STABILITY OF CALDERÓN'S INVERSE CONDUCTIVITY PROBLEM IN THE PLANE FOR DISCONTINUOUS CONDUCTIVITIES

Albert Clop Daniel Faraco Alberto Ruiz

Abstract

It is proved that, in two dimensions, the Calderón inverse conductivity problem in Lipschitz domains is stable in the L^p sense when the conductivities are uniformly bounded in any fractional Sobolev space $W^{\alpha,p}$ $\alpha > 0, 1 < p < \infty$.

Mathematics Subject Classification (2000): 35R30, 35J15, 30C62.

1 Introduction

Calderón inverse problem, see [22], consists in the determination of an isotropic L^∞ conductivity coefficient γ on Ω from boundary measurements. These measurements are given by the Dirichlet to Neumann map Λ_γ, defined for a function f on $\partial\Omega$ as the Neumann value

$$\Lambda_\gamma(f) = \gamma \frac{\partial}{\partial \nu} u,$$

where u is the solution of the Dirichlet boundary value problem

$$\begin{cases} \nabla \cdot (\gamma \nabla u) = 0 \\ u_{|\partial\Omega} = f \end{cases} \tag{1.1}$$

and $\frac{\partial}{\partial \nu}$ denotes the outer normal derivative. For general domain and conductivities where the pointwise definition $\gamma \frac{\partial}{\partial \nu} u$ has no meaning, the Dirichlet to Neumann map

$$\Lambda_\gamma : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega) \tag{1.2}$$

can be defined by

$$\langle \Lambda_\gamma(f), \varphi_0 \rangle = \int_\Omega \gamma \nabla u \cdot \nabla \varphi \tag{1.3}$$

where $\varphi \in W^{1,2}(\Omega)$ is a function such that $\varphi_{|\partial\Omega} = \varphi_0$ in the sense of traces.

Since the foundational work of Calderón, research on this question has been very intense but it is not until 2006 when, by means of quasiconformal mappings, K. Astala and L. Päivärinta in [12], see also [11], were able to establish the injectivity of the map

$$\gamma \to \Lambda_\gamma$$

for an arbitrary L^∞ function bounded away from zero. Previous planar results were obtained in [35], [45] and [20]. In higher dimensions, the known results on uniqueness require some extra a priori regularity on γ (basically some control on $\frac{3}{2}$ derivatives of γ, see [44], [17], [38] and [19].)

A relevant question (specially in applications and in the development of recovery algorithms, see [30] and [16]) is the stability of the inverse problem, that is, the continuity of the inverse map

$$\Lambda_\gamma \to \gamma.$$

For dimension $n > 2$, the known results are due to Alessandrini [4], [5]. There the author proved stability under the extra assumption $\gamma \in W^{2,\infty}$. In the planar case, $n = 2$, the situation is different. Liu proved stability for conductivities in $W^{2,p}$ with $p > 1$ in [32]. In [13], stability was obtained when $\gamma \in \mathcal{C}^{1+\alpha}$ with $\alpha > 0$. Recently, Barceló, Faraco and Ruiz [14] obtained stability under the weaker assumption $\gamma \in \mathcal{C}^\alpha$, $0 < \alpha < 1$. Precisely, they prove that for any two conductivities γ_1, γ_2 on a Lipschitz domain Ω, with a priori bounds $\frac{1}{K} \leq \gamma_i \leq K, K \geq 1$ and $\|\gamma_i\|_{\mathcal{C}^\alpha} \leq \Lambda_0$, the following estimate holds:

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)} \leq V(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)})$$

with $V(t) = C \log(\frac{1}{t})^{-a}$. Here $C, a > 0$ depend only on K, α and Λ_0, and

$$\|f\|_{\mathcal{C}^\alpha} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}$$

is the seminorm of the class $\mathcal{C}^\alpha$ of Hölder continuous functions.

An example, due to Alessandrini [4], shows that in absence of continuity L^∞ estimates do not hold. Namely, if we denote by $B_{r_0} = \{x \in \mathbb{R}^2, |x| < r_0\}$ the ball centered at the origin with radius r_0, take $\Omega = B_1$ the unit ball in $\mathbb{R}^2$, $\gamma_1 = 1$ and $\gamma_2 = 1 + \chi_{B_{r_0}}$, then $\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)} = 1$, but $\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{\frac{1}{2}} \to H^{-\frac{1}{2}}} \leq 2r_0 \to 0$ as $r_0 \to 0$.

A closer look to the previous example shows that $\lim_{r_0 \to 0} \|\gamma_1 - \gamma_2\|_{L^2(\Omega)} = 0$. Therefore one could conjecture that, in absence of continuity, average stability (in the L^2 sense) might hold. However, it is well known that some control on the oscillation of γ is needed to obtain stability. Namely, let

γ be defined in the unit square and extended periodically, and denote $\gamma_j(x) = \gamma(jx)$.. Then the sequence $\{\gamma_j\}_{j=1}^{\infty}$ G-converges to a matrix γ_0 (see for example [46] for the notion of G-convergence). Since G-convergence implies the convergence of the fluxes [46, Proposition 9], we get that if u_j, u_0 solve the corresponding Dirichlet problems for a fixed function $f \in H^{\frac{1}{2}}(\partial\Omega)$,

$$\begin{cases} \nabla \cdot (\gamma_j \nabla u_j) = 0 \\ u_j\big|_{\partial\Omega} = f \end{cases} \tag{1.4}$$

then the fluxes satisfy that $\gamma_j \nabla u_j \rightharpoonup \gamma \nabla u$. Thus, by (1.3)

$$\lim_{j_1, j_2 \to \infty} \langle \Lambda_{\gamma_{j_1}} - \Lambda_{\gamma_{j_2}})(f), \varphi_0 \rangle \tag{1.5}$$

for each $f, \varphi_0 \in H^{\frac{1}{2}}$. However, γ_j has no convergent subsequence in L^2. Notice that γ_j can be chosen even being C^∞, so the problem here is not so much a matter of regularity but rather a control on the oscillation. In [6] it is provided a specific choice of γ where the pointwise convergence (1.5) is strengthened to convergence in the operator norm $H^{\frac{1}{2}} \to H^{-\frac{1}{2}}$.

In this paper we prove that L^2 stability holds if we prescribe a bound of γ in any fractional Sobolev space $W^{\alpha,2}$. By the relation with Besov spaces this could be interpreted as controlling the average oscillation of the function. Thus average control on the oscillation of the coefficients yields average stability of the inverse problem.

Theorem 1.1. *Let Ω be a Lipschitz domain in the plane. Let $\gamma = \gamma_1, \gamma_2$ be two planar conductivities in Ω satisfying*

- *(I) Ellipticity: $\frac{1}{K} \leq \gamma(x) \leq K$.*
- *(II) Sobolev regularity: $\gamma_i \in W^{\alpha,p}(\Omega)$ with $\alpha > 0, 1 < p < \infty$, and $\|\gamma_i\|_{W^{\alpha,p}(\Omega)} \leq \Gamma_0$.*

Let $\tilde{\alpha} = \min\{\alpha, \frac{1}{2}\}$. Then there exists two constants $c(K,p)$, $C(K,\alpha,p) > 0$, such that:

$$\|\gamma_1 - \gamma_2\|_{L^2(\Omega)} \leq \frac{C(1+\Gamma_0)}{|\log(\rho)|^{c\tilde{\alpha}^2}} \tag{1.6}$$

where $\rho = \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)}$.

The theorem is specially interesting for $\alpha \to 0$. Then we are close to obtainning stability for conductivities in L^∞ and we allow all sort of wild discontinuities. Arguing by interpolation one can also obtain L^p stability estimates. Concerning the logarithmic modulus of continuity, the arguments of Mandache [34] can be adapted to the L^2 setting. Namely we can consider the same set of conductivities with the obvious replacement of the C^m function by a normalized $W^{\alpha,2}$ function. The argument shows the existence of two conductivities such that $\|\gamma_1-\gamma_2\|_{L^\infty(\mathbb{D})}\leq\epsilon$, $\|\gamma_i\|_{W^{\alpha,p}(\Omega)}\leq\Gamma_0$, but

$$\|\gamma_1-\gamma_2\|_{L^2(\mathbb{D})}\geq\frac{1}{C|\log(\rho)|^{\frac{3(1+\alpha)}{2\alpha}}}. \tag{1.7}$$

Here C is a constant depending on all the parameters. Notice that the power is better than in the L^∞ setting but still the modulus of continuity is far from being satisfactory.

In our way to prove Theorem 1.1 we have dealt with several questions related to quasiconformal mappings of independent interest. More precisely, we have needed to understand how quasiconformal mappings interact with fractional Sobolev spaces. In particular we analyze the regularity of Beltrami equations with Sobolev bounds on the coefficients which has been a recent topic of interest in the theory. See [23, 24] where the case $\mu\in W^{1,p}$ is investigated in relation with the size of removable sets. We prove the following regularity result.

Theorem 1.2. *Let $\alpha\in(0,1)$, and suppose that $\mu,\nu\in W^{\alpha,2}(\mathbb{C})$ are Beltrami coefficients, compactly supported in $\mathbb{D}$, such that*

$$|\mu(z)|+|\nu(z)|\leq\frac{K-1}{K+1}.$$

at almost every $z\in\mathbb{D}$. Let $\phi:\mathbb{C}\to\mathbb{C}$ be the only homeomorphism satisfying

$$\overline{\partial}\phi=\mu\,\partial\phi+\nu\,\overline{\partial\phi}$$

and $\phi(z)-z=\mathcal{O}(1/z)$ as $|z|\to\infty$. Then, $\phi(z)-z$ belongs to $W^{1+\theta\alpha,2}(\mathbb{C})$ for every $\theta\in(0,\frac{1}{K})$, and

$$\|D^{1+\theta\alpha}(\phi-z)\|_{L^2(\mathbb{C})}\leq C_K\,\left(\|\mu\|^\theta_{W^{\alpha,2}(\mathbb{C})}+\|\nu\|^\theta_{W^{\alpha,2}(\mathbb{C})}\right)$$

for some constant C_K depending only on K.

Many corollaries can be obtained from this theorem by interpolation. An interesting case is for example what do you obtain if μ is a function of bounded variation. We have contented ourselves with the L^2 setting but similar results hold in L^p. As a consequence of this theorem, we obtain the corresponding regularity of the complex geometric optics solutions.
The other crucial ingredient in our proof is the regularity of $\mu\circ\psi$ where ψ is a normalized quasiconformal mapping. It is well known that quasiconformal mappings preserve BMO and $\dot{W}^{1,2}$ (see [39]). Then an interpolation argument is used in [40] to prove that the same happens with $\dot{W}^{\alpha,\frac{2}{\alpha}}$, $0<\alpha<1$. For more general fractional spaces, we prove the following statement:

$$\mu\in W^{\alpha,2}\qquad\Rightarrow\qquad\mu\circ\psi\in W^{\beta,2},\quad\text{for every }\beta<\frac{\alpha}{K} \tag{1.8}$$

which suffices for our purposes. The proof relies on the precise bounds for the powers that Jacobians of quasiconformal mappings to be Muckenhoupt weights obtained in [10].

The Lipschitz regularity of the domain Ω is used to reduce the problem to the unit disk $\mathbb{D}$. This reduction relies on two facts. First, any Lispchitz domain Ω is an extension domain for fractional Sobolev spaces. Secondly, the characteristic function χ_Ω belongs to $W^{\alpha,2}(\mathbb{C})$ for any $\alpha < \frac{1}{2}$. Indeed, this is responsible also of the constraint $\tilde{\alpha} < \frac{1}{2}$ at Theorem 1.1. In fact, a stability result holds as well if Ω is any simply connected extension domain. To see this, recall that planar simply connected extension domains Ω are quasidisks ([26]), that is, $\Omega = \phi(\mathbb{D})$ where $\phi : \mathbb{C} \to \mathbb{C}$ is quasiconformal. Therefore, for instance by our results in Section 4, $\chi_\Omega = \chi_{\mathbb{D}} \circ \phi^{-1}$ belongs to some space $W^{\tilde{\alpha},2}$, and then use Theorem 1.1.

The rest of the paper is organized as follows. In Section 2 we recall previous facts from [12, 14] which will be needed in the present paper, and describe the strategy of our proof. In Section 3 we reduce the problem to conductivities γ such that $\gamma - 1 \in W_0^{\alpha,2}(\mathbb{D})$. In Section 4 we study the interaction between quasiconformal mappings and fractional Sobolev spaces. Finally in Section 5 we prove the subexponetial growth of the complex geometric optic solutions and in Section 6 we prove the theorem.

In closing we remark several issues raised by our work. The first one is to improve the logarithmic character of the stability. It was proved by Alesssandrini and Vesella that often a logarithmic estimate yields Lipschitz stability for some finite dimensional spaces of conductivities. However, to achieve the desired estimates in our setting seems to require a more subtle understanding of the Beltrami equation and we leave it for the future. It will also be desirable to obtain L^p estimates in terms of $W^{\alpha,p}$ with constants independent of p, so that the $\mathcal{C}^\alpha$ situation in [14] could be understood as a limit of this paper. Finally, from the quasiconformal point of view, there seems to be room for improvement in our estimates specially concerning the composition which is far from being optimal when $\alpha \nearrow 1$, since $\dot{W}^{1,2}$ is invariant under composition with quasiconformal maps. This will also be the issue for further investigations.

Notation. For any multiindex $\alpha = (\alpha_1, \alpha_2)$, we write $\partial^\alpha = \partial_x^{\alpha_1} \partial_y^{\alpha_2}$ and $|\alpha| = \alpha_1 + \alpha_2$. The complex derivatives are then

$$\partial_{\overline{z}} = \overline{\partial} = \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\partial_x + i\partial_y\right)$$
$$\partial_z = \partial = \frac{\partial}{\partial z} = \frac{1}{2}\left(\partial_x - i\partial_y\right)$$

where $z = x + iy$. For a mapping $\phi : \Omega \to \mathbb{C}$, $|\nabla\phi(z)| = |D\phi(z)| = |\partial\phi(z)| + |\overline{\partial}\phi(z)|$ is the operator norm of the differential matrix $D\phi$, and $J(z,\phi) = |\partial\phi(z)|^2 - |\overline{\partial}\phi(z)|^2$ is the Jacobian determinant. The fractional derivatives $D^\alpha f$ are defined in (3.2), along the work we denote the ordinary differential by Df but, when this notation is not clear, we will denote it by ∇f. Given a Banach space X we denote the operator norm of $T\colon X \to X$ by $\|T\|_X$. By C or a we denote constants which may change at each occurrence. We will indicate, when necessary, the dependence of the constants on parameters K, Γ, etc, by writing $C = C(K, \Gamma, ...)$. This tracking of the constants is essential for stability results. By $X \lesssim Y$ we mean that there exists a harmless constant C such that $X \leq CY$.

Acknowledgements

Part of this work was done in several research visits of A. Clop to the Department of Mathematics of the Universidad Autónoma de Madrid, to which he is indebted for their hospitality. A. Clop is partially supported by projects *Conformal Structures and Dynamics*, GALA (contract no. 028766), 2005-SGR-00774 (Generalitat de Catalunya) and MTM2007-62817 (Spain). D. Faraco wants to thank C. Sbordone for inspiring remarks concerning G convergence. D. Faraco and A. Ruiz are partially supported by projects MTM2005-07652-C02-01 and MTM2008-02568 of Ministerio de Educación y Ciencia, Gobierno de España and by the project SIMUMAT from CAM. Finally we are indebted to the referees for their carefully reading of the manuscript as well as for the many suggestions to improve the previous version of our manuscript.

2 Scheme of the proof

We will follow the strategy of [14]. This work focuses on the approach based on the Beltrami equation initiated in [12]. The starting point is the answer to Calderón conjecture in the plane obtained by Astala and Päivärinta.

Theorem 2.1 (Astala-Päivärinta). *Let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected domain, and let $\gamma_i \in L^\infty(\Omega)$, $i = 1, 2$. Suppose that there exist a constant $K > 1$ such that $\frac{1}{K} \leq \gamma_i \leq K$. If*

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$$

then $\gamma_1 = \gamma_2$.

In other words, the mapping $\gamma \mapsto \Lambda_\gamma$ is injective. We recall the basic elements from [12] needed in the sequel, also the strategies for uniqueness and stability, and what we will need in the current paper.

Equivalence between Beltrami and conductivity equation: Let $\mathbb{D}$ be the unit disc. If a function u is γ-harmonic in $\mathbb{D}$, then there exists another function v, called its γ-harmonic conjugate (and actually γ^{-1}-harmonic in Ω), unique modulo constants, such that $f = u + iv$ satisfies the $\mathbb{R}$-linear Beltrami type equation

$$\overline{\partial} f = \mu \, \overline{\partial f} \tag{2.1}$$

with

$$\mu = \frac{1-\gamma}{1+\gamma} \in \mathbb{R}. \tag{2.2}$$

Then if $K \geq 1$ is the ellipticity constant of γ we denote by

$$\kappa = \frac{K-1}{K+1}.$$

It is easy to see that $\|\mu\|_\infty \leq \kappa$ and thus the Beltrami equation is elliptic if and only if the conductivity equation is elliptic. Moreover, for $x \in (\frac{1}{K}, K)$, the function $F(x) = \frac{1-x}{1+x}$ satisfies $\frac{2}{1+K} \leq |F'(x)| \leq \frac{2K}{1+K}$. Thus, it also follows that

$$\frac{1}{C} \|\gamma\|_{W^{\alpha,p}(\Omega)} \leq \|\mu\|_{W^{\alpha,p}(\Omega)} \leq C \|\gamma\|_{W^{\alpha,p}(\Omega)},$$

where the constant C only depends on K (see Lemma 3.1). Therefore, bounds in terms of μ and γ are equivalent.
We can argue as well in the reverse direction. If $f \in W^{1,2}_{loc}(\mathbb{D})$ satisfies (2.1) for real μ with $\|\mu\|_\infty \leq \kappa$, then we can write $f = u + iv$ where u and v satisfy

$$\operatorname{div}(\gamma \nabla u) = 0 \qquad \text{and} \qquad \operatorname{div}\left(\gamma^{-1} \nabla v\right) = 0.$$

Thus, it is equivalent to determine either γ or μ, and throughout the paper we will work with either of them interchangeably.

As for holomorphic functions, u and v are related by the corresponding Hilbert transform

$$\mathcal{H}_\mu \colon H^{\frac{1}{2}}(\partial\mathbb{D}) \to H^{\frac{1}{2}}(\partial\mathbb{D})$$

defined as

$$\mathcal{H}_\mu(u|_{\partial\mathbb{D}}) = v|_{\partial\mathbb{D}}$$

for real functions, and $\mathbb{R}$-linearly extended to $\mathbb{C}$-valued functions by setting $\mathcal{H}_\mu(iu) = i\,\mathcal{H}_{-\mu}(u)$. Since $\partial_T \mathcal{H}_\mu = \Lambda_\gamma$ it follows [12, Proposition 2.7] that $\mathcal{H}_\mu$, $\mathcal{H}_{-\mu}$ and $\Lambda_{\gamma^{-1}}$ are uniquely determined by Λ_γ. Accordingly in [14, Proposition 2.2] it is shown that

$$\|\mathcal{H}_{\mu_1} - \mathcal{H}_{\mu_2}\| \lesssim \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|,$$

with respect to the corresponding operator norms.

Existence of complex geometric optics solutions, scattering transform and ∂_k equations: The theory of quasiconformal mappings and Beltrami operators allows to combine in an efficient way ideas from complex analysis, singular integral operators and degree arguments to prove the existence of *complex geometric optics solutions* with no assumptions on the coefficients.

Theorem 2.2. *Let $\kappa \in (0,1)$, and let μ be a real Beltrami coefficient satisfying $|\mu| \leq \frac{K-1}{K+1}\chi_{\mathbb{D}}$. For every $k \in \mathbb{C}$ and $p \in (2, 1+\frac{1}{\kappa})$ the equation*

$$\overline{\partial} f = \mu\, \overline{\partial f}$$

admits a unique solution $f_\mu \in W^{1,p}_{loc}(\mathbb{C})$ of the form

$$f_\mu(z) = e^{ikz}\, M_\mu(z,k) \tag{2.3}$$

such that $M_\mu(z,k) - 1 = \mathcal{O}(1/z)$ as $|z| \to \infty$. Moreover,

$$\mathrm{Re}\left(\frac{M_{-\mu}}{M_\mu}\right) > 0$$

and $f_\mu(z,0) = 1$.

In this context, the proper definition of scattering transform of μ (or of γ) is

$$\tau_\mu(k) = \frac{i}{4\pi}\int_{\mathbb{D}} \frac{\partial}{\partial z}\left(e^{i\overline{kz}}(\overline{f_\mu(z)} - \overline{f_{-\mu}(z)})\right) dA(z). \tag{2.4}$$

Alternatively the scattering transform is given by the asymptotics of the scattering solutions. Namely,

$$\tau_\mu(k) = \lim_{z\to\infty} \frac{1}{2} z(\overline{M_\mu(z,k) - M_{-\mu}(z,k)}) \tag{2.5}$$

The complex geometric optics solutions $\{u_\gamma, \tilde{u}_\gamma\}$ to the divergence type equation (1.1) are then obtained from the corresponding ones from the Beltrami equation by

$$\begin{aligned} u_\gamma &= \mathrm{Re}(f_\mu) + i\,\mathrm{Im}(f_{-\mu}) \\ \tilde{u}_\gamma &= \mathrm{Im}(f_\mu) + i\,\mathrm{Re}(f_{-\mu}), \end{aligned}$$

and they uniquely determine the pair $\{f_\mu, f_{-\mu}\}$ (and viceversa) in a stable way. We consider u_γ as a function of (z,k). In the z plane, u_γ satisfies the complex γ-harmonic equation,

$$\mathrm{div}(\gamma\, \nabla u_\gamma) = 0.$$

As a function of k, u_γ is a solution to the following $\overline{\partial}$-type equation

$$\frac{\partial u_\gamma}{\partial \overline{k}}(z,k) = -i\,\tau_\mu(k)\,\overline{u(z,k)}. \tag{2.6}$$

Let us emphasize that $\tau_\mu(k)$ is independent of z.

Strategy for uniqueness: Let γ_1, γ_2 be two conductivities. In [12], the strategy for uniqueness is divided in the following steps:

(i) Reduction to $\mathbb{D}$.

(ii) If $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, then $\tau_{\mu_1} = \tau_{\mu_2}$.

(iii) Step (ii) and (2.6) imply that $u_{\gamma_1} = u_{\gamma_2}$.

(iii) Finally, condition $u_{\gamma_1} = u_{\gamma_2}$ is equivalent to $Du_{\gamma_1} = Du_{\gamma_2}$, which holds as well if and only if $\gamma_1 = \gamma_2$

The first step is relatively easy since there is no regularity of γ to preserve and thus one can extend by 0 in $\mathbb{D} \setminus \Omega$. Second step is dealt with in [12, Proposition 6.1]. It is shown that $\mathcal{H}_{\mu_1} = \mathcal{H}_{\mu_2}$ implies $f_{\mu_1}(z,k) = f_{\mu_2}(z,k)$ for all $k \in \mathbb{C}$ and $|z| > 1$. As a consequence (ii) follows from the characterization of τ_μ (2.5).

The step (iii) is more complex because uniqueness results and a priori estimates for pseudoanalytic equations in $\mathbb{C}$ like (2.6) only hold if either the coefficients or the solutions decay fast enough at ∞. Unfortunately the needed decay properties for τ seem to require roughly one derivative for γ. However in [12] it is shown that in the measurable setting at least one can obtain subexponential decay for the solutions. That is,

$$u_\gamma(z,k) = e^{ik(z+\epsilon_\mu(z,k))} \tag{2.7}$$

for some function $\epsilon = \epsilon_\mu(z,k)$ satisfying

$$\lim_{k\to\infty} \|\epsilon_\mu(z,k)\|_{L^\infty(\mathbb{C})} = 0.$$

This would not be enough if we would consider equation (2.6) for a single z. However, in [12] it is used that $u(z,k)$ solves an equation for each z. Further, one has asymptotic estimates for u both in the k (as above) and z variables. Then, a clever topological argument in both variables shows that, with these estimates, τ_μ determines the solution to (2.6).

Strategy for stability: In order to obtain stability, the natural idea is to try to quantify in an uniform way the arguments for uniqueness. This was done in [14] for $\mathcal{C}^\alpha$ conductivities. Let us recall the argument and specially the results which did not require regularity of γ and would be instrumental for the current work. Let $\rho = \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|$. First one reduces to the unit disk by an argument which involves the Whitney extension operator, the weak formulation (1.3) and a result of Brown about recovering continuous conductivities at the boundary ([18]). Next we investigate the relation between the corresponding scattering transforms.

Theorem 2.3 (Stability of the scattering transforms)**.** *Let γ_1, γ_2 be conductivities in $\mathbb{D}$, with $\frac{1}{K} \leq \gamma_i \leq K$, and denote $\mu_i = \frac{1-\gamma_i}{1+\gamma_i}$. Then, for every $k \in \mathbb{C}$ it holds that*

$$|\tau_{\mu_1}(k) - \tau_{\mu_2}(k)| \leq c\, e^{c|k|}\, \rho. \tag{2.8}$$

where the constant c depends only on K.

The estimate is just pointwise but on the positive side it holds for L^∞ conductivities. In [14, Theorem 4.6] there is an explicit formula for the difference of scattering transforms which might be of independent interest. Next we state a result that is implicitly proved in [14, Theorem 5.1]. There it is stated as a property of solutions to regular conductivities. However, in the proof the regularity is only used to obtain the decay in the k variable. Because of this, here we state it separately as condition (2.9).

Theorem 2.4 (A priori estimates in terms of scattering transform)**.** *Let $K \geq 1$ and γ_1, γ_2 be conductivities on $\mathbb{D}$, with $\frac{1}{K} \leq \gamma_i \leq K$. Let*

$$u_{\gamma_j}(z,k) = e^{ik\left(z+\epsilon_{\mu_j}(z,k)\right)},$$

denote, as in (2.7)*, the complex geometric optics solutions to* (1.1)*. Let us assume that there exist positive constants α, B such that for each $z, k \in \mathbb{C}$,*

$$|\epsilon_{\mu_i}(z,k)| \leq \frac{B}{|k|^\alpha}. \tag{2.9}$$

Then it follows that:

A *There exists new constants $b = b(K)$, $C = C(K, B)$, such that for every $z \in \mathbb{C}$ there exists $w \in \mathbb{C}$ satisfying:*

(a) $|z - w| \leq CB \left|\log \frac{1}{\rho}\right|^{-b\alpha}$*, where* $\rho = \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|$.

(b) $u_{\gamma_1}(z,k) = u_{\gamma_2}(w,k)$.

B *For each $k \in \mathbb{C}$, there exists new constants $b = b(K)$ and $C = C(k, K)$ such that*

$$\|u_{\gamma_1}(z,k) - u_{\gamma_2}(z,k)\|_{L^\infty(\mathbb{D}, dA(z))} \leq \frac{CB^{\frac{1}{K}}}{|\log(\rho)|^{b\alpha}}. \tag{2.10}$$

Proof. The proof of **A** follows from [14, Proposition 5.2] and [14, Proposition 5.3]. Let us prove **B**. Given $z \in \mathbb{C}$, let $w \in \mathbb{C}$ be given by part **A**. Then

$$|u_{\gamma_1}(z,k) - u_{\gamma_2}(z,k)| = |u_{\gamma_1}(z,k) - u_{\gamma_1}(w,k)|.$$

By the Hölder continuity of K-quasiregular mappings, together with (a), we get

$$|u_{\gamma_1}(z,k) - u_{\gamma_2}(z,k)| \leq C(k,K)\, |z-w|^{\frac{1}{K}} \leq C(k,K)\, C^{\frac{1}{K}}\, B^{\frac{1}{K}} \left|\log \frac{1}{\rho}\right|^{-\frac{b\alpha}{K}}$$

and the desired estimate follows after renaming the constants. ☐

Unlike in the uniqueness arguments, estimating $D(u_{\gamma_1} - u_{\gamma_2})$ in terms of $u_{\gamma_1} - u_{\gamma_2}$ is more delicate in the stability setting, since functions do not control their derivatives in general. This is solved in [14], under Hölder regularity.

Theorem 2.5 (Schauder estimates)**.** *Let γ_i, $i = 1, 2$ be conductivities on $\mathbb{D}$, such that $\frac{1}{K} \leq \gamma_i \leq K$ and $\|\gamma_1\|_{\mathcal{C}^\alpha(\mathbb{D})} \leq \Gamma_0$. As always, denote $\mu_i = \frac{1-\gamma_i}{1+\gamma_i}$, and let $f_{\mu_i}(z,k)$ be the corresponding complex geometric optics solutions to (2.1). Then*

1. *For each $k \in \mathbb{C}$ there esists a constant $C = C(k) > 0$ with*

$$\|f_{\mu_1}(\cdot, k) - f_{\mu_2}(\cdot, k)\|_{\mathcal{C}^{1+\alpha}(\mathbb{D})} \leq C(k). \tag{2.11}$$

2. *The jacobian determinant of $f_{\mu_i}(z,k)$ has a positive lower bound*

$$J(z, f_{\mu_i}(\cdot, k)) \geq C(K, k, \Gamma_0) > 0.$$

Now, to finish the proof of stability for Hölder continuous conductivities, just note that an interpolation argument between L^∞ and $C^{1+\alpha}$ gives Lipschitz bounds for Df_{μ_i}. Thus, by $\mu = \frac{\overline{\partial} f}{\overline{\partial f}}$ and the second statement above, one obtains L^∞ stability for $\mu_1 - \mu_2$. The corresponding result for $\gamma_1 - \gamma_2$ comes due to (2.2).

Strategy for stability under Sobolev regularity In the current work we will try to push the previous strategy to obtain L^2 stability. The previous analysis shows that we can rely on many of the results from [12, 14]. In particular, we only have to prove that $\tau_\mu \mapsto \mu$ is continuous.
For this, we start by reducing the problem in Section 3. We replace the assumption $\gamma_i \in W^{\alpha,p}(\Omega)$ by $\gamma_i \in W_0^{\beta,2}(\mathbb{D})$, where $0 < \beta < \min\{\frac{1}{2}, \alpha\}$. For this, it is used there that characteristic functions of Lipschitz domains belong to $W^{\beta,q}(\mathbb{C})$ whenever $\beta q < 1$.
Then we proceed by investigating the regularity of solutions of Beltrami equations with coefficients in fractional Sobolev spaces in order to obtain an estimate like (2.11), with the $\mathcal{C}^{1+\alpha}$ norm replaced by the sharp Sobolev norm attainable under our assumption on the Beltrami coefficient (see Theorem 4.7). It is also needed here to understand how composition with quasiconformal mappings affects fractional Sobolev spaces. As far as we know, the estimates here are new and of their own interest.
Afterwards we prove that our Sobolev assumption on μ suffices to get the uniform subexponential growth of the geometric optics solutions needed in condition (2.9) in Theorem 2.4 (this is done in Section 5, see Theorem 5.7). In fact we obtain a very clean expression for the precise growth, achieving that the exponent depends linearly on α. Finally, in Section 6 we do the interpolation argument. Here we do not have enough regularity to control

$W^{1,\infty}$ norms and here is where one sees why we need to be happy with the control on $\|\mu_1 - \mu_2\|_{L^2(\mathbb{D})}$. Also we do not have a pointwise lower bound for the corresponding Jacobians which causes also difficulties.

3 Fractional Sobolev spaces and Reduction to $\mu \in W_0^{\alpha,2}(\mathbb{D})$

3.1 On fractional Sobolev Spaces

Here and in the rest of the section we consider $1 \leq p < \infty$. Following [1, p.21], for any domain Ω, we denote by $\dot{W}^{1,p}(\Omega)$ the class of distributions f with $L^p(\Omega)$ distributional derivatives of first order. This means that for any constant coefficients first order differential operator D there exists an $L^p(\Omega)$ function Df such that

$$\langle f, D\varphi\rangle = -\int_\Omega Df\,\varphi.$$

whenever $\varphi \in \mathcal{C}^\infty$ is compactly supported inside of Ω. We also denote $W^{1,p}(\Omega) = L^p(\Omega) \cap \dot{W}^{1,p}(\Omega)$. Similarly one can define the Sobolev spaces $W^{m,p}(\Omega)$ and $\dot{W}^{m,p}(\Omega)$ of general integer order $m \geq 1$. These are Banach spaces with the norms

$$\|f\|_{W^{m,p}(\Omega)} = \sum_{|\alpha|\leq m} \|\partial^\alpha f\|_{L^p(\Omega)} \quad \text{and} \quad \|f\|_{\dot{W}^{m,p}(\Omega)} = \sum_{|\alpha|=m} \|\partial^\alpha f\|_{L^p(\Omega)}.$$

Let us introduce for general domains Ω and any real number $0 < \alpha < 1$ the complex interpolation space

$$W^{\alpha,p}(\Omega) = [L^p(\Omega), W^{1,p}(\Omega)]_\alpha, \tag{3.1}$$

and similarly for the homogeneous case $\dot{W}^{\alpha,p}(\Omega) = [L^p(\Omega), \dot{W}^{1,p}(\Omega)]_\alpha$. Then the closure of $\mathcal{C}_0^\infty(\Omega)$ ($\mathcal{C}^\infty$ functions with compact support contained in Ω) in $W^{\alpha,p}(\Omega)$ is denoted by $W_0^{\alpha,p}(\Omega)$. Functions in $W_0^{\alpha,p}(\Omega)$ can be extended by zero to the whole plane, and the extension belongs to $W^{\alpha,p}(\mathbb{C})$, so we can identify any function in $W_0^{\alpha,p}(\Omega)$ with its extension in $W^{\alpha,p}(\mathbb{C})$. For simplicity, $H^1(\Omega) = W^{1,2}(\Omega)$, $H_0^1(\Omega) = W_0^{1,2}(\Omega)$ and $H^{\frac{1}{2}}(\partial\Omega) = H^1(\Omega)/H_0^1(\Omega)$.
It comes from the work of Calderón (see [2, p.7] or [42]) that every Lipschitz domain Ω is an *extension domain.* Given $\Omega' \supset \Omega$, we denote the corresponding extension operator by E,

$$E : W^{m,p}(\Omega) \to W_0^{m,p}(\Omega')$$

When Ω is an extension domain, an interpolation argument (see [1, p.222]) shows that $W^{\alpha,p}(\Omega)$ coincides with the space of restrictions to Ω of functions in $W^{\alpha,p}(\mathbb{C})$. That is, to each function $u \in W^{\alpha,p}(\Omega)$ one can associate a function $\tilde{u} \in W^{\alpha,p}(\mathbb{C})$ such that $\tilde{u}_{|\Omega} = u$ and $\|\tilde{u}\|_{W^{\alpha,p}(\mathbb{C})} \leq C\,\|u\|_{W^{\alpha,p}(\Omega)}$. We have chosen just one way to introduce the fractional Sobolev spaces. Next, we discuss alternative characterizations and further properties of these spaces needed in the rest of the paper. Two good sources for the basics of this theory are [1, Chapter 7], [42, Chapter 4].

Fourier side. By denoting $e_k(z) = e^{ikz+i\overline{kz}}$, the Fourier transform can be defined as

$$\hat{f}(k) = \int_{\mathbb{C}} e_{-k}(z) f(z) dA(z).$$

Then, if $f \in W^{\alpha,p}(\mathbb{C})$ we introduce the fractional derivative of order α as

$$\widehat{D^\alpha f}(\xi) = |\xi|^\alpha \hat{f}(\xi). \tag{3.2}$$

When $p = 2$, it is easy to see that

$$W^{\alpha,2}(\mathbb{C}) = \left\{ f \in L^2(\mathbb{C}); (1+|\xi|^2)^{\frac{\alpha}{2}}\, \widehat{f}(\xi) \in L^2(\mathbb{C}) \right\},$$

and that this agrees with the space of Bessel potentials

$$W^{\alpha,2}(\mathbb{C}) = G_\alpha * L^2(\mathbb{C}) = \left\{ f = G_\alpha * g; g \in L^2(\mathbb{C}) \right\},$$

where G_α is the Bessel kernel [2, p.10]. Similarly $\dot{W}^{\alpha,2}(\mathbb{C}) = I_\alpha * L^2(\mathbb{C})$ for the Riesz kernels I_α. If $p \neq 2$, the situation is more complicated but it can be shown that

$$W^{\alpha,p}(\mathbb{C}) = \left\{ f \in L^p(\mathbb{C}); \left((1+|\xi|^2)^{\frac{\alpha}{2}}\, \widehat{f}(\xi) \right)^{\vee} \in L^p(\mathbb{C}) \right\}.$$

Integral modulus of continuity We define the L^p-difference of a function f by

$$\omega_p(f)(y) = \|f(\cdot + y) - f(\cdot)\|_{L^p(\mathbb{C})}. \tag{3.3}$$

(see [42, Chapter V]). Then the Besov spaces $B^{p,q}_\alpha(\mathbb{C})$ are defined by

$$B^{p,q}_\alpha(\mathbb{C}) = \{ f \in L^p(\mathbb{C}) : \|f\|^q_{B^{p,q}_\alpha} = \int_{\mathbb{C}} \omega_p(f)(y)^q |y|^{-(n+\alpha q)} dy < \infty \}.$$

There are many relations between Besov and fractional Sobolev spaces. We will need the following two facts,

$$B^{2,2}_\alpha = W^{\alpha,2}, \qquad\qquad W^{\alpha,p} \subset B^{p,2}_\alpha \quad (p<2). \tag{3.4}$$

For a proof see [1, Chapter 7] or [42, Chapter V].

Generalized Leibniz Rule The following result is shown in [29]. See also [27] and [47].

Lemma 3.1. *Let $f, g \in \mathcal{C}_0^\infty(\mathbb{C})$.*

(a) Let $\alpha_1, \alpha_2 \in [0, \alpha] \subset [0, 1)$ be such that $\alpha_1 + \alpha_2 = \alpha$. Let also $p_1, p_2 \in (1, \infty)$ satisfy $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$. Then

$$\|D^\alpha(fg) - f\, D^\alpha(g) - g\, D^\alpha(f)\|_{L^p} \leq C\, \|D^{\alpha_1}(f)\|_{L^{p_1}}\, \|D^{\alpha_2}(g)\|_{L^{p_2}},$$

for some constant $C = C(\alpha_1, \alpha_2, \alpha, p_1, p_2, p) > 0$.

(b) If $0 < \alpha \leq 1$, $p_1 \in (1, \infty]$ and $p_2 \in (1, \infty)$ satisfy $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$, then

$$\|D^\alpha(f \circ g)\|_{L^p} \leq C\, \|Df(g)\|_{L^{p_1}}\, \|D^\alpha g\|_{L^{p_2}},$$

for some constant $C = C(\alpha, p_1, p_2, p)$.

(c) If $0 < \alpha < 1$ and $1 < p < \infty$ then

$$\|D^\alpha(fg) - f\, D^\alpha(g) - g\, D^\alpha(f)\|_{L^p} \leq C\, \|D^\alpha(f)\|_{L^p}\, \|g\|_{L^\infty},$$

for some constant $C = C(\alpha, p) > 0$.

Remark 3.2. From property (a) and (c) it follows the generalized Leibnitz rule

$$\|D^\alpha(f\, g)\|_{L^p} \leq C_0 \|D^\alpha f\|_{L^{p_1}}\, \|g\|_{L^{p_2}} + \|D^\alpha g\|_{L^{p_3}}\, \|f\|_{L^{p_4}} \tag{3.5}$$

whenever $1 < p_1, p_3 < \infty$ and $1 \leq p_2, p_4 \leq \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$. Moreover, we can localize the support in (3.5) in the following way. Let us assume $\operatorname{supp} f \subset \mathbb{D}$, then

$$\|D^\alpha(f\, g)\|_{L^p(\mathbb{D})} \leq C_0 \|f\|_{W^{\alpha, p_1}(2\mathbb{D})}\, \|g\|_{L^{p_2}} + \|D^\alpha g\|_{L^{p_3}}\, \|f\|_{L^{p_4}(2\mathbb{D})} \tag{3.6}$$

The key point is to use a cutoff function ϕ with $\phi = 1$ on $\mathbb{D}$ and supported on $2\mathbb{D}$, and by using (a) and (c) above, we can write

$$\begin{aligned} \|D^\alpha(f)\|_{L^p(\mathbb{D})} &= \|D^\alpha(f\phi)\|_{L^p(\mathbb{D})} \\ &\leq C\|D^\alpha \phi\|_{L^{p_1}}\, \|f\|_{L^{p_2}} + \|D^\alpha \phi\|_{L^\infty(\mathbb{D})}\, \|f\|_{L^p(\mathbb{D})} + \|\phi\, D^\alpha f\|_{L^p}. \end{aligned}$$

We need to take $p_2 > p$. This can be achieved by using Sobolev embedding

$$W^{\alpha, p}(\mathbb{D}) \subset L^{p_2}(\mathbb{D}), \qquad \text{with } p_2 \leq \frac{2p}{2 - \alpha p}$$

to finally obtain that

$$\|D^\alpha f\|_{L^p} \leq C\, \|f\|_{W^{\alpha, p}(2\mathbb{D})}. \tag{3.7}$$

Pointwise Inequalities

Lemma 3.3. *[[43]] If $f \in W^{\alpha,p}(\mathbb{C})$, $\alpha > 0$, $1 < p < \infty$, then for each $0 \le \lambda \le \alpha$ there exists a function $g = g_\lambda \in L^{p_\lambda}(\mathbb{C})$, $p_\lambda = \frac{2p}{2-(\alpha-\lambda)p}$ such that*

$$|f(z) - f(w)| \le |z - w|^\lambda \; (g(z) + g(w)) \tag{3.8}$$

for almost every $z, w \in \mathbb{C}$. Furthermore, we have that

$$\|g\|_{L^{p_\lambda}} \le C \, \|f\|_{W^{\alpha,p}(\mathbb{C})},$$

for some constant $C > 0$.

3.2 Reduction to $p = 2$

This reduction relies on the fact that $\mu \in L^\infty(\mathbb{C}) \cap W^{\alpha,p}(\mathbb{C})$ and the following interpolation Lemma.

Lemma 3.4. *Let $f \in W^{\alpha_0,p_0} \cap W^{\alpha_1,p_1}$, where $1 < p_0, p_1 < \infty$, $0 \le \alpha_0, \alpha_1 \le 1$, and $\theta \in (0,1)$. Then,*

$$\|f\|_{W^{\alpha,p}} \le \|f\|^{\theta}_{W^{\alpha_0,p_0}} \, \|f\|^{1-\theta}_{W^{\alpha_1,p_1}}$$

where

$$\alpha = \theta \, \alpha_0 + (1-\theta) \, \alpha_1 \qquad \text{and} \qquad \frac{1}{p} = \frac{\theta}{p_0} + \frac{1-\theta}{p_1}.$$

Furthermore, if either $p_0 = \infty$ or $p_1 = \infty$, then the above inequality holds true by replacing W^{α_i,p_i} by the Riesz potentials space $I_{\alpha_i} * BMO$.

Proof. It is well known that the complex interpolation method gives

$$[W^{\alpha_0,p_0}, W^{\alpha_1,p_1}]_\theta = W^{\alpha,p}$$

whenever $1 < p < \infty$ (for the proof of this, see for instance [48]). For $p = \infty$, the same result holds true if we replace $W^{\alpha,\infty}$ by the space of Riesz potentials $I_\alpha * BMO$ of BMO functions (for this, see [40]). □

Let μ be a compactly supported Beltrami coefficient. Then, it belongs both to $L^1(\mathbb{C})$ and $L^\infty(\mathbb{C})$. If we also assume that $\mu \in W^{\alpha,p}(\mathbb{C})$ for some α, p, then we can use the above interpolation to see that $\mu \in W^{\beta,q}(\mathbb{C})$, for any $1 < q < \infty$ and some $0 < \beta < \alpha$. We are particularly interested in $q = 2$.

Lemma 3.5. *Suppose that $\mu \in W^{\alpha,p}(\Omega) \cap L^\infty(\Omega)$ for some $p > 1$ and $0 < \alpha < 1$. Then,*

- *For any $0 \le \theta \le 1$,*

$$\|\mu\|_{W^{\alpha\theta,\frac{p}{\theta}}(\Omega)} \le \|\mu\|^{1-\theta}_{L^\infty(\Omega)} \, \|\mu\|^{\theta}_{W^{\alpha,p}(\Omega)}.$$

- *For any* $0 \leq \theta \leq 1$,

$$\|\mu\|_{W^{\theta\alpha,\frac{p}{(1-\theta)p+\theta}}(\Omega)} \leq \|\mu\|_{L^1(\Omega)}^{1-\theta} \|\mu\|_{W^{\alpha,p}(\Omega)}^{\theta}.$$

- *One always has*

$$\|\mu\|_{W^{\beta,2}(\Omega)} \leq C(K,p) \|\mu\|_{W^{\alpha,p}(\Omega)}^{p^*/2},$$

where $\beta = \frac{\alpha p^*}{2}$ *and* $p^* = \min\{p, \frac{p}{p-1}\}$.

Proof. The first inequality comes easily interpolating between $L^\infty(\Omega)$ and $W^{\alpha,p}(\Omega)$ (the L^∞ norm can even be replaced by the BMO norm, which is smaller, see [40] for more details). For the second, simply notice that compactly supported Beltrami coefficients belong to all $L^p(\Omega)$ spaces, $p > 1$, so one can do the same between $L^{1+\varepsilon}(\Omega)$ (ε as small as desired) and $W^{\alpha,p}(\Omega)$. The last statement is obtained by letting $\theta = \frac{p^*}{2}$ above. □

3.3 Reduction to $\Omega = \mathbb{D}$ and $\mu \in W_0^{\alpha,p}(\mathbb{D})$

The proof of the following lemma relies in the fact that characteristic functions of Lipschitz belong to $W^{\alpha,2}$ for each $\alpha < \frac{1}{2}$.

Theorem 3.6. *Let* Ω *be a Lipschitz domain, strictly included in* $\mathbb{D}$. *Let* $\mu \in W^{\alpha,2}(\Omega)$. *Define*

$$\tilde{\mu} = \begin{cases} \mu & \Omega \\ 0 & \mathbb{C} \setminus \Omega \end{cases}.$$

Then, $\tilde{\mu} \in W_0^{\beta,2}(\mathbb{C})$ *for* $\beta < \min\{\alpha, \frac{1}{2}\}$ *and*

$$\|\tilde{\mu}\|_{W^{\beta,2}(\mathbb{C})} \leq C \|\mu\|_{W^{\alpha,2}(\mathbb{C})}.$$

Analogous results can be stated for the extensions by 1 *of* γ_i.

Proof. Since Ω is an extension domain, there is an extension μ_0 of μ belonging to $W^{\alpha,2}(\mathbb{C})$. Of course, such extension μ_0 need not be supported in Ω any more. Now $\tilde{\mu}$ can be introduced as the pointwise multiplication

$$\tilde{\mu} = \chi_\Omega \, \mu_0.$$

By virtue Lemma 3.1 it is enough to study the smoothness of the characteristic function χ_Ω. A way to see this is to recall that fractional Sobolev spaces are invariant under composition with bilipschitz maps [50]. Now, the characteristic function of the half plane belongs to $W_{loc}^{\alpha,p}(\mathbb{C})$ whenever $\alpha p < 1$. Therefore, by a partition of unity argument, we get that $\chi_\Omega \in W^{\alpha,p}(\mathbb{C})$ when $\alpha p < 1$. The proof is concluded. □

Now we need to compare the original Dirichlet-to-Neumann maps with the Dirichlet-to-Neumann maps of the extensions.

Lemma 3.7. *Let Ω be a domain strictly included in $\mathbb{D}$. Let $\gamma_1, \gamma_2 \in L^\infty(\Omega)$ be conductivities in Ω. Further, assume that*

$$\frac{1}{K} \leq \gamma_i(z) \leq K$$

for almost every $z \in \Omega$. Let $\tilde{\gamma_i}$ denote the corresponding extensions by 1 to all of $\mathbb{C}$. Then,

$$\|\Lambda_{\tilde{\gamma_1}} - \Lambda_{\tilde{\gamma_2}}\|_{H^{\frac{1}{2}}(\partial\mathbb{D}) \to H^{-\frac{1}{2}}(\partial\mathbb{D})} \leq C\,\rho,$$

where $\rho = \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)}$.

Proof. We follow the ideas of [14, Theorem 6.2], although the stability result from [18] is not needed in our situation. Let $\varphi_0 \in H^{\frac{1}{2}}(\partial\mathbb{D})$. Let $\tilde{u}_j \in H^1(\mathbb{D})$ be the solution to

$$\begin{cases} \nabla \cdot (\tilde{\gamma}_j \nabla \tilde{u}_j) = 0 & \text{in } \mathbb{D} \\ \tilde{u}_j = \varphi_0 & \text{in } \partial\mathbb{D}. \end{cases}$$

Let also u_2 be defined by

$$\begin{cases} \nabla \cdot (\gamma_2 \nabla u_2) = 0 & \text{in } \Omega \\ u_2 = \tilde{u}_1 & \text{in } \partial\Omega. \end{cases}$$

Define now $\tilde{v}_2 = u_2\,\chi_\Omega + \tilde{u}_1\,\chi_{\mathbb{D}\setminus\Omega}$. As in [14], we first control $\tilde{u}_2 - \tilde{v}_2$ in terms of ρ. To do this,

$$\begin{aligned} \int_{\mathbb{D}} |\nabla(\tilde{v}_2 - \tilde{u}_2)|^2 &\leq c \int_{\mathbb{D}} \tilde{\gamma}_2\, \nabla(\tilde{v}_2 - \tilde{u}_2) \cdot \nabla(\tilde{v}_2 - \tilde{u}_2) \\ &= c \int_{\mathbb{D}} \tilde{\gamma}_2\, \nabla \tilde{v}_2 \cdot \nabla(\tilde{v}_2 - \tilde{u}_2) \end{aligned}$$

because $\tilde{v}_2 - \tilde{u}_2 \in H^1_0(\mathbb{D})$ and the $\tilde{\gamma}_2$-harmonicity of $\tilde{u}_2$ in $\mathbb{D}$. By adding and substracting $\int_{\mathbb{D}} \tilde{\gamma}_1 \nabla \tilde{u}_1 \cdot \nabla(\tilde{v}_2 - \tilde{u}_2)$, and using that $\tilde{\gamma}_1 = \tilde{\gamma}_2 = 1$ off Ω, the right hand side above is bounded by a constant times

$$\left| \int_{\mathbb{D}} \tilde{\gamma}_1 \nabla \tilde{u}_1 \cdot \nabla(\tilde{v}_2 - \tilde{u}_2) \right| + \left| \int_\Omega (\gamma_1\, \nabla \tilde{u}_1 - \gamma_2\, \nabla u_2) \cdot \nabla(\tilde{v}_2 - \tilde{u}_2) \right|.$$

Here the first term vanishes because $\tilde{u}_1$ is $\tilde{\gamma}_1$-harmonic on $\mathbb{D}$ and $\tilde{v}_2 - \tilde{u}_2 \in H^1_0(\mathbb{D})$. For the second, we observe that $\tilde{u}_1$ is γ_1-harmonic in Ω, u_2 is γ_2-harmonic in Ω, and $u_2 - \tilde{u}_1 \in H^1_0(\Omega)$. Thus,

$$\begin{aligned} \left| \int_\Omega (\gamma_1\, \nabla \tilde{u}_1 - \gamma_2\, \nabla u_2) \cdot \nabla(\tilde{v}_2 - \tilde{u}_2) \right| &= \left| \langle (\Lambda_{\gamma_1} - \Lambda_{\gamma_2})(\tilde{u}_{1|\partial\Omega}), (\tilde{v}_2 - \tilde{u}_2)_{|\partial\Omega} \rangle \right| \\ &\leq \rho\, \|\tilde{u}_1\|_{H^{\frac{1}{2}}(\partial\Omega)}\, \|\tilde{v}_2 - \tilde{u}_2\|_{H^{\frac{1}{2}}(\partial\Omega)} \\ &\leq \rho\, \|\nabla \tilde{u}_1\|_{L^2(\Omega)}\, \|\nabla(\tilde{v}_2 - \tilde{u}_2)\|_{L^2(\Omega)} \end{aligned}$$

Summarizing, we get

$$\left(\int_{\mathbb{D}} |\nabla(\tilde{v}_2 - \tilde{u}_2)|^2\right)^{\frac{1}{2}} \leq c\,\rho\,\|\nabla\tilde{u}_1\|_{L^2(\Omega)} \leq c\,\rho\,\|\nabla\tilde{u}_1\|_{L^2(\mathbb{D})} \\ \leq c\,\rho\,\|\varphi_0\|_{H^{\frac{1}{2}}(\partial\mathbb{D})}. \tag{3.9}$$

We will use this to compare the Dirichlet-to-Neumann maps at $\partial\mathbb{D}$. If $\psi_0 \in H^{\frac{1}{2}}(\partial\mathbb{D})$ is any testing function, and ψ is any $H^1(\mathbb{D})$ extension,

$$\langle(\Lambda_{\tilde{\gamma}_1} - \Lambda_{\tilde{\gamma}_2})(\varphi_0), \psi_0\rangle = \int_{\mathbb{D}} (\tilde{\gamma}_1\,\nabla\tilde{u}_1 - \tilde{\gamma}_2\,\nabla\tilde{u}_2)\cdot\nabla\psi. \tag{3.10}$$

We will divide the bound of this quantity in two steps. For the first,

$$\left|\int_{\mathbb{D}} (\tilde{\gamma}_1\,\nabla\tilde{u}_1 - (\gamma_2\,\chi_\Omega + \tilde{\gamma}_1\,\chi_{\mathbb{D}\setminus\Omega})\,\nabla\tilde{v}_2)\cdot\nabla\psi\right| = \left|\langle(\Lambda_{\gamma_1} - \Lambda_{\gamma_2})(\tilde{u}_{1|\partial\Omega}), \psi_{|\partial\Omega}\rangle\right|$$

which is bounded by

$$\begin{aligned} \rho\,\|\tilde{u}_{1|\partial\Omega}\|_{H^{\frac{1}{2}}(\partial\Omega)}\,\|\psi_{|\partial\Omega}\|_{H^{\frac{1}{2}}(\partial\Omega)} &\leq \rho\,\|\nabla\tilde{u}_1\|_{L^2(\Omega)}\,\|\nabla\psi\|_{L^2(\Omega)} \\ &\leq \rho\,\|\nabla\tilde{u}_1\|_{L^2(\mathbb{D})}\,\|\nabla\psi\|_{L^2(\mathbb{D})} \\ &\leq \rho\,\|\varphi_0\|_{H^{\frac{1}{2}}(\partial\mathbb{D})}\,\|\psi_0\|_{H^{\frac{1}{2}}(\partial\mathbb{D})}. \end{aligned}$$

We are left with

$$\left|\int_{\mathbb{D}} \big((\gamma_2\,\chi_\Omega + \tilde{\gamma}_1\,\chi_{\mathbb{D}\setminus\Omega})\nabla\tilde{v}_2 - \tilde{\gamma}_2\,\nabla\tilde{u}_2\big)\cdot\nabla\psi\right|$$

which is equal to

$$\left|\int_{\Omega} \gamma_2\,\nabla(\tilde{v}_2 - \tilde{u}_2)\cdot\nabla\psi + \int_{\mathbb{D}\setminus\Omega} \nabla(\tilde{v}_2 - \tilde{u}_2)\cdot\nabla\psi\right|$$

which in turn is controlled, using (3.9), by a multiple of

$$\begin{aligned} \int_{\mathbb{D}} |\nabla(\tilde{v}_2 - \tilde{u}_2)|\,|\nabla\psi| &\leq \|\nabla(\tilde{v}_2 - \tilde{u}_2)\|_{L^2(\mathbb{D})}\,\|\nabla\psi\|_{L^2(\mathbb{D})} \\ &\leq c\,\rho\,\|\varphi\|_{H^{\frac{1}{2}}(\partial\mathbb{D})}\,\|\psi_0\|_{H^{\frac{1}{2}}(\partial\mathbb{D})}. \end{aligned}$$

This gives for (3.10) that the difference of Dirichlet-to-Neumann maps satisfies

$$|\langle(\Lambda_{\tilde{\gamma}_1} - \Lambda_{\tilde{\gamma}_2})(\varphi_0), \psi_0\rangle| \leq c\,\rho\,\|\varphi\|_{H^{\frac{1}{2}}(\partial\mathbb{D})}\,\|\psi_0\|_{H^{\frac{1}{2}}(\partial\mathbb{D})}$$

as desired. □

Remark 3.8. The trivial extension of the conductivities by 1 simplifies the arguments but has the price of losing regularity if $\alpha \geq 1/2$. An argument similar to that in [14] would need an L^2 version of the boundary recovery result of Brown (see also [5]) of the type

$$\|\gamma_1 - \gamma_2\|_{L^2(\partial\Omega)} \leq C\rho.$$

4 Beltrami equations and fractional Sobolev spaces

This section is devoted to investigate how quasiconformal mappings interplay with fractional Sobolev spaces. We face three different goals. First, given a Beltrami coefficient $\mu \in W^{\alpha,2}_0(\mathbb{C})$, we find $\beta \in (0,\alpha)$ such that for any K-quasiconformal mapping ϕ the composition $\mu \circ \phi$, which is another Beltrami coefficient with the same ellipticity bound, belongs to $W^{\beta,2}(\mathbb{C})$. Secondly, we obtain the optimal (at least when $\alpha \approx 1$) Sobolev regularity for the homeomorphic solutions to the equation

$$\overline{\partial} f = \mu\, \partial f + \nu\, \overline{\partial f}$$

under the assumptions of ellipticity and Sobolev regularity for the coefficients. Finally, we obtain bounds for the complex geometric optics solutions. Many properties of planar quasiconformal mappings rely on two precise integral operators, the Cauchy transform,

$$\mathcal{C}\varphi(z) = \frac{-1}{\pi} \int \frac{\varphi(w)}{(w-z)}\, dA(w). \tag{4.1}$$

and the Beurling transform,

$$T\varphi(z) = \frac{-1}{\pi} \lim_{\varepsilon \to 0} \int_{|w-z|\geq \varepsilon} \frac{\varphi(w)}{(w-z)^2}\, dA(w). \tag{4.2}$$

Their basic mapping properties are well known and can be found in any reference concerning planar quasiconformal mappings, see for instance [3, 9, 12]. For $s \in (1,\infty)$ we will denote by $\|T\|_{L^s(\mathbb{C})}$ the norm of T as a bounded operator in $L^s(\mathbb{C})$. We recall also their relation with complex derivatives

$$\overline{\partial}\mathcal{C}\varphi = \varphi,$$
$$T(\overline{\partial}\varphi) = \partial\varphi$$

which holds for any $\varphi \in C^\infty_0(\mathbb{C})$.

4.1 Composition with quasiconformal mappings

Let μ be a compactly supported Beltrami coefficient, satisfying

$$|\mu| \leq \frac{K-1}{K+1}\chi_{\mathbb{D}} = \kappa\, \chi_{\mathbb{D}}.$$

Further, assume that

$$\mu \in W^{\alpha,2}(\mathbb{C}) \text{ and } \|\mu\|_{W^{\alpha,2}(\mathbb{C})} \leq \Gamma_0$$

for some $\alpha > 0$ and some $\Gamma_0 > 0$. Let $\phi : \mathbb{C} \to \mathbb{C}$ be a planar K-quasiconformal mapping. In this section, we look for those $\beta > 0$ such that $\mu \circ \phi \in W^{\beta,2}(\mathbb{C})$.

We need to recall a local version of a lemma due to Fefferman and Stein, see [36] and [25, Proposition 2.24]. The proof follows from Vitali covering Lemma, exactly as in [36]. By Mf we denote the Hardy-Littlewood maximal function,

$$Mf(x) = \sup \frac{1}{|D|} \int_D |f(z)| \, dA(z),$$

where the supremum runs over all disks D with $x \in D$, while $M_\Omega f$ denote its local version, that is,

$$M_\Omega f(x) = \sup \frac{1}{|D|} \int_D |f(z)| \, dA(z),$$

where the supremum is taken over all discs D with $x \in D \subset \Omega$.

Lemma 4.1. *Let $1 < p < \infty$ and $w \geq 0$ a locally integrable function. Then*

$$\int_\Omega |M_\Omega f(x)|^p \omega(x) dA(x) \leq \int_\Omega |f(x)|^p \, M\omega(x) \, dA(x).$$

We can now prove the main result of this section.

Proposition 4.2. *Let $K \geq 1$. Let $\mu \in W^{\alpha,2}(\mathbb{C})$ for some $\alpha \in (0,1)$, and assume that $|\mu| \leq \frac{K-1}{K+1} \chi_{\mathbb{D}}$. Let $\phi : \mathbb{C} \to \mathbb{C}$ be any K-quasiconformal mapping, conformal out of a compact set, and normalized so that $|\phi(z) - z| \to 0$ as $|z| \to \infty$. Then*

$$\mu \circ \phi \in W^{\beta,2}(\mathbb{C})$$

whenever $\beta < \frac{\alpha}{K}$. Moreover,

$$\|\mu \circ \phi\|_{W^{\beta,2}(\mathbb{C})} \leq C \, \|\mu\|_{W^{\alpha,2}(\mathbb{C})}^{\frac{1}{K}},$$

for some constant $C > 0$ depending only on α, β and K.

Proof. It is clear that $\mu \circ \phi$ belongs to $L^2(\mathbb{C})$, so since $W^{\alpha,2}$ agrees with the Besov space $B^{2,2}_\alpha$, it suffices to show the convergence of the integral

$$\int_\mathbb{C} \int_\mathbb{C} \frac{|\mu(\phi(z+w)) - \mu(\phi(z))|^2}{|w|^{2+2\beta}} \, dA(z) \, dA(w)$$

for every $\beta < \frac{\alpha}{K}$. First of all, for large w there is nothing to say since

$$\begin{aligned}
&\int_{|w|>1} \int_\mathbb{C} \frac{|\mu(\phi(z+w)) - \mu(\phi(z))|^2}{|w|^{2+2\beta}} \, dA(z) \, dA(w) \\
&\qquad = \int_{|w|>1} \frac{1}{|w|^{2+2\beta}} \int_\mathbb{C} |\mu(\phi(z+w)) - \mu(\phi(z))|^2 \, dA(z) \, dA(w) \\
&\qquad \leq \int_{|w|>1} \frac{4 \, \|\mu\|^2_{L^\infty(\mathbb{C})} \, |\phi^{-1}(\mathbb{D})|}{|w|^{2+2\beta}} dA(w) = \frac{4\pi \, \|\mu\|^2_{L^\infty(\mathbb{C})} \, |\phi^{-1}(\mathbb{D})|}{\beta}.
\end{aligned}$$

Then we are left to bound the integral

$$\int_{|w|\le 1}\int_{\mathbb{C}} \frac{|\mu(\phi(z+w))-\mu(\phi(z))|^2}{|w|^{2+2\beta}}\, dA(z)\, dA(w).$$

As μ has support in $\mathbb{D}$, and $|w|\le 1$, the difference $|\mu(\phi(z+w))-\mu(\phi(z))|$ is supported in the 1-neighbourhood of $\phi^{-1}(\mathbb{D})$, that is, $F=\{z\in\mathbb{C} : d(z,\phi^{-1}(\mathbb{D}))\le 1\}$. Indeed, $\phi(z)\in\mathbb{D}$ if and only if $z\in\phi^{-1}(\mathbb{D})\subset F$, while $\phi(z+w)\in\mathbb{D}$ if and only if $z\in\phi^{-1}(\mathbb{D})-w$. But if $z=\phi^{-1}(\zeta)-w$ for some $|\zeta|<1$,

$$\begin{aligned} d(z,\phi^{-1}(\mathbb{D})) &= \inf_{|\xi|<1}|z-\phi^{-1}(\xi)| = \inf_{|\xi|<1}|\phi^{-1}(\zeta)-w-\phi^{-1}(\xi)| \\ &\le |\phi^{-1}(\xi)-w-\phi^{-1}(\xi)| = |w|\le 1, \end{aligned}$$

so also $\phi^{-1}(\mathbb{D})-w\subset F$. In other words, if $z\notin F$ then $\mu(\phi(z))=\mu(\phi(z+w))=0$, and we are reduced to bound

$$\int_{|w|\le 1}\int_{F} \frac{|\mu(\phi(z+w))-\mu(\phi(z))|^2}{|w|^{2+2\beta}}\, dA(z)\, dA(w). \tag{4.3}$$

Note also that, by Koebe's $\frac{1}{4}$ Theorem, we have the inclusions $\phi(\mathbb{D})\subset 4\mathbb{D}$ and $\phi^{-1}(4\mathbb{D})\subset 16\mathbb{D}$, so that $F\subset 17\mathbb{D}$.

To bound (4.3) the local behavior of μ is important, so we will use condition (3.8) for the function μ. But before, recall that $\mu\in W^{\alpha,2}(\mathbb{C})\cap L^\infty(\mathbb{C})$, so that by interpolation we obtain $\mu\in W^{\alpha\theta,\frac{2}{\theta}}(\mathbb{C})$ for each $\theta\in(0,1)$, with the estimates

$$\|\mu\|_{W^{\alpha\theta,\frac{2}{\theta}}(\mathbb{C})}\le C\,\|\mu\|_{L^\infty}^{1-\theta}\,\|\mu\|_{W^{\alpha,2}(\mathbb{C})}^{\theta}.$$

Thus, by (3.8), for every $\lambda\in(0,\alpha\theta)$ there exists a function $g=g_\lambda\in L^{p_\lambda}(\mathbb{C})$, $p_\lambda=\frac{2}{\lambda+(1-\alpha)\theta}$, such that

$$|\mu(\zeta)-\mu(\xi)|\le|\zeta-\xi|^\lambda\,\left(g(\zeta)+g(\xi)\right)$$

at almost every $\zeta,\xi\in\mathbb{C}$. The choice $\theta=1/K$ and Lemma 3.3 also gives us L^{p_λ} estimates,

$$\begin{aligned} \|g_\lambda\|_{L^{p_\lambda}(\mathbb{C})} &\le C\,\|\mu\|_{W^{\frac{\alpha}{K},2K}(\mathbb{C})} \\ &\le C\,\|\mu\|_{L^\infty}^{1-\frac{1}{K}}\,\|\mu\|_{W^{\alpha,2}(\mathbb{C})}^{\frac{1}{K}}. \end{aligned} \tag{4.4}$$

with $C>0$.
It follows that

$$\frac{|\mu(\phi(z+w))-\mu(\phi(z))|}{|w|^\lambda}\le\left(\frac{|\phi(z+w)-\phi(z)|}{|w|}\right)^\lambda\,\left(g(\phi(z+w))+g(\phi(z))\right). \tag{4.5}$$

Next we recall that quasiconformal mappings are quasisymmetric (see for instance [9] or [33]). That is, for $K \geq 1$ there exists an increasing homeomorphism $\eta_K : \mathbb{R} \to \mathbb{R}$ such that for any K-quasiconformal mapping ϕ, and for any $a, z_1, z_2 \in \mathbb{C}$, we have

$$\frac{|\phi(z_2) - \phi(a)|}{|\phi(z_1) - \phi(a)|} \leq \eta_K \left(\frac{|z_2 - a|}{|z_1 - a|} \right).$$

Thus

$$\begin{aligned}
(\operatorname{diam} \phi(D(a,r)))^2 &\leq 4 \left(\max_{|z_2 - a| = r} |\phi(z_2) - \phi(a)| \right)^2 \\
&\leq 4\eta_K(1)^2 \left(\min_{|z_1 - a| = r} |\phi(z_1) - \phi(a)| \right)^2 \qquad (4.6) \\
&\leq 4 \frac{\eta(1)^2}{\pi} |\phi(D(a,r))| = C_K \int_{D(a,r)} J(z,\phi)\, dA(z),
\end{aligned}$$

for some $C_K > 1$ depending only on K. We now plug (4.6) into (4.5), and use that $\lambda < 1$,

$$\begin{aligned}
\left(\frac{|\phi(z+w) - \phi(z)|}{|w|} \right)^\lambda &\leq \left(C_K \frac{\operatorname{diam} \phi(D(z,|w|))}{\operatorname{diam} D(z,|w|)} \right)^\lambda \\
&\leq \left(\frac{C_K}{|D(z,|w|)|} \int_{D(z,|w|)} J(\zeta,\phi)\, dA(\zeta) \right)^{\frac{\lambda}{2}} \\
&\leq C_K \left(\frac{1}{|D(z,|w|)|} \int_{D(z,|w|)} J(\zeta,\phi)^\lambda\, dA(\zeta) \right)^{\frac{1}{2}}.
\end{aligned}$$

At the last step we used the reverse Hölder inequality for Jacobians of quasiconformal mappings, which holds uniformly in λ because $\lambda \in (0,1)$ (for a precise result see [10, Theorem 12]). Thus, if $\Omega = \{ z \in \mathbb{C} : d(z, \phi^{-1}(\mathbb{D})) \leq 2 \}$ then

$$\left(\frac{|\phi(z+w) - \phi(z)|}{|w|} \right)^\lambda \leq C_K \, (M_\Omega J_\lambda(z))^{\frac{1}{2}}$$

where $M_\Omega J_\lambda(z)$ denotes the local Hardy-Littlewood maximal function M_Ω at the point z of $J(\cdot,\phi)^\lambda$. Note also that $\Omega \subset 18\mathbb{D}$ by Koebe's Theorem. By symmetry, we could also write $M_\Omega J_\lambda(z+w)$ instead of $M_\Omega J_\lambda(z)$, so the integral at (4.3) is bounded from above by

$$C_K \int_{|w| \leq 1} \int_F \frac{M_\Omega J_\lambda(z+w)\, g(\phi(z+w))^2 + M_\Omega J_\lambda(z)\, g(\phi(z))^2}{|w|^{2+2\beta-2\lambda}}\, dA(z)\, dA(w), \tag{4.7}$$

and this reduces our job to find bounds for

$$\frac{C_K}{\lambda - \beta} \int_\Omega M_\Omega J_\lambda(z)\, g(\phi(z))^2\, dA(z), \tag{4.8}$$

whenever $\lambda > \beta$. Indeed, we simply divide the integral of (4.7) into two terms, one in z (for which the bound (4.8) is obvious) and one in $z+w$. For the second one, we note that if $z \in F$ and $|w| \leq 1$ then $z+w \in \Omega$ and after a change of coordinates we obtain

$$\begin{aligned} C_K \int_{|w|\leq 1} \int_F & \frac{M_\Omega J_\lambda(z+w)\, g(\phi(z+w))^2}{|w|^{2+2\beta-2\lambda}} dA(z)\, dA(w) \\ &\leq C_K \int_{|w|\leq 1} \int_\Omega M_\Omega J_\lambda(\zeta)\, g(\phi(\zeta))^2\, dA(\zeta) \frac{dA(w)}{|w|^{2+2\beta-2\lambda}} \\ &= \frac{C_K}{\lambda-\beta} \int_\Omega M_\Omega J_\lambda(\zeta)\, g(\phi(\zeta))^2\, dA(\zeta) \end{aligned}$$

provided that $\lambda > \beta$, and where C_K may have changed, but still depends only on K, as claimed in (4.8).

To finish the proof, we will use Lemma 4.1 and the fact that jacobians of quasiconformal mappings are A_∞ weights. This requires two auxiliar indexes $r, s > 1$, chosen as follows:

- For each $\lambda \in (\beta, \frac{\alpha}{K})$, we have $K\lambda + (1-\alpha) < 1$, whence there exists numbers s such that

  $$1 < s < \frac{1}{K\lambda + (1-\alpha)}. \tag{4.9}$$

  For instance,

  $$s = 1 + \frac{1}{2}\left(\frac{1}{K\lambda + (1-\alpha)} - 1\right). \tag{4.10}$$

  Further, since $\beta < \lambda$ we get

  $$1 < \frac{1}{K\lambda + (1-\alpha)} < 1 + \frac{\alpha - K\beta}{1-(\alpha-K\beta)}$$

  therefore, by choosing $\alpha - K\beta < 1/2$, we can assume that $s < 2$. Note that $s-1$ and $\alpha - K\beta$ are comparable quantities.

- Recall that $p_\lambda = \frac{2K}{K\lambda+(1-\alpha)}$. Now, the choice (4.10) guarantees us that

  $$\frac{p_\lambda}{2Ks} = 2 - \frac{1}{s} > 1. \tag{4.11}$$

  Hence we can find numbers r satisfying

  $$1 < \frac{r}{1+\lambda s(K-1)} < \frac{p_\lambda}{2Ks}, \tag{4.12}$$

  as for instance

  $$\frac{r}{1+\lambda s(K-1)} = \frac{3}{2} - \frac{1}{2s}. \tag{4.13}$$

  Again, the difference $\frac{r}{1+\lambda s(K-1)} - 1$ is comparable to $\alpha - \lambda K$.

By denoting $\alpha - K\beta = \epsilon > 0$, the particular choice

$$\lambda = \beta + \frac{\alpha - K\beta}{2K}$$

gives us the following parameters:

$$\begin{aligned} s = 1 + \frac{\epsilon}{4 - 2\epsilon} \qquad \Rightarrow \qquad 1 + \lambda s(K-1) &= 1 + (K-1)\beta + M_1\epsilon \\ &\simeq 1 + (K-1)\beta \text{ for small enough } \epsilon \end{aligned}$$

and similarly

$$\begin{aligned} r = 1 + (K-1)\beta + M_2\epsilon \qquad \Rightarrow \qquad \frac{r}{r-1} &= 1 + \frac{1}{(K-1)\beta + M_2\epsilon} \\ &\simeq 1 + \frac{1}{(K-1)\beta} \text{ for small enough } \epsilon \end{aligned}$$

where M_1, M_2 are positive constants depending only on K. Once the parameters have been chosen, we can start bounding the integral at (4.8). Since we can not work in L^1, we first bring s into the estimates by Hölder's inequality,

$$\begin{aligned} \int_\Omega M_\Omega J_\lambda(z) \,(g \circ \phi(z))^2 \, dA(z) &= \int_\Omega M_\Omega J_\lambda(z) \,(g \circ \phi(z))^2 \, \chi_\Omega(z) \, dA(z) \\ &\le \Bigg(\underbrace{\int_\Omega (M_\Omega J_\lambda(z))^s \,(g \circ \phi(z))^{2s} \, \chi_\Omega(z) \, dA(z)}_{\mathbf{I}} \Bigg)^{\frac{1}{s}} |\Omega|^{1-\frac{1}{s}}. \end{aligned}$$

Now Lemma 4.1 provide us with a constant $C_1(s)$ to obtain

$$\mathbf{I} \le C_1(s) \int_\Omega J_\lambda(z)^s \, M((g \circ \phi)^{2s} \, \chi_\Omega)(z) \, dA(z).$$

Note that $C_1(s)$ blows up only as $s \to 1$, that is, as $\lambda \to \frac{\alpha}{K}$ due to (4.10). Now, by Hölder's inequality with exponent r, one gets

$$\mathbf{I} \le \left(\int_\Omega J_\lambda(z)^s \, \left(M((g \circ \phi^{2s}) \, \chi_\Omega)(z) \right)^r dA(z) \right)^{\frac{1}{r}} \left(\int_\Omega J_\lambda(z)^s dA(z) \right)^{1-\frac{1}{r}}.$$

The first inequality at (4.12) guarantees that the weight $J_\lambda(z)^s = J(z,\phi)^{\lambda s}$ belongs to the Muckenhoupt class A_r (see [10] or [9, Theorem 13.4.2]), with constant

$$\|J_\lambda^s\|_{A_r} \le \frac{C(K)}{r - 1 - \lambda s(K-1)} < \frac{C(K)}{\epsilon} \tag{4.14}$$

due to (4.9), (4.10) and (4.13). We can use the weighted L^r inequality for the maximal function and a change of coordinates to see that

$$\begin{aligned} \int_{\mathbb{C}} J_\lambda(z)^s \, \left(M((g \circ \phi)^{2s} \, \chi_\Omega)(z) \right)^r dA(z) &\le C_2 \int_\Omega J_\lambda(z)^s \,(g \circ \phi(z))^{2sr} \, dA(z) \\ &= C_2 \int_{\phi(\Omega)} J(w, \phi^{-1})^{1-\lambda s} \, g(w)^{2sr} \, dA(w). \end{aligned}$$

The precise behavior for C_2 comes from [21] (see also [37, Theorem 1.1]),

$$\begin{aligned} C_2^{\frac{1}{r}} &= \|M\|_{L^r(J_\lambda^s\, dA)} \leq C\,\frac{r}{r-1}\,\|J_\lambda^s\|_{A_r}^{\frac{1}{r-1}} \\ &\leq C\left(1+\frac{1}{(K-1)\beta+M_2\epsilon}\right)\frac{C(K)}{\epsilon} \leq \frac{C(K)}{\beta\epsilon}, \end{aligned}$$

where $C(K)$ is a positive constant that depends only on K. Summarizing, we get for the integral at (4.8) the bound

$$C_2^{1/r}\,|\Omega|^{1-\frac{1}{s}}\left(\int_\Omega J(z,\phi)^{\lambda s} dA(z)\right)^{\frac{1}{s}-\frac{1}{sr}} \underbrace{\left(\int_{\phi(\Omega)} J(w,\phi^{-1})^{1-\lambda s}\, g(w)^{2sr}\, dA(w)\right)^{\frac{1}{rs}}}_{\mathbf{II}}.$$

Now, the second inequality at (4.12) gives us that $p_\lambda > 2rs$. Thus Hölder's inequality is justified and we get

$$\mathbf{II} \leq \left(\int_{\phi(\Omega)} g(w)^{p_\lambda} dA(w)\right)^{\frac{2}{p_\lambda}} \left(\int_{\phi(\Omega)} J(w,\phi^{-1})^{\frac{p_\lambda(1-\lambda s)}{p_\lambda-2rs}}\, dA(w)\right)^{\frac{p_\lambda-2rs}{p_\lambda\, rs}}.$$

The first integral above is finite since $g \in L^{p_\lambda}$. To see the finiteness of the second integral, observe that ϕ^{-1} is a K-quasiconformal mapping, hence by Astala's Theorem [8] the $\frac{p_\lambda(1-\lambda s)}{p_\lambda-2rs}$-th power of its Jacobian determinant $J(\cdot,\phi^{-1})$ will be locally integrable provided that this exponent does not exceed $\frac{K}{K-1}$. But

$$\frac{p_\lambda(1-\lambda s)}{p_\lambda-2rs} < \frac{K}{K-1} \qquad \Leftrightarrow \qquad r < \frac{p_\lambda}{2sK}(1+\lambda s(K-1))$$

which comes again from the second inequality at (4.12). Furthermore,

$$\frac{K}{K-1} - \frac{p_\lambda(1-\lambda s)}{p_\lambda-2rs} \leq M_3\epsilon$$

where $M_3 > 0$ depends only on K. Thus we have that

$$\mathbf{II} \leq C(K)\epsilon^{\frac{1}{K}-1}\,\|g\|^2_{L^{p_\lambda}(\phi(\Omega))}$$

where the constant $C(K)$ depends only on K. This means that (4.8) has the upper bound

$$\begin{aligned} &\frac{C_2^{1/r}\,|\Omega|^{1-\frac{1}{s}}}{\frac{\alpha}{K}-\beta}\left(\int_\Omega J(z,\phi)^{\lambda s} dA(z)\right)^{\frac{1}{s}-\frac{1}{sr}} C(K)\,\epsilon^{\frac{1}{K}-1}\,\|g\|^2_{L^{p_\lambda}(\phi(\Omega))} \\ &\qquad \leq \frac{C(K)}{\beta\epsilon^{3-1/K}}\,|\Omega|^{1-\frac{1}{rs}}\left(\frac{1}{|\Omega|}\int_\Omega J(z,\phi)^{\lambda s} dA(z)\right)^{\frac{1}{s}-\frac{1}{sr}} \|g\|^2_{L^{p_\lambda}(\phi(\Omega))} \\ &\qquad \leq \frac{C(K)}{\beta\epsilon^{3-1/K}}\,|\Omega|^{1-\frac{1}{rs}}\left(\frac{1}{|\Omega|}\int_\Omega J(z,\phi) dA(z)\right)^{\lambda\left(1-\frac{1}{r}\right)} \|g\|^2_{L^{p_\lambda}(\phi(\Omega))} \\ &\qquad \leq \frac{C(K)}{\beta\,\epsilon^{3-1/K}}\,\|g\|^2_{L^{p_\lambda}(\phi(\Omega))} \end{aligned}$$

where we have used that $\lambda s < 1$, the area distortion theorem of Astala [8] and the fact that $|\Omega| \le C(K)$. Using (4.4), one finally obtains for the square root of the integral at (4.3) the bound

$$\frac{C(K)}{\beta^{1/2}\,\epsilon^{\frac{1}{2}(3-1/K)}}\,\|\mu\|_{L^\infty(\mathbb{C})}^{1-1/K}\,\|\mu\|_{W^{\alpha,2}(\mathbb{C})}^{1/K}.$$

Since $\|\mu\|_{L^\infty(\mathbb{C})}^{1-1/K} < 1$, the obtained inequality for the nonhomogeneous norms is

$$\|\mu\circ\phi\|_{W^{\beta,2}(\mathbb{C})} \le \frac{C(K)}{\sqrt{\beta(\alpha-K\beta)^{3-1/K}}}\,\|\mu\|_{W^{\alpha,2}(\mathbb{C})}^{\frac{1}{K}},$$

as desired. □

Remark 4.3. The condition $\beta < \frac{\alpha}{K}$ is by no means sharp. This is clear when α is close to 1. As promised in the introduction this will be a matter of a forthcoming work.

4.2 Regularity of homeomorphic solutions

We start by recalling the basic result on the existence of homeomorphic solutions to Beltrami type equations. In absence of extra regularity the integrability of the solutions comes from the work of Astala [8]. We recall the proof in terms of Neumann series since it will be used both in this section and in the sequel.

Lemma 4.4. *Let μ,ν be bounded functions, compactly supported in $\mathbb{D}$, such that $||\mu(z)|+|\nu(z)|| \le \frac{K-1}{K+1}$ at almost every $z\in\mathbb{C}$. The equation*

$$\overline{\partial} f = \mu\,\partial f + \nu\,\overline{\partial f} \tag{4.15}$$

admits only one homeomorphic solution $\phi:\mathbb{C}\to\mathbb{C}$, such that $|\phi(z)-z| = \mathcal{O}(1/|z|)$ as $|z|\to\infty$. Further, if $p\in(\frac{2K}{K+1},\frac{2K}{K-1})$ then the quantity

$$\|\partial\phi - 1\|_{L^p(\mathbb{C})} + \|\overline{\partial}\phi\|_{L^p(\mathbb{C})}$$

is bounded by a constant $C = C(K,p)$ that depends only on K and p.

Proof. Put $\phi(z) = z + \mathcal{C}h(z)$, where h is defined by

$$(I - \mu\,T - \nu\,\overline{T})h = \mu + \nu.$$

and $\mathcal{C}$ and T denote, respectively, Cauchy and Beurling transforms. Since T is an isometry in $L^2(\mathbb{C})$, one can construct such a function h as Neumann series

$$h = \sum_{n=0}^{\infty}(\mu T + \nu\overline{T})^n(\mu+\nu)$$

which obviously defines an $L^2(\mathbb{C})$ function. By Riesz-Thorin interpolation theorem,

$$\lim_{p\to 2}\|T\|_{L^p(\mathbb{C})} = 1,$$

it then follows that $h \in L^p(\mathbb{C})$ for every $p > 2$ such that $\|T\|_{L^p(\mathbb{C})} < \frac{K+1}{K-1}$. Hence, the Cauchy transform $\mathcal{C}h$ is Hölder continuous (with exponent $1-\frac{2}{p}$). Further, since h is compactly supported, we get $|\phi(z) - z| = |\mathcal{C}h(z)| \leq \frac{C}{|z|}$, and in fact $\phi - z$ belongs to $W^{1,p}(\mathbb{C})$ for such values of p. A usual topological argument (see for instance [9, Chapter 5]) proves that ϕ is a homeomorphism. For the uniqueness, note that if we are given two solutions ϕ_1, ϕ_2 as in the statement then $\overline{\partial}(\phi_1 \circ \phi_2^{-1}) = 0$ so that $\phi_1 \circ \phi_2^{-1}(z) - z$ is holomorphic on $\mathbb{C}$ and vanishes at infinity.

Now we recall a remarkable result from [10], which says that $I - \mu T - \nu \overline{T} : L^p(\mathbb{C}) \to L^p(\mathbb{C})$ defines a bounded invertible operator whenever $p \in (\frac{2K}{K+1}, \frac{2K}{K-1})$. Further, for the norm of the inverse operator we have the following estimate,

$$\|(I - \mu T - \nu \overline{T})^{-1}\|_{L^p(\mathbb{C})} \leq C(K,p).$$

Thus, if $p \in (\frac{2K}{K+1}, \frac{2K}{K-1})$

$$\|h\|_{L^p(\mathbb{C})} \leq C(K,p)\,\|\mu + \nu\|_{L^p(\mathbb{C})} \leq C(K,p).$$

Therefore

$$\|\partial\phi - 1\|_{L^p(\mathbb{C})} + \|\overline{\partial}\phi\|_{L^p(\mathbb{C})} = \|Th\|_{L^p(\mathbb{C})} + \|h\|_{L^p(\mathbb{C})} \leq C(K,p)$$

since T is a bounded operator in $L^p(\mathbb{C})$. □

Once we know about the existence of homeomorphic solutions, it is time to check their regularity when the coefficients belong to some fractional Sobolev space.

Theorem 4.5. *Let $\alpha \in (0,1)$, and suppose that $\mu, \nu \in W^{\alpha,2}(\mathbb{C})$ are Beltrami coefficients, compactly supported in $\mathbb{D}$, such that*

$$|\mu(z)| + |\nu(z)| \leq \frac{K-1}{K+1}.$$

at almost every $z \in \mathbb{D}$. Let $\phi : \mathbb{C} \to \mathbb{C}$ be the only homeomorphism satisfying

$$\overline{\partial}\phi = \mu\,\partial\phi + \nu\,\overline{\partial\phi}$$

and $\phi(z) - z = \mathcal{O}(1/z)$ as $|z| \to \infty$. Then, $\phi(z) - z$ belongs to $W^{1+\theta\alpha,2}(\mathbb{C})$ for every $\theta \in (0, \frac{1}{K})$, and

$$\|D^{1+\theta\alpha}(\phi - z)\|_{L^2(\mathbb{C})} \leq C\,\left(\|\mu\|^{\theta}_{W^{\alpha,2}(\mathbb{C})} + \|\nu\|^{\theta}_{W^{\alpha,2}(\mathbb{C})}\right)$$

for some constant $C = C(K,\theta,\alpha)$.

Proof. We consider a $\mathcal{C}^\infty$ function ψ, compactly supported inside of $\mathbb{D}$, such that $0 \le \psi \le 1$ and $\int \psi = 1$. For $n = 1, 2, \dots$ let $\psi_n(z) = n^2\, \psi(nz)$. Put

$$\mu_n(z) = \int_{\mathbb{C}} \mu(w)\, \psi_n(z-w)\, dA(w),$$

and

$$\nu_n(z) = \int_{\mathbb{C}} \nu(w)\, \psi_n(z-w)\, dA(w).$$

It is clear that both μ_n, ν_n are compactly supported in $\frac{n+1}{n}\mathbb{D}$, $|\mu_n(z)| + |\nu_n(z)| \le \frac{K-1}{K+1}$, $\|\mu_n - \mu\|_{W^{\alpha,2}(\mathbb{C})} \to 0$ and $\|\nu_n - \nu\|_{W^{\alpha,2}(\mathbb{C})} \to 0$ as $n \to \infty$. Indeed there is convergence in L^p for all $p \in (1, \infty)$. Thus, by interpolation we then get that for any $0 < \theta < 1$

$$\lim_{n\to\infty} \|\mu_n - \mu\|_{W^{\alpha\theta,\frac{2}{\theta}}(\mathbb{C})} + \|\nu_n - \nu\|_{W^{\alpha\theta,\frac{2}{\theta}}(\mathbb{C})} = 0$$

and in particular, the sequences $D^{\alpha\theta}\mu_n$ and $D^{\alpha\theta}\nu_n$ are bounded in $L^{\frac{2}{\theta}}(\mathbb{C})$. Let ϕ_n be the only K-quasiconformal mapping $\phi_n : \mathbb{C} \to \mathbb{C}$ satisfying

$$\overline{\partial}\phi_n = \mu_n\, \partial\phi_n + \nu_n \overline{\partial\phi_n} \tag{4.16}$$

and normalized by $\phi_n(z) - z = \mathcal{O}_n(1/z)$ as $|z| \to \infty$. By the construction in Lemma 4.4, $\phi_n(z) = z + \mathcal{C}h_n(z)$ where h_n is the only $L^2(\mathbb{C})$ solution to

$$h_n = \mu_n\, Th_n + \nu_n\, \overline{Th_n} + (\mu_n + \nu_n),$$

and $\mathcal{C}h_n$ denotes the Cauchy transform. As in Lemma 4.4, h_n belongs to $L^p(\mathbb{C})$ for all $p \in (\frac{2K}{K+1}, \frac{2K}{K-1})$ and

$$\|h_n\|_{L^p(\mathbb{C})} \le C(K, p) \tag{4.17}$$

with a constant $C(K, p)$ that depends on K and the product $(\frac{2K}{K-1} - p)(p - \frac{2K}{K+1})$. In particular, $\phi_n - z$ is a bounded sequence in $W^{1,p}(\mathbb{C})$.

Let us denote $H_n(z) = \mathcal{C}h_n(z) = \phi_n(z) - z$. We now write equation (4.16) as

$$\overline{\partial} H_n = \mu_n \partial H_n + \nu_n \overline{\partial H_n} + \mu_n + \nu_n$$

and take fractional derivatives. If $\beta = \alpha\theta$, we can use Lemma 3.1 (a) to find two functions E_β, F_β such that

$$\begin{aligned} D^\beta \overline{\partial} H_n - \mu_n\, D^\beta \partial H_n - \nu_n\, D^\beta \overline{\partial H_n} &= \\ D^\beta \mu_n\, \partial H_n + E_\beta + D^\beta \nu_n\, \overline{\partial H_n} &+ F_\beta. \end{aligned}$$

Now recall that we have $D^\beta \partial \varphi = \partial D^\beta \varphi$ and similarly for $\overline{\partial}$. Further, if φ is real then $D^\beta \varphi$ is also real. Thus

$$\begin{aligned} &\overline{\partial} D^\beta H_n - \mu_n \partial D^\beta H_n - \nu_n \overline{\partial D^\beta H_n} \\ &\qquad = D^\beta \mu_n\, \partial H_n + E_\beta + D^\beta \nu_n\, \overline{\partial H_n} + F_\beta. \end{aligned}$$

Equivalently, since $T\overline{\partial} = \partial$,

$$(I - \mu_n T - \nu_n \overline{T}) \left(\overline{\partial} D^\beta H_n \right) = D^\beta \mu_n \, \partial H_n + D^\beta \nu_n \, \overline{\partial H_n} + E_\beta + F_\beta. \tag{4.18}$$

For E_β and F_β we have precise L^2 estimates. To see this, choose $p_1 = \frac{2}{\theta}$, and then let p_2 be such that $2 < p_2 < \frac{2K}{K-1}$ and $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{2}$. Observe that this forces $0 < \theta < \frac{1}{K}$, and further

$$\frac{2K}{K-1} - p_2 = \frac{2}{(K-1)(1-\theta)} (1 - K\theta) \leq C_K \left(\frac{1}{K} - \theta \right). \tag{4.19}$$

Now, by Lemma 3.1 there exists $C_0 = C_0(\beta, p_1, p_2)$ such that

$$\|E_\beta\|_{L^2(\mathbb{C})} \leq C_0 \, \|D^\beta \mu\|_{L^{p_1}(\mathbb{C})} \, \|\partial H_n\|_{L^{p_2}(\mathbb{C})}, \tag{4.20}$$

and similarly

$$\|F_\beta\|_{L^2(\mathbb{C})} \leq C_0 \, \|D^\beta \nu\|_{L^{p_1}(\mathbb{C})} \, \|\partial H_n\|_{L^{p_2}(\mathbb{C})}. \tag{4.21}$$

This says us that the right term at (4.18) is in fact an $L^2(\mathbb{C})$ function, whose $L^2(\mathbb{C})$ norm is bounded from above by

$$(C_0 + 1) \left(\|D^\beta \mu_n\|_{L^{p_1}(\mathbb{C})} + \|D^\beta \nu_n\|_{L^{p_1}(\mathbb{C})} \right) \|\partial H_n\|_{L^{p_2}(\mathbb{C})}.$$

Now, recall that the operator $I - \mu_n T - \nu_n \overline{T}$ is continuously invertible in $L^2(\mathbb{C})$, and a Neumann series argument shows that the norm of its inverse is bounded by $\frac{1}{2}(K+1)$. Thus,

$$\begin{aligned}
&\|\overline{\partial} D^\beta H_n\|_{L^2(\mathbb{C})} \\
&\quad \leq (C_0 + 1) \, \frac{K+1}{2} \left(\|D^\beta \mu_n\|_{L^{\frac{2}{\theta}}(\mathbb{C})} + \|D^\beta \nu_n\|_{L^{\frac{2}{\theta}}(\mathbb{C})} \right) \|\partial H_n\|_{L^{p_2}(\mathbb{C})} \\
&\quad \leq (C_0 + 1) \, \frac{K+1}{2} \left(\|\mu_n\|^\theta_{W^{\alpha,2}(\mathbb{C})} + \|\nu_n\|^\theta_{W^{\alpha,2}(\mathbb{C})} \right) \|\partial H_n\|_{L^{p_2}(\mathbb{C})}
\end{aligned}$$

where $C_0 = C_0(\beta, p_1, p_2)$ is the constant in (4.20). As $n \to \infty$, we have the uniform bound (4.17),

$$\|\partial H_n\|_{L^{p_2}(\mathbb{C})} = \|T h_n\|_{L^{p_2}(\mathbb{C})} \leq C_{p_2} \, \|h_n\|_{L^{p_2}(\mathbb{C})} \leq C_1$$

where now the constant $C_1 = C_1(K, \theta)$ depends on K and $\frac{1}{K} - \theta$. Thus, we obtain for $\|\overline{\partial} D^\beta H_n\|_{L^2(\mathbb{C})}$ the upper bound

$$(C_0 + 1) C_1(K, \theta) \left(\|\mu\|^\theta_{W^{\alpha,2}(\mathbb{C})} + \|\nu\|^\theta_{W^{\alpha,2}(\mathbb{C})} \right).$$

By passing to a subsequence we see that $D^\beta H_n$ converges in $W^{1,2}(\mathbb{C})$, and as a consequence $\phi - z$ belongs to $W^{1+\beta,2}(\mathbb{C})$. Further, we have the bounds

$$\|D^{1+\theta\alpha}(\phi - z)\|_{L^2(\mathbb{C})} \leq C \left(\|\mu\|^\theta_{W^{\alpha,2}(\mathbb{C})} + \|\nu\|^\theta_{W^{\alpha,2}(\mathbb{C})} \right)$$

with C depending only on K, α and $\frac{\alpha}{K} - \beta$. $\square$

4.3 Regularity of complex geometric optics solutions

We are now ready to give precise bounds on the Sobolev regularity of the complex geometric optics solutions to the equation $\overline{\partial} f = \mu\, \overline{\partial f}$ introduced in Theorem 2.2. For this, the following lemma will be needed.

Lemma 4.6. *Let $\Omega \subset \mathbb{C}$ be any domain. If $f \in W^{\alpha,p}(\Omega)$ and $\varphi \in \mathcal{C}^\infty(\mathbb{C})$ is bounded, then the multiplier*

$$f \mapsto \varphi f$$

is bounded from $W^{\alpha,p}(\Omega)$ to itself, and

$$\|\varphi f\|_{W^{\alpha,p}(\Omega)} \leq \|\varphi\|_{L^\infty(\Omega)} \left(1 + \frac{\|D\varphi\|_{L^\infty(\Omega)}}{\|\varphi\|_{L^\infty(\Omega)}}\right)^\alpha \|f\|_{W^{\alpha,p}(\Omega)},$$

whenever $0 \leq \alpha \leq 1$, $1 < p < \infty$.

The proof follows easily by interpolation.

Theorem 4.7. *Let $\mu \in W^{\alpha,2}(\mathbb{C})$ be such that $|\mu| \leq \frac{K-1}{K+1}\chi_{\mathbb{D}}$ and $\|\mu\|_{W^{\alpha,2}(\mathbb{C})} \leq \Gamma_0$. Let $f = f_\mu(z,k)$ the complex geometric optics solutions to the equation*

$$\overline{\partial} f = \mu\, \overline{\partial f}.$$

For any $0 < \theta < \frac{1}{K}$ we have that

$$f \in W^{1+\theta\alpha,2}_{loc}(\mathbb{C}).$$

Further, we have the estimate

$$\|D^{1+\alpha\theta}(f_\mu)(\cdot,k)\|_{L^2(\mathbb{D})} \leq e^{C(K)|k|} \left(1 + \Gamma_0^\theta\right)$$

whenever $0 < \theta < \frac{1}{K}$.

Proof. The case $k = 0$ is trivial. For $k \neq 0$, the existence and uniqueness of the complex geometric optics solutions comes from [12, Theorem 4.2] (see Theorem 2.2 in the present paper). It is shown in [12, Lemma 7.1] that f may be represented as

$$f(z,k) = e^{ik\phi(z,k)}$$

where $\phi : \mathbb{C} \to \mathbb{C}$ is the only $W^{1,2}_{loc}(\mathbb{C})$ homeomorphism solving

$$\overline{\partial}\phi(z) = -\mu(z)\, \frac{\overline{k}}{k}\, e_{-k}(\phi(z,k))\, \overline{\partial\phi}(z) \tag{4.22}$$

and such that

$$|\phi(z) - z| \leq \frac{C_1(K)}{|z|}, \qquad |z| \geq 1. \tag{4.23}$$

Let us recall that $e_{-k}(w) = e^{-ikw-i\bar{k}\bar{w}}$ is a unimodular function, whence $|e_{-k}(\phi(z))| = 1$.
We will first deduce the smoothness of ϕ from that of its Beltrami coefficient $\mu\, e_{-k}(\phi)$, see (4.22), with the help of Theorem 4.5. For the ellipticity there is nothing to say since

$$|\mu(z)\, e_{-k}(\phi(z,k))| = |\mu(z)| \leq \frac{K-1}{K+1}.$$

For the Sobolev regularity, we will use that

$$e_{-k}(\phi(z,k)) = e_{-k}(\phi(z,k) - z)\, e_{-k}(z),$$

which is more convenient since for $\phi(z,k) - z$ we have global estimates. We then describe the Beltrami coefficient of (4.22) as

$$\mu\, e_{-k}(\phi) = \mu \left(e_{-k}(\phi - z) - 1 \right) e_{-k}(z) + \mu\, e_{-k}(z).$$

Using Lemma 4.6 with $\varphi(z) = e_{-k}(z)$,

$$\begin{aligned} \|\mu\, e_{-k}(\phi)\|_{W^{\alpha,2}(\mathbb{C})} &\leq \|\mu\,(e_{-k}(\phi - z) - 1)e_{-k}\|_{W^{\alpha,2}(\mathbb{C})} + \|\mu\, e_{-k}\|_{W^{\alpha,2}(\mathbb{C})} \\ &\leq (1+|k|)^{\alpha}\, \|\mu\, g\|_{W^{\alpha,2}(\mathbb{C})} + (1+|k|)^{\alpha}\, \|\mu\|_{W^{\alpha,2}(\mathbb{C})} \end{aligned}$$

where $g = e_{-k}(\phi - z) - 1$. For the first term above we use the fractional Leibniz rule (Lemma 3.1) to get that

$$\|D^{\alpha}(\mu g)\|_{L^2(\mathbb{C})} \leq \|D^{\alpha} g\|_{L^2(\mathbb{C})}\, \|\mu\|_{L^{\infty}(\mathbb{C})} + (C_0 + 1)\|D^{\alpha}\mu\|_{L^2(\mathbb{C})}\, \|g\|_{L^{\infty}(\mathbb{C})}.$$

The bound for $\|D^{\alpha} g\|_{L^2(\mathbb{C})}$, will be found by interpolation. Bounds for $D^1 g$ come easily from Lemma 4.4. Indeed, $\phi(z,k) - z$ belongs to $W^{1,p}(\mathbb{C})$ for every $\frac{2K}{K+1} < p < \frac{2K}{K-1}$, and in fact, by using the chain rule

$$|\nabla g(z)| \leq |k|\, |\nabla(\phi(z,k) - z)|\, (1 + |g(z)|) \leq 3\, |k|\, |D(\phi(z,k) - z)|,$$

whence

$$\|D^1 g\|_{L^p(\mathbb{C})} \leq |k|\, C(K,p)$$

for every $\frac{2K}{K+1} < p < \frac{2K}{K-1}$. The L^p bounds for g (now with $p > 2$) follow from the decay estimate (4.23) and the global boundednes of g. We obtain that $g \in L^p(\mathbb{C})$ for any $p > 2$, with

$$\|g\|_{L^p(\mathbb{C})} \leq (1+|k|)^2\, C(K,p).$$

If we now let $\max\{\frac{2K}{K+1}, 2\alpha\} < p_1 < 2$ and

$$\frac{1}{2} = \frac{\alpha}{p_1} + \frac{1-\alpha}{p_2},$$

then $p_2 > 2$. Therefore $g \in L^{p_2}(\mathbb{C})$, $D^1 g \in L^{p_1}(\mathbb{C})$ and

$$\|D^\alpha g\|_{L^2(\mathbb{C})} \le \|D^1 g\|^{\alpha}_{L^{p_1}(\mathbb{C})} \|g\|^{1-\alpha}_{L^{p_2}(\mathbb{C})} \le C(K)(1+|k|)^{2-\alpha}.$$

Summarizing, $\|D^\alpha(\mu g)\|_{L^2(\mathbb{C})} \le C\,\Gamma_0 + C(K)\,(1+|k|)^{2-\alpha}$. By Theorem 4.5, the smoothness of $\phi(\cdot,k)$ can be recovered by that of its coefficient, and we get

$$\|D^{1+\alpha\theta}(\phi(z,k)-z)\|_{L^2(\mathbb{C})} \le C(K,\theta,\alpha)\,(\Gamma_0 + (1+|k|)^{2-\alpha})^\theta, \tag{4.24}$$

for any $0<\theta<\frac{1}{K}$. Now the job is to use the above estimates to get local bounds for the derivative $D(e^{ik\phi})$ in the $W^{\alpha\theta,2}$ norm. For this, we write again

$$e^{ik\phi} = e^{ik(\phi-z)}e^{ikz} \tag{4.25}$$

By letting $h(z,k) = e^{ik(\phi(z,k)-z)}$, by the chain rule one gets, for the first order derivatives

$$\nabla(e^{ik\phi}) = ik\,e^{ikz}\left(h\,\nabla(\phi-z) + h\nabla z\right).$$

But 1 and h have obvious local Sobolev bounds, and furthermore e^{ikz} is $\mathcal{C}^\infty$. Thus by Lemma 4.6 we get for any disk D, for some $C=C(D)$, that

$$\|\nabla(e^{ik\phi})\|_{W^{\alpha\theta,2}(D)} \le |k|(1+|k|)^{\alpha\theta}e^{C|k|}\|h\,\nabla(\phi-z)+h\nabla z\|_{W^{\alpha\theta,2}(D)} \tag{4.26}$$

and only local $W^{\alpha\theta,2}$ bounds for $h\,D(\phi-z)$ and h are needed. To find these bounds, we use again Lemma 3.1,

$$\begin{aligned}&\|D^{\alpha\theta}\left(h\,\nabla(\phi-z)\right)\|_{L^2(\mathbb{C})}\\ &\qquad\le \|D^{\alpha\theta}h\,\nabla(\phi-z)\|_{L^2(\mathbb{C})} + (C_0+1)\|h\|_{L^\infty(\mathbb{C})}\,\|D^{\alpha\theta}(\nabla(\phi-z))\|_{L^2(\mathbb{C})}.\end{aligned}$$

For the second term above we use (4.24) and also the fact that

$$|h(z,k)| \le 1 + e^{|k||\phi(z,k)-z|} \le e^{C(K)|k|}, \qquad z\in\mathbb{C}, \tag{4.27}$$

which holds for an appropriate constant $C(K)\ge 0$. For the first term, an interpolation is needed. By using the chain rule for the gradient and (4.27) we get

$$\|D^1 h\|_{L^2(\mathbb{C})} \le |k|\,e^{C(K)|k|}\,\|D^1(\phi-z)\|_{L^2(\mathbb{C})}.$$

From Lemma 4.4, since for any $0<\theta<\frac{1}{K}$ we have $2<\frac{2}{1-\alpha\theta}<\frac{2K}{K-1}$, we obtain

$$\begin{aligned}\|D^{\alpha\theta}h\,\nabla(\phi-z)\|_{L^2(\mathbb{C})} &\le \|D^{\alpha\theta}h\|_{L^{\frac{2}{\alpha\theta}}(\mathbb{C})}\,\|D^1(\phi-z)\|_{L^{\frac{2}{1-\alpha\theta}}(\mathbb{C})}\\ &\le \|h\|^{1-\alpha\theta}_{L^\infty(\mathbb{C})}\,\|D^1 h\|^{\alpha\theta}_{L^2(\mathbb{C})}\,C(K,\alpha,\theta)\\ &\le C(K,\alpha,\theta)\,|k|^{\alpha\theta}\,e^{C(K)|k|}.\end{aligned}$$

Summarizing,

$$\|h\,\nabla(\phi - z)\|_{W^{\alpha\theta,2}(\mathbb{C})} \le C(K,\alpha,\theta)\, e^{C(K)|k|} \left((\Gamma_0 + (1+|k|)^{2-\alpha})^\theta + |k|^{\alpha\theta} \right)$$
$$\le C(K,\alpha,\theta)\, e^{C(K)|k|} \left(\Gamma_0^\theta + (1+|k|)^{(2-\alpha)\theta} \right).$$

The desired $W^{1+\alpha\theta,2}(\mathbb{D})$ estimates for $e^{ik\phi}$ come then easily from (4.26). $\square$

We will also need the following bounds in Section 6.

Lemma 4.8. *Let μ be such that $|\mu| \le \frac{K-1}{K+1}\chi_{\mathbb{D}}$. Let $f = f_\mu(z,k)$ denote the complex geometric optics solutions to*

$$\overline{\partial} f = \mu\, \overline{\partial f}$$

with $k \neq 0$, and let $0 < p < \frac{2}{K-1}$. Then,

$$\int_{\mathbb{D}} \left| \frac{1}{\partial f(z)} \right|^p dA(z) \le C$$

where the constant C depends only on k, p and K.

Proof. The function f can be represented as $f = e^{ik\phi}$. Since $\mathrm{supp}(\mu) \subset \mathbb{D}$, ϕ is conformal in $\mathbb{C} \setminus \mathbb{D}$. Thus by Koebe $\frac{1}{4}$ Theorem $\phi(\mathbb{D}) \subset 4\mathbb{D}$ and therefore in $\mathbb{D}$ one has

$$\|\phi(\cdot,k)\|_{L^\infty(\mathbb{D})} \le 4. \tag{4.28}$$

Now, by the chain rule we get that $\frac{1}{|\partial\phi|} \le |\partial\phi^{-1}(\phi)|$. Therefore

$$\left| \frac{1}{\partial f} \right| = \frac{1}{|e^{ik\phi}|} \frac{1}{|ik\partial\phi|} \le \frac{e^{4|k|}}{|k|} \frac{1}{|\partial\phi|} \le \frac{e^{4|k|}}{|k|} |\partial\phi^{-1}(\phi)|.$$

It then follows from the regularity theory of quasiconformal mappings [8] that

$$\int_{\mathbb{D}} \left| \frac{1}{\partial f(z)} \right|^p dA(z) \le \frac{e^{4p|k|}}{|k|^p} \int_{\mathbb{D}} |\partial\phi^{-1}(\phi(z))|^p dA(z)$$
$$= \frac{e^{4p|k|}}{|k|^p} \int_{\phi(\mathbb{D})} |\partial\phi^{-1}(w)|^p J(w,\phi^{-1}) dA(w)$$
$$\le \frac{e^{4p|k|}}{|k|^p} \int_{\phi(\mathbb{D})} |\partial\phi^{-1}(w)|^{p+2} dA(w).$$

As ϕ^{-1} is also K-quasiconformal, the above integral is bounded whenever $\frac{2K}{K+1} < p+2 < \frac{2K}{K-1}$. Further, by the reverse Hölder inequality for K-quasidisks (see [10, Corollary 11]) we obtain

$$\int_{\mathbb{D}} \left| \frac{1}{\partial f(z)} \right|^p dA(z) \le |\phi(\mathbb{D})|\, \frac{C(K,p)}{\frac{2}{K-1} - p} \left(\frac{\pi}{|\phi(\mathbb{D})|} \right)^{1+\frac{p}{2}}$$
$$= C(K,p)\pi^{1+p/2}\, |\phi(\mathbb{D})|^{-p/2}.$$

The claim follows after noticing that $|\phi(\mathbb{D})|$ is bounded (from above and from below) by constants that depend only on K, which can be seen as in (4.6). $\square$

5 Uniform subexponential decay

We investigate the decay property of complex geometric optic solutions to the equation

$$\overline{\partial} f_\lambda = \lambda \mu \, \overline{\partial f_\lambda},$$

where $\lambda \in \partial\mathbb{D}$ is a fixed complex parameter, and $\mu \in W_0^{\alpha,2}(\mathbb{C})$ is such that $|\mu| \leq \frac{K-1}{K+1}\,\chi_{\mathbb{D}}$. It turns out that f_λ admits the representation

$$f_\lambda(z,k) = e^{ik\phi_\lambda(z,k)}$$

where ϕ_λ satisfies the following properties (see [12, Lemma 7.1] or the proof of Theorem 4.7 above):

1. $\phi_\lambda(\cdot,k) : \mathbb{C} \to \mathbb{C}$ is a quasiconformal mapping.
2. $\phi_\lambda(z,k) = z + \mathcal{O}_k(1/z)$ as $|z| \to \infty$
3. ϕ_λ satisfies the *nonlinear* equation

$$\overline{\partial}\phi_\lambda(z,k) = -\lambda\, \mu(z)\, \frac{\overline{k}}{k}\, e_{-k}(\phi_\lambda(z,k))\, \overline{\partial\phi_\lambda(z,k)} \tag{5.1}$$

As was explained in Section 2, our goal is to obtain a uniform decay of the type

$$|\phi_\lambda(z,k) - z| \leq \frac{C}{|k|^{b\alpha}} \tag{5.2}$$

The precise statement can be found at Theorem 5.7. For the proof, we will mainly follow the lines of both [12, 14]. This consists on investigating first the behaviour of *linear* Beltrami equations with the rapidly oscillating coefficients $\mu(z)\, e_{-k}(z)$, and then treat the nonlinearity as a perturbation, by passing to the inverse quasiconformal mapping $\psi_\lambda = \phi_\lambda^{-1}$, which satisfies a Beltrami equation with coefficient $\mu(z)\, e_{-k}(\psi_\lambda(z,k))$, see (5.17).

5.1 Estimates for the linear equation

As usually, μ denotes a Beltrami coefficient, with $|\mu| \leq \frac{K-1}{K+1}\chi_{\mathbb{D}}$ and the smoothness assumption

$$\|\mu\|_{W^{\alpha,2}(\mathbb{C})} = \|\mu\|_{L^2(\mathbb{C})} + \|D^\alpha \mu\|_{L^2(\mathbb{C})} \leq \Gamma_0$$

for some $0 < \alpha < \frac{1}{2}$ and $\Gamma_0 > 0$. For each complex numbers $k \in \mathbb{C}$ and $\lambda \in \partial\mathbb{D}$, let $\psi = \psi_\lambda(z,k)$ be the only homeomorphic solution to the problem,

$$\begin{cases} \overline{\partial}\psi(z,k) = \frac{\overline{k}}{k}\,\lambda\, e_{-k}(z)\,\mu(z)\,\partial\psi(z,k) \\ \psi(z,k) - z = \mathcal{O}(1/z), \qquad z \to \infty \end{cases} \tag{5.3}$$

Then, by a Neumann series argument as in Lemma 4.4, ψ can be represented by means of a Cauchy transform

$$\psi(z,k) - z = \frac{1}{\pi}\int_{\mathbb{C}} \overline{\partial}\psi(w,k)\Phi(z,w)dA(w), \tag{5.4}$$

where $\Phi(z,w) = \frac{\psi_{\mathbb{D}}(w)}{z-w}$ for a smooth cutoff function $\psi_{\mathbb{D}} = 1$ on $\mathbb{D}$ (in particular on the support of $\overline{\partial}\psi$). We need subtle properties for both terms. The first two properties of the next lemma where already proved in [12, Lemma 7.5] but the regularity on μ allows to prove 3.

Lemma 5.1. *Let n_0 be given, and let $s \geq 2$ be such that*

$$\kappa\,\|T\|_{L^s(\mathbb{C})} < 1.$$

There exists a decomposition $\overline{\partial}\psi_\lambda(z,k) = g_\lambda(z,k) + h_\lambda(z,k)$ satisfying the following properties:

1. $\|h_\lambda(\cdot,k)\|_{L^s} \leq C(\kappa,s)\,\left(\kappa\,\|T\|_{L^s(\mathbb{C})}\right)^{n_0}$.
2. $\|g_\lambda(\cdot,k)\|_{L^s} \leq C(\kappa)$.
3. *If* $1 < p < 2$, $q = \frac{p}{p-1}$, $R > 0$ *and* $|k| > 2R$, *then*

$$\left(\int_{|\xi|<R} |\widehat{g}_\lambda(\xi,k)|^q dA(\xi)\right)^{\frac{1}{q}} \leq \frac{C(p)}{\log\frac{1}{\kappa}}\,\frac{\Gamma_0}{|k|^\alpha}\,\left(M(p)\right)^{n_0}$$

where $\widehat{g}_\lambda(\xi,k) = (g_\lambda(\cdot,k))^\wedge(\xi)$ *and* $M(p) = \|T\|_{L^{\frac{2p}{2-p}}(\mathbb{C})} + \|T\|_{L^p(\mathbb{C})}$ *is as in Lemma 5.2.*

The proof is based on an idea from [12]. It relies on the Neumann series expression of $\overline{\partial}\psi$. For this, we consider the unimodular factors

$$e_k(z) = e^{i(kz+i\overline{k}\overline{z})}.$$

Then one writes

$$\overline{\partial}\psi(z) = \sum_{n=0}^{\infty}\left(\frac{-\overline{k}}{k}\lambda\right)^{n+1} e_{-(n+1)k}(z) f_n(z) \tag{5.5}$$

where

$$\begin{cases} f_0 = \mu \\ f_n = \mu\, T_n(f_{n-1}),\ n = 1, 2, \ldots \end{cases} \tag{5.6}$$

Here by T_n we denote a singular integral operator defined by the rule

$$T_n(\varphi) = e_{nk}\, T(e_{-nk}\varphi)$$

where T is the usual Beurling transform (4.2). It is not hard to see that T_n is represented, at the frequency side, by a unimodular multiplier of the form

$$\widehat{T_n\varphi}(\xi) = \frac{\xi - nk}{\overline{\xi - nk}}\widehat{\varphi}(\xi).$$

Thus,

$$\|T_n\|_{L^2(\mathbb{C})} = \|T_n\|_{L^2(\mathbb{C})\to L^2(\mathbb{C})} = 1$$

and T_n is an isometry of $L^2(\mathbb{C})$. In fact, for any $1 < p < \infty$,

$$\|T_n(\varphi)\|_{L^p(\mathbb{C})} = \|T(e_{-nk}\varphi)\|_{L^p(\mathbb{C})} \leq \|T\|_{L^p(\mathbb{C})}\, \|\varphi\|_{L^p(\mathbb{C})}$$

because $|e_{nk}(z)| = 1$, so that $\|T_n\|_{L^p(\mathbb{C})} = \|T\|_{L^p(\mathbb{C})}$. As T_n is given by a Fourier multiplier, it commutes with any constant coefficients differential operator D and thus

$$\|T_n\varphi\|_{W^{1,p}} = \|T_n\varphi\|_{L^p} + \|T_n(D\varphi)\|_{L^p} \leq \|T\|_p\, \|\varphi\|_{W^{1,p}(\mathbb{C})}.$$

Therefore $\|T_n\|_{W^{1,p}(\mathbb{C})} \leq \|T\|_{L^p}$. Finally, the complex interpolation method gives that for any $0 < \beta < 1$

$$\|T_n\|_{W^{\beta,p}(\mathbb{C})} \leq \|T\|_{L^p(\mathbb{C})}. \tag{5.7}$$

Lemma 5.2. *For any $1 < p < 2$ there exists a constant $C(p)$ such that*

$$\|f_n\|_{W^{\alpha,p}(\mathbb{C})} \leq C(p)\, \Gamma_0\, \kappa^n\, (M(p))^n, \tag{5.8}$$

for any $n = 1, 2, \ldots$, where $\kappa = \frac{K-1}{K+1}$, $M(p) = \|T\|_{L^{\frac{2p}{2-p}}} + \|T\|_{L^p}$.

Proof. We start by recalling Leibniz rule (Remark 3.2). Let $g \in W^{\alpha,p}$. Then it holds that

$$\|D^\alpha(\mu g)\|_{L^p(\mathbb{C})} \leq C_0\, \Gamma_0 \|g\|_{L^{\frac{2p}{2-p}}(\mathbb{C})} + \kappa\|D^\alpha g\|_{L^p(\mathbb{C})} \tag{5.9}$$

for some positive constant $C_0 = C_0(p, \alpha)$ as in (3.5). Since $0 < \alpha < \frac{1}{2}$ the constant can be choosen uniform in α. Thus, $C_0 = C_0(p)$.

First of all we proceed to prove (5.8) for $n = 1$. Recall that $f_1 = \mu T_1 \mu$ and denote $X_1 = \|f_1\|_{W^{\alpha,p}(\mathbb{C})}$, then

$$X_1 \leq \kappa\|T_1\mu\|_{L^p} + \|D^\alpha f_1\|_{L^p}.$$

By (5.9) we have

$$\|D^\alpha f_1\|_{L^p} \le \kappa \|D^\alpha T_1\mu\|_{L^p(\mathbb{C})} + C_0\,\Gamma_0 \|T_1\mu\|_{L^{\frac{2p}{2-p}}}$$

We can use Hölder's inequality to get

$$\|T_1\mu\|_{L^p} \le \|T\|_{L^p}\|\mu\|_{L^2}\,\pi^{\frac{1}{p}-\frac{1}{2}} \le \|T\|_{L^p}\Gamma_0\,\pi^{\frac{1}{p}-\frac{1}{2}},$$

and

$$\|T_1\mu\|_{L^{\frac{2p}{2-p}}} \le \|T\|_{L^{\frac{2p}{2-p}}}\kappa\pi^{\frac{1}{p}-\frac{1}{2}}.$$

We also have by using (3.7) with a universal constant C_1 and Hölder inequality

$$\|D^\alpha T_1\mu\|_{L^p} \le \|T\|_{L^p}\,\|D^\alpha\mu\|_{L^p(\mathbb{C})}$$

$$\le \|T\|_{L^p}\,(C_1\|\mu\|_{W^{\alpha,p}(2\mathbb{D})}) \le C_2\|T\|_{L^p}\,\Gamma_0\pi^{\frac{1}{p}-\frac{1}{2}}$$

Thus, we obtain (5.8)

$$X_1 \le C(p)\,M(p)\,\kappa\,\Gamma_0,$$

where $C(p) = C_0(1+C_2)\pi^{\frac{1}{p}-\frac{1}{2}}$

Now we study the L^p norm of f_n. Recalling that μ is compactly supported inside of $\mathbb{D}$, we first see that

$$\|f_n\|_{L^p(\mathbb{C})} = \|f_n\|_{L^p(\mathbb{D})} \le \kappa\,\|T_n f_{n-1}\|_{L^p(\mathbb{D})}.$$

Next, (5.9) yields that,

$$\begin{aligned}\|D^\alpha f_n\|_{L^p(\mathbb{C})} &= \|D^\alpha(\mu\,T_n f_{n-1})\|_{L^p(\mathbb{C})}\\ &\le C_0\,\Gamma_0\,\|T_n f_{n-1}\|_{L^{\frac{2p}{2-p}}(\mathbb{C})} + \kappa\,\|D^\alpha T_n f_{n-1}\|_{L^p(\mathbb{C})}\end{aligned}$$

Hence, for any $n>1$, denoting $X_n = \|f_n\|_{W^{\alpha,p}(\mathbb{C})}$

$$\begin{aligned}X_n &= \|f_n\|_{L^p(\mathbb{C})} + \|D^\alpha f_n\|_{L^p(\mathbb{C})}\\ &\le C\,\Gamma_0\,\|T_n f_{n-1}\|_{L^{\frac{2p}{2-p}}(\mathbb{C})} + \kappa\|T_n f_{n-1}\|_{W^{\alpha,p}(\mathbb{C})}\end{aligned}$$

To control the first term above, we see that

$$\begin{aligned}\|T_n f_{n-1}\|_{L^{\frac{2p}{2-p}}(\mathbb{C})} &\le \|T\|_{L^{\frac{2p}{2-p}}}\,\|f_{n-1}\|_{L^{\frac{2p}{2-p}}(\mathbb{C})} \le (\|T\|_{L^{\frac{2p}{2-p}}}\kappa)\,\|T_{n-1}f_{n-2}\|_{L^{\frac{2p}{2-p}}(\mathbb{C})}\\ &\le (\|T\|_{L^{\frac{2p}{2-p}}}\kappa)^{n-1}\,\|T_1 f_0\|_{L^{\frac{2p}{2-p}}} \le (\|T\|_{L^{\frac{2p}{2-p}}}\kappa)^n\,\pi^{\frac{1}{p}-\frac{1}{2}}\end{aligned}$$

and for the second, if $n>1$

$$\|T_n f_{n-1}\|_{W^{\alpha,p}(\mathbb{C})} \le \|T\|_{L^p}\,\|f_{n-1}\|_{W^{\alpha,p}(\mathbb{C})}.$$

We have just proved the recursive relation

$$X_n \le C_0\,\Gamma_0\,\pi^{\frac{1}{p}-\frac{1}{2}}\,(\kappa\,\|T\|_{L^{\frac{2p}{2-p}}})^n + (\kappa\,\|T\|_{L^p})\,X_{n-1} \tag{5.10}$$

whenever $n > 1$.

If we assume (5.8) for $n-1$ with $C(p) = C_0(1+C_2)\pi^{\frac{1}{p}-\frac{1}{2}}$, i.e.,

$$X_{n-1} \le C(p)\,\Gamma_0\,\kappa^{n-1}\,(M(p))^{n-1},$$

we obtain after (5.10),

$$\begin{aligned} X_n &\le C_0\,\Gamma_0\,\pi^{\frac{1}{p}-\frac{1}{2}}\,(\kappa\,\|T\|_{L^{\frac{2p}{2-p}}})^n + (\kappa\,\|T\|_{L^p})\,C(p)\,\Gamma_0\,\kappa^{n-1}\,(M(p))^{n-1} \\ &\le C(p)\Gamma_0\,\kappa^n\,(M(p))^{n-1}\left(\|T\|_{L^{\frac{2p}{2-p}}(\mathbb{C})}\left(\frac{\|T\|_{L^{\frac{2p}{2-p}}}}{M(p)}\right)^{n-1} + \|T\|_{L^p}\right) \\ &\le C(p)\Gamma_0\,\kappa^n\,(M(p))^n, \end{aligned}$$

and the proof is concluded. □

In particular, every function f_n of the Neumann series is compactly supported and belongs to $L^p(\mathbb{C})$ for any $p \in (1,\infty)$, and also to $W^{\alpha,p}(\mathbb{C})$ for any $p < 2$.

Lemma 5.3. *If h belongs to $W^{\alpha,p}(\mathbb{C})$ for some $1 < p < 2$, then if q is such that $1/p + 1/q = 1$*

$$\left(\int_{|\xi|>R} |\widehat{h}(\xi)|^q\,dA(\xi)\right)^{\frac{1}{q}} \le C(p)\,\frac{\|h\|_{W^{\alpha,p}(\mathbb{C})}}{R^\alpha}$$

Proof. We wil use the characterization in terms of Bessel potentials of $W^{\alpha,p}(\mathbb{C})$. Since the Fourier transform maps continuously $L^p(\mathbb{C})$ into $L^q(\mathbb{C})$, we get that

$$\left(\int_{\mathbb{C}}\left((1+|\xi|^2)^{\frac{\alpha}{2}}\,|\widehat{h}(\xi)|\right)^q\,dA(\xi)\right)^{\frac{1}{q}} \le C(p)\,\|h\|_{W^{\alpha,p}}$$

Thus, a simple computation yields

$$\begin{aligned} \left(\int_{|\xi|>R} |\widehat{h}(\xi)|^q\,dA(\xi)\right)^{\frac{1}{q}} &\le \left(\int_{|\xi|>R}\left(\frac{(1+|\xi|^2)^{\frac{\alpha}{2}}}{|\xi|^\alpha}\right)^q |\widehat{h}(\xi)|^q dA(\xi)\right)^{\frac{1}{q}} \\ &\le \frac{1}{R^\alpha}\left(\int_{\mathbb{C}}(1+|\xi|^2)^{\frac{\alpha q}{2}}\,|\widehat{h}(\xi)|^q dA(\xi)\right)^{\frac{1}{q}} \\ &\le C(p)\,\frac{\|h\|_{W^{\alpha,p}}}{R^\alpha} \end{aligned}$$

and the result follows. □

Proof of Lemma 5.1. We recall how to obtain properties *1* and *2*. We use the Neumann series

$$\overline{\partial}\psi(z) = \sum_{n=0}^{\infty} \left(-\frac{\overline{k}}{k}\lambda\right)^{n+1} e_{-(n+1)k}(z) f_n(z) \tag{5.11}$$

introduced before. Then, take $g = \sum_{n=0}^{n_0} \left(-\frac{\overline{k}}{k}\,\lambda\, e_{-nk}\,\mu T\right)^n \left(-\frac{\overline{k}}{k}\,\lambda\, e_{-nk}\,\mu\right)$ and $h = \overline{\partial}\psi - g$. In this way, properties *1* and *2* follow easily from the general theory of the Beltrami equation, since

$$\begin{aligned}
&\Big\| \left(\frac{\overline{k}}{k}\,\lambda\, e_{-nk}\,\mu T\right)^n \left(\frac{\overline{k}}{k}\,\lambda\, e_{-nk}\,\mu\right) \Big\|_{L^s(\mathbb{C})} \\
&\qquad \le \kappa\, \|T\|_{L^s(\mathbb{C})} \Big\| \left(\frac{\overline{k}}{k}\,\lambda\, e_{-nk}\,\mu T\right)^{n-1} \left(\frac{\overline{k}}{k}\,\lambda\, e_{-nk}\,\mu\right) \Big\|_{L^s(\mathbb{C})} \\
&\qquad \le (\kappa\, \|T\|_{L^s(\mathbb{C})})^n\, \|\mu\|_{L^s(\mathbb{C})} = (\kappa\, \|T\|_{L^s(\mathbb{C})})^n\, \kappa\, \pi^{\frac{1}{s}}.
\end{aligned}$$

For the proof of *3*, we must use the regularity of μ. By (5.11) we can write $g(z,k) = \sum_{n=0}^{n_0} G_n(k,z)$ where $G_n(z,k) = \left(-\frac{\overline{k}}{k}\lambda\right)^{n+1} e_{-(n+1)k}\, f_n$. Then, we apply Lemma 5.2 to f_n. The Fourier transform of $G_n(z,k)$ (with respect to the z variable) reads as

$$\widehat{G_n}(\xi,k) = \left(\frac{-\overline{k}}{k}\lambda\right)^{n+1} \widehat{f_n}(\xi - (n+1)k).$$

Hence, for $|k| > R$, lemma 5.3 implies that

$$\begin{aligned}
\left(\int_{|\xi|<R} |\widehat{g}(\xi,k)|^q\, dA(\xi)\right)^{\frac{1}{q}} &\le \sum_{n=0}^{n_0} \left(\int_{|\xi|<R} |\widehat{G_n}(\xi,k)|^q\, dA(\xi)\right)^{\frac{1}{q}} \\
&= \sum_{n=0}^{n_0} \left(\int_{|\xi|<R} |\widehat{f_n}(\xi - (n+1)k)|^q\, dA(\xi)\right)^{\frac{1}{q}} \\
&= \sum_{n=0}^{n_0} \left(\int_{|\zeta+(n+1)k|<R} |\widehat{f_n}(\zeta)|^q\, dA(\zeta)\right)^{\frac{1}{q}} \\
&\le \sum_{n=0}^{n_0} \left(\int_{|\zeta|>(n+1)|k|-R} |\widehat{f_n}(\zeta)|^q\, dA(\zeta)\right)^{\frac{1}{q}} \\
&\le C(p) \sum_{n=0}^{n_0} \frac{\|f_n\|_{\alpha,p}}{((n+1)|k|-R)^{\alpha}}
\end{aligned}$$

where $C(p)$ is the constant from Lemma 5.3. Now, using Lemma 5.2, and recalling that $|k| \geq 2R$,

$$\left(\int_{|\xi|<R} |\widehat{g}(\xi,k)|^q \, dA(\xi)\right)^{\frac{1}{q}} \leq C(p)\,\Gamma_0 \sum_{n=0}^{n_0} \frac{(\kappa\, M(p))^n}{((n+1)|k|-R)^\alpha}$$
$$\leq C(p)\,(\kappa\, M(p))^{n_0}\,\Gamma_0 \sum_{n=0}^{n_0} \frac{1}{((n+1)|k|-R)^\alpha}$$
$$\leq C(p)\,(\kappa\, M(p))^{n_0}\,\frac{\Gamma_0}{|k|^\alpha} \sum_{n=0}^{n_0} \frac{1}{(n+\frac{1}{2})^\alpha}$$
$$\leq C(p)\, M(p)^{n_0}\,\frac{\Gamma_0}{|k|^\alpha} n_0 \kappa^{n_0}.$$

Now the claim follows since $\sup_n n\kappa^n \leq \frac{1}{\log\frac{1}{\kappa}}$. $\square$

The Cauchy kernel is not in L^2 but it belongs locally to $W^{\epsilon,p}$ for $1 < p < 2$, $\epsilon < \frac{2-p}{p}$. Thus we can work with a mollification of it, provided that good estimates are available. However we need to choose carefully the mollification kernel (see [49] vol 1 &V.1).

Lemma 5.4. *There exists $C_* > 0$ such that for any $N > 0$, there exists a C^∞ function ϕ_N in $\mathbb{C}$ having the following properties:*

- $0 \leq \phi_N \leq 1$, *$\phi_N = 1$ on $\mathbb{D}$ and $\phi_N = 0$ outside $2\mathbb{D}$.*
- $\int \phi_N = 1$.
- $|D^\alpha \phi_N| \leq (C_* N)^{|\alpha|}$ *for any $\alpha \in \mathbb{Z}^2_+$ with $|\alpha| \leq N$.*

Lemma 5.5. *Let $\Phi(z,w) = \Phi_z(w) = \frac{\psi_{\mathbb{D}}}{z-w}$ and $1 < p < 2$.*

(a) $\|\Phi(\cdot,z)\|_{L^p(\mathbb{D})} \leq C(p)$ *for all $z \in \mathbb{C}$.*

(b) $\Phi(\cdot,z) \in W^{\epsilon,p}$ *for $\epsilon < \frac{2-p}{p}$ uniformly in z.*

(c) *For any $N > 0$, there exists a mollification $\Phi_{\delta,N}$ such that*

$$\|\Phi(\cdot,z) - \Phi_{\delta,N}(\cdot,z)\|_{L^p(\mathbb{D})} \leq C(\epsilon,p)\,\delta^\epsilon$$

whenever $z \in \mathbb{C}$ and $\epsilon < \frac{2-p}{p}$.

(d) $\|\Phi_{\delta,N}\|_{L^2(\mathbb{C})}$ *blows up as a power of δ, i.e.*

$$\|\Phi_{\delta,N}(\cdot,z)\|_{L^2(\mathbb{C})} \leq C(p)\,\delta^{1-\frac{2}{p}}$$

(e) *For each $R > \frac{1}{\delta}$ and $m > 0$, there exists a universal constant C_* and $C = C(p)$ such that for any $m \leq N$*

$$\|\widehat{\Phi_{\delta,N}}(\cdot,z)\|_{L^2(|\xi|\geq R)} \leq C(p)(C_*N)^m\,\delta^{1-\frac{2}{p}}\,(\delta R)^{-m}$$

Proof. Claims (a) and (b) follow by the compactness of the support and Lemma 3.1. Now define

$$\widehat{\Phi_{\delta,N}(z,\cdot)}(\xi) = \widehat{\phi_N}(\delta\xi)\widehat{\Phi(z,\cdot)}(\xi).$$

Claim (c) follows from the fact that since $p < 2$, $W^{\epsilon,p} \subset B_\epsilon^{p,2}$, see (3.4). Namely, by denoting $\phi_\delta(x) = \delta^{-2}\phi_N(\delta^{-1}x)$,

$$\|\Phi_z(\cdot) - \Phi_{\delta,N}(z,\cdot)\|_{L^p} \le \int_{\mathbb{C}} \omega_p(\Phi_z)(w)\phi_\delta(w)dw$$

$$\le \|\Phi_z\|_{B_\epsilon^{p,2}} \left(\int (\phi_\delta(w))^2|w|^{2+\epsilon 2}dw\right)^{\frac{1}{2}} \le C(p)\delta^\epsilon(\int \phi^2(y)|y|^{2+\epsilon 2}dy)^{\frac{1}{2}} \le \delta^\epsilon\|\phi\|_{L^2(\mathbb{C})}$$

For claim (d), using Plancherel, Hölder, Hausdorff-Young inequalities and (a), we obtain, for $1/p - 1/q = 1/2$, that

$$\|\Phi_{\delta,N}\|_{L^2} \le \|\Phi_z\|_{L^p}\||\widehat{\phi_N}(\delta\cdot)\|_{L^q} \le C\delta^{1-\frac{2}{p}}.$$

For the last claim, write again

$$\begin{aligned}\|\widehat{\Phi_{\delta,N}}\|_{L^2(|\xi|>R_0)} &\le \|\Phi_z\|_{L^p}\||\widehat{\phi_N}(\delta\xi)\|_{L^q(|\xi|>R_0)} \\ &\le \|\Phi_z\|_{L^p}\delta^{1-2/p}\|\widehat{\phi_N}(\xi)\|_{L^q(|\xi|>\delta R_0)}\end{aligned}$$

Now

$$\begin{aligned}&\|\widehat{\phi_N}(\xi)\|_{L^q(|\xi|>\delta R_0)} \le \||\frac{(\xi_1+i\xi_2)^m}{|\xi|^m}|\widehat{\phi_N}(\xi)\|_{L^q(|\xi|>\delta R_0)} \\ &\le (\delta R_0)^{-m}\|\sum_{|\alpha|=m}\frac{m!}{\alpha!}|\widehat{D^\alpha\phi_N}(\xi)|\|_{L^q} \le (\delta R_0)^{-m}\sum_{|\alpha|=m}\frac{m!}{\alpha!}\|D^\alpha\phi_N\|_{L^{q'}} \\ &\le (\delta R_0)^{-m}\sum_{|\alpha|=m}\frac{m!}{\alpha!}(C_*N)^m \le (\delta R_0)^{-m}(2C_*N)^m\end{aligned}$$

for $m \le N$ from where (d) follows. □

Now we combine the above estimates to obtain a precise decay for the solutions to the linear equation.

Proposition 5.6. *Let $0 < \alpha < \frac{1}{2}$ and assume that $\mu \in W^{\alpha,2}(\mathbb{C})$ satisfies $|\mu| \le \frac{K-1}{K+1}\chi_{\mathbb{D}}$ and $\|\mu\|_{W^{\alpha,2}(\mathbb{C})} \le \Gamma_0$. For each $\lambda \in \partial\mathbb{D}$ and each $k \in \mathbb{C}$, let $\psi = \psi_\lambda(z,k)$ be the quasiconformal mapping satisfying*

$$\overline{\partial}\psi_\lambda(z,k) = \frac{\overline{k}}{k}\,\lambda\, e_{-k}(z)\,\mu(z)\,\partial\psi_\lambda(z,k) \tag{5.12}$$

and normalized by

$$\psi_\lambda(z,k) - z = \mathcal{O}(1/z), \qquad z \to \infty.$$

There exists positive constants $C = C(\kappa)$ and $b = b(\kappa)$ such that

$$|\psi_\lambda(z,k) - z| \le \frac{C\,\Gamma_0}{|k|^{b\,\alpha}}$$

for every $z, k \in \mathbb{C}$ and every $\lambda \in \partial\mathbb{D}$.

Proof. Since $\|\psi_\lambda(z,k) - z\|_{L^\infty} \le C(\kappa)$, it will be enough to prove the proposition for $|k| \ge C(\kappa)$. Let $b > 0$ a constant to be defined, and let $n_0 \in \mathbb{N}$. From (5.4), we can represent

$$\begin{aligned}\psi_\lambda(z,k) - z &= C \int_{\mathbb{D}} \frac{\overline{\partial}\psi_\lambda(w,k)}{w-z} dA(w) \\ &= C \int_{\mathbb{C}} \Phi(w,z)\,(g(w,k) + h(w,k)) dA(w)\end{aligned}$$

with $g = g_\lambda(z,k)$ and $h = h_\lambda(z,k)$ as in Lemma 5.1. Recall that we have control on $\widehat{g}$ for low frequencies by property 3 in Lemma 5.1, whereas h will be bounded with the help of ellipticity. It is also convenient to consider the mollification $\Phi_{\delta,N}$ of Φ given in Lemma 5.5 for N to be chosen along the proof. We will therefore estimate the following four terms separately. The first three are dealt with by the usual ellipticity theory and the Sobolev regularity of the Cauchy kernel. Hence the estimates will depend on a suitable exponent $s = s(\kappa)$. It is in the last term where the α-Sobolev control of μ will appear. These are the precise terms to be bounded:

$$\mathrm{I} = \int_{\mathbb{D}} \Phi(w,z)\, h(w,k)\, dA(w),$$

$$\mathrm{II} = \int_{\mathbb{D}} (\Phi(w,z) - \Phi_{\delta,N}(w,z)) g(w,k)\, dA(w),$$

$$\mathrm{III} = \int_{|\xi| \ge R} \widehat{\Phi_{\delta,N}}(\xi,z)\, \widehat{g}(\xi,k)\, dA(\xi),$$

$$\mathrm{IV} = \int_{|\xi| < R} \widehat{\Phi_{\delta,N}}(\xi,z)\, \widehat{g}(\xi,k)\, dA(\xi)$$

I: The tail. Fix $s = s(\kappa) > 2$ such that $\kappa\,\|T\|_s < 1$. Then we have

$$\begin{aligned}\left|\int_{\mathbb{D}} \Phi(w,z)\, h(w)\, dA(w)\right| &\le \|\Phi(\cdot,z)\|_{L^{\frac{s}{s-1}}(\mathbb{D})} \|h\|_{L^s(\mathbb{D})} \\ &\le C(\kappa,s)\ (\kappa\,\|T\|_s)^{n_0}\end{aligned}$$

since by Lemma 5.5 (a), the norm $\|\Phi(\cdot,z)\|_{L^{\frac{s}{s-1}}(\mathbb{D})}$ does not depend on z. Take now,

$$n_0 \ge \frac{\log C(\kappa,s) + b\log(|k|)}{-\log(\kappa\,\|T\|_s)} = C(\kappa)(1 + b\log(|k|)) \tag{5.13}$$

so that,

$$C(\kappa, s)\ (\kappa \, \|T\|_s)^{n_0} \leq |k|^{-b} \tag{5.14}$$

and hence

$$|\mathbf{I}| \leq |k|^{-b}.$$

II: The error of mollification. We will use Lemma 5.5 (c) with $p = s'$, $s' \in (1, 2)$. If $\frac{1}{s} + \frac{1}{s'} = 1$, $\epsilon = \frac{1}{2}(1 - \frac{2}{s})$ and $\delta = \frac{1}{2}|k|^{\frac{-b}{\epsilon}}$, then it follows from Lemma 5.5 (c) and Lemma 5.1 that

$$|\mathbf{II}| \leq \|g\|_{L^s(\mathbb{D})} \|\Phi(\cdot, z) - \Phi_{\delta,N}(\cdot, z)\|_{L^{\frac{s}{s-1}}(\mathbb{D})} \leq C(\kappa, s, \epsilon)\, \delta^\epsilon \leq C(\kappa)|k|^{-b},$$

and we still have to determine N.

III: The mollification at high frequencies. Fix $b < \frac{\epsilon}{3}$ and declare

$$R = C|k|^{2\frac{b}{\epsilon}} \tag{5.15}$$

with a constant $C > 1$ to be fixed later. Firstly $R\delta = C|k|^{\frac{b}{\epsilon}} > 1$. Thus Lemma 5.5 (e) with $m = N$ in cooperation with Plancherel's Theorem imply that

$$\begin{aligned} \mathbf{III} \leq \left| \int_{|\xi| \geq R} \widehat{\Phi_{\delta,N}}(\xi, z)\, \widehat{g}(\xi, k)\, dA(\xi) \right| &\leq \|g\|_{L^2(\mathbb{C})} \, \|\widehat{\Phi_{\delta,N}}(\cdot, z)\|_{L^2(|\xi| \geq R)} \\ &\leq C(\kappa)(C_* N)^N \, \delta^{\frac{2}{s}-1} \, (\delta R)^{-N}. \end{aligned} \tag{5.16}$$

Notice that $\delta^{\frac{2}{s}-1} = |k|^{2b}$. Thus $\mathbf{III} < |k|^{-b}$ if

$$R^N \geq C(\kappa)(C_* N)^N |k|^{3b + \frac{Nb}{\epsilon}}$$

or, with a different constant $C(\kappa)$,

$$R \ \geq C(\kappa)|k|^{\frac{b}{\epsilon}} N |k|^{\frac{3b}{N}}.$$

With the optimal $N = [3b \log |k|] + 1]$ we get the condition

$$R \geq C(\kappa) 3b |k|^{\frac{b}{\epsilon}} \log |k|.$$

Since $|k|^{b/\epsilon} > \frac{b}{\epsilon} \log |k|$, it suffices to choose $C = 6\epsilon C(\kappa)$ in the definition of R (5.15).

IV: The mollification at low frequencies. The final term is the crucial one. Take $1 < p < 2$, and $q = \frac{p}{p-1}$. Then

$$\left| \int_{|\xi|<R} \widehat{g}(\xi,k)\, \widehat{\Phi_{\delta,N}}(\xi,z)\, dA(w) \right| \le \left(\int_{|\xi|<R} |\widehat{g}(\xi,k)|^q \, dA(\xi) \right)^{\frac{1}{q}} \|\widehat{\Phi_{\delta,N}}(\cdot,z)\|_{L^p(|\xi|<R)}.$$

Now by our choice of R, (5.15) and $b < \epsilon/3$ it follows that $|k| \ge 2R$ for $|k| \ge C(\epsilon)$. Thus we can use Lemma 5.1 and obtain

$$\left(\int_{|\xi|<R} |\widehat{g}(\xi,k)|^q \, dA(\xi) \right)^{\frac{1}{q}} \le C(\kappa,p)\, M(p)^{n_0}\, \frac{\Gamma_0}{|k|^\alpha}.$$

At the same time, the other factor is bounded with the help of Lemma 5.5 (d), which is allowed since $p < 2$. More precisely, we have

$$\begin{aligned}
\|\widehat{\Phi_{\delta,N}}(\cdot,z)\|_{L^p(|\xi|<R)} &= \left(\int_{|\xi|<R} |\widehat{\Phi_{\delta,N}}(\xi,z)|^p \, dA(\xi) \right)^{\frac{1}{p}} \\
&\le C(p)\, R^{\frac{2}{p}-1} \left(\int_{|\xi|<R} |\widehat{\Phi_{\delta,N}}(\xi,z)|^2 \, dA(\xi) \right)^{\frac{1}{2}} \\
&\le C(p)\, R^{\frac{2}{p}-1}\, \|\Phi_{\delta,N}(\cdot,z)\|_{L^2(\mathbb{C})} \\
&\le C(p) \left(\frac{R}{\delta} \right)^{\frac{2}{p}-1} \le |k|^{(\frac{3b}{\epsilon})(\frac{2}{p}-1)}
\end{aligned}$$

Here we have inserted the values of R and δ from **II** and **III**. Thus,

$$\left| \int_{|\xi|<R} \widehat{g}(\xi,k)\, \widehat{\Phi_{\delta,N}}(\xi,z)\, dA(w) \right| \le C(\kappa,p)\, \frac{\Gamma_0}{|k|^\alpha}\, M(p)^{n_0}\, |k|^{(\frac{3b}{\epsilon})(\frac{2}{p}-1)}$$

Now, since $\|T\|_{L^r(\mathbb{C})} \le C(r^*-1)$, $r^* = \max\{r, r/(r-1)\}$, for any $1 < r < \infty$, it follows that the best choice for $M(p)$ is $p = 4/3$. Inserting this and the value of n_0 from (5.13) in the previous equation we arrive to the estimate

$$\mathbf{IV} \le C(\kappa)\, \Gamma_0\, \frac{1}{|k|^\alpha}\, |k|^{C(\kappa)\, b}\, |k|^{\frac{3b}{2\epsilon}} \le C(\kappa)\, \Gamma_0 |k|^{bC(\kappa)\epsilon^{-1}-\alpha}$$

To conclude the proof we need that (**IV**) is controlled by k^{-b} as well. Since $\epsilon = \epsilon(\kappa) < 1$ and we already required $b < \frac{\epsilon}{3}$, we end up getting that it suffices that

$$b < \min\left\{ \frac{\epsilon\alpha}{C}, \frac{\epsilon}{3} \right\} = \frac{\epsilon\alpha}{C} = C\alpha$$

Here $C = C(\kappa) > 1$ and we have used that $\alpha < 1/2$. The proof is concluded. $\square$

5.2 Estimates for the nonlinear equation

Now that the behavior at $k \to \infty$ of the solutions to the linearized equation (5.12) is known, it is time to study the behavior of the complex geometric optics solutions.

Theorem 5.7. *Let $0 < \alpha < \frac{1}{2}$ and $\mu \in W^{\alpha,2}(\mathbb{C})$ be real valued, compactly supported in $\frac{1}{4}\mathbb{D}$, such that $\|\mu\|_\infty \leq \frac{K-1}{K+1}$ and $\|\mu\|_{W^{\alpha,2}(\mathbb{C})} \leq \Gamma_0$. Let $\phi = \phi_\lambda(z,k)$ be the solution to*

$$\begin{cases} \overline{\partial}\phi_\lambda(z,k) = -\frac{\overline{k}}{k}\,\lambda\,\mu(z)\,e_{-k}(\phi_\lambda(z,k))\,\overline{\partial\phi_\lambda(z,k)} \\ \phi_\lambda(z,k) - z = \mathcal{O}(1/|z|) \text{ as } |z| \to \infty. \end{cases}$$

There exists constants $C = C(K) > 0$ and $b = b(K)$ such that

$$|\phi_\lambda(z,k) - z| \leq \frac{C\,\Gamma_0^{\frac{1}{K}}}{|k|^{b\alpha}}$$

for every $z \in \mathbb{C}$, $k \in \mathbb{C}$ and $\lambda \in \partial\mathbb{D}$.

Proof. Since the estimate we look for is uniform in z and λ, it suffices to show equivalent decay for the inverse mapping $\psi_\lambda = \phi_\lambda^{-1}$. But ψ_λ is the only quasiconformal mapping on the plane that satisfies both the equation

$$\overline{\partial}\psi_\lambda(z,k) = \frac{\overline{k}}{k}\,\lambda\,e_{-k}(z)\,\mu(\psi_\lambda(z,k))\,\partial\psi_\lambda(z,k) \tag{5.17}$$

(compare with (5.12)) and the condition $\psi_\lambda(z,k) - z = \mathcal{O}(1/|z|)$ as $|z| \to \infty$. Then, we just need to show that the coefficient

$$\mu \circ \psi_\lambda(\cdot, k))$$

satisfies the assumptions of Proposition 5.6. First, it is obvious that

$$\|\mu \circ \psi_\lambda(\cdot,k)\|_\infty \leq \frac{K-1}{K+1}$$

and it is also obvious that $\operatorname{supp}(\mu \circ \psi_\lambda(\cdot,k)) \subset \mathbb{D}$ (this follows from Koebe's $\frac{1}{4}$ Theorem). Then, it remains to prove that $\mu \circ \psi_\lambda \in W^{\beta,2}(\mathbb{C})$ for some $\beta \in (0,1)$. But this follows from Proposition 4.2. Indeed, since $\mu \in W^{\alpha,2}(\mathbb{C}) \cap L^\infty(\mathbb{C})$, we have $\mu \circ \psi_\lambda \in W^{\beta,2}(\mathbb{C})$ with

$$\|\mu \circ \psi_\lambda\|_{W^{\beta,2}(\mathbb{C})} \leq C\,\Gamma_0^{\frac{1}{K}}$$

for any $0 < \beta < \frac{\alpha}{K}$, where $C = C(\alpha,\beta,K)$. Note also that β behaves linearly as a function of α, with constant depending only on K. So the result follows. $\square$

Remark 5.8. In the above result, the assumption $\mathrm{supp}(\mu) \in \frac{1}{4}\mathbb{D}$ is not restrictive. Indeed, if $\mathrm{supp}(\mu) \subset D(0,R)$ for some $R>0$ then the function $\mu_R(z)=\mu(4Rz)$ defines a new Beltrami coefficient, compactly supported in $\frac{1}{4}\mathbb{D}$, does not change the ellipticity bound, and

$$\|D^\alpha \mu_R\|_{L^2(\mathbb{C})} = (4R)^{1-\alpha}\,\|D^\alpha \mu\|_{L^2(\mathbb{C})}.$$

One can then apply the previous Theorem to this coefficient μ_R and obtain estimates for the complex geometric optics solutions. But $f_{\mu_R}(z,k) = f_\mu(4Rz, \frac{k}{4R})$ and in fact if we represent these solutions as $f_\mu(z,k) = \exp(ik\phi_\mu(z,k))$, then

$$\phi_{\mu_R}(z,k) = \frac{1}{4R}\,\phi_\mu\left(4Rz, \frac{k}{4R}\right).$$

so the estimates for ϕ_{μ_R} coming from the previous theorem give similar estimates for ϕ_μ, modulo a power of R.

Now as discovered in [12] the unimodular complex parameter λ allows to push the decay estimates to complex geometric optics solutions to the γ-harmonic equation. As always, given a real Beltrami coefficient ν we denote by $f_\nu(z,k) = e^{ikz}\,M_\nu(z,k)$ the complex geometric optics solutions to $\overline{\partial} f = \nu\,\overline{\partial f}$.

Theorem 5.9. *Let μ be as in Theorem 5.7, and define, see [12],*

$$u = \mathrm{Re}(f_\mu) + i\ \mathrm{Im}(f_{-\mu}).$$

There exist a function $\epsilon = \epsilon(z,k)$ and positive constants $C = C(K)$ and $b = b(K)$ such that

(a) $u(z,k) = e^{ik(z+\epsilon(z,k))}$.

(b) $|\epsilon(z,k)| \le \dfrac{C\,\Gamma_0^{\frac{1}{K}}}{|k|^{b\,\alpha}}$ *for each* $z,k\in\mathbb{C}$.

Further, a similar estimate holds for $\tilde{u} = \mathrm{Re}(f_{-\mu}) + i\ \mathrm{Im}(f_\mu)$.

Proof. By means of Theorem 5.7 for each $\lambda\mu$ we can write the corresponding $f_{\lambda\mu} = e^{ik(z+\epsilon_{\lambda,\mu}(z,k))}$ with the right estimates. It is shown in [12] how to push these estimates to u. Namely, a calculation shows that u may be rewritten as

$$u = f_\mu\,\frac{1+\dfrac{\overline{f_\mu}-\overline{f_{-\mu}}}{f_\mu+f_{-\mu}}}{1+\dfrac{f_\mu-f_{-\mu}}{f_\mu+f_{-\mu}}}.$$

Thus, it suffices to find a function $\epsilon_1(z,k)$ such that $|\epsilon_1(z,k)| \leq \frac{C\,\Gamma_0^{\frac{1}{K}}}{|k|^{b\,\alpha}}$ and

$$\left|\frac{f_\mu - f_{-\mu}}{f_\mu + f_{-\mu}}\right| \leq 1 - e^{-|k\,\epsilon_1(z,k)|}$$

Then the theorem will holds with $\epsilon = \epsilon_\mu + \epsilon_1$. Following [12, Lemma 8.2], it suffices to see that

$$\inf_t \left|\frac{f_\mu - f_{-\mu}}{f_\mu + f_{-\mu}} + e^{it}\right| \geq e^{-|k\,\epsilon_1(z,k)|}$$

For this, define $\Phi_t(z,k) = e^{-\frac{it}{2}}\,(f_\mu\,\cos(t/2) + i\,f_{-\mu}\,\sin(t/2))$. It follows easily that for each fixed k,

$$\begin{cases} |e^{-ikz}\,\Phi_t(z,k) - 1| = \mathcal{O}(1/z) & \text{as } |z| \to \infty \\ \overline{\partial}\Phi_t = e^{-it}\mu\,\overline{\partial\Phi_t} \end{cases}$$

Thus, by uniqueness in Theorem 2.2, Φ_t is nothing but the complex geometric optics solution $\Phi_t = f_{\lambda\mu}$ with $\lambda = e^{-it}$. But then,

$$\frac{f_\mu - f_{-\mu}}{f_\mu + f_{-\mu}} + e^{it} = \frac{2\,e^{it}\,\Phi_t}{f_\mu + f_{-\mu}} = \frac{f_{\lambda\mu}}{f_\mu}\,\frac{2\,e^{it}}{1 + \frac{M_{-\mu}}{M_\mu}},$$

where M_μ is defined in (2.3). On the other hand, from Theorem 5.7 we get that

$$e^{-|k\,\epsilon_\mu(z,k)|} \leq |M_\mu(z,k)| = |e^{ik(\phi_\mu(z,k)-z)}| \leq e^{|k\,\epsilon_\mu(z,k)|},$$

where $|\epsilon_\mu(z,k)| \leq \frac{C\,\Gamma_0^{\frac{1}{K}}}{|k|^{b\,\alpha}}$ and

$$e^{-2|k\,|\epsilon_\mu|+|\epsilon_{\lambda\mu}||} \leq \frac{|f_{\lambda\mu}(z,k)|}{|f_\mu(z,k)|} \leq e^{2|k\,|\epsilon_\mu|+|\epsilon_{\lambda\mu}||}$$

uniformly for $\lambda \in \partial\mathbb{D}$. Finally, by Theorem 2.2, we also have $\operatorname{Re}\left(\frac{M_{-\mu}}{M_\mu}\right) > 0$, so that the result follows with $|\epsilon_1| \leq 2\max_\lambda |\epsilon_{\lambda\mu}|$. $\square$

6 Proof of Theorem 1.1

First we recall that by Lemma 3.7 we can reduce to the situation μ compactly supported in $\mathbb{D}$ and $0 < \alpha < \frac{1}{2}$. The stability from of μ in terms Λ_γ will follow from the stability of the derivatives of the the complex geometric optics solutions. This will be obtained as an interpolating consequence of the L^∞ stability result given at Theorem 5.9 and of the regularity of the solutions to a Beltrami equation with Sobolev coefficients (see Theorem 4.7). We denote $\rho = \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{\frac{1}{2}}(\partial\mathbb{D}) \to H^{-\frac{1}{2}}(\partial\mathbb{D})}$.

Theorem 6.1. *Let μ_1, μ_2 be such that $|\mu_j| \le \frac{K-1}{K+1}\chi_{\mathbb{D}}$ and $\|\mu_j\|_{W^{\alpha,2}(\mathbb{C})} \le \Gamma_0$. Let f_{μ_j} denote the complex geometric optics solutions to $\overline{\partial} f_{\mu_j} = \mu_j \, \overline{\partial f_{\mu_j}}$. Then, for each $q < 2 + \alpha/K$ we have*

$$\|f_{\mu_1} - f_{\mu_2}\|_{W^{1,q}(\mathbb{D})} \le C\ (1+\Gamma_0) \left|\log \frac{1}{\rho}\right|^{-b\alpha^2}$$

for come constants $C = C(|k|, \alpha, K) > 0$ and $b = b(K) > 0$. In particular, the same bound holds with the $W^{1,2}(\mathbb{D})$-norm.

Proof. The subexponential growth obtained in Theorem 5.9 entitled us to apply Theorem 2.4 **B** to the solutions u_{γ_i} and $\bar{u}_{\gamma_i}$. Since they are equivalent to the corresponding f_μ we achieve the estimate

$$\|f_{\mu_1}(\cdot,k) - f_{\mu_2}(\cdot,k)\|_{L^\infty(\mathbb{D})} \le \frac{C\,\Gamma_0^{\frac{1}{K^2}}}{|\log(\rho)|^{b\alpha}} \tag{6.1}$$

for some positive constants $C = C(k, K)$ and $b = b(K)$. On the other hand, from Theorem 4.7, for every $\theta \in (0, \frac{1}{K})$ we have

$$\begin{aligned} \|f_{\mu_1}(\cdot,k) - f_{\mu_2}(\cdot,k)\|_{\dot{W}^{1+\alpha\theta,2}(\mathbb{D})} &= \|D^{1+\theta\alpha}\left(f_{\mu_1}(\cdot,k) - f_{\mu_2}(\cdot,k)\right)\|_{L^2(\mathbb{D})} \\ &\le e^{C(K)|k|}\left(1+\Gamma_0^\theta\right). \end{aligned} \tag{6.2}$$

As in Theorem 4.7, here $C = C(K, \alpha, \theta)$. From the exponential growth of f_{μ_1} and f_{μ_2}, we get that

$$\|\left(f_{\mu_1}(\cdot,k) - f_{\mu_2}(\cdot,k)\right)\|_{W^{1+\alpha\theta,2}(\mathbb{D})} \le e^{C(K)|k|}\left(1+\Gamma_0^\theta\right)$$

where for $1 < p < \infty$, $\|\cdot\|_{W^{s,p}(\mathbb{D})}$ denotes the nonhomogeneous Sobolev norm. Now, we use the definition of Sobolev spaces as interpolation spaces (3.1). We obtain that for each $0 < \beta < 1$ and $q < \frac{2}{\beta}$

$$\begin{aligned} &\|(f_{\mu_1}(\cdot,k) - f_{\mu_2}(\cdot,k))\|_{W^{(1+\alpha\theta)\beta,2/\beta}(\mathbb{D})} \\ &\qquad \le C\,e^{\beta C(K)|k|}\left(1+\Gamma_0^\theta\right)^\beta \frac{\Gamma_0^{\frac{1-\beta}{K^2}}}{|\log(\rho)|^{b\alpha(1-\beta)}} \\ &\qquad \le C\ e^{\beta C(K)|k|}\,(1+\Gamma_0)^{\frac{1}{K^2}+\beta(\theta-\frac{1}{K^2})}\ \frac{1}{|\log(\rho)|^{b\alpha(1-\beta)}} \end{aligned}$$

where $C = C(K, \theta, \alpha)$. In particular for $\beta = \frac{1}{1+\alpha\theta}$, we get that for $q < 2(1+\alpha\theta)$,

$$\begin{aligned} &\|f_{\mu_1} - f_{\mu_2}\|_{\dot{W}^{1,q}(\mathbb{D})} \\ &\qquad \le C\ e^{C(K)|k|}\,(1+\Gamma_0)^{\frac{1}{K^2}+\frac{1}{1+\alpha\theta}(\theta-\frac{1}{K^2})}\ \frac{1}{|\log(\rho)|^{\frac{b\alpha^2\theta}{1+\alpha\theta}}}. \end{aligned}$$

Now the choice $\theta = \frac{1}{2K}$ gives us, for $q < 2 + \alpha/K$,

$$\|f_{\mu_1} - f_{\mu_2}\|_{\dot{W}^{1,q}(\mathbb{D})} \le C(|k|, \alpha, K)\,(1+\Gamma_0)\,\frac{1}{|\log(\rho)|^{\frac{b\alpha^2}{2K+\alpha}}}$$

and the claim follows for $b(K) = \frac{b}{(2K+1)}$. □

It just remains to see how the previous estimate drives us to the final stability bounds for the Beltrami coefficients (and therefore for the conductivities). To do this, the following interpolation Lemma will be needed. Note that it includes L^p spaces with $p < 1$.

Lemma 6.2 (Interpolation). *Let $0 < p_0 \le 2$ and $2 < p_1 \le \infty$. Let θ be such that*

$$\frac{1}{2} = \frac{\theta}{p_0} + \frac{1-\theta}{p_1}.$$

Then

$$\|f\|_{L^2} \le \|f\|_{L^{p_0}}^{\theta} \|f\|_{L^{p_1}}^{1-\theta}$$

for any $f \in L^{p_0} \cap L^{p_1}$.

Proof. The proof is adapted for the usual Riesz method for interpolation with a little extra care when $p_0 < 1$. We choose $r < p_0$ and define exponents q_0, q_1, q_2 such that

$$\frac{1}{r} = \frac{1}{p_0} + \frac{1}{q_0} \qquad \frac{1}{r} = \frac{1}{2} + \frac{1}{q_2} \qquad \frac{1}{r} = \frac{1}{p_1} + \frac{1}{q_1}$$

Let $g \in L^{q_2}$ any test function. For $\omega = x + iy$ in the strip $\Omega = \{z = x + iy; 0 \le y \le 1\}$, and z fixed, we define the analytic function

$$G_z(\omega) = |g(z)|^{q_2(\frac{\omega}{q_0} + \frac{1-\omega}{q_1})}\,\frac{g(z)}{|g(z)|}.$$

Notice that $|G_z(iy)|^{q_1} = |g(z)|^{q_2}$, $|G_z(1+iy)|^{q_0} = |g(z)|^{q_2}$, and $|G_z(\theta+iy)| = |g(z)|$. Now we introduce the function

$$I(\omega) = \left(\int |f(z)|^r\, |G_z(\omega)|^r dA(z)\right)^{\frac{1}{r}}.$$

Using Hölder's inequality, we can estimate its values at the boundary of the strip,

$$|I(iy)| \le \|f\|_{L^{p_1}} \left(\int |g(z)|^{q_2} dA(z)\right)^{\frac{1}{q_1}},$$

$$|I(1+iy)| \le \|f\|_{L^{p_0}} \left(\int |g(z)|^{q_2} dA(z)\right)^{\frac{1}{q_0}}.$$

Then we apply the three lines Theorem to the function $I(\omega)$ obtaining that

$$I(\theta + iy) \le |I(iy)|^{1-\theta}\, |I(1+iy)|^{\theta} \le \|g\|_{L^{q_2}} \|f\|_{L^{p_0}}^{\theta}\, \|f\|_{L^{p_1}}^{1-\theta}$$

But $I(\theta + iy) = \|fg\|_{L^r}$, so the result follows. □

We are finally led to obtain the desired stability in L^2 norm of the Beltrami coefficients.

Corollary 6.3 (Proof of Theorem 1.1). *Let μ_1, μ_2 be such that $|\mu_j| \leq \frac{K-1}{K+1}\chi_{\mathbb{D}}$ and $\|\mu_j\|_{W^{\alpha,2}(\mathbb{C})} \leq \Gamma_0$. Then*

$$\|\mu_1 - \mu_2\|_{L^2(\mathbb{D})} \leq C\ (1+\Gamma_0) \left|\log \frac{1}{\rho}\right|^{-b\alpha^2}$$

for some constants $b = b(K) > 0$ and $C = C(\alpha, K) > 0$.

Proof. Denote by f_i the complex geometric optics solution f_{μ_i} of $\overline{\partial} f = \mu_i \overline{\partial f}$ with $k = 1$. Then,

$$\begin{aligned} |\mu_1 - \mu_2| &= \left| \frac{\overline{\partial} f_1 \, \overline{\partial f_2} - \overline{\partial} f_2 \, \overline{\partial f_1}}{\overline{\partial f_1}\,\overline{\partial f_2}} \right| = \left| \frac{-\overline{\partial} f_1 \, (\overline{\partial f_1} - \overline{\partial f_2}) + (\overline{\partial} f_1 - \overline{\partial} f_2) \overline{\partial f_1}}{\overline{\partial f_1}\,\overline{\partial f_2}} \right| \\ &\leq \frac{|\overline{\partial} f_1 - \overline{\partial} f_2|}{|\partial f_2|} + |\mu_1| \, \frac{|\partial f_1 - \partial f_2|}{|\partial f_2|} \leq 2\, \frac{|Df_1 - Df_2|}{|\partial f_2|} \end{aligned}$$

because $|Df_j| = |\partial f_j| + |\overline{\partial} f_j|$. Therefore, for any $s > 0$

$$\|\mu_1 - \mu_2\|_{L^s(\mathbb{D})} \leq 2 \left\| \frac{Df_1 - Df_2}{\overline{\partial f_2}} \right\|_{L^s(\mathbb{D})}.$$

Now, let $p \in (0, \frac{2}{K-1})$ and $q < 2 + \alpha/K$. Then put $\frac{1}{s} = \frac{1}{q} + \frac{1}{p}$. An application of Hölder's inequality gives us that

$$\begin{aligned} \left\| \frac{Df_1 - Df_2}{\overline{\partial f_2}} \right\|_{L^s(\mathbb{D})} &\leq C \, \|Df_1 - Df_2\|_{L^q(\mathbb{D})} \left\| \frac{1}{\overline{\partial f_2}} \right\|_{L^p(\mathbb{D})} \\ &\leq C) \, \|f_1 - f_2\|_{W^{1,q}(\mathbb{D})} \left\| \frac{1}{\overline{\partial f_2}} \right\|_{L^p(\mathbb{D})}. \end{aligned}$$

Let us choose the complex parameter$k = 1$ and the exponents $q = 2 + \frac{\alpha}{2K}$ and $p = \frac{K(4K+\alpha-4)}{2(4K+\alpha)} < \frac{2}{K-1}$, so that $s = \frac{2}{K}$. Then using Lemma 4.8 and Theorem 6.1, we obtain the estimate

$$\|\mu_1 - \mu_2\|_{L^s(\mathbb{D})} \leq C\ (1+\Gamma_0) \left|\log \frac{1}{\rho}\right|^{-b\alpha^2}$$

where $C > 0$ depends on α, and K, and $b > 0$ depends on K. Finally, in case $s < 2$ we get the L^2 estimates combining the L^s and the trivial L^∞ bound. Namely, for $s = \frac{2}{K}$

$$\|\mu_1 - \mu_2\|_{L^2(\mathbb{D})} \leq \|\mu_1 - \mu_2\|_{L^s(\mathbb{D})}^{\frac{s}{2}} \, \|\mu_1 - \mu_2\|_{L^\infty(\mathbb{D})}^{\frac{2-s}{2}}$$

and now the stability estimate looks like

$$\|\mu_1 - \mu_2\|_{L^2(\mathbb{D})} \leq C\ (1+\Gamma_0) \left|\log \frac{1}{\rho}\right|^{-b\alpha^2}$$

where $C = C(\alpha, K), b = b(K)$ as desired. □

References

[1] D.R. Adams, *Sobolev spaces*, Academic Press, New York, 1975.

[2] D.R. Adams, L.I. Hedberg, *Function spaces and potential theory*, Springer (1996).

[3] L. V. Ahlfors, *Lectures on quasiconformal mappings.* Manuscript prepared with the assistance of Clifford J. Earle, Jr. Van Nostrand Mathematical Studies, No. 10 D. Van Nostrand Co., Inc., Toronto, Ont.-New York-London 1966

[4] G. Alessandrini, *Stable determination of conductivity by boundary measurements.* Appl. Anal. 27 (1988), no. 1-3, 153–172.

[5] G. Alessandrini, *Singular solutions of elliptic equations and the determination of conductivity by boundary measuraments*, J. Diff. Eq. 84, (1990), 252-272.

[6] G.Alessandrini, E.Cabib, *EIT and average conductivity* J. Inv. Ill-Posed Problems 15 (2007), 1-10.

[7] G. Alessandrini, S.Vesella *Lipschitz stability for the inverse conductivity problem.* Adv. in Appl. Math. 35 (2005), no. 2, 207–241.

[8] K. Astala, *Area distortion of quasiconformal mappings*, Acta Math. 173 (1994), 37–60.

[9] K. Astala, T. Iwaniec, G. Martin, *Elliptic partial differential equations and quasiconformal mappings in the plane*, Princeton Mathematical Series, 48. Princeton University Press, Princeton, NJ, 2009.

[10] K. Astala, T. Iwaniec, E. Saksman, *Beltrami operators in the plane*, Duke Math. J. 107 (2001), no.1, 27–56.

[11] K. Astala, M. Lassas, L. Päivärinta, *Calderón's inverse problem for anisotropic conductivity in the plane.* Comm. Partial Differential Equations 30 (2005), no. 1-3, 207–224.

[12] K. Astala, L. Päivärinta, *Calderón Inverse Conductivity problem in plane.* Ann. of Math. (2) 163 (2006), no. 1, 265–299.

[13] J.A. Barceló, T. Barceló, A. Ruiz, *Stability ot the inverse conductivity problem in the plane for less regular conductivities*, J. of Diff. Eq. 173, 2001, 231-270.

[14] T. Barceló, D. Faraco, A. Ruiz, *Stability of Calderón inverse conductivity problem in the plane*, J. Math. Pur. Appl., 2007; 88 (6), 522–556.

[15] R. Beals, R. Coifman, *Multidimensional inverse scattering and non linear partial differential equations.* Pseudodifferential Operators and Applications (F. Treves ed.) Proc Sympos. pure math. Vol 43, pp 45-70. AMS Providence 1985.

[16] L. Borcea Electrical impedance tomography. Inverse Problems 18 (2002), no. 6, R99–R136.

[17] R. Brown, *Global uniqueness in the impedance-imaging problem for less regular conductivities,* SIAM J. Math. Anal. 27 (1996), no. 4, 1049–1056.

[18] R. Brown, *Recovering the conductivity at the boundary from the Dirichlet to Neumann map: A pointwise result,* J. Inverse Ill-Posed Probl. 9 (2001), no. 6, 567–574.

[19] R. Brown, R. Torres, *Uniqueness in the inverse conductivity problem for conductivities with 3/2 derivatives in L^p.* J. Fourier Analysis and Appl.,9, 6 (2003) 563-574.

[20] R. Brown, G. Uhlmann, *Uniqueness in the inverse conductivity problem for nonsmooth conductivities in two dimensions*, Comm. in PDE, 22, (1997), 1009-1027.

[21] S. Buckley, *Estimates for operator norms on weighted spaces and reverse Jensen inequalities*, Trans. Amer. Math. Soc. 340 (1993), no. 1, 253–272.

[22] A.P. Calderón, *On an inverse boundary value problem*, Seminar on Numerical Analysis and its Applications to Continuum Physics, Soc. Brasileira de Matematica, Rio de Janeiro, (1980), 65-73.

[23] A. Clop, D. Faraco, J. Mateu, J. Orobitg, X. Zhong, *Beltrami equations with coefficient in the Sobolev space $W^{1,p}$*, Publ. Mat. 53 (2009), no. 1, 197–230.

[24] A. Clop, X. Tolsa, *Analytic capacity and quasiconformal mappings with $W^{1,2}$ Beltrami coefficient*, . Math. Res. Lett. 15 (2008), no. 4, 779–793.

[25] J. Duoandikoetxea, *Fourier analysis*, Graduate Studies in Mathematics 29, American Mathematical Society, Providence (2000).

[26] F. Gehring. *Characteristic properties of quasidisks.* Séminaire de Mathématiques Supérieures [Seminar on Higher Mathematics], 84. Presses de l'Université de Montréal, Montréal, Québec, 1982. 97

[27] S. Hofmann, *An off-diagonal $T1$ theorem and applications*, with an appendix *The Mary Weiss lemma* by Loukas Grafakos and the author, J. Funct. Anal. 160 (1998), no. 2, 581–622.

[28] T. Iwaniec, G.Martin, *Geometric function theory and non-linear analysis*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 2001.

[29] C.E. Kenig, A. Ponce, L. Vega *Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle*, Comm. Pure Appl. Math. 46 (1993), no. 4, 527–620.

[30] K. Knudsen, M. Lassas, J. Mueller, S. Siltanen, D-bar method for electrical impedance tomography with discontinuous conductivities. SIAM J. Appl. Math. 67 (2007), no. 3, 893–913

[31] M.Lassas, J.L.Mueller, S.Siltanen, Mapping properties of the nonlinear Fourier transform in dimension two, Comm. Partial Differential Equations 32 (2007), no. 4-6, 591–610.

[32] L. Liu, *Stability estimates for the two-dimensional inverse conductivity problem*, Ph. D. Thesis, Dep. of Mathematics, University of Rochester, New York, (1997).

[33] O. Lehto, K. I.Virtanen, *Quasiconformal mappings in the plane*, second ed. Springer-Verlag, New York, 1973. Translated from the German by K. W. Lucas, Die Grundlehren der mathematischen Wissenschaften, Band 126.

[34] N. Mandache, *Exponential instability in an inverse problem for the Schrödinger equation,* Inverse Problems 17, 2001. 1435-1444.

[35] A. Nachman, *Global uniqueness for a two dimensional inverse boundary problem,* Ann. of Math. 143 (1995) 71-96.

[36] C. Pérez, *Banach function spaces and the two-weight problem for maximal functions*, Function spaces, differential operators and nonlinear analysis (Paseky nad Jizerou, 1995), 141–158, Prometheus, Prague, 1996.

[37] C. Pérez, *The growth of the A_p constant on classical estimates*, Proceeding of the CIMPA-UNESCO-ARGENTINA School on Real Analysys and its Applications, Córdoba, Argentina (2008).

[38] L. Päivärinta, A. Panchenko G. Uhlmann, *Complex geometrical optics solutions for Lipschitz conductivities,* Rev. Mat. Iberoamericana 19 (2003), no. 1, 57–72.

[39] H. M. Reimann, *Functions of bounded mean oscillation and quasiconformal mappings*, Comment. Math. Helv. 49 (1974), 260–276.

[40] H. M. Reimann, T. Rychener, *Funktionen beschränkter mittlerer oszillation*, Lect. Notes in Math. 487 Springer-Verlag, Berlin (1975).

[41] H. Renelt, *Elliptic systems and quasiconformal mappings.* John Wiley and Sons. New York 1988.

[42] E.M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Math. Series 30. Princeton University Press. N.J. 1970.

[43] D. Swanson, *Pointwise inequalities and approximation in fractional Sobolev spaces*, Studia Math. 149 (2) (2002), 147–174.

[44] J. Sylvester, G. Uhlmann, *A global uniqueness theorem for an inverse boundary value problem*, Annals of Math., 125, (1987), 153-169.

[45] J. Sylvester, G. Uhlmann, *A uniqueness theorem for an inverse boundary value problem in electrical prospection*, Comm. Pure and Appl. Math 39,(1986) 91-112.

[46] L.Tartar *Lecture notes in homogenization and optimal design* http://www.math.cmu.edu/ nw0z/CIMECNA.dvi

[47] M.E. Taylor, *Tools for PDE: pseudodifferential operators, paradifferential operators, and Layer potentials*, Mathematical Surveys and Monographs, vol.81, American Mathematical Society 2000.

[48] H. Triebel, *Theory of function spaces*, Monographs in Mathematics, 78. Birkhauser Verlag, Basel, 1983.

[49] F. Treves, *Introduction to Pseudo Differential and Fourier Integral Operators*, University Series in Mathematics, Plenum Publ. Co. 1981.

[50] W. P. Ziemer, *Weakly Differentiable Functions: Sobolev Spaces and Functions of Bounded Variation*, Springer Verlag.

A. Clop
www.mat.uab.cat/∼albertcp
Department of Mathematics and Statistics,
P.O.Box 35 (MaD)
FI-40014 University of Jyväskylä
Finland

and
Department of Mathematics and Statistics,
P.O.Box 68
FI-00014 University of Helsinki
Finland

D. Faraco,
Instituto de Ciencias Matemáticas,
CSIC-UAM-UC3M-UCM
and
Departamento de Matemáticas, Universidad Autónoma de Madrid
Campus de Cantoblanco, s/n
28049-Madrid
Spain

A. Ruiz
Departmento de Matemáticas
Universidad Autónoma de Madrid
Campus de Cantoblanco, s/n
28049-Madrid
Spain