Improved Painlevé Removability for Bounded Planar Quasiregular Mappings

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Abstract. We review some aspects of the classical Painlevé problem (characterize geometrically the removable sets for bounded analytic functions) which has been recently solved by Tolsa. It is natural to try to understand the analogous problem in the quasiconformal world, i.e. understand the removable sets for bounded solutions of the Beltrami equation, which is the next simplest elliptic PDE after $\overline{\partial} f = 0$.

We also review some of the Hausdorff dimension distortion results for quasiconformal mappings and announce some results we have recently obtained. In particular, we show that compact sets of finite $\frac{2}{K+1}$-dimensional Hausdorff measure are mapped to compact sets of σ-finite length and, somewhat surprisingly, that they are removable for bounded K-quasiregular mappings.

1. Introduction

The classical Painlevé problem (characterize geometrically the removable sets for bounded analytic functions, or equivalently, sets of zero analytic capacity, see section 2 for the appropriate definitions) has been recently solved by Tolsa (with previous partial results by Guy David, and many others). Since analytic functions are solutions to the elliptic PDE $\overline{\partial} f = 0$, it is natural to try to understand the analogous problem in the quasiconformal world, i.e. understand the removable sets for bounded solutions to the Beltrami equation, which is the next simplest elliptic PDE, namely $\overline{\partial} f = \mu \partial f$, where $\| \mu \|_\infty = k < 1$ (refer to section 3 for the pertinent definitions). The solutions to the Beltrami equation are known to have many properties similar to those of analytic mappings, and there are many well-known interactions between classical complex analysis and the quasiconformal world. One

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of the objectives of this work is to further our understanding of these interactions and see how much of that parallelism holds in the removability problems. And one of our main results is that, perhaps somewhat surprisingly, the parallelism breaks down in some points.

In the Painlevé problem (i.e. removability for bounded analytic functions in the plane) the critical dimension is 1. More precisely, any compact set of dimension strictly less than 1 is removable, and any compact set of dimension strictly larger than 1 is not removable. At the critical dimension, if the compact set E has zero length, then it is removable (this is the classical result of Painlevé.) However, if E has sigma-finite length, it is removable if and only if it is purely unrectifiable. If E has dimension 1 but does not have sigma-finite length, the characterization is more complicated, and was given by Tolsa [**Tol03**]. See section 2 for a more detailed explanation of these results and references.

In this paper we consider, among other problems, the analogous quasiconformal problem (removability for bounded K-quasiregular mappings). Of course if $K = 1$ we recover the removability problem for bounded analytic functions. See section 3 for the appropriate definitions.

It is natural to expect some parallelism between the removability for bounded K-quasiregular mappings and the removability for bounded analytic functions. The content of theorems 5.4 and 5.5 below could be understood as a parallelism with the situation of analytic capacity breaking down (at least in some points). Namely, the critical dimension for K-removability is $\frac{2}{K+1}$. At the critical dimension, compacta with Hausdorff measure zero are K-removable (so far it is analogous to Painlevé's theorem). This was proven in a preprint by Astala ([**Ast**].) (It can also be immediately deduced from Theorems 5.1 (a) and 2, but Astala proved it before using other methods.)

However, at the critical dimension, if $K > 1$, sets with sigma finite Hausdorff measure are always removable, as opposed to the classical case $K = 1$ (Theorem 5.4), this being one of our main results. Also, the behaviour of some sets being removable and some sets not being removable in the classical case (analytic capacity) occurs only at the critical dimension, as we mentioned before. However, in K-removability, for $K > 1$, this behaviour appears strictly "above" (sigma finite sets at) the critical dimension, spread out in a range of dimensions (Theorem 5.5), at least within the hypotheses of Theorem 5.5. So another critical dimension appears, namely d_c (as in Theorem 5.5), below which some sets are removable and some not, and above which all sets are non-removable. This one is another one of our main results. We conjecture that the value of d_c is $1 - k^2$ for bounded K-quasiregular mappings, where as usual, $k = \frac{K-1}{K+1}$.

The techniques we use come from complex analysis and quasiconformal mappings (conformal welding, integral means estimates, Makarov's compression and expansion for conformal mappings), multifractal analysis, nonlinear potential theory (Riesz and Bessel capacities), harmonic analysis (Calderón-Zygmund theory, Hörmander-Mihlin multiplier theorem), geometric measure theory, etc.

The Euclidean ball of center 0 and radius R is denoted by $B(0, R)$. The unit disk is denoted by $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Its complement is denoted by $\mathbb{D}^c$ (and similarly with the complement of other sets.) The boundary of $\mathbb{D}$ will be denoted by $\partial\mathbb{D}$, $\mathbb{S}^1$, or $\mathbb{T}$, indistinctly. The diameter of a ball (or a set) A will be denoted by $diam(A)$.

Throughout this paper, $\dim(F)$ denotes the Hausdorff dimension of the set F.

The notation $A \approx B$ means that there exist constants $C_1, C_2 > 0$ such that $A \leq C_1 B \leq C_2 A$. Also, $A \lesssim B$ means that there is a constant $C > 0$ such that $A \leq CB$.

2. Analytic Capacity

A classical problem in complex analysis is to determine which compact sets $E \subset \mathbb{C}$ are removable for bounded analytic functions in the following sense:

DEFINITION 2.1. *Let $U \subset \mathbb{C}$ be an open set, with $E \subset U$, and let $f : U \setminus E \to \mathbb{C}$ be a bounded analytic function. We say E is removable for bounded analytic functions if all such f have an analytic extension to U.*

There is an extensive literature in removability problems for various classes of functions (see [**Mat95**] for a nice introduction, from where we have taken some of these introductory remarks, and also see, e.g. [**Gar72**], [**Mat95**], [**Gam69**] for analytic capacity.) If the boundedness is replaced by the Hölder continuity with exponent α, $0 < \alpha \leq 1$, then the necessary and sufficient condition for the removability of E is that $H^{1+\alpha}(E) = 0$, see [**Dol63**] and [**Uy79**]. For the corresponding question regarding harmonic functions see [**Car67**]. If the boundedness condition is replaced by BMO, then the removable sets E are characterized by the condition $H^1(E) = 0$ (see [**Krá84**] and [**Kau82**].)

Some other removability problems related to these pertain to removability with respect to Riesz kernels in $\mathbb{R}^n$ and have been discussed, e.g. in [**Pra04b**], [**Pra04a**], [**MPV05**]. For harmonic Lipschitz capacity see [**Vol03**]. There are many more references and important problems in this area that we are not citing. Those we cited (a necessarily incomplete list) were some examples of those that are more directly related to the theorems we prove.

Going back to removable sets in the sense of Definition 2.1, Ahlfors [**Ahl47**], [**AB50**] introduced an important set function called *analytic capacity* and denoted by γ. (See also the real analytic capacity and Cauchy capacity introduced by S. Khavinson [**Kha03**].) For a compact set $E \subset \mathbb{C}$,

$$\gamma(E) = \sup\{|f'(\infty)| : f \text{ is analytic in } \mathbb{C} \setminus E, \| f \|_\infty \leq 1, f(\infty) = 0\} \tag{2.1}$$

where $f'(\infty) = \lim_{z\to\infty} z(f(z) - f(\infty))$, and $f(\infty) = \lim_{z\to\infty} f(z)$. (Note that, usually, $f'(\infty) \neq \lim_{z\to\infty} f'(z)$).

Recall also that, for an open set $\Omega \subset \mathbb{C}$, the notation $H^\infty(\Omega)$ stands for the space of bounded analytic functions defined in Ω with the supremum norm. The open set Ω is frequently omitted if it is clear from the context.

A classical result is the following

THEOREM 1. *For a compact set $E \subset \mathbb{C}$, the following conditions are equivalent:*

(a) $\gamma(E) = 0$
(b) *Every bounded analytic function $f : \mathbb{C} \setminus E \to \mathbb{C}$ is constant*
(c) *E is removable in the sense of Definition 2.1*

Moreover, these conditions imply that E is totally disconnected.

The geometric characterization of the compacta with $\gamma(E) = 0$ is the so-called Painlevé problem, which has taken a good number of years and efforts to solve. The critical dimension for this problem is 1, in the sense that if $\dim(E) < 1$, then E is removable, and if $\dim(E) > 1$, then E is non-removable (see [**Ahl47**].)

However, the solution to this problem does not only depend on the size of sets, and in dimension 1, the situation gets more complicated. A classical result by Painlevé (see e.g. [**Pai88**], [**Bes29**], [**Gar72**] or [**AIM**]) states that

THEOREM 2. *Let $E \subset \mathbb{C}$ be a compact set. Then $\gamma(E) \lesssim H^1(E)$.*

So, in particular, sets of zero length are removable for bounded analytic functions.

If $0 < H^1(E) < \infty$, then by a classical theorem of Besicovitch (later generalized by Federer in higher dimensions), (see e.g. [**Bes28**], [**Bes38**], [**Bes39**], [**Fed47**], [**Fed69**] or the excellent book by Mattila [**Mat95**] for this and related results), E can be decomposed as a disjoint union of three sets: $E = G \cup R \cup N$. Here $H^1(G) = 0$, R is a rectifiable set (i.e. $\exists \Gamma_i$ Lipschitz or C^1 curves so that $H^1(R \setminus \cup_{i=1}^{\infty} \Gamma_i) = 0$), and N is a purely unrectifiable set (i.e. $H^1(\Gamma \cap N) = 0$ for every rectifiable set Γ). A typical example of a purely unrectifiable set is a planar Cantor set. Then $\gamma(E) = 0 \iff E = G \cup N$ (see [**Dav98**] for the final step in the direction we need, namely E purely unrectifiable $\Rightarrow \gamma(E) = 0$. Some important milestones for establishing this direction should be mentioned such as [**Chr90**], [**Jon90**], [**DS91**], [**MMV96**], and previous work by J. C. Leger on one hand and by David and Mattila on the other hand. The other direction, so called Denjoy conjecture, follows from [**Cal77**], and work of Garabedian [**Gar49**], who established the equivalence for removability for H^∞ functions and removability for functions in the Hardy space H^2.) A complete description of sets with $\gamma(E) = 0$ was given by Tolsa in [**Tol03**].

An important property of the analytic capacity is $\gamma(\cup_{i=1}^{\infty} A_i) \leq C \sum_{i=1}^{\infty} \gamma(A_i)$. (This inequality is only needed in our argument for the case that $\gamma(A_i) = 0$. In this case it was well-known (see [**Gam69**], [**Gar72**]), but only when A_i are compact sets. It was proven in the general case by Tolsa [**Tol03**].)

Summarizing the aspects we consider relevant for the parallelism we want to draw later, at the critical dimension (i.e. 1), a set with Hausdorff measure (length) zero is removable for bounded analytic functions and a set of sigma finite measure (length) may be removable or not, depending on the geometry.

3. Quasiconformal and Quasiregular Mappings

If Ω, Ω' are open connected sets in $\mathbb{C}$, recall that a map $f : \Omega \to \Omega'$ which is orientation preserving, continuous, is in $W^{1,2}_{loc}(\Omega) = \left\{ f \in L^2_{loc}(\Omega) : |\nabla f| \in L^2_{loc}(\Omega) \right\}$, and satisfies the distortion inequality $\max_\alpha |\partial_\alpha f(z)| \leq K \min_\alpha |\partial_\alpha f(z)|$ a.e. $z \in \Omega$ is called a K-quasiregular mapping. Here $\partial_\alpha f(z)$ stands for the directional derivative of f in the direction α. If, on top of the previous conditions, f is a homeomorphism, then it is called a K-quasiconformal mapping.

K-quasiregular mappings are precisely the solutions to the Beltrami equation $\overline{\partial} f = \mu \partial f$, a.e. z, where, as usual, $\overline{\partial} f = f_{\overline{z}} = \frac{1}{2}(f_x + i f_y)$ and $\partial f = f_z = \frac{1}{2}(f_x - i f_y)$, and $\| \mu \|_\infty \leq k < 1$, with $K = \frac{1+k}{1-k}$. For general references on quasiconformal mappings, see, e.g. [**Ahl66**], [**LV73**], [**IM01**], [**AIM**].

As we mentioned above, quasiconformal and quasiregular mappings, i.e. the solutions of the Beltrami equation (the next simplest elliptic PDE to $\overline{\partial} f = 0$) are known to have many properties similar to those of analytic mappings. There are many well-known interactions between classical complex analysis and the quasiconformal world.

Conformal mappings preserve Hausdorff dimension, whereas quasiconformal mappings need not. However, there are precise bounds as to how much a K quasiconformal mapping can distort Hausdorff dimension. Namely, a well-known result in the paper [**Ast94**] on area distortion for quasiconformal mappings states that for a planar K-quasiconformal mapping $\phi : \mathbb{C} \to \mathbb{C}$, if dim denotes Hausdorff dimension, and E is a compact planar set, one has

$$\frac{1}{K}\left(\frac{1}{\dim(E)} - \frac{1}{2}\right) \leq \frac{1}{\dim(\phi(E))} - \frac{1}{2} \leq K\left(\frac{1}{\dim(E)} - \frac{1}{2}\right) \tag{3.1}$$

This result is sharp, but it gives no information on the Hausdorff measures at the corresponding dimensions. In particular, if $0 < \tau < 2$ and $\delta = \delta_K(\tau) = 2K\tau/(2 + \tau(K-1))$ (i.e. the maximum stretching allowed by (3.1)), and ϕ is a planar K-quasiconformal mapping, then a question in [**Ast94**] is whether

$$H^\tau(E) = 0 \Rightarrow H^\delta(\phi E) = 0 \tag{3.2}$$

where H^τ is the Hausdorff measure at dimension τ.

An important tool to understand K quasiregular mappings is Stoilow's theorem, according to which any such mapping f decomposes as $f = h \circ g$, where g is K quasiconformal (hence a homeomorphism) and h is analytic. Consequently, the analogous problem to removability for bounded analytic functions in the "quasi-world" is removability for bounded K quasiregular mappings, (what we will call K-removability in Definition 3.1). Notice that if $K = 1$, f reduces to a bounded analytic map, so that K-removability becomes removability for bounded analytic functions.

So, let us agree on the following definition for the purposes of this paper. (Compare with Theorem 1 (b).)

Definition 3.1. *Let $E \subset \mathbb{C}$ be compact set. We say that E is K-removable if for every bounded K-quasiregular mapping $f : \mathbb{C} \setminus E \to \mathbb{C}$, one has that f is a constant map.*

Regarding the critical dimension for K-removability, let us briefly recall why it is $\frac{2}{K+1}$. As we saw above, by Stoilow's theorem, any bounded K-quasiregular $f : \mathbb{C} \setminus E \to \mathbb{C}$ decomposes as $f = h \circ g$, where g is K quasiconformal (hence a homeomorphism) and h is analytic. The sets E we are considering have zero area, so the Beltrami coefficient μ_g is defined a.e. and hence, solving the Beltrami equation, g extends to $\mathbb{C}$. So the question is when can we extend h, which is precisely the removability question for bounded analytic functions. So

$$E \text{ is } K-\text{removable} \iff \gamma(gE) = 0 \text{ for all } K-\text{quasiconformal maps } g : \mathbb{C} \to \mathbb{C}. \tag{3.3}$$

Incidentally, in particular Theorem 5.4 below says that if E has sigma finite $H^{\frac{2}{K+1}}$ measure, then, for all K quasiconformal mappings $g : \mathbb{C} \to \mathbb{C}$ either $H^1(gE) = 0$ or gE is purely unrectifiable.

An important element in (a proof of) (3.1) is the integrability of Jacobians of quasiconformal mappings, namely the fact that if $f : \mathbb{C} \to \mathbb{C}$ is K quasiconformal, then its Jacobian $J(z, f) \in L^{\frac{K}{K-1}, \infty}$, this estimate being sharp in the class of general K quasiconformal mappings. (Recall that $L^{p,\infty}(d\mu) = \{f \text{ measurable} : \sup_{\lambda > 0} \lambda^p \mu\{x : |f| > \lambda\} < \infty\}$, which contains the space $L^p(d\mu)$ by Chebychev's inequality.) However, if the K quasiconformal mapping f happens to be conformal on a certain measurable set E (i.e. $\overline{\partial} f(z) = 0$ for $z \in E$), then one has a slightly better integrability, namely that $J(z,f) \in L^{\frac{K}{K-1}}(E)$. This was proven in [**AN03**], and is an important element in the proof of our results. We use it once a quasiconformal map f has been decomposed, by Stoilow, into a map which is conformal in a certain region and another map.

A particular case of equation (3.1) is when $\dim(E) = \frac{2}{K+1}$. Then for any K-quasiconformal mapping $g : \mathbb{C} \to \mathbb{C}$, $\dim(gE) \leq 1$, and this is sharp. In particular, if $\dim(E) < \frac{2}{K+1}$, then $\dim(gE) < 1$ and hence $\gamma(gE) = 0$. And also, for any $d > \frac{2}{K+1}$, $\exists E \subset \mathbb{C}$ compact, with $\dim(E) = d$, and a K quasiconformal map g with $\dim(gE) > 1$, so that E is not K-removable. (An example of such a set E is an appropriately chosen Cantor set.) This was proven by Astala in [**Ast94**]. So, consequently, $\frac{2}{K+1}$ is the critical dimension for K-removability.

Concerning other properties of quasiconformal mappings, if $\Gamma = g(\mathbb{S}^1)$ is a K-quasicircle (i.e. the image of S^1 under a K-quasiconformal map) for g a K-quasiconformal map (often denoted K-q.c. map), then, with $k = \frac{K-1}{K+1}$,

$$1 \leq \dim(\Gamma) \leq 1 + ck^2 \leq 1 + k \tag{3.4}$$

The lower bound is a consequence of a quasicircle being a continuum. The rightmost upper bound is what area distortion results predict for a general set of dimension 1 (see equation (3.1).) This rightmost dimension distortion estimate is sharp in the class of general compact sets. However, in the specific class of sets of quasicircles, the topology of the line somehow does not allow maximum distortion and one gets the leftmost upper bound (which is due to Becker and Pommerenke [**BP87**]), with a quadratic behaviour in k.

Incidentally, a non-published theorem by Smirnov [**Smi**] shows that in the Becker-Pommerenke bound, c can be taken to be 1. This is conjectured to be sharp. Unpublished results by Astala, Rohde and Schramm [**ARS**] obtained by holomorphically deforming the von Koch snowflake give examples with $c = 0.69$.

4. Review of Nonlinear Potential Theory and Hausdorff Measures

A good general reference for Hausdorff measures is [**Mat95**], and for Riesz and Bessel capacities, a good general reference is [**AH96**].

Let us first recall some notations we will use pertaining to Hausdorff measures and related quantities. For a measure function h, we denote for a set $E \subset \mathbb{C}$ and $0 < \rho \leq \infty$,

$$\Lambda_h^\rho(E) = \inf \sum_1^\infty h(r_i)$$

where the infimum is taken over all coverings of E by countable unions of (open or closed) balls $\{B(x_i, r_i)\}_{i=1}^\infty$ with radii $\{r_i\}_{i=1}^\infty$, and so that $\sup r_i \leq \rho$.

Since $\Lambda_h^\rho(E)$ is a decreasing function of ρ, then $\lim_{\rho\to 0} \Lambda_h^\rho(E)$ exists ($\leq \infty$), and we can define the Hausdorff measure of E with respect to the function h as

$$\Lambda_h(E) = \lim_{\rho\to 0} \Lambda_h^\rho(E).$$

If $h(r) = r^\alpha$, we write $\Lambda_\alpha = H^\alpha$ for Λ_{r^α}, and H^α_ρ for Λ^ρ_h. Recall that $H^\alpha_\infty(E) = 0 \iff H^\alpha(E) = 0$. $H^\alpha_\infty(E)$ is the so-called Hausdorff content. In many ways it behaves like a capacity, which makes it a very useful quantity. As a brief remark, let us mention that the notation H^α has already appeared in this paper when referring to the spaces H^2 and H^∞. However, it should be clear from the context what it means. Also, H^∞ cannot be a Hausdorff measure, and we only consider in this paper H^α as a Hausdorff measure when $\alpha < 2$.

Recall that in $\mathbb{C}$, for $0 < \alpha < 2$, the Riesz kernel I_α is defined as the inverse Fourier transform of $|\xi|^{-\alpha}$ (in the sense of distributions), and one can show that, for a certain constant $c_\alpha > 0$,

$$I_\alpha(x) = \frac{c_\alpha}{|x|^{2-\alpha}} \tag{4.1}$$

The Riesz potential of a function g is defined as

$$I_\alpha g(x) = (I_\alpha * g)(x) = c_\alpha \int_{\mathbb{C}} \frac{g(y)}{|x-y|^{2-\alpha}} dy, \tag{4.2}$$

In a similar fashion, recall that the Bessel kernel (for $\alpha \in \mathbb{C}$) is

$$G_\alpha = \mathcal{F}^{-1}\left(\left(1+|\xi|^2\right)^{-\alpha/2}\right) \tag{4.3}$$

And similarly for the Bessel potential.

The Bessel potential spaces $L^{\alpha,p} = L^{\alpha,p}(\mathbb{C})$ are defined by

$$L^{\alpha,p}(\mathbb{C}) = \{f : f = G_\alpha * g, g \in L^p(\mathbb{C})\} \tag{4.4}$$

with norm $\| f \|_{\alpha,p} = \| g \|_p$.

A fundamental theorem of A.P. Calderón ([**Cal61**], [**AH96**] p.13) that explains part of the importance of these spaces states that

THEOREM 3. ***(A.P. Calderón).*** *If $\alpha \in \mathbb{N}$, for $1 < p < \infty$, $W^{\alpha,p}(\mathbb{C}) = L^{\alpha,p}(\mathbb{C})$ with equivalence of norms. (Here $W^{\alpha,p}(\mathbb{C})$ denotes the usual Sobolev space of distributions with derivatives up to order α in L^p).*

If $1 \le p < \infty$, $E \subset \mathbb{C}$, dm denotes Lebesgue measure, and $L^p_+(dm)$ denotes the nonnegative functions in $L^p(dm)$, we denote

$$\Omega_E = \Omega_E(I_\alpha) = \{f : f \in L^p_+(dm), I_\alpha f(x) \ge 1, \text{ for all } x \in E\}$$

and the Riesz capacity is defined as:

$$\dot{C}_{\alpha,p}(E) = \inf\left\{\int_{\mathbb{C}} f^p dm : f \in \Omega_E\right\} \tag{4.5}$$

Similar definitions hold for the Bessel capacity, which is denoted by $C_{\alpha,p}(E)$. Assuming $1 < p < \infty$, $E \subset \mathbb{C}$, $\dot{C}_{\alpha,p}(E) < \infty$, there is a unique extremal function, denoted f^E, such that $f^E \in L^p_+(\mathbb{C})$ and $I_\alpha f^E(x) \ge 1$ (α,p)-q.e. on E, and

$$\int_{\mathbb{C}} (f^E)^p = \dot{C}_{\alpha,p}(E). \tag{4.6}$$

where a property holds (α,p)-q.e. if it holds except on a set A so that $\dot{C}_{\alpha,p}(A) = 0$ (or the corresponding capacity being zero instead of $\dot{C}_{\alpha,p}$). Similarly, there is a unique extremal function for the Bessel capacity.

On the other hand, if $\alpha p < 2$, and $E \subseteq B(0,R)$, one has

$$\dot{C}_{\alpha,p}(E) \approx C_{\alpha,p}(E) \tag{4.7}$$

with constants that depend only on α, p and R. (See [**AH96**].)

Another important fact about capacities is that they satisfy an equivalent dual definition, namely that for $\alpha > 0$, $p > 1$, and E a compact set in $\mathbb{C}$,

(4.8)
$$\dot{C}_{\alpha,p}(E) = \sup\{\mu(E)^p : \mu \text{ is a positive measure supported on E }, \| I_\alpha * \mu \|_{p'} \leq 1\}$$

And a similar equivalent definition holds for Bessel capacities. In both cases (Riesz and Bessel), analogously to (4.6), there is an extremal measure that realizes the supremum, denoted by μ^E. For some capacities, one can use distributions supported on E instead of restricting oneself to positive measures supported on E, and get a comparable capacity.

There are well-known sharp relations between capacities and Hausdorff measures (see e.g. [**AH96**]), but we only use one of those, which is written as the second part of Theorem 5.2.

5. Our Results

Our first theorem gives a positive answer to question (3.2) in one particular case, which is the case needed for our removability result, Theorem 5.4 below. Namely, we have

THEOREM 5.1. *Let $\phi : \mathbb{C} \to \mathbb{C}$ be a K-quasiconformal mapping, and $E \subset \mathbb{C}$ a compact set.*

(a) *If $H^{\frac{2}{K+1}}(E) = 0$, then $H^1(\phi(E)) = 0$.*
(b) *If $H^{\frac{2}{K+1}}(E) < \infty$, then $\phi(E)$ is a set of σ-finite length.*

We also get another result towards the conjecture (3.2) for some other dimensions, namely our second theorem:

THEOREM 5.2. *Let $\phi : \mathbb{C} \to \mathbb{C}$ be a K-quasiconformal mapping, and $E \subset \mathbb{C}$ a compact set. Let also $1 < \delta = \delta_K(\tau) < 2$. Then $H^\tau(E) = 0 \Rightarrow C_{\alpha,p}(\phi E) = 0$, where $\alpha = \frac{2}{\delta} - 1, p = \delta$ and $C_{\alpha,p}(A)$ is the Bessel capacity associated to the Bessel kernel G_α.*

In particular, the generalized Hausdorff measure $\Lambda^h(\phi E)$ is equal to 0, for all measure functions h so that

$$\int_0^1 \left(\frac{h(r)}{r^\delta}\right)^{\frac{1}{\delta-1}} \frac{dr}{r} < \infty$$

REMARK 5.3. *The statement of Theorem 5.2 may seem somewhat unnatural as compared to that of Theorem 5.1. The method of our proof uses as an important tool the invariance of certain spaces under composition with quasiconformal mappings. In particular BMO (see [**Rei74**]) and $\dot{W}^{1,2} = \dot{L}^{1,2}$, which is defined as $L^{\alpha,p}$ in (4.4), but using the Riesz kernel (4.1) instead of the Bessel kernel (4.3) (and analogously for other indices α, p.) These lead to the homogeneous Sobolev spaces $\dot{L}^{s,\frac{2}{s}} = \dot{H}^s_{\frac{2}{s}}$ also being invariant under composition with quasiconformal mappings (by interpolation, see [**RR75**], [**FJ90**].) The invariance of these homogeneous Sobolev spaces is related to certain capacities being invariant under composition with quasiconformal mappings, which is what we use to prove Theorem 5.2. (We mentioned all those notations for the homogeneous Sobolev spaces for the convenience of the reader, since they are all classical.)*

However, let us explain another intuitive reason hinting at why these methods lead to the formulation of Theorem 5.2 with Riesz capacities being zero as opposed to Hausdorff measures (or equivalently, Hausdorff contents) being zero. Recall (see [**AH96**] *p.151), that the fractional maximal function for $0 < \alpha < 2$ for a positive Radon measure μ is defined by*

$$M_\alpha(\mu)(x) = \sup_{r>0} \frac{1}{|B(x,r)|^{\frac{2-\alpha}{2}}} \int_{B(x,r)} d\mu(y)$$

Denote the space of positive Radon measures supported on a compact set K by $\mathcal{M}^+(K)$. Then, on the one hand by Frostman's lemma, for a compact $K \subset \mathbb{C}$ and $0 < \alpha < 2$,

$$H^\infty_{2-\alpha}(K) > 0 \Longleftrightarrow \exists \mu \in \mathcal{M}^+(K), 0 <\| M_\alpha \mu \|_\infty < \infty.$$

On the other hand, for $p > 1$ and $0 < \alpha p \leq 2$,

$$C_{\alpha,p}(K) > 0 \Longleftrightarrow \exists \mu \in \mathcal{M}^+(K), 0 <\| M_\alpha \mu \|_{p'} < \infty.$$

Also, by Corollary 3.6.3 in [**AH96**], *$\| M_\alpha * \mu \|_p \approx \| I_\alpha * \mu \|_p$, for $0 < \alpha < 2$, and $1 < p < \infty$.*

Hence, when interpolating between BMO and $\dot{W}^{1,2}$, thinking of BMO morally as $p = \infty$, the parameter p runs through all values between 2 and ∞, and this "indicates why" we get $C_{\alpha,p}$ and not $H^\infty_{2-\alpha}$. (Think of (4.2) *and the equivalent of* (4.4) *for I_α.)*

As we mentioned in the introduction, the content of theorems 5.4 and 5.5 below could be understood as a parallelism with the situation of analytic capacity breaking down (at least in certain points). Namely, the critical dimension for K-removability is $\frac{2}{K+1}$ as we saw. At the critical dimension, compacta with Hausdorff measure zero are K-removable (this is analogous to Painlevé's theorem). This was proven first by Astala ([**Ast**] using different methods than those of Theorem 5.1.) (It can also be immediately deduced from Theorems 5.1 (a) and 2.)

However, at the critical dimension, if $K > 1$, sets with sigma finite Hausdorff measure are always removable, as opposed to the classical case $K = 1$ (Theorem 5.4). Also, the behaviour of some sets being removable and some sets not being removable in the classical case (analytic capacity) occurs only at the critical dimension. However, in K-removability, $K > 1$, this behaviour appears strictly "above" (sigma finite sets at) the critical dimension, spread out in a range of dimensions (Theorem 5.5), at least within the hypotheses of Theorem 5.5. So another critical dimension appears, namely d_c (as in Theorem 5.5), below which some sets are removable and some are non-removable, and above which all sets are non-removable. We conjecture that the value of d_c is $1 - k^2$ for bounded K-quasiregular mappings, where, as usual, $k = \frac{K-1}{K+1}$.

The theorems we mentioned before are the following:

Theorem 5.4. *Let $E \subset \mathbb{C}$ be compact set with sigma finite $H^{\frac{2}{K+1}}$ measure. Let $f : \mathbb{C} \setminus E \to \mathbb{C}$ be a bounded K-quasiregular mapping, with $K > 1$. Then f is a constant map. In other words, E is K-removable.*

Theorem 5.5. *Let $1 < K \leq 1 + \varepsilon_0$. Then $\exists d_c > \frac{2}{K+1}$ such that for all $d < d_c$, $\exists E \subset \mathbb{C}$ compact with $\dim(E) = d$, and E is K-removable.*

We expected (and proved) a quadratic behaviour for lower bounds similar to that of (3.4), i.e. for compression of dimension of subsets of the circle under a quasiconformal mapping. (It is important to consider subsets in order to avoid the trivial lower bound in (3.4) which is due to connectedness.) Philosophically, that is precisely one of the reasons to conjecture that the value of d_c is $1-k^2$: if Smirnov's result is sharp, and the expanding behaviour is the same as the contracting one, then one should get that value for d_c.

As we mentioned before, Theorem 5.4 above has the nice geometric consequence that if E has sigma finite $H^{\frac{2}{K+1}}$ measure, then, for all K quasiconformal mappings $g:\mathbb{C}\to\mathbb{C}$ either $H^1(gE)=0$ or gE is purely unrectifiable.

Very recently, István Prause [**Pra**] obtained an independent proof of a compression estimate that was a key component in our proof of Theorem 5.4. He also obtained a proof of Theorem 5.5, also taking subsets of $\mathbb{R}$.

As we mentioned above, if in the analytic capacity problem the boundedness condition is replaced by BMO, then the removable sets E are characterized by the condition $H^1(E)=0$ (see [**Krá84**] and [**Kau82**].) A problem related to K-removability is to identify the removable sets for BMO-quasiregular mappings. Hence we obtain immediately from Theorem 5.1 the following

Corollary 5.6. *Let E be a compact subset of the plane. Assume $H^{\frac{2}{K+1}}(E)=0$. Then, E is removable for BMO K-quasiregular mappings.*

Details of the proofs and some other related questions are contained in [**ACM**$^+$].

References

[AB50] Lars Ahlfors and Arne Beurling. Conformal invariants and function-theoretic null-sets. *Acta Math.*, 83:101–129, 1950.

[ACM$^+$] Kari Astala, Albert Clop, Joan Mateu, Joan Orobitg, and Ignacio Uriarte-Tuero. Distortion of Hausdorff measures and improved Painlevé removability for quasiregular mappings. *[Preprint]*.

[AH96] David R. Adams and Lars Inge Hedberg. *Function spaces and potential theory*, volume 314 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1996.

[Ahl47] Lars V. Ahlfors. Bounded analytic functions. *Duke Math. J.*, 14:1–11, 1947.

[Ahl66] Lars V. Ahlfors. *Lectures on quasiconformal mappings.* Manuscript prepared with the assistance of Clifford J. Earle, Jr. Van Nostrand Mathematical Studies, No. 10. D. Van Nostrand Co., Inc., Toronto, Ont.-New York-London, 1966.

[AIM] Kari Astala, Tadeusz Iwaniec, and Gaven Martin. *Elliptic Partial Differential Equations and Quasiconformal Mappings in the Plane.* [In preparation].

[AN03] Kari Astala and Vincenzo Nesi. Composites and quasiconformal mappings: new optimal bounds in two dimensions. *Calc. Var. Partial Differential Equations*, 18(4):335–355, 2003.

[ARS] Kari Astala, Steffen Rohde, and Oded Schramm. Dimension of quasicircles. *[Preprint]*.

[Ast] Kari Astala. Painleve's theorem and removability properties of planar quasiregular mappings. *[Preprint]*.

[Ast94] Kari Astala. Area distortion of quasiconformal mappings. *Acta Math.*, 173(1):37–60, 1994.

[Bes28] A. S. Besicovitch. On the fundamental geometrical properties of linearly measurable plane sets of points. *Math. Ann.*, 98(1):422–464, 1928.

[Bes29] A. S. Besicovitch. On sufficient conditions for a function to be analytic, and on behavior of analytic functions in the neighbourhood of non-isolated singular points. *Proc. Lond. Math. Soc.*, 32:1–9, 1929.

[Bes38] A. S. Besicovitch. On the fundamental geometrical properties of linearly measurable plane sets of points (II). *Math. Ann.*, 115(1):296–329, 1938.

[Bes39] A. S. Besicovitch. On the fundamental geometrical properties of linearly measurable plane sets of points (III). *Math. Ann.*, 116(1):349–357, 1939.

[BP87] J. Becker and Ch. Pommerenke. On the Hausdorff dimension of quasicircles. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 12(2):329–333, 1987.

[Cal61] A.-P. Calderón. Lebesgue spaces of differentiable functions and distributions. In *Proc. Sympos. Pure Math., Vol. IV*, pages 33–49. American Mathematical Society, Providence, R.I., 1961.

[Cal77] A.-P. Calderón. Cauchy integrals on Lipschitz curves and related operators. *Proc. Nat. Acad. Sci. U.S.A.*, 74(4):1324–1327, 1977.

[Car67] Lennart Carleson. *Selected problems on exceptional sets.* Van Nostrand Mathematical Studies, No. 13. D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto, Ont.-London, 1967.

[Chr90] Michael Christ. A $T(b)$ theorem with remarks on analytic capacity and the Cauchy integral. *Colloq. Math.*, 60/61(2):601–628, 1990.

[Dav98] Guy David. Unrectifiable 1-sets have vanishing analytic capacity. *Rev. Mat. Iberoamericana*, 14(2):369–479, 1998.

[Dol63] E. P. Dolženko. The removability of singularities of analytic functions. *Uspehi Mat. Nauk*, 18(4 (112)):135–142, 1963.

[DS91] G. David and S. Semmes. Singular integrals and rectifiable sets in $\mathbf{R}^n$: Beyond Lipschitz graphs. *Astérisque*, (193):152, 1991.

[Fed47] Herbert Federer. The (φ, k) rectifiable subsets of n-space. *Trans. Amer. Soc.*, 62:114–192, 1947.

[Fed69] Herbert Federer. *Geometric measure theory.* Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969.

[FJ90] Michael Frazier and Björn Jawerth. A discrete transform and decompositions of distribution spaces. *J. Funct. Anal.*, 93(1):34–170, 1990.

[Gam69] Theodore W. Gamelin. *Uniform algebras.* Prentice-Hall Inc., Englewood Cliffs, N. J., 1969.

[Gar49] P. R. Garabedian. Schwarz's lemma and the Szegö kernel function. *Trans. Amer. Math. Soc.*, 67:1–35, 1949.

[Gar72] John Garnett. *Analytic capacity and measure.* Springer-Verlag, Berlin, 1972. Lecture Notes in Mathematics, Vol. 297.

[IM01] Tadeusz Iwaniec and Gaven Martin. *Geometric function theory and non-linear analysis.* Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 2001.

[Jon90] Peter W. Jones. Rectifiable sets and the traveling salesman problem. *Invent. Math.*, 102(1):1–15, 1990.

[Kau82] Robert Kaufman. Hausdorff measure, BMO, and analytic functions. *Pacific J. Math.*, 102(2):369–371, 1982.

[Kha03] S. Ya. Khavinson. Duality relations in the theory of analytic capacity. *Algebra i Analiz*, 15(1):3–62, 2003.

[Krá84] Josef Král. Semielliptic singularities. *Časopis Pěst. Mat.*, 109(3):304–322, 1984.

[LV73] O. Lehto and K. I. Virtanen. *Quasiconformal mappings in the plane.* Springer-Verlag, New York, second edition, 1973. Translated from the German by K. W. Lucas, Die Grundlehren der mathematischen Wissenschaften, Band 126.

[Mat95] Pertti Mattila. *Geometry of sets and measures in Euclidean spaces*, volume 44 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1995. Fractals and rectifiability.

[MMV96] Pertti Mattila, Mark S. Melnikov, and Joan Verdera. The Cauchy integral, analytic capacity, and uniform rectifiability. *Ann. of Math. (2)*, 144(1):127–136, 1996.

[MPV05] Joan Mateu, Laura Prat, and Joan Verdera. The capacity associated to signed Riesz kernels, and Wolff potentials. *J. Reine Angew. Math.*, 578:201–223, 2005.

[Pai88] Paul Painlevé. Sur les lignes singulières des fonctions analytiques. *Ann. Fac. Sci. Toulouse Sci. Math. Sci. Phys.*, 2:B1–B130, 1888.

[Pra] István Prause. A remark on quasiconformal dimension distortion on the line. *[Preprint].*

[Pra04a] Laura Prat. Null sets for the capacity associated to Riesz kernels. *Illinois J. Math.*, 48(3):953–963, 2004.

[Pra04b] Laura Prat. Potential theory of signed Riesz kernels: capacity and Hausdorff measure. *Int. Math. Res. Not.*, (19):937–981, 2004.

[Rei74] H. M. Reimann. Functions of bounded mean oscillation and quasiconformal mappings. *Comment. Math. Helv.*, 49:260–276, 1974.

[RR75] Hans Martin Reimann and Thomas Rychener. *Funktionen beschränkter mittlerer Oszillation*. Springer-Verlag, Berlin, 1975. Lecture Notes in Mathematics, Vol. 487.

[Smi] Stanislav Smirnov. *[Preprint]*.

[Tol03] Xavier Tolsa. Painlevé's problem and the semiadditivity of analytic capacity. *Acta Math.*, 190(1):105–149, 2003.

[Uy79] Nguyen Xuan Uy. Removable sets of analytic functions satisfying a Lipschitz condition. *Ark. Mat.*, 17(1):19–27, 1979.

[Vol03] Alexander Volberg. *Calderón-Zygmund capacities and operators on nonhomogeneous spaces*, volume 100 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2003.

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