thanks for your attention!