Strange chaotic triangular maps

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- $(X, \rho) \ldots$ compact metric space
- $f \in C(X) \ldots$ continuous map $f : X \to X$
- $I = [0, 1]$
- triangular map ... a continuous map $F : I^2 \to I^2$ of the form $F(x, y) = (f(x), g_x(y))$
- $\mathcal{T} \ldots$ the class of triangular maps

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- *UR(f)* . . . the set of *uniformly recurrent points* of f

  $x \in UR(f)$ if, for every neighborhood V of x there is a positive integer $K = K(V)$ such that every interval $N \subset [0, \infty)$ of length K contains an integer j such that $f^j(x) \in V$.

  $UR(f)$ coincides with the union of all *minimal sets* of f, i.e., nonempty compact sets $M \subseteq X$ such that $f(M) = M$ and no proper compact subset of M has this property.

Li-Yorke chaos

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$f \in C(X)$ is **Li-Yorke chaotic (LYC)**, if there is an uncountable set $\emptyset \neq S \subset X$ such that $\forall x, y \in S$, $x \neq y$

$$\liminf_{n\to\infty} \rho(f^n(x), f^n(y)) = 0,$$
$$\limsup_{n\to\infty} \rho(f^n(x), f^n(y)) > 0.$$

Distributional chaos

- **Schweizer and Smítal, TAMS 1994**
- **Smítal and Štefánková, ChSF 2004**
- **Balibrea, Smítal and Štefánková, ChSF 2005**

Let $f \in C(X)$, $n \in \mathbb{N}$, $t \in \mathbb{R}$. Put

$$\Phi_{xy}^{(n)}(t) = \frac{1}{n} \#\{m;\ 0 \leq m < n \text{ and } \rho(f^m(x), f^m(y)) < t\}.$$

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$\Phi_{xy}(t) := \liminf_{n\to\infty} \Phi_{xy}^{(n)}(t) \ldots$ **lower distribution** of x and y

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$$\Phi_{xy} \text{ and } \Phi^*_{xy} \text{ are nondecreasing}$$

$$\Phi_{xy}(t) \leq \Phi^*_{xy}(t),\ \forall t \in \mathbb{R}$$

$$\Phi_{xy}(t) = \Phi^*_{xy}(t) = 0,\ \forall t \leq 0$$

$$\Phi_{xy}(t) = \Phi^*_{xy}(t) = 1,\ \forall t > \mathrm{diam}(X)$$

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$$\textbf{DC1} \Rightarrow \textbf{DC2} \Rightarrow \textbf{DC3}$$

Theorem 1. *There is a nonempty family of maps $\mathcal{F}_1 \subseteq \mathcal{T}$ nondecreasing on the fibres and without DC2 pairs such that every $F \in \mathcal{F}_1$, restricted to the set of uniformly recurrent points, is Li-Yorke chaotic.*
(Every $F \in \mathcal{F}_1$ is of type 2^∞ and has zero topological entropy.)

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Proof.
We use the parametric family of maps introduced by BSŠ in 2005 and formalized by M. Mlíchová in 2006.

$$Q \times I \to Q \times I, \ (x, y) \mapsto (\tau(x), g_x(y))$$

$Q = \{0, 1\}^{\mathbb{N}} \ldots$ the middle-third Cantor set
$\tau \ldots$ the (binary) *adding machine* on Q;
$\tau(x_1 x_2 x_3 \cdots) = x_1 x_2 x_3 \cdots + 1000 \cdots$, where the adding is mod 2 with carry; e. g., $\tau(11010 \cdots) = 00110 \cdots$

$\{n_k\}_{k=1}^{\infty} \ldots$ an increasing sequence of positive integers of the form $n_k = 2^{c_k}$, $k, c_k \in \mathbb{N}$, with $c_k \geq 2$.
Write any $x = x_1 x_2 x_3 \cdots \in Q$ in blocks as

$$x = x^1 x^2 x^3 \cdots, \text{ where } x^j \text{ is the block of } c_j \text{ digits of } x. \tag{1}$$

For any finite block $\alpha = x_s x_{s+1} \cdots x_{s+k}$ the *evaluation* of α is $e(\alpha) = x_s + 2x_{s+1} + 2^2 x_{s+2} + \cdots + 2^k x_{s+k}$.
For any family of continuous maps $I \to I$

$$\{\varphi_k^{(j)}; 0 \leq j \leq n_k - 2\}_{k=1}^{\infty} \tag{2}$$

define $F(x, y) = (\tau(x), y)$ if $x = 1^{\infty}$ (i.e., if x contains no zero digit). Otherwise, let x^k be the first block in (1) containing a zero digit, and let

$$F(x, y) = (\tau(x), \varphi_k^{(p)}(y)), \text{ where } p = e(x^k). \tag{3}$$

If the maps $\varphi_k^{(j)}$ in (2) are taken such that

$$\lim_{k\to\infty} \max_j ||\varphi_k^{(j)} - Id|| = 0, \tag{4}$$

where Id denotes the identity map on I then F is continuous, and if

$$\varphi_k^{(n_k-2)} \circ \varphi_k^{(n_k-3)} \circ \cdots \circ \varphi_k^{(1)} \circ \varphi_k^{(0)} = \varphi_k^{(0)} = Id, \ k \in \mathbb{N}, \tag{5}$$

then some recurrence formulas are valid.
For $x \in Q$, $y \in I$, and a nonnegative integer i, let $y_x(i)$ be the second coordinate of $F^i(x, y)$. Then we have

Lemma (Čiklová 2006). *For any $j, k \in \mathbb{N}$ such that $1 \leq j < n_{k+1}$, (5) implies*

$$y_0(j \cdot m_k) = \varphi_{k+1}^{(j-1)} \circ \varphi_{k+1}^{(j-2)} \circ \cdots \circ \varphi_{k+1}^{(1)} \circ \varphi_{k+1}^{(0)}(y),$$

where $m_k := n_1 n_2 n_3 \cdots n_k$. In particular, $y_0(m_k) = y_0(0)$ $(= y)$.

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Proof.
STAGE 1. Define F on $Q \times I$ and show that it has unique (infinite) minimal set M, and that $F|_M$ is Li-Yorke chaotic.
Let $\{r_k\}_{k \geq 1}$ be a sequence in $(0,1)$ such that

$$r_k < r_{k+1}, \ k \in \mathbb{N}, \text{ and } \lim_{k \to \infty} r_k = 1. \tag{6}$$

Then there is an increasing sequence $\{n_k\}_{k \geq 1}$ of positive integers being powers 2^{c_k} of 2 such that

$$r_k^{n_k/2} > r_{k+1}^{n_{k+1}/2}, \ k \in \mathbb{N}, \quad \text{and} \quad \lim_{k \to \infty} r_k^{n_k/2} = 0. \tag{7}$$

For every $k \in \mathbb{N}$ and every $t \in I$ let
$\theta_k(t) = r_k t$ and $\overline{\theta_k}(t) = \min\{1, t/r_k\}$,
$\psi_k(t) = 1 - r_k(1-t)$, and $\overline{\psi_k}(t) = \max\{0, (t + r_k - 1)/r_k\}$.

It is easy to see that $\overline{\theta}_k \circ \theta_k = \overline{\psi}_k \circ \psi_k = Id$.

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Define the family (2) by

$$\varphi_k^{(j)} = \begin{cases} Id & \text{if } j = 0, \\ \psi_k & \text{if } 0 < j \leq n_k/2 - 1, \\ \overline{\psi_k} & \text{if } n_k/2 - 1 < j \leq n_k/2, \end{cases} \qquad \text{if } k \text{ is odd,} \tag{8}$$

and

$$\varphi_k^{(j)} = \begin{cases} Id & \text{if } j = 0, \\ \theta_k & \text{if } 0 < j \leq n_k/2 - 1, \\ \overline{\theta_k} & \text{if } n_k/2 - 1 < j \leq n_k/2. \end{cases} \qquad \text{if } k \text{ is even,} \tag{9}$$

Then (4) and (5) are satisfied.

Using Lemma it is easy to verify that

$$F^{jm_{k-1}}(0,1) = (\tau^{jm_{k-1}}(0), r_k^j), \quad j,k \in \mathbb{N}, \ k \text{ even}, \tag{10}$$

$$F^{jm_{k-1}}(0,0) = (\tau^{jm_{k-1}}(0), 1 - r_k^j), \quad j,k \in \mathbb{N}, \ k \text{ odd}. \tag{11}$$

This gives that $M = \omega_F(0,0)$ is the unique minimal set and $F|_M$ is LYC.

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STAGE 2. We show that parameters n_k can be chosen such that $F|_{Q\times I}$, or equivalently (since (Q,τ) is distal), no I_x with $x \in Q$ contains a *DC2*-pair. So it suffices to show that

$$\Phi_{uv}(t) = \Phi^*_{uv}(t) = 1, \text{ for every } u,v \in I_x, \quad x \in Q, \quad \text{and} \quad t > 0.$$

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$$\Phi_{uv}(t) = \Phi_{uv}^*(t) = 1, \text{ for every } u,v \in I_x, \quad x \in Q, \quad \text{and} \quad t > 0.$$

STAGE 3. Extend the map F from $Q \times I$ in the affine manner onto a map $(x,y) \to (f(x), g_x(y))$ in $\mathcal{T}$.

Remarks on open(?) problems

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