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Smooth chaotic interval maps and indecomposable planar attractors

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- $f : I \to I$ continuous
- $I_f = \lim_{\leftarrow}\{f, I\}$ an inverse limit space

The well known connection established by M. Barge and J. Martin in the 1980's states that chaotic (in the sense of positive entropy) interval maps generate planar dynamical systems with attractors that must contain an indecomposable subcontinuum.

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Must weak chaos (i.e. chaos in the sense of Li and Yorke) imply indecomposability in the inverse limit space?

In [Boronski, Oprocha, to appear] showed that there exists a Li-Yorke chaotic interval map f such that the inverse limit space $I_f = \lim_{\leftarrow}\{f, I\}$ does not contain an indecomposable subcontinuum.

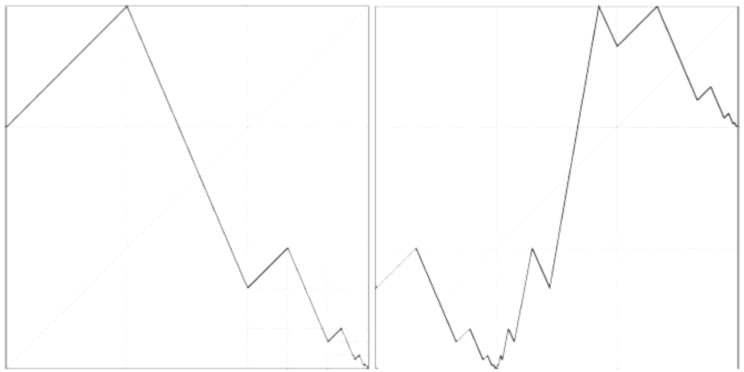

FIGURE 2. Graph of f and f^2

Question

Is there a $k > 0$ and a Li-Yorke chaotic interval map f such that f is C^k-smooth and the inverse limit space $I_f = \lim_{\leftarrow}\{f, I\}$ does not contain an indecomposable subcontinuum?

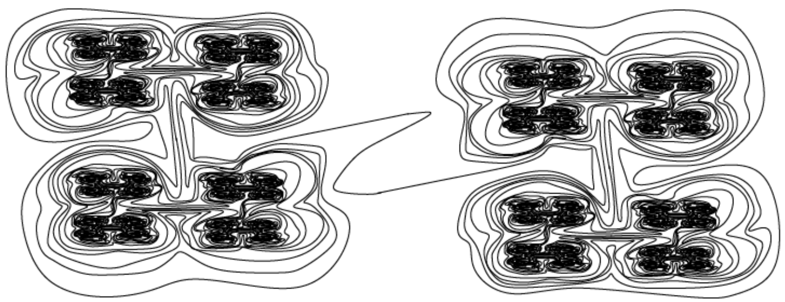

FIGURE 1. An hereditarily decomposable attractor X_F.

Question

Must I_f have a periodic structure similar to the continua described in [Boronski, Oprocha, to appear]?

A **continuum** X is a compact and connected metric space that contains at least two points. A continuum is **decomposable** if it can be written as the union of two proper subcontinua. It is **hereditarily decomposable** if every subcontinuum is decomposable. An **indecomposable** continuum is a continuum that is not decomposable, and it is **hereditarily indecomposable** if every subcontinuum is indecomposable.

Suppose a map $f : I \to I$ is given. The **inverse limit space** $I_f = \varprojlim\{f, I\}$ is the space given by

$$I_f = \{(x_1, x_2, x_3, \dots) \in I^{\mathbb{N}} : f(x_{i+1}) = x_i\}.$$

The topology of I_f is induced from the product topology of $I^{\mathbb{N}}$, with the basic open sets in I_f given by

$$U_{\leftarrow} = (f^{i-1}(U), f^{i-2}(U), \dots, U, f^{-1}(U), f^{-2}(U), \dots),$$

where U is an open subset of the ith factor space I.

There is a natural homeomorphism $\sigma_f : I_f \to I_f$, called the *shift homeomorphism*, given by

$$\sigma_f(x_1, x_2, x_3, \dots) = (f(x_1), f(x_2), f(x_3), \dots) = (f(x_1), x_1, x_2, \dots).$$

The shift homeomorphism σ_f preserves topological entropy of f, as well as many other dynamical properties such as existence of periodic orbits of given period, shadowing property, and topological mixing [Chen, Li, 1993].

Let ρ denote the metric on I. A map $f : I \to I$ is **Li-Yorke chaotic** if there is an uncountable set $S \subset I$ such that

$$\liminf_{n\to\infty} \rho(f^n(x), f^n(y)) = 0$$

and

$$\limsup_{n\to\infty} \rho(f^n(x), f^n(y)) > 0$$

for any distinct points $x, y \in S$.

Let us recall Bowen's definition of the topological entropy. Let $K \subset X$ be a compact subset, and fix $\varepsilon > 0$ and $n \in \mathbb{N}$. We say that a set $E \subset K$ is (n, ε, K, f)-**separated** (by the map f) if for any $x, y \in E$, $x \neq y$, there is $k \in \{0, 1, ..., n-1\}$ such that

$$\rho(f^k(x), f^k(y)) > \varepsilon.$$

Denote by $s_n(\varepsilon, K, f)$ the cardinality of any maximal (n, ε, K, f)-separated set in K and define

$$s(\varepsilon, K, f) = \limsup_{n\to\infty} \frac{1}{n} \log s_n(\varepsilon, K, f).$$

Then, the **topological entropy** of f is

$$h(f) = \sup_K \lim_{\varepsilon\to 0} s(\varepsilon, K, f).$$

An interval map f is called **unimodal** if there exists a **turning** point $c \in I$, $0 < c < 1$, such that $f|_{[0,c]}$ is strictly increasing and $f|_{[c,1]}$ is strictly decreasing.

A map f is **weakly unimodal** if there exists a $c \in I$, $0 < c < 1$, such that $f|_{[0,c]}$ is nondecreasing and $f|_{[c,1]}$ is nonincreasing.

We say that an interval map (or graph map) $f : G \to f(G)$ is **monotone** if $f^{-1}(x)$ is connected for every $x \in f(G)$. We say that f is **piecewise monotone** on G if there is a finite set of points $A = \{a_1, \ldots, a_n\} \subseteq G$ such that f is monotone on each component of $G \setminus A$.

Lemma

(Barge & Diamond, 1994) Suppose $f : G \to G$ is a piecewise monotone graph map. f has zero topological entropy if and only if $\lim_{\leftarrow}\{f, G\}$ does not contain an indecomposable subcontinuum.

Note that every weakly unimodal map is piecewise monotone.

Let us consider a system $\mathcal{F} \subseteq C(I)$ of weakly unimodal interval maps satisfying the following conditions

1. the set
$$J_f := \{x \in I \mid f(y) \leq f(x) \text{ for every } y \in I\}$$
consists of more than one point,
2. for each $n \in \mathbb{N}$, f has a periodic point of period 2^n,
3. f has no periodic points of other periods.

It is well known that the family $\mathcal{F}$ is nonempty and any map that satisfies the properties (2) and (3) is said to be **of type** 2^∞.

Lemma (Misiurewicz, Smítal, 1988)

Any map $f \in \mathcal{F}$ has zero topological entropy and is chaotic in the sense of Li and Yorke.

Lemma (Misiurewicz, Smítal, 1988)

Let $\mathcal{F}_0 \subseteq \mathcal{F}$ be a family of C^∞ interval maps f satisfying $f(0) = f(1) = 0$. Then $\mathcal{F}_0 \neq \emptyset$.

Lemma

For every positive integer k there exists a weakly unimodal map $f : [0,1] \to [0,1]$ such that

(i) $f(0) = f(1) = 0$.

(ii) *f is C^k-smooth,*

(iii) *f is not C^{k+1}-smooth*

Moreover, for any $c \in (0,1]$, the map $c \cdot f$ satisfies (i)-(iii) and

(iv) *there exists $\bar{c} \in (0,1]$ such that $\bar{c} \cdot f \in \mathcal{F}$.*

Remark

Differentiability is important here.

Main Theorem

For every positive integer k there exists an interval map $f : I \to I$ such that

1. *f is Li-Yorke chaotic,*
2. *$I_f = \lim_{\leftarrow}\{f, I\}$ does not contain an indecomposable subcontinuum,*
3. *f is C^k-smooth,*
4. *f is not C^{k+1}-smooth.*

Main Theorem

There exists a C^∞-smooth interval map $f : I \to I$ such that

1. *f is Li-Yorke chaotic,*
2. *$I_f = \lim_{\leftarrow}\{f, I\}$ does not contain an indecomposable subcontinuum,*

A (topological) **ray** is a homeomorphic image of the half-line $[0, +\infty)$ and a (topological) **line** is a homeomorphic image of $(-\infty, +\infty)$.

Theorem

(Bennett) (the proof can be found in [Ingram, 1995]) *Suppose that $g : [a,b] \to [a,b]$ is continuous and $d \in (a,b)$ is such that*

1. *$g([d,b]) \subset [d,b]$,*
2. *$g|_{[a,d]}$ is monotone, and*
3. *there is $n \in \mathbb{N}$ such that $g^n([a,d]) = [a,b]$.*

Then continuum $K = \lim_{\leftarrow}\{g, [a,b]\}$ is the union of a topological ray R and a continuum $C = \lim_{\leftarrow}\{g, [d,b]\}$ such that $\overline{R} \setminus R = C$.

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Lemma

For every $f \in \mathcal{F}$ *with* $f(0) = 0$ *there are* $a, b, d \in I$ *such that*

1. $f([d, b]) \subset [d, b]$,
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3. *there is an* $n \in \mathbb{N}$ *such that* $f^n([a, d]) = [a, b]$.

Remark

The assumption $f(0) = 0$ *is necessary.*

Theorem

For every $f \in \mathcal{F}$ *with* $f(0) = 0$*, there is a topological ray* L *such that* $\overline{L} = I_f$.

We are able to specify inner structure inside those inverse limit spaces.

Theorem

Suppose $f : I \to I$ *is a map of type* 2^∞*. Then the shift homeomorphism* σ_f *has a* 2^i*-periodic subcontinuum of* I_f*, for every integer* $i > 0$.

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Any map $f \in \mathcal{F}$ possesses the unique infinite ω-limit set $\bar{\omega}$ such that

$$\bar{\omega} \subseteq \bigcap_{i \in \mathbb{N}} \bigcup_{n=1}^{2^i} f^n(J_i),$$

where each J_i is a nondegenerate 2^i-periodic interval (i.e. $f^{2^i}(J_i) = J_i$). Intervals J_i are called *generating*.

Theorem

Let $f \in \mathcal{F}$. Then there exists a system $\{J_i\}_{i \geq 0}$ of generating intervals such that, for any $i \geq 0$, $\lim_{\leftarrow}\{f^{2^i}|_{J_i}, J_i\}$ is a compactification of a topological ray.

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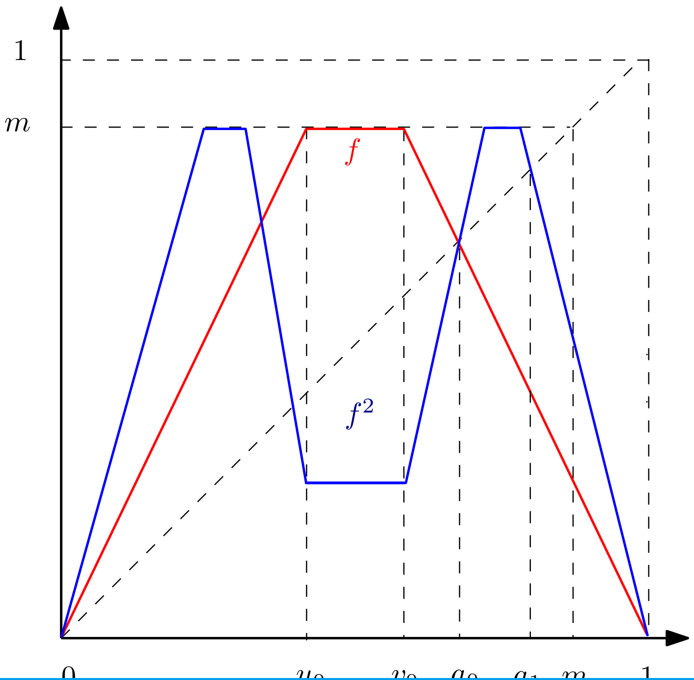

Main Theorem

Suppose $f : I \to I$ is a Li-Yorke chaotic zero entropy weakly unimodal map. Then I_f contains, for every i, a subcontinuum C_i with the following two properties:

(i) *C_i is 2^i-periodic under the shift homeomorphism, and*

(ii) *C_i is a compactification of a topological ray.*

- We found a class of weakly chaotic interval maps with zero topological entropy whose inverse limit spaces do not contain an indecomposable subcontinuum.
- We found smooth maps within this class.
- We described periodic structure of those inverse limit spaces.
- Inverse limit spaces constructed within this work are topologically distinct from those mentioned in [Boroński, Oprocha, to appear].

Question

Suppose f and g are two Li-Yorke chaotic weakly unimodal maps of type 2^∞, that are in two different differentiability classes, as guaranteed by our Main theorems. Are I_f and I_g homeomorphic?

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Suppose g is a Li-Yorke chaotic weakly unimodal map of type 2^∞. Is I_g homeomorphic to a ray limiting onto one of the attractors described in [Boroński, Oprocha, to appear] or a subcontinuum of one of them?

Conclusions

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Thanksgiving

Thank You for Your Attention

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