Beltrami equations with coefficient in the Sobolev space $W^{1,p}$

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Abstract

We study the removable singularities for solutions to the Beltrami equation $\overline{\partial} f = \mu\, \partial f$, where μ is a bounded function, $\|\mu\|_\infty \leq \frac{K-1}{K+1} < 1$, and such that $\mu \in W^{1,p}$ for some $p \leq 2$. Our results are based on an extended version of the well known Weyl's lemma, asserting that distributional solutions are actually true solutions. Our main result is that quasiconformal mappings with compactly supported Beltrami coefficient $\mu \in W^{1,p}$, $\frac{2K^2}{K^2+1} < p \leq 2$, preserve compact sets of σ-finite length and vanishing analytic capacity, even though they need not be bilipschitz.

1 Introduction

A homeomorphism between planar domains $\phi : \Omega \to \Omega'$ is called μ-quasiconformal if it is of class $W^{1,2}_{loc}(\Omega)$ and satisfies the Beltrami equation,

$$\overline{\partial}\phi(z) = \mu(z)\, \partial\phi(z) \tag{1}$$

for almost every $z \in \Omega$. Here μ is the Beltrami coefficient, that is, a measurable bounded function with $\|\mu\|_\infty < 1$. More generally, any $W^{1,2}_{loc}(\Omega)$ solution is called μ-quasiregular. When $\mu = 0$, we recover conformal mappings and analytic functions, respectively.

When $\|\mu\|_\infty \leq \frac{K-1}{K+1} < 1$ for some $K \geq 1$, then clearly μ-quasiregular mappings are K-quasiregular [22]. Quasiconformal and quasiregular mappings are a central tool in modern geometric function theory and have had a strong impact in other areas such as differential geometry, material science, calculus of variations, complex dinamics or partial differential

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equations. The later is particularly evident since as holomorphic mappings are linked to harmonic functions so are quasiregular mappings with elliptic equations. In particular, it is well known that if $f = u + iv$ solves (1), then u is a solution to

$$\operatorname{div}(\sigma \nabla u) = 0 \tag{2}$$

where for almost every z $(\sigma_{ij}(z)) \in \mathbb{S}(2)$, the space of symmetric matrices with $\det(\sigma) = 1$. Moreover, μ and σ are related by $\mu = \frac{\sigma_{11}-\sigma_{22}+2i\sigma_{12}}{\sigma_{11}+\sigma_{22}+2}$. A similar equation holds for v.

The basic theory of quasiregular mappings is established for example in the monographs [1, 20, 22, 34]. For many of the recent developments and their implications in the theory of planar elliptic equations see the coming monograph [6].

One of the basic facts in both the theory of quasiregular mappings and in elliptic systems is the self-improvement of regularity. Namely, solutions a priori in $W^{1,2}_{loc}$ belong to $W^{1,p}_{loc}$ with some $p > 2$. The theory of planar quasiregular mappings is one of the few instances where this gain can be made quantitative: solutions in $W^{1,2}_{loc}$ belong to $W^{1,p}_{loc}$ for all $p < \frac{2K}{K-1}$, and the result is sharp. This result comes directly from the celebrated Astala Theorem of 1994 on the area distortion of quasiconformal mappings [3]. Astala Theorem has a lot of consequences on quasiconformal distortion, answering questions on how quasiconformal mapping distort Hausdorff dimension, Hausdorff measures, or the size of removable singularities. Recently some of these theorems have been improved and taken to the limits (see for instance [27], or [4]).

The examples showing that these results are sharp (see for example [3, 20, 6, 5]) are typically based on Beltrami coefficients which are highly oscillating. On the other hand, if the coefficients are Hölder continous, it can be shown (by means of Schauder estimates and the equations satisfied by the inverse of quasiconformal mappings) that bijective solutions are bilipschitz, so that all the above questions are rather trivial.

In this paper we start the investigation of what happens in between. We have some control on the oscillation of Beltrami coefficients but just in the category of the Sobolev space $W^{1,p}$. We prove that in this situation one can achieve optimal regularity results, a surprising analogous to the classical Weyl lemma and removability results very close to those of analytic functions. The precise nature of the results is explained through the rest of the introduction.

We say that a compact set E is *removable for bounded μ-quasiregular mappings* if for every open set Ω, every bounded function f, μ-quasiregular on $\Omega \setminus E$, admits an extension

μ-quasiregular in all of Ω. When $\mu = 0$, this is the classical Painlevé problem. It is also a natural question to replace bounded functions by others, such as BMO (bounded mean oscillation), VMO (vanishing mean oscillation) or $\mathrm{Lip}_\alpha(\Omega)$ (Hölder continuous with exponent α). We want to give geometric characterizations of these sets. Of special interest is the case $\mu \in W^{1,2}$, which is at the borderline.

An important feature of μ-quasiregular mappings is the Stoilow factorization (see for instance [20, p. 254]). Namely, a function $f : \Omega \to \mathbb{C}$ is μ-quasiregular if and only if $f = h \circ \phi$ where $\phi : \Omega \to \phi(\Omega)$ is μ-quasiconformal and $h : \phi(\Omega) \to \mathbb{C}$ is holomorphic. Thus a compact set E is removable for bounded μ-quasiregular mappings if and only if the compact set $\phi(E)$ is removable for bounded analytic functions. Therefore, one easily sees that the way μ-quasiconformal mappings distort sets is very related to removability problems. We have the precise bounds for distortion of Hausdorff dimension from Astala [3], which apply to any K-quasiconformal mapping,

$$\dim(\phi(E)) \leq \frac{2K \, \dim(E)}{2 + (K-1) \, \dim(E)}.$$

In the particular case $\dim(E) = \frac{2}{K+1}$ we also have absolute continuity of measures [4], that is,

$$\mathcal{H}^{\frac{2}{K+1}}(E) = 0 \qquad \Longrightarrow \qquad \mathcal{H}^1(\phi(E)) = 0,$$

where $\mathcal{H}^s$ denotes the s-dimensional Hausdorff measure (e.g. [23]). If the coefficient is more regular one improves these estimates as well. For instance, if the Beltrami coefficient μ lies in VMO, then every μ-quasiconformal mapping ϕ has distributional derivatives in L^p_{loc} for every $p \in (1, \infty)$ (see for instance [7] or [17]). Thus, $\phi \in \mathrm{Lip}_\alpha$ for every $\alpha \in (0, 1)$, and as a consequence,

$$\dim(\phi(E)) \leq \dim(E).$$

Moreover, actually for such μ one has $\dim(\phi(E)) = \dim(E)$.

However if we further know that $\mu \in W^{1,2}$ we obtain more precise information. An important reason is the following: We first recall that for $\mu = 0$ we have the well known Weyl's Lemma, which asserts that if T is any (Schwartz) distribution such that

$$\langle \overline{\partial} T, \varphi \rangle = 0$$

for each test function $\varphi \in \mathcal{D}$ (by $\mathcal{D}$ we mean the algebra of compactly supported $\mathcal{C}^\infty$ functions), then T agrees with a holomorphic function. In other words, *distributional* solutions to Cauchy-Riemann equation are actually *strong* solutions. When trying to extend this kind of result to the Beltrami equation, one first must define the distribution $(\overline{\partial} - \mu \, \partial)T = \overline{\partial} T - \mu \, \partial T$. It need

not to make sense, because bounded functions in general do not multiply distributions nicely. However, if the multiplier is asked to exhibit some regularity, and the distribution T is an integrable function, then something may be done. Namely, one can write

$$\langle(\overline{\partial} - \mu\,\partial)f, \varphi\rangle = -\langle f, \overline{\partial}\varphi\rangle + \langle f, \partial\mu\,\varphi\rangle + \langle f, \mu\,\partial\varphi\rangle$$

whenever each term makes sense. For instance, this is the case if $\mu \in W^{1,p}_{loc}$ and $f \in L^q_{loc}$, $\frac{1}{p} + \frac{1}{q} = 1$. Hence we can call $\overline{\partial} f - \mu\,\partial f$ the *Beltrami distributional derivative* of f, and we can say that a function $f \in L^q_{loc}$ is *distributionally μ-quasiregular* precisely when $(\overline{\partial} - \mu\,\partial)f = 0$ as a distribution. Of course, a priori such functions f could be not quasiregular, since it is not clear if the distributional equation actually implies $f \in W^{1,2}_{loc}$. Thus it is natural to ask when this happens.

Theorem 1. *Let $f \in L^p_{loc}(\Omega)$ for some $p > 2$, and let $\mu \in W^{1,2}$ be a compactly supported Beltrami coefficient. Assume that*

$$\langle(\overline{\partial} - \mu\,\partial)f, \varphi\rangle = 0$$

for any $\varphi \in \mathcal{D}(\Omega)$. Then, f is μ-quasiregular. In particular, $f \in W^{1,2}_{loc}(\Omega)$.

We must point out that this selfimprovement of regularity is even stronger, because of the factorization theorem for μ-quasiregular mappings, as well as the regularity of homeomorphic solutions when the Beltrami coefficient is *nice*. Namely, when $\mu \in W^{1,2}$ is compactly supported, it can be shown that any μ-quasiconformal mapping is actually in $W^{2,q}_{loc}$ whenever $q < 2$. Hence, every $L^{2+\varepsilon}_{loc}$ distributional solution to the corresponding Beltrami equation is actually a $W^{2,q}_{loc}$ solution, for every $q < 2$. Further, we can show that the above Weyl's Lemma holds, as well, when $\mu \in W^{1,p}$ for $p \in (\frac{2K}{K+1}, 2)$ (see Section 6).

One may use this selfimprovement to give removability results and study distortion problems for μ-quasiconformal mappings. The conclusions we obtain encourage us to believe that Beltrami equation with $W^{1,2}$ Beltrami coefficient is not so far from the classical planar Cauchy-Riemann equation. For instance, we shall show that for any $0 < \alpha < 1$, any set E with $\mathcal{H}^{1+\alpha}(E) = 0$ is removable for Lip_α μ-quasiregular mappings, precisely as it is when $\mu = 0$ [12]. Nevertheless, this Lip_α removability problem does not imply in general any result on μ-quasiconformal distortion of Hausdorff measures, since there are examples of $\mu \in W^{1,2}$ for which the space Lip_α is not μ-quasiconformally invariant. Therefore, to get results in terms of distortion we study the removability problem with the BMO norm. Then we get that E is removable for BMO μ-quasiregular mappings, if and only if $\mathcal{H}^1(E) = 0$. More precisely, this is what happens for $\mu = 0$ [21]. Moreover, E is removable for VMO μ-quasiregular mappings

if and only if $\mathcal{H}^1(E)$ is σ-finite, again as in the analytic case [35]. In distortion terms, this reads as $\mathcal{H}^1(E) = 0$ if and only if $\mathcal{H}^1(\phi(E)) = 0$, and $\mathcal{H}^1(E)$ is σ-finite if and only if $\mathcal{H}^1(\phi(E))$ is.

μ-quasiconformal distortion of analytic capacity is somewhat deeper, since the rectifiable structure of sets plays an important role there. We show in Lemma 14 that if $\mu \in W^{1,2}$ is compactly supported, then ϕ maps rectifiable sets to rectifiable sets. As a consequence, purely unrectifiable sets are mapped to purely unrectifiable sets. Therefore, we get from [11] our following main result.

Theorem 2. *Let $\mu \in W^{1,2}$ be a compactly supported Beltrami coefficient, and let ϕ be μ-quasiconformal. If E has σ-finite length, then*

$$\gamma(E) = 0 \qquad \Longleftrightarrow \qquad \gamma(\phi(E)) = 0,$$

where $\gamma(E)$ denotes the analytic capacity of E.

Recall that one defines $\gamma(E)$ as

$$\gamma(E) = \sup\left\{|f'(\infty)|\right\}$$

where the supremum runs over functions f which are holomorphic and bounded in $\mathbb{C} \setminus E$, with $\|f\|_\infty \leq 1$ (e.g. [23, p.265]). Removable sets for bounded analytic functions are precisely those having zero analytic capacity. Let's mention that in [32] Tolsa proved that an homeomorphism ϕ is a bilipschitz map if and only if it preserves the analytic capacity of sets, that is, there exists $C > 0$ such that

$$\frac{1}{C}\,\gamma(E) \leq \gamma(\phi(E)) \leq C\,\gamma(E)$$

for all sets E. On the other hand, the radial stretching $g(z) = z|z|^{\frac{1}{K}-1}$ is not bilipschitz but clearly it preserves sets of zero analytic capacity. Theorem 2 asserts that μ-quasiconformal mappings , $\mu \in W^{1,2}$, also preserve sets of zero analytic capacity having also σ-finite length.

As a natural question, one may ask wether these distortion results apply also for compactly supported Beltrami coefficients $\mu \in W^{1,p}$ when $p \in (\frac{2K}{K+1}, 2)$. In this case, we study the same removability problems and we obtain analogous results. For instance, if $\mathcal{H}^{1+\alpha}(E) = 0$, then E is removable for Lip_α μ-quasiregular mappings, as well as for the analytic case [26]. Again, this does not translate to the distortion problem for Hausdorff measures, since Lip_α is not quasiconformally invariant. However, this has some interesting consequences in terms of distortion of Hausdorff dimension. Namely, it follows that

$$\dim(E) \leq 1 \qquad \Longrightarrow \qquad \dim(\phi(E)) \leq 1 \tag{3}$$

Moreover, when letting $\alpha = 0$ we get the corresponding BMO and VMO removability problems. Due to our Weyl type Lemma, we show that even when $\mu \in W^{1,p}$, $\frac{2K}{K+1} < p < 2$, we actually have absolute continuity of measures, i.e.

$$\mathcal{H}^1(E) = 0 \qquad \Longrightarrow \qquad \mathcal{H}^1(\phi(E)) = 0 \tag{4}$$

for any μ-quasiconformal mapping ϕ. This improves the absolute continuity results in [4].

We do not know if implication (4) is an equivalence. Indeed, if $\frac{2K}{K+1} < p < 2$ and $\mu \in W^{1,p}$ then the Beltrami coefficient ν of inverse mapping ϕ^{-1} need not belong to the same Sobolev space $W^{1,p}$ (this is true for $p = 2$). However, if p ranges the smaller interval $(\frac{2K^2}{K^2+1}, 2)$ then a calculation shows that $\nu \in W^{1,r}$ for some $r > \frac{2K}{K+1}$. As a consequence, we obtain the following result.

Theorem 3. *Let $\frac{2K^2}{K^2+1} < p < 2$. Let $\mu \in W^{1,p}$ be a compactly supported Beltrami coefficient, and let ϕ be μ-quasiconformal. Then,*

$$\gamma(E) = 0 \qquad \Longleftrightarrow \qquad \gamma(\phi(E)) = 0,$$

for any compact set E with σ-finite length.

This paper is structured as follows. In Section 2 we study the regularity of μ-quasiregular mappings. In Section 3 we prove Theorem 1. In Section 4, we study the BMO and VMO removability problems for $\mu \in W^{1,2}$, and deduce distortion theorems for $\mathcal{H}^1$. In Section 5 we study μ-quasiconformal distortion of rectifiable sets, and prove Theorem 2. In Section 6 we study Beltrami equations with coefficient in $W^{1,p}$, $p \in (\frac{2K}{K+1}, 2)$, and prove Theorem 3.

Concerning the notation, we write $A \lesssim B$ if there exists a constant C such that $A \leq C\,B$. Similarly, $A \simeq B$ means $B \lesssim A \lesssim B$. By $D(a, r)$ we denote the disk of center a and radius r, and λD denotes the disc concentric with D whose radius is λ times that of D. We use ∂ and $\overline{\partial}$ to denote the usual complex derivatives,

$$\partial = \frac{1}{2}\,(\partial_x - i\,\partial_y) \qquad\qquad \overline{\partial} = \frac{1}{2}(\partial_x + i\,\partial_y).$$

By Cf we denote the solid Cauchy transform of f,

$$Cf(z) = \frac{-1}{\pi} \int \frac{f(w)}{w - z}\, dA(w)$$

while by Bf we mean the Beurling transform of the function f,

$$Bf(z) = \frac{-1}{\pi} \lim_{\varepsilon \to 0} \int_{|w-z| \geq \varepsilon} \frac{f(w)}{(w - z)^2}\, dA(w).$$

For compactly supported $\mathcal{C}^\infty$ functions f, we have $C(\overline{\partial} f) = f$ and $B(\overline{\partial} f) = \partial f$. The main properties of B and V may be found at [1, 34].

2 Regularity of μ-quasiconformal mappings

It is well known (see for instance [8]) that any K-quasiregular mapping belong to better Sobolev spaces than the usual $W^{1,2}_{loc}$ appearing in its definition. More precisely, $Df \in L^{\frac{2K}{K-1},\infty}$ [3], and this is sharp. However, if we look not only at K but also at the regularity of the Beltrami coefficient, something better may be said. This situation is given when the Beltrami coefficients are in Lip_α. In this case, every homeomorphic solution (and hence the corresponding μ-quasiregular mappings) have first order derivatives also in Lip_α. In particular, ϕ is locally bilipschitz. The limiting situation in terms of continuity is obtained when assuming $\mu \in VMO$. In this case, as mentioned before, every μ-quasiconformal mapping has derivatives in $L^p_{loc}(\mathbb{C})$ for every $p \in (1,\infty)$. Let us discuss the situation in terms of the Sobolev regularity of μ. If $\mu \in W^{1,p}$, $p > 2$, then $D\phi \in \text{Lip}_{1-\frac{2}{p}}$, as shows [34]. Actually, it comes from [1] that $\phi \in W^{2,p}_{loc}$. In the next lemma we study what happens for an arbitrary $1 < p < \infty$.

Proposition 4. *Let $\mu \in W^{1,p}$ be a compactly supported Beltrami coefficient, and assume that $\|\mu\|_\infty \leq \frac{K-1}{K+1}$. Let ϕ be μ-quasiconformal.*

(a) If $p > 2$, then $\phi \in W^{2,p}_{loc}(\mathbb{C})$.

(b) If $p = 2$, then $\phi \in W^{2,q}_{loc}(\mathbb{C})$ for every $q < 2$.

(c) If $\frac{2K}{K+1} < p < 2$, then $\phi \in W^{2,q}_{loc}(\mathbb{C})$ for every $q < q_0$, where $\frac{1}{q_0} = \frac{1}{p} + \frac{K-1}{2K}$.

Proof. There is no restriction if we suppose that μ has compact support included in $\mathbb{D}$. Assume first that $p > 2$. In this case, it comes from the Sobolev embedding Theorem that μ has a Hölder continuous representative. Arguing as in Ahlfors [1, Lemma 5.3], we can find a continuous function g such that $\partial\phi = e^g$. Indeed, this function g is a solution to $\overline{\partial} g = \mu\, \partial g + \partial\mu$, which may be constructed as

$$g = \frac{1}{z} * (I - \mu B)^{-1}(\partial\mu)$$

where B denotes the Beurling transform. We have that $g \in W^{1,p}_{loc}(\mathbb{C})$. To see this, we first recall that for continuous μ (in fact, for $\mu \in VMO$, see [19]) the operator $I - \mu B$ is continuously invertible in $L^r(\mathbb{C})$ for all $1 < r < \infty$. Thus, g is the Cauchy transform of a compactly supported $L^p(\mathbb{C})$ function. Hence, g is continuous and bounded and therefore there exists a constant $C > 0$ such that

$$\frac{1}{C} \leq |e^{g(z)}| \leq C \tag{5}$$

for a.e. $z \in \mathbb{C}$. Finally, since $\partial\partial\phi = \partial(e^g) = e^g\, \partial g$ (similarly for the other derivatives) we get that $\partial\phi, \overline{\partial}\phi \in W^{1,p}_{loc}(\mathbb{C})$.

Let now $p \le 2$. Let $\psi \in \mathcal{C}^\infty(\mathbb{C})$, $0 \le \psi \le 1$, $\int \psi = 1$, supported on $\mathbb{D}$, and let $\psi_n(z) = n^2\,\psi(nz)$. Define

$$\mu_n(z) = \mu * \psi_n(z) = \int n^2\,\psi(nw)\,\mu(z-w)\,dA(w)$$

Then μ_n is of class $\mathcal{C}^\infty$, has compact support inside of $2\mathbb{D}$, $\|\mu_n\|_\infty \le \|\mu\|_\infty$ and $\mu_n \to \mu$ in $W^{1,p}(\mathbb{C})$, *that is,*

$$\lim_{n\to\infty} \|\mu_n - \mu\|_{W^{1,p}(\mathbb{C})} = 0.$$

As in [1], the corresponding principal solutions ϕ_n and ϕ can be written as $\phi(z) = z + Ch(z)$ and $\phi_n(z) = z + Ch_n(z)$, where h, h_n are respectively defined by $h = \mu Bh + \mu$ and $h_n = \mu_n Bh_n + \mu_n$. We then get $\phi_n \to \phi$ as $n \to \infty$ with convergence in $W^{1,r}$ for every $r < \frac{2K}{K-1}$. Now observe that ϕ_n is a $\mathcal{C}^\infty$ diffeomorphism and conformal outside of $2\mathbb{D}$. This allows us to take derivatives in the equation $\overline{\partial}\phi_n = \mu_n\,\partial\phi_n$. We get

$$\overline{\partial}\partial\phi_n - \mu_n\,\partial\partial\phi_n = \partial\mu_n\,\partial\phi_n.$$

This may be written as

$$(\overline{\partial} - \mu_n\,\partial)(\log\partial\phi_n) = \partial\mu_n$$

or equivalently

$$(I - \mu_n B)(\overline{\partial}\log(\partial\phi_n)) = \partial\mu_n \tag{6}$$

so that

$$\overline{\partial}\partial\phi_n = \partial\phi_n\,(I - \mu_n B)^{-1}(\partial\mu_n) \tag{7}$$

Fix $\frac{2K}{K+1} < p < 2$. In this case [7], the norm of $\|(I-\mu_n B)^{-1}\|_{L^p(\mathbb{C})\to L^p(\mathbb{C})}$ depends only on K and p. Now recall that $\partial\mu_n \to \partial\mu$ in $L^p(\mathbb{C})$ and $\partial\phi_n \to \partial\phi$ in $L^r(\mathbb{C})$ for $r < \frac{2K}{K-1}$. Then if $q < q_0$, $\frac{1}{q_0} = \frac{1}{p} + \frac{K-1}{2K}$, the right hand side in (7) converges to $(I-\mu B)^{-1}(\partial\mu)\,\partial\phi$ in $L^q(\mathbb{C})$. Hence, the sequence $(\overline{\partial}\partial\phi_n)_n$ is uniformly bounded in $L^q(\mathbb{C})$. Taking a subsequence, we get that ϕ_n converges in $W^{2,q}_{loc}(\mathbb{C})$, and obviously the limit is ϕ, so that $\phi \in W^{2,q}_{loc}(\mathbb{C})$.

Assume finally that $p = 2$. Repeating the argument above, we get $\phi \in W^{2,q}$ for every $q < \frac{2K}{2K-1} < 2$, which is weaker than the desired result. To improve it, we first show that $\phi_n \to \phi$ in $W^{1,r}_{loc}(\mathbb{C})$ for every $r \in (1,\infty)$. To do that, notice that both $I - \mu_n B$ and $I - \mu B$ are invertible operators in $L^r(\mathbb{C})$ for all $r \in (1,\infty)$, since both $\mu_n, \mu \in VMO$ (see for instance [19]). Further, from the Sobolev Embedding Theorem, $\mu_n \to \mu$ in $L^r(\mathbb{C})$. Thus,

$$\lim_{n\to\infty} \|(I-\mu_n B) - (I-\mu B)\|_{L^r\to L^r} = \lim_{n\to\infty} \|(\mu_n - \mu)B\|_{L^r\to L^r} = 0$$

for any $r \in (1,\infty)$. Now recall that the set of bounded operators $L^r(\mathbb{C}) \to L^r(\mathbb{C})$ defines a complex Banach algebra, in which the invertible operators are an open set and, moreover, the

inversion is continuous. As a consequence,

$$\lim_{n\to\infty} \|(I-\mu_n B)^{-1}\|_{L^r\to L^r} = \|(I-\mu B)^{-1}\|_{L^r\to L^r}$$

for each $r \in (1,\infty)$. This implies that $h_n \to h$ in $L^r(\mathbb{C})$ so that $\phi_n \to \phi$ in $W^{1,r}_{loc}(\mathbb{C})$. Going back to (7), the right hand side converges to $\partial\phi\,(I-\mu B)^{-1}(\partial\mu)$ in the norm of $L^q(\mathbb{C})$, provided that $q<2$, and now the result follows. □

If $p>2$, $D^2\phi$ cannot have better integrability than $D\mu$, because $J(\cdot,\phi) = (1-|\mu|^2)\,e^{2g}$ is by (5) a continuous function bounded from above and from below. Further, the radial stretching $f(z) = z|z|^{\frac{1}{K}-1}$ has Beltrami coefficient in $W^{1,p}$ for every $p<2$ and, however, $D^2 f$ lives in no better space than $L^{\frac{2K}{2K-1},\infty}$. If $p=2$, the sharpness of the above proposition may be stated as a consequence of the following example [33, p.142].

Example. The function

$$\phi(z) = z\,(1-\log|z|) \tag{8}$$

is μ-quasiconformal in a neighbourhood of the origin, with Beltrami coefficient

$$\mu(z) = \frac{z}{\overline{z}}\,\frac{1}{2\log|z|-1}$$

In particular, we have $\mu \in W^{1,2}$ in a neighbourhood of the origin. Thus, we have $\phi \in W^{2,q}_{loc}$ whenever $q<2$. However,

$$|D^2\phi(z)| \simeq \frac{1}{|z|}$$

so that $\phi \notin W^{2,2}_{loc}$.

In order to study distortion results, we need information about the integrability of the inverse of a μ-quasiconformal mapping. This can be done by determining the Sobolev regularity of the corresponding Beltrami coefficient to ϕ^{-1}.

Proposition 5. *Let $\mu \in W^{1,2}$ be a compactly supported Beltrami coefficient, and let ϕ be μ-quasiconformal. Then, ϕ^{-1} has Beltrami coefficient*

$$\nu(z) = -\mu(\phi^{-1}(z))\,\frac{\partial\phi}{\overline{\partial\phi}}(\phi^{-1}(z))$$

In particular, $\nu \in W^{1,2}$.

Proof. An easy computation shows that

$$\nu(z) = \frac{\overline{\partial}\phi^{-1}(z)}{\partial\phi^{-1}(z)} = -\left(\mu\,\frac{\partial\phi}{\overline{\partial\phi}}\right)(\phi^{-1}(z))$$

For compactly supported $\mu \in W^{1,2}$, it follows from equation (6) that the normalized solution ϕ is such that $\log \partial\phi \in W^{1,2}$. Hence,

$$\partial\phi = e^{\lambda}$$

for a function $\lambda \in W^{1,2}(\mathbb{C})$ (in fact, $\lambda = \log \partial\phi$). Thus, in terms of λ, we get

$$\nu \circ \phi = -\mu \, e^{2i \, \mathrm{Im}(\lambda)}$$

where $\mathrm{Im}(\lambda)$ is the imaginary part of the function λ. Hence,

$$D(\nu \circ \phi) = -D\mu \, e^{2i \, \mathrm{Im}(\lambda)} - \mu \, 2i \, e^{2i \, \mathrm{Im}(\lambda)} \, D(\mathrm{Im}(\lambda))$$

so that

$$|D(\nu \circ \phi)| \leq |D\mu| + 2 \, |\mu| \, |D(\mathrm{Im}(\lambda))|$$

In particular, $\nu \circ \phi$ has derivatives in $L^2(\mathbb{C})$. Now, from the identity

$$\int |D\nu(z)|^2 dA(z) = \int |D\nu(\phi(w))|^2 J(w, \phi) \, dA(w) \leq \int |D(\nu \circ \phi)(w)|^2 dA(w)$$

the result follows. □

Remark 1. *As shown above, if μ belongs to $W^{1,p}$ for some $p \geq 2$, then the same can be said for ν. If μ is only in $W^{1,p}$, $\frac{2K}{K+1} < p < 2$, the situation is different. More precisely, an argument as above shows that $\nu \in W^{1,r}$ for every r such that*

$$r < \frac{2p}{2K - (K-1)p}$$

In particular, for $p > \frac{2K}{K+1}$ we always have $\nu \in W^{1,1}$, but ν does not fall, in general, in the same Sobolev space $W^{1,p}$ than μ. However, for $p > \frac{2K^2}{K^2+1}$, we always have $\nu \in W^{1,r}$ for some $r > \frac{2K}{K+1}$.

The above regularity results can be applied to study distortion properties of μ-quasiconformal mappings. For instance, if μ is a compactly supported $W^{1,2}$ Beltrami coefficient, then both ϕ and ϕ^{-1} are $W^{2,q}_{loc}$ functions, for every $q < 2$. Therefore, $\phi, \phi^{-1} \in \mathrm{Lip}_{\alpha}$ for every $\alpha \in (0, 1)$ (notice that this is true under the more general assumption $\mu \in VMO$). Thus,

$$\dim(\phi(E)) = \dim(E) \tag{9}$$

On the other hand, we may ask if this identity can be translated to Hausdorff measures. As a matter of fact, observe that the mapping in Example 8 is not Lipschitz continuous. Thus, is not clear how μ-quasiconformal mappings with $W^{1,2}$ Beltrami coefficient distort Hausdorff measures or other set functions, such as analytic capacity, even preserving Hausdorff dimension. Further, we do not know if for Beltrami coefficients $\mu \in W^{1,p}$, $p < 2$, the corresponding μ-quasiconformal mappings satisfy equation (9) or not. Questions related with this will be treated in Sections 4, 5 and 6.

3 Distributional Beltrami equation with $\mu \in W^{1,2}$

A typical feature in the theory of quasiconformal mappings is the selfimprovement of regularity. Namely, it is known that weakly K-quasiregular mappings in $W^{1,\frac{2K}{K+1}}_{loc}$ are actually K-quasiregular [3, 27]. This improvement is stronger for $K = 1$, since in this case we do not need any Sobolev regularity as a starting point. The classical Weyl's Lemma establishes that if f is a distribution such that

$$\langle \overline{\partial} f, \varphi \rangle = 0$$

for every testing function $\varphi \in \mathcal{D}$, then f agrees at almost every point with a holomorphic function. Our following goal is to deduce an extension to this result for the Beltrami operator, provided that $\mu \in W^{1,2}$ is compactly supported.

Let $f \in L^p_{loc}$ for some $p \in (2, \infty)$. Such a function f admits distributional first order derivatives. For instance, ∂f is defined by

$$\langle \partial f, \varphi \rangle = -\int f \, \partial \varphi \, dA$$

for any compactly supported $\varphi \in W^{1,\frac{p}{p-1}}$. In fact, if $\varphi_n \in W^{1,\frac{p}{p-1}}$ have compact support inside a fixed disk D, then $\langle \partial f, \varphi_n \rangle \to 0$ if $\varphi_n \to 0$ in $W^{1,\frac{p}{p-1}}$, in other words,

$$\|\varphi_n\|_{W^{1,\frac{p}{p-1}}} \to 0 \qquad \Rightarrow \qquad \langle \partial f, \varphi_n \rangle \to 0.$$

Analogously happens with $\overline{\partial} f$. In this situation, it makes sense to multiply the distribution ∂f by the Beltrami coefficient μ,

$$\langle \mu \, \partial f, \, \varphi \rangle = \langle \partial f, \, \mu \, \varphi \rangle.$$

Indeed, if $\varphi \in W^{1,\frac{p}{p-1}}$, then also $\mu\varphi \in W^{1,\frac{p}{p-1}}$ with control on the norms. Hence, we can define a linear functional

$$\langle \overline{\partial} f - \mu \, \partial f, \varphi \rangle = -\langle f, (\overline{\partial} - \partial \mu)\varphi \rangle = -\langle f, \overline{\partial}\varphi \rangle + \langle f, \partial(\mu \, \varphi) \rangle$$

for each compactly supported $\varphi \in \mathcal{C}^\infty$. Clearly, $\overline{\partial} f - \mu \, \partial f$ defines a distribution, which will be called the *Beltrami distributional derivative* of f.

We say that a function $f \in L^p_{loc}$ is *distributionally μ-quasiregular* if its Beltrami distributional derivative vanishes, that is,

$$\langle \overline{\partial} f - \mu \, \partial f, \varphi \rangle = 0$$

for every testing function $\varphi \in \mathcal{D}$. It turns out that one may take then a bigger class of testing functions φ.

Lemma 6. *Let $p > 2$, $q = \frac{p}{p-1}$, and let $\mu \in W^{1,2}$ be a compactly supported Beltrami coefficient. Assume that $f \in L^p_{loc}$ satisfies*

$$\langle \overline{\partial} f - \mu\, \partial f, \varphi \rangle = 0$$

for every $\varphi \in \mathcal{D}$. Then, it also holds for compactly supported $\varphi \in W^{1,q}_0$.

Proof. When $\mu \in W^{1,2}$ is compactly supported and $f \in L^p_{loc}$ for some $p > 2$, the Beltrami distributional derivative $\overline{\partial} f - \mu\, \partial f$ acts continuously on compactly supported $W^{1,q}$ functions, since

$$\begin{aligned} |\langle \overline{\partial} f - \mu\, \partial f, \varphi \rangle| &\leq |\langle f, \overline{\partial} \varphi \rangle| + |\langle f, \partial(\mu\, \varphi) \rangle| \\ &\leq \int |f|\, |\overline{\partial} \varphi| + \int |f|\, |\partial \mu|\, |\varphi| + \int |f|\, |\mu|\, |\partial \varphi| \\ &\leq \|f\|_p\, \|\overline{\partial} \varphi\|_q + \|f\|_p\, \|\partial \mu\|_2\, \|\varphi\|_{\frac{2q}{2-q}} + \|f\|_p\, \|\mu\|_\infty\, \|\partial \varphi\|_q \end{aligned}$$

Hence, if $\overline{\partial} f - \mu\, \partial f$ vanishes when acting on $\mathcal{D}$, it will also vanish on $W^{1,q}$. □

Theorem 7. *Let $f \in L^p_{loc}$ for some $p > 2$. Let $\mu \in W^{1,2}$ be a compactly supported Beltrami coefficient. Assume that*

$$\langle \overline{\partial} f - \mu\, \partial f, \psi \rangle = 0$$

for each $\psi \in \mathcal{D}$. Then, f is μ-quasiregular.

Proof. Let ϕ be any μ-quasiconformal mapping, and define $g = f \circ \phi^{-1}$. Since $\phi \in W^{2,q}_{loc}$ for any $q < 2$, then $J(\cdot, \phi) \in L^q_{loc}$ for every $q \in (1, \infty)$ so that $g \in L^{p-\varepsilon}_{loc}$ for every $\varepsilon > 0$. Thus, we can define $\overline{\partial} g$ as a distribution. We have for each $\varphi \in \mathcal{D}$

$$\begin{aligned} \langle \overline{\partial} g, \varphi \rangle &= -\langle g, \overline{\partial} \varphi \rangle \\ &= -\int g(w)\, \overline{\partial} \varphi(w)\, dA(w) \\ &= -\int f(z)\, \overline{\partial} \varphi(\phi(z))\, J(z, \phi)\, dA(z) \\ &= -\int f(z) \left(\partial \phi(z)\, \overline{\partial} (\varphi \circ \phi)(z) - \overline{\partial} \phi(z)\, \partial (\varphi \circ \phi)(z) \right) dA(z). \end{aligned}$$

On one hand,

$$-\int f(z)\, \overline{\partial} \phi(z)\, \partial (\varphi \circ \phi)(z) dA(z) = \langle \partial f, \overline{\partial} \phi \cdot\ \varphi \circ \phi \rangle + \int f(z)\, \partial \overline{\partial} \phi(z)\, \varphi \circ \phi(z)\, dA(z).$$

and here everything makes sense. On the other hand,

$$-\int f(z)\, \partial \phi(z)\, \overline{\partial} (\varphi \circ \phi)(z)\, dA(z) = \langle \overline{\partial} f, \partial \phi \cdot \varphi \circ \phi \rangle + \int f(z)\, \overline{\partial} \partial \phi(z)\, \varphi \circ \phi(z)\, dA(z).$$

Therefore,

$$\langle \overline{\partial} g, \varphi \rangle = \langle \overline{\partial} f, \partial \phi \cdot \varphi \circ \phi \rangle - \langle \partial f, \overline{\partial} \phi \cdot \varphi \circ \phi \rangle.$$

But if $\varphi \in \mathcal{D}$ then the function $\psi = \partial\phi \cdot \varphi \circ \phi$ belongs to $W_0^{1,q}$ for every $q < 2$ and, in particular, for $q = \frac{p}{p-1}$, provided that $p > 2$. Hence, also $\mu\psi \in W^{1,q}$. Thus,

$$\begin{aligned}\langle \overline{\partial} g, \varphi \rangle &= \langle \overline{\partial} f, \partial\phi \ \varphi \circ \phi \rangle - \langle \partial f, \mu\, \partial\phi \ \varphi \circ \phi \rangle \\ &= \langle \overline{\partial} f, \partial\phi \ \varphi \circ \phi \rangle - \langle \mu\, \partial f, \partial\phi \ \varphi \circ \phi \rangle \\ &= \langle \overline{\partial} f - \mu\, \partial f, \partial\phi \ \varphi \circ \phi \rangle.\end{aligned}$$

By Lemma 6, the last term vanishes. Hence, g is holomorphic and therefore f is μ-quasiregular. □

Remark 2. *The key tool in the above proof is that, thanks to Stoilow factorization Theorem, we can reduce the situation to the classical Weyl lemma. Solutions to the generalized Beltrami equation $\overline{\partial} f = \mu\, \partial f + \nu\, \overline{\partial f}$ with $\| |\mu| + |\nu| \|_\infty < 1$ are not conjugate to the semigroup of holomorphic functions and hence a different strategy should be used to obtain this type of result for this more general equation.*

From the above theorem, if f is an L^p_{loc} function for some $p > 2$ whose Beltrami distributional derivative vanishes, then f may be written as $f = h \circ \phi$ with holomorphic h and μ-quasiregular ϕ. As a consequence, we get $f \in W^{2,q}_{loc}$ for every $q < 2$, so we actually gain not 1 but 2 degrees of regularity.

4 μ-quasiconformal distortion of Hausdorff measures

Let E be a compact set, and let μ be any compactly supported $W^{1,2}$ Beltrami coefficient. If ϕ is μ-quasiconformal, then it follows already from the fact that $\mu \in VMO$ that $\dim(\phi(E)) = \dim(E)$. However we do not know how Hausdorff measures are distorted. In this section we answer this question when $\dim(E) = 1$, but in an indirect way. Our arguments go through some removability problems for μ-quasiregular mappings. For solving these problems, the Weyl's Lemma for the Beltrami equation (Theorem 7) plays an important role.

Given a compact set E and two real numbers $t \in (0, 2)$ and $\delta > 0$, we denote

$$\mathcal{M}^t_\delta(E) = \inf \left\{ \sum_j \operatorname{diam}(D_j)^t ; E \subset \cup_j D_j, \operatorname{diam}(D_j) \leq \delta \right\}.$$

Then, $\mathcal{M}^t(E) = \mathcal{M}^t_\infty(E)$ is the t-dimensional Hausdorff content of E, and

$$\mathcal{H}^t(E) = \lim_{\delta \to 0} \mathcal{M}^t_\delta(E)$$

is the t-dimensional Hausdorff measure of E. Recall that $\mathcal{H}^t(E) = 0$ if and only if $\mathcal{M}^t(E) = 0$. Analogously, for any nondecreasing function $h : [0, \infty) \to [0, \infty)$ with $h(0) = 0$, we denote

$$\mathcal{M}^h_\delta(E) = \inf \left\{ \sum_j h(\mathrm{diam}(D_j))^t; E \subset \cup_j D_j, \mathrm{diam}(D_j) \leq \delta \right\}.$$

and $\mathcal{M}^h(E) = \mathcal{M}^h_\infty(E)$. Then,

$$\mathcal{M}^t_*(E) = \sup \left\{ \mathcal{M}^h(E); h(s) \leq s^t \, , \, \lim_{s \to 0} \frac{h(s)}{s^t} = 0 \right\}$$

is called the t-dimensional lower Hausdorff content of E.

If f is locally integrable on $\mathbb{C}$ and D is a disk, we denote by $f_D = \frac{1}{|D|} \int_D f$ the mean value of f on D. Then, $\|f\|_*$ denotes the BMO norm of the function f, that is,

$$\|f\|_* = \sup_{D \subset \mathbb{C}} \frac{1}{|D|} \int_D |f - f_D| \, dA.$$

For such functions, using the well known John-Niremberg property one has

$$\sup_{D \subset \mathbb{C}} \frac{1}{|D|} \int_D |f - f_D| \, dA \simeq \sup_{D \subset \mathbb{C}} \left(\frac{1}{|D|} \int_D |f - f_D|^p \, dA \right)^{\frac{1}{p}}$$

for any $p \in (1, \infty)$, with constants that may depend on p but not on D. One denotes by VMO the closure of compactly supported $\mathcal{C}^\infty$ functions in BMO. We write $f \in \mathrm{Lip}_\alpha(\mathbb{C})$ to say that f is locally Hölder continuous with exponent α. This means that there are constants $C > 0$ and $r > 0$ such that

$$|z - w| \leq r \qquad \Rightarrow \qquad \frac{|f(z) - f(w)|}{|z - w|^\alpha} \leq C$$

and $\|f\|_\alpha$ denotes the best (i.e. smallest) constant C.

Lemma 8. *Let E be a compact set, and $\mu \in W^{1,2}$ a Beltrami coefficient, with compact support inside of $\mathbb{D}$. Suppose that f is μ-quasiregular on $\mathbb{C} \setminus E$, and $\varphi \in \mathcal{D}$.*

(a) If $f \in BMO(\mathbb{C})$, then

$$|\langle \overline{\partial} f - \mu \, \partial f, \varphi \rangle| \leq C \, (1 + \|\mu\|_\infty + \|\partial \mu\|_2) \, (\|\varphi\|_\infty + \|D\varphi\|_\infty) \, \|f\|_* \, \mathcal{M}^1(E)$$

(b) If $f \in VMO(\mathbb{C})$, then

$$|\langle \overline{\partial} f - \mu \, \partial f, \varphi \rangle| \leq C \, (1 + \|\mu\|_\infty + \|\partial \mu\|_2) \, (\|\varphi\|_\infty + \|D\varphi\|_\infty) \, \|f\|_* \, \mathcal{M}^1_*(E)$$

(c) If $f \in \mathrm{Lip}_\alpha(\mathbb{C})$, then

$$|\langle \overline{\partial} f - \mu\, \partial f, \varphi\rangle| \le C\ (1 + \|\mu\|_\infty + \|\partial\mu\|_2)\ (\|\varphi\|_\infty + \|D\varphi\|_\infty)\ \|f\|_\alpha\, \mathcal{M}^{1+\alpha}(E)$$

Proof. We consider the function $\delta = \delta(t)$ defined for $0 \le t \le 1$ by

$$\delta(t) = \sup_{\mathrm{diam}(D)\le 2t} \left(\frac{1}{|D|}\int_D |f - f_D|^2\right)^{\frac{1}{2}}$$

and

$$\delta(t) = \sup_{D\subset\mathbb{C}} \left(\frac{1}{D}\int_D |f - f_D|^2\, dA\right)^{\frac{1}{2}}$$

for any $t \ge 1$. By construction, for each disk $D \subset \mathbb{C}$ we have

$$\left(\frac{1}{|D|}\int_D |f - f_D|^2\right)^{\frac{1}{2}} \le \delta\left(\frac{\mathrm{diam}(D)}{2}\right).$$

Now consider the measure function $h(t) = t\,\delta(t)$. Let $\{D_j\}_{j=1}^n$ be a covering of E by disks, such that

$$\sum_j h(\mathrm{diam}(D_j)) \le \mathcal{M}^h(E) + \varepsilon$$

By a Lemma of Harvey and Polking [18, p.43], we can construct functions ψ_j in $\mathcal{C}^\infty$, compactly supported in $2D_j$, satisfying $|D\psi_j(z)| \le \frac{C}{\mathrm{diam}(2D_j)}$ and $0 \le \sum_j \psi_j \le 1$ on $\mathbb{C}$. In particular, $\sum_j \psi_j = 1$ on $\cup_j D_j$. Since f is μ-quasiregular on $\mathbb{C}\setminus E$, we have that for every test function $\varphi \in \mathcal{D}$,

$$-\langle \overline{\partial} f - \mu\,\partial f, \varphi\rangle = \sum_{j=1}^n \langle f - c_j, \overline{\partial}(\varphi\,\psi_j)\rangle - \sum_{j=1}^n \langle f - c_j, \partial(\mu\,\varphi\,\psi_j)\rangle \tag{10}$$

where $c_j = \frac{1}{|2D_j|}\int_{2D_j} f(z)\, dA(z)$. For the first sum, we have

$$\begin{aligned}
\left|\sum_{j=1}^n \langle f - c_j, \overline{\partial}(\varphi\,\psi_j)\rangle\right| &\le \sum_j \int_{2D_j} |f - c_j|\ \left(|\overline{\partial}\varphi|\,|\psi_j| + |\varphi|\,|\overline{\partial}\psi_j|\right) \\
&\le \sum_j \left(\int_{2D_j} |f - c_j|^2\right)^{\frac{1}{2}}\ \left(\|\overline{\partial}\varphi\|_\infty \mathrm{diam}(2D_j) + C\|\varphi\|_\infty\right) \\
&\lesssim \sum_j h(\mathrm{diam}(D_j))\ \left(\|\overline{\partial}\varphi\|_\infty \mathrm{diam}(2D_j) + C\|\varphi\|_\infty\right)
\end{aligned}$$

and this sum may be bounded by $\left(\mathcal{M}^h(E) + \varepsilon\right)(\|\varphi\|_\infty + \|D\varphi\|_\infty)$. The second sum in (10) is divided into two terms,

$$\left|\sum_{j=1}^n \langle f - c_j, \partial(\mu\,\varphi\,\psi_j)\rangle\right| \le \sum_j \int_{2D_j} |f - c_j|\,|\partial\mu|\,|\varphi\,\psi_j| + \sum_j \int_{2D_j} |f - c_j|\,|\mu|\,|\partial(\varphi\,\psi_j)|.$$

The second term can be bounded as before,

$$\sum_j \int_{2D_j} |f-c_j|\,|\mu|\,|\partial(\varphi\,\psi_j)| \lesssim \|\mu\|_\infty \left(\mathcal{M}^h(E)+\varepsilon\right)\left(\|\varphi\|_\infty + \|D\varphi\|_\infty\right).$$

Finally, for the first term, and using that $0 \le \sum_j \psi_j \le 1$,

$$\begin{aligned}
\sum_j \int |f-c_j|\,|\partial\mu|\,|\varphi\,\psi_j| &\le \|\varphi\|_\infty \sum_j \left(\int |f-c_j|^2\,|\psi_j|\right)^{\frac{1}{2}} \left(\int |\partial\mu|^2\,|\psi_j|\right)^{\frac{1}{2}} \\
&\le \|\varphi\|_\infty \left(\sum_j |2D_j|\delta(\operatorname{diam}(D_j))^2\right)^{\frac{1}{2}} \left(\sum_j \int |\partial\mu|^2\,\psi_j\right)^{\frac{1}{2}} \\
&\lesssim \|\varphi\|_\infty \left(\sum_j \operatorname{diam}(D_j)^2\,\delta(\operatorname{diam}(D_j))^2\right)^{\frac{1}{2}} \left(\int_{\mathbb{D}} |\partial\mu|^2\right)^{\frac{1}{2}} \\
&\le \|\varphi\|_\infty \left(\sum_j \operatorname{diam}(D_j)\,\delta(\operatorname{diam}(D_j))\right) \|\partial\mu\|_2 \\
&\le \|\varphi\|_\infty \left(\mathcal{M}^h(E)+\varepsilon\right) \|\partial\mu\|_2.
\end{aligned}$$

It just remains to distinguish in terms of the regularity of f. If $f \in BMO(\mathbb{C})$ then we can say that

$$\delta(t) \lesssim \|f\|_*$$

for all $t>0$, so that $\mathcal{M}^h(E) \le \mathcal{M}^1(E)\|f\|_*$. Secondly, functions in VMO have the additional property that their mean oscillation over small disks is small. Thus, if $f \in VMO(\mathbb{C})$, then

$$\lim_{t\to 0} \frac{h(t)}{t} = \lim_{t\to 0} \delta(t) = 0$$

and hence $\mathcal{M}^h(E) \le \mathcal{M}^1_*(E)\|f\|_*$. Finally, if $f \in \operatorname{Lip}_\alpha$, then

$$\delta(t) \le \|f\|_\alpha t^\alpha$$

and therefore $\mathcal{M}^h(E) \le \mathcal{M}^{1+\alpha}(E)\|f\|_\alpha$. □

Lemma 8 has very interesting consequences, related to μ-quasiconformal distortion. First, we show that our μ-quasiconformal mappings preserve sets of zero length.

Corollary 9. *Let $E \subset \mathbb{C}$ be a compact set. Let $\mu \in W^{1,2}$ be a compactly supported Beltrami coefficient, and ϕ a μ-quasiconformal mapping. Then,*

$$\mathcal{H}^1(E) = 0 \qquad \Longleftrightarrow \qquad \mathcal{H}^1(\phi(E)) = 0$$

Proof. By Proposition 5, it will suffice to prove that $\mathcal{H}^1(E) = 0$ implies $\mathcal{H}^1(\phi(E)) = 0$. Assume, thus, that $\mathcal{H}^1(E) = 0$. Let $f \in BMO(\mathbb{C})$ be holomorphic on $\mathbb{C} \setminus \phi(E)$. Then $g = f \circ \phi$ belongs also to $BMO(\mathbb{C})$. Moreover, g is μ-quasiregular on $\mathbb{C} \setminus E$ so that, by Lemma 8, $\langle \overline{\partial} g - \mu\, \partial g, \varphi \rangle = 0$ whenever $\varphi \in \mathcal{D}$. As a consequence, by Theorem 7, g is μ-quasiregular on the whole of $\mathbb{C}$ and hence f admits an entire extension. This says that the set $\phi(E)$ is removable for BMO holomorphic functions. But these sets are characterized [21] by the condition $\mathcal{H}^1(\phi(E)) = 0$. □

Another consequence is the complete solution of the removability problem for BMO μ-quasiregular mappings. Recall that a compact set E is said *removable for BMO μ-quasiregular mappings* if every function $f \in BMO(\mathbb{C})$, μ-quasiregular on $\mathbb{C} \setminus E$, admits an extension which is μ-quasiregular on $\mathbb{C}$.

Corollary 10. *Let $E \subset \mathbb{C}$ be compact. Let $\mu \in W^{1,2}$ be a compactly supported Beltrami coefficient. Then, E is removable for BMO μ-quasiregular mappings if and only if $\mathcal{H}^1(E) = 0$.*

Proof. Assume first that $\mathcal{H}^1(E) = 0$, and let $f \in BMO(\mathbb{C})$ be μ-quasiregular on $\mathbb{C} \setminus E$. Then, by Lemma 8, we have $\langle \overline{\partial} f - \mu\, \partial f, \varphi \rangle = 0$ for every $\varphi \in \mathcal{D}$. Now by Theorem 7 we deduce that f is μ-quasiregular. Consequently, E is removable. Conversely, if $\mathcal{H}^1(E) > 0$, then by Corollary 9, $\mathcal{H}^1(\phi(E)) > 0$, so that $\phi(E)$ is not removable for BMO analytic functions [21]. Hence, there exists a function h belonging to $BMO(\mathbb{C})$, holomorphic on $\mathbb{C} \setminus \phi(E)$, non entire. But therefore $h \circ \phi$ belongs to $BMO(\mathbb{C})$, is μ-quasiregular on $\mathbb{C} \setminus E$, and does not admit any μ-quasiregular extension on $\mathbb{C}$. Consequently, E is not removable for BMO μ-quasiregular mappings. □

A second family of consequences of Lemma 8 comes from the study of the VMO case. First, we prove that μ-quasiconformal mappings preserve compact sets with σ-finite length.

Corollary 11. *Let $\mu \in W^{1,2}$ be a compactly supported Beltrami coefficient, and ϕ any μ-quasiconformal mapping. For every compact set E,*

$$\mathcal{H}^1(E) \text{ is } \sigma\text{-finite} \qquad \Longleftrightarrow \qquad \mathcal{H}^1(\phi(E)) \text{ is } \sigma\text{-finite}$$

Proof. Again, we only have to show that $\mathcal{M}^1_*(E) = 0$ implies $\mathcal{M}^1_*(\phi(E)) = 0$. Assume, thus, that $\mathcal{M}^1_*(E) = 0$. By Verdera's work [35], the set $\phi(E)$ satisfies $\mathcal{M}^1_*(\phi(E)) = 0$ if and only if it is removable for VMO analytic functions. Thus, given $f \in VMO(\mathbb{C})$, analytic on $\mathbb{C} \setminus \phi(E)$, we have to prove that f extends holomorphically on $\mathbb{C}$. To do that, we first observe that $g = f \circ \phi$ also belongs to $VMO(\mathbb{C})$. Further, g is μ-quasiregular on $\mathbb{C} \setminus E$, and since $\mathcal{M}^1_*(E) = 0$, by Lemma 8 we get that $\overline{\partial} g - \mu\, \partial g = 0$ on $\mathcal{D}'$. Consequently, from Theorem 7, g is μ-quasiregular on the whole of $\mathbb{C}$ and hence f extends holomorphically on $\mathbb{C}$. □

As in the BMO setting, the removability problem for VMO μ-quasiregular functions also gets solved. A compact set E is said to be *removable for VMO μ-quasiregular mappings* if every function $f \in VMO(\mathbb{C})$ μ-quasiregular on $\mathbb{C} \setminus E$ admits an extension which is μ-quasiregular on $\mathbb{C}$.

Corollary 12. *Let $E \subset \mathbb{C}$ be compact. Let $\mu \in W^{1,2}$ be a compactly supported Beltrami coefficient. Then E is removable for VMO μ-quasiregular mappings if and only if $\mathcal{H}^1(E)$ is σ-finite.*

Proof. If $\mathcal{H}^1(E)$ is σ-finite, then $\mathcal{M}^1_*(E) = 0$, so that from Lemma 8 every function $f \in VMO(\mathbb{C})$ μ-quasiregular on $\mathbb{C} \setminus E$ satisfies $\overline{\partial} f = \mu \, \partial f$ on $\mathcal{D}'$. By Theorem 7, f extends μ-quasiregularly and thus E is removable.
If $\mathcal{H}^1(E)$ is not σ-finite, we have just seen that $\mathcal{H}^1(\phi(E))$ must not be σ-finite. Thus, it comes from Verdera's work [35] that there exists a function $h \in VMO(\mathbb{C})$, analytic on $\mathbb{C} \setminus \phi(E)$, non entire. But therefore $h \circ \phi$ belongs to VMO, is μ-quasiregular on $\mathbb{C} \setminus E$, and does not extend μ-quasiregularly on $\mathbb{C}$. □

The class Lip_α has, in comparison with BMO or VMO, the disadvantage of being not quasiconformally invariant. This means that we cannot read any removability result for Lip_α in terms of distortion of Hausdorff measures, and therefore for $\mathcal{H}^{1+\alpha}$ we cannot obtain results as precise as Lemmas 9 or 11. Hence question remains unsolved. However, Theorem 7 can be used to study the Lip_α removability problem. Recall that a compact set E is *removable for* Lip_α *μ-quasiregular mappings* if every function $f \in \mathrm{Lip}_\alpha(\mathbb{C})$, μ-quasiregular on $\mathbb{C} \setminus E$, has a μ-quasiregular extension on $\mathbb{C}$.

Corollary 13. *Let E be compact, and assume that $\mathcal{H}^{1+\alpha}(E) = 0$. Then, E is removable for* Lip_α *μ-quasiregular mappings.*

Proof. As before, if $f \in \mathrm{Lip}_\alpha$ is μ-quasiregular outside of E, then Lemma 8 tells us that its Beltrami distributional derivative vanishes. By Theorem 7, we get that f is μ-quasiregular. □

The above result is sharp, in the sense that if $\mathcal{H}^{1+\alpha}(E) > 0$ then there is a compactly supported Beltrami coefficient $\mu \in W^{1,2}$ such that E is not removable for Lip_α μ-quasiregular mappings (take simply $\mu = 0$, [26]).

5 μ-quasiconformal distortion of analytic capacity

If $\mu \in W^{1,2}(\mathbb{C})$ is a Beltrami coefficient, compactly supported on $\mathbb{D}$, and $E \subset \mathbb{D}$ is compact, we say that E is *removable for bounded μ-quasiregular functions*, if and only if any bounded

function f, μ-quasiregular on $\mathbb{C} \setminus E$, is actually a constant function. As it is in the BMO case, just 1-dimensional sets are interesting, because of the Stoilow factorization, together with the fact that μ-quasiconformal mappings with $\mu \in W^{1,2}$ do not distort Hausdorff dimension.
As we know from Corollary 9, if E is such that $\mathcal{H}^1(E) = 0$ then also $\mathcal{H}^1(\phi(E)) = 0$ whenever ϕ is μ-quasiconformal. Thus, also $\gamma(\phi(E)) = 0$. This shows that zero length sets are removable for bounded μ-quasiregular mappings.
Now the following step consists of understanding what happens with sets of positive and finite length. It is well known that those sets can be decomposed as the union of a rectifiable set, a purely unrectifiable set, and a set of zero length (see for instance Mattila [23, p.205]). Hence, we may study them separately.

Lemma 14. *Let $\phi : \mathbb{C} \to \mathbb{C}$ be a planar homeomorphism, such that $\phi \in W^{2,1+\varepsilon}_{loc}(\mathbb{C})$ for some $\varepsilon > 0$. Suppose also that*

$$\mathcal{H}^1(E) = 0 \qquad \Rightarrow \qquad \mathcal{H}^1(\phi(E)) = 0.$$

for any set $E \subset \mathbb{C}$. Then,

$$\Gamma \text{ rectifiable} \qquad \Rightarrow \qquad \phi(\Gamma) \text{ rectifiable.}$$

Proof. Assume that Γ is a rectifiable set. Since $\phi \in W^{2,1+\varepsilon}_{loc}$, then ϕ is strongly differentiable $C_{1,1+\varepsilon}$-almost everywhere (see for instance [13]). In particular, ϕ is differentiable $\mathcal{H}^1$-almost everywhere, so that the quantity

$$\limsup_{w \to z} \frac{\phi(z) - \phi(w)}{z - w}$$

is finite at $\mathcal{H}^1$-almost every $z \in \Gamma$. For each $k = 1, 2, \ldots$ define

$$E_k = \left\{ z \in \Gamma : \frac{|\phi(z) - \phi(w)|}{|z - w|} \leq k \text{ whenever } 0 < |z - w| < \frac{1}{k} \right\}.$$

This sets E_k are rectifiable, since $E_k \subset \Gamma$. Furthermore, the set $Z = \Gamma \setminus \cup_{k=1}^{\infty} E_k$ has zero length. By our assumptions, it then follows that

$$\mathcal{H}^1(\phi(Z)) = 0.$$

However,

$$\phi(Z) = \phi(\Gamma) \setminus \cup_{k=1}^{\infty} \phi(E_k)$$

which means that the sets $\phi(E_k)$ cover $\phi(\Gamma)$ up to a subset of zero length. Moreover, each $\phi(E_k)$ is a rectifiable set. To see this, notice that E_k can be divided into countably many pieces over which ϕ is Lipschitz contiuous with constant k. Thus, $\phi(\Gamma)$ is covered by a countable

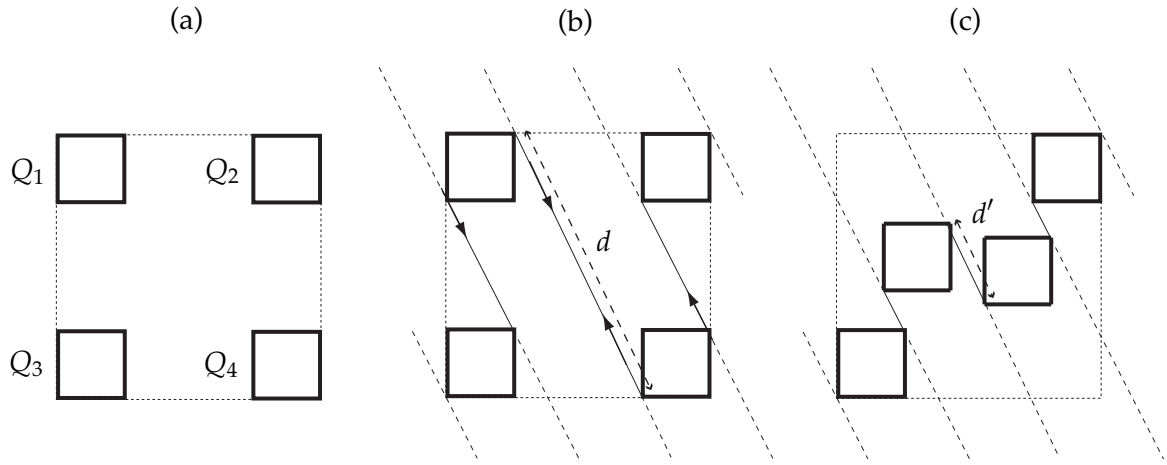
union of Lipschitz images of rectifiable sets, modulo a set of zero length. Therefore, $\phi(\Gamma)$ is a rectifiable set.

□

In this lemma, the regularity assumption is necessary. In the following example, due to J. B. Garnett [16], we construct a homeomorphism of the plane that preserves sets of zero length and, at the same time, maps a purely unrectifiable set to a rectifiable set.

Example. Denote by E the planar $\frac{1}{4}$-Cantor set. Recall that this set is obtained as a countable intersection of a decreasing family of compact sets E_N, each of which is the union of 4^N squares of sidelength $\frac{1}{4^N}$, and where every father has exactly 4 identic children.

At the first step, the unit square has 4 children Q_1, Q_2, Q_3 and Q_4. The corners of the



squares Q_j are connected with some parallel lines. The mapping ϕ_1 consists on displacing along these lines the squares Q_2 and Q_3, while Q_1 and Q_4 remain fixed. This displacement must be done in such a way that de distance between the images of Q_2 and Q_3 is positive, since ϕ_1 must be a homeomorphism. However, we can do this construction with d' as small as we wish. Our final mapping ϕ will be obtained as a uniform limit $\phi = \lim_{N\to\infty} \phi_N$. The other mappings ϕ_N are nothing else but copies of ϕ_1 acting on every one of the different squares in all generations. The only restriction is that the sum of distances d' must be finite. It is clear that this procedure gives a sequence of homeomorphisms ϕ_N which converge uniformly to a homeomorphism ϕ. Further, it can be shown that $\mathcal{H}^1(F) = 0$ if and only if $\mathcal{H}^1(\phi(F)) = 0$. On the other hand, the image of E under the mapping ϕ is included in a compact connected set, whose length is precisely the sum of the distances d', which we have chosen to be finite. Therefore, $\phi(E)$ is rectifiable.

If $\mu \in W^{1,2}$ is a compactly supported Beltrami coefficient, then we know that every μ-quasiconformal mapping belongs to the local Sobolev space $W^{2,q}_{loc}(\mathbb{C})$ for all $q < 2$. Furthermore, we also know that ϕ preserves the sets of zero length (even σ-finite length are preserved), and the same happens to ϕ^{-1}. Under these hypotheses, we can use Lemma 14 both for ϕ and ϕ^{-1} and what we actually have is that

$$\Gamma \text{ rectifiable} \qquad \Leftrightarrow \qquad \phi(\Gamma) \text{ rectifiable .}$$

This may be stated as follows.

Corollary 15. *Let $\mu \in W^{1,2}$ be a compactly supported Beltrami coefficient, and ϕ a μ-quasiconformal mapping.*

(a) If E is a rectifiable set, then $\phi(E)$ is also rectifiable.

(b) If E is a purely unrectifiable set, then, $\phi(E)$ is also purely unrectifiable.

Proof. The first statement comes from the above lemma. Indeed, in this situation μ-quasiconformal maps send sets of zero length to sets of zero length, and have the needed Sobolev regularity. For the second, let Γ be a rectifiable curve. Then, also $\phi^{-1}(\Gamma)$ is rectifiable. To see this, notice that ϕ^{-1} is under the assumptions of Lemma 14. Thus,

$$\mathcal{H}^1(\phi(E) \cap \Gamma) = 0 \Leftrightarrow \mathcal{H}^1(E \cap \phi^{-1}(\Gamma)) = 0$$

but since E is purely unrectifiable, all rectifiable sets intersect E in a set of zero length. Thus, the result follows. □

Theorem 16. *Let $\mu \in W^{1,2}$ be a compactly supported Beltrami coefficient, and ϕ a μ-quasiconformal mapping. Let E be such that $\mathcal{H}^1(E)$ is σ-finite. Then,*

$$\gamma(E) = 0 \qquad \Longleftrightarrow \qquad \gamma(\phi(E)) = 0$$

Proof. By Corollaries 9 and 11, if $\mathcal{H}^1(E)$ is positive and σ-finite, then $\mathcal{H}^1(\phi(E))$ is positive and σ-finite. Hence, we may decompose $\phi(E)$ as

$$\phi(E) = \bigcup_n R_n \cup N_n \cup Z_n$$

with R_n rectifiable sets, N_n purely unrectifiable sets, and Z_n zero length sets. Notice that $\gamma(N_n) = 0$ because purely unrectifiable sets of finite length are removable for bounded analytic

functions [11], and also $\gamma(Z_n) = 0$ since $\mathcal{H}^1(Z_n) = 0$. Thus, due to the semiadditivity of analytic capacity [31], we get

$$\gamma(\phi(E)) \leq C \sum_n \gamma(R_n)$$

However, each R_n is a rectifiable set, so that $\phi^{-1}(R_n)$ is also rectifiable. Now, since E has σ-finite length, the condition $\gamma(E) = 0$ forces that E cannot contain any rectifiable subset of positive length, so that $\mathcal{H}^1(\phi^{-1}(R_n)) = 0$ and hence $\mathcal{H}^1(R_n) = 0$. Consequently, $\gamma(\phi(E)) = 0$. □

The above theorem is an exclusively qualitative result. Therefore, we must not hope for any improvement in a quantitative sense. Namely, in the bilipschitz invariance of analytic capacity by Tolsa [32], it is shown that a planar homeomorphism $\phi : \mathbb{C} \to \mathbb{C}$ satisfies $\gamma(\phi(E)) \simeq \gamma(E)$ for every compact set E if and only if it is a bilipschitz mapping, while Example 8 shows that there exist μ-quasiconformal mappings ϕ in the above hypotheses, which are not Lipschitz continuous.

6 Beltrami coefficient in $W^{1,p}$, $\frac{2K}{K+1} < p < 2$

In this section, we will try to understand the situation when the Beltrami coefficient μ lies in the Sobolev space $W^{1,p}$, for some $p \in (\frac{2K}{K+1}, 2)$, where as usually we assume $\|\mu\|_\infty \leq \frac{K-1}{K+1}$. As we showed in Proposition 4, under this assumption every μ-quasiconformal mapping ϕ belongs to $W^{2,q}_{loc}$ for each $q < q_0$, where

$$\frac{1}{q_0} = \frac{1}{p} + \frac{K-1}{2K}.$$

Note that we always have $1 < q_0 < \frac{2K}{2K-1} < 2$. We will also denote $p_0 = \frac{q_0}{q_0-1}$, so that

$$\frac{1}{p_0} = \frac{K+1}{2K} - \frac{1}{p}.$$

Here we always have $p_0 \in (2K, \infty)$.

Our first goal is to prove an analogous result to Theorem 7 (the Weyl's Lemma for the Beltrami operator) in this weaker situation. We start by introducing the class of functions

$$E^{p,q} = E^{p,q}(\mathbb{C}) = W^{1,q}_{loc}(\mathbb{C}) \cap L^{\frac{pq}{p-q}}_{loc}(\mathbb{C})$$

defined for $1 < q < p$. Given a sequence of functions $f_n \in E^{p,q}$, we say that $f_n \to f$ if

$$\|f_n - f\|_{L^{\frac{pq}{p-q}}(D)} + \|Df_n - Df\|_{L^q(D)} \to 0$$

as $n \to \infty$, for any disk $D \subset \mathbb{C}$.

Among the main properties of the class $E^{p,q}$, we first notice that

$$p < 2 \qquad \Leftrightarrow \qquad \frac{2q}{2-q} < \frac{pq}{p-q}.$$

Thus, if $p < 2$ the intersection $W^{1,q} \cap L^{\frac{pq}{p-q}}$ is proper, so that $E^{p,q} \subsetneq W^{1,q}$. Further, it is not hard to see that every function in $E^{p,q}$ is the limit in $E^{p,q}$ of a sequence φ_n of functions $\varphi_n \in \mathcal{D}$, in other words, $\mathcal{D}$ is dense in $E^{p,q}$.

Given a compactly supported Beltrami coefficient $\mu \in W^{1,p}$, we can define (as it was done in Section 3) the distributional Beltrami derivative of any function $f \in L^{\frac{p}{p-1}}$ by the rule

$$\langle \overline{\partial} f - \mu\, \partial f, \varphi \rangle = -\langle f, \overline{\partial}\varphi \rangle + \langle f, \mu\, \partial\varphi \rangle + \langle f, \varphi\, \partial\mu \rangle.$$

This expression converts $\overline{\partial} f - \mu\, \partial f$ into a continuous linear functional on $\mathcal{D}$. The following proposition shows the precise reasons for introducing the class $E^{p,q}$ to study this functional.

Lemma 17. *Let $1 < q < p < 2$ be fixed. Let $f \in L^{\frac{q}{q-1}}_{loc}$. Let $\mu \in W^{1,p}$ be a Beltrami coefficient with compact support inside of $\mathbb{D}$. Then,*

1. *The mapping $\varphi \mapsto \mu\,\varphi$ is continuous in $E^{p,q}$.*
2. *The distributional derivatives ∂f and $\overline{\partial} f$ act continuously on compactly supported functions of $E^{p,q}$.*
3. *The distribution $\overline{\partial} f - \mu \partial f$ acts continuously on $E^{p,q}$ functions.*

Proof. First of all, let D be a disk. Given $\varphi \in E^{p,q}$,

$$\begin{aligned}
\|\mu\varphi\|_{L^{\frac{pq}{p-q}}(D)} + \|D(\mu\varphi)\|_{L^q(D)} &\leq \|\mu\|_\infty \|\varphi\|_{L^{\frac{pq}{p-q}}(D)} + \|D\mu\,\varphi\|_{L^q(D)} + \|\mu\|_\infty \|D\varphi\|_{L^q(D)} \\
&\leq \|\mu\|_\infty \left(\|\varphi\|_{L^{\frac{pq}{p-q}}(D)} + \|D\varphi\|_{L^q(D)} \right) + \|D\mu\|_p \|\varphi\|_{L^{\frac{pq}{p-q}}(D)} \\
&\leq (\|\mu\|_\infty + \|D\mu\|_p) \left(\|\varphi\|_{L^{\frac{pq}{p-q}}(D)} + \|D\varphi\|_{L^q(D)} \right)
\end{aligned}$$

In other words, if $1 < q < p$ then the class $E^{p,q}$ is stable under multiplication by bounded functions in $W^{1,p}$.

For the second statement, let $\varphi_n \in E^{p,q}$ be compactly supported on a disk D, and assume that

$$\|\varphi_n\|_{L^{\frac{pq}{p-q}}(D)} + \|D\varphi_n\|_{L^q(D)} \to 0$$

as $n \to \infty$. Then, Hölder's inequality implies that

$$|\langle \partial f, \varphi_n \rangle| = |\int_D f\, \partial\varphi_n\, dA| \leq \left(\int_D |f|^{\frac{q}{q-1}}\, dA \right)^{\frac{q-1}{q}} \left(\int_D |\partial\varphi_n|^q\, dA \right)^{\frac{1}{q}} \to 0$$

and analogously for $\overline{\partial}$.

Finally, let $\varphi_n \in E^{p,q}$ be compactly supported in a disk D, and such that $\|\varphi\|_{L^{\frac{pq}{p-q}}(D)} + \|D\varphi\|_{L^q(D)} \to 0$ as $n \to \infty$. From the first statement, also

$$\|\mu\varphi\|_{L^{\frac{pq}{p-q}}(D)} + \|D(\mu\varphi)\|_{L^q(D)} \to 0.$$

Further, from the second statement also $\langle \partial f, \varphi_n \rangle$ and $\langle \overline{\partial} f, \mu\varphi_n \rangle$ converge to 0. Hence,

$$|\langle \overline{\partial} f - \mu\, \partial f, \varphi \rangle| \leq |\langle \overline{\partial} f, \varphi_n \rangle| + |\langle \partial f, \mu\, \varphi_n \rangle| \to 0$$

and the statement follows. □

Theorem 18. *Let $\mu \in W^{1,p}(\mathbb{C})$ be a compactly supported Beltrami coefficient, $\|\mu\|_\infty \leq \frac{K-1}{K+1}$ and $p > \frac{2K}{K+1}$. Let f be in $L^{p_0+\varepsilon}_{loc}$ for some $\varepsilon > 0$, and assume that*

$$\langle \overline{\partial} f - \mu\, \partial f, \psi \rangle = 0$$

for each $\psi \in \mathcal{D}$. Then, f is μ-quasiregular.

Proof. The proof repeats somehow the computations of Theorem 7, with a bit more care on the indexes. Let $\phi : \mathbb{C} \to \mathbb{C}$ be a μ-quasiconformal mapping, and define $g = f \circ \phi^{-1}$. Clearly g is a locally integrable function. Thus we may define $\overline{\partial} g$ as a distribution and for each $\varphi \in \mathcal{D}$ we have

$$\begin{aligned}
\langle \overline{\partial} g, \varphi \rangle &= -\langle g, \overline{\partial}\varphi \rangle \\
&= -\int g(w)\, \overline{\partial}\varphi(w)\, dA(w) \\
&= -\int f(z)\, \overline{\partial}\varphi(\phi(z))\, J\phi(z)\, dA(z) \\
&= -\int f(z) \left(\partial\phi(z)\, \overline{\partial}(\varphi \circ \phi)(z) - \overline{\partial}\phi(z)\, \partial(\varphi \circ \phi)(z) \right) dA(z)
\end{aligned}$$

This expression makes sense, because $f \in L^{p_0+\varepsilon}$, and both ϕ and $\varphi \circ \phi$ belong to $W^{2,q}$ for each $q < q_0$, so that the integrand is an L^q function. We have

$$-\int f(z)\, \overline{\partial}\phi(z)\, \partial(\varphi \circ \phi)(z) dA(z) = \langle \partial f, \overline{\partial}\phi \cdot\ \varphi \circ \phi \rangle + \int f(z)\, \partial\overline{\partial}\phi(z)\, \varphi \circ \phi(z)\, dA(z)$$

and here everything makes sense. Indeed, the chain rule shows that the function $\partial\phi \cdot (\varphi \circ \phi)$ belongs not only to $L^{\frac{2K}{K-1},\infty}$ but also to $W^{1,q}_{loc}$ for each q satisfying $q < \min\{q_0, \frac{K}{K-1}\}$. However,

$$\frac{2K}{K+1} < p < 2 \qquad \Leftrightarrow \qquad 1 < q_0 < \frac{2K}{2K-1}$$

and for $K > 1$ we have $\frac{2K}{2K-1} < \frac{K}{K-1}$. Thus, $\partial\phi \cdot (\varphi \circ \phi) \in W^{1,q}$ for any $q < q_0$. Moreover, clearly it holds that $p > q_0$, and

$$q < q_0 \qquad \Rightarrow \qquad \frac{pq}{p-q} < \frac{2K}{K-1}$$

so that also $\partial\phi \cdot (\varphi \circ \phi) \in L^{\frac{pq}{p-q}}_{loc}$. Thus, what we have is

$$\partial\phi \cdot (\varphi \circ \phi) \in E^{p,q}, \qquad \forall q < q_0.$$

Arguing analogously,

$$-\int f(z)\,\partial\phi(z)\,\overline{\partial}(\varphi\circ\phi)(z)\,dA(z) = \langle \overline{\partial} f, \partial\phi \cdot \varphi \circ \phi\rangle + \int f(z)\,\overline{\partial}\partial\phi(z)\,\varphi\circ\phi(z)\,dA(z)$$

Thus, for any $\varphi \in \mathcal{D}$,

$$\langle \overline{\partial} g, \varphi\rangle = \langle \overline{\partial} f, \partial\phi \cdot \varphi \circ \phi\rangle - \langle \partial f, \overline{\partial}\phi \cdot\ \varphi \circ \phi\rangle. \tag{11}$$

Now, assume that the Beltrami derivative of f vanishes as a linear functional acting on $\mathcal{D}$. Then, we get from Lemma 17 that

$$\langle \overline{\partial} f - \mu\,\partial f, \psi\rangle = 0$$

for every compactly supported function $\psi \in E^{p,q}$, and any $1 < q < q_0$. In particular, this holds if we take

$$\psi = \partial\phi\,\varphi\circ\phi.$$

Thus, we have

$$\langle \overline{\partial} f, \psi\rangle = \langle \partial f, \mu\,\psi\rangle,$$

or equivalently

$$\langle \overline{\partial} f, \partial\phi\,\varphi\circ\phi\rangle = \langle \partial f, \mu\,\partial\phi\,\varphi\circ\phi\rangle.$$

But $\overline{\partial}\phi = \mu\,\partial\varphi$, so that

$$\langle \overline{\partial} f, \partial\phi\,\varphi\circ\phi\rangle = \langle \partial f, \overline{\partial}\phi\,\varphi\circ\phi\rangle.$$

Hence, by equation (11),

$$\langle \overline{\partial} g, \varphi\rangle = 0$$

whenever $\varphi \in \mathcal{D}$, which means by the classical Weyl Lemma that g is holomorpic. Therefore, f is μ-quasiregular. □

Once we know that distributional solutions are strong solutions, also under the weaker assumption $\frac{2K}{K+1} < p < 2$, it then follows that some removability theorems can be obtained. The arguments in Section 4 may be repeated to obtain similar estimates for the BMO, VMO and Lip_α problems. In fact, an analogous result to Lemma 8 holds as well under these weaker assumptions.

Lemma 19. *Let $\frac{2K}{K+1} < p < 2$ and $q = \frac{p}{p-1}$. Let E be a compact set, and $\mu \in W^{1,p}$ a Beltrami coefficient, with compact support inside of $\mathbb{D}$. Suppose that f is μ-quasiregular on $\mathbb{C} \setminus E$, and $\varphi \in \mathcal{D}$.*

(a) If $f \in BMO(\mathbb{C})$, then

$$|\langle \overline{\partial} f - \mu\, \partial f, \varphi \rangle| \leq C\ (1 + \|\mu\|_\infty + \|\partial \mu\|_p)\ (\|\varphi\|_\infty + \|D\varphi\|_\infty)\ \|f\|_*\ \left(\mathcal{M}^1(E) + \frac{\mathcal{M}^1(E)^{\frac{2}{q}}}{\|f\|_*^{\frac{2}{q}}} \right)$$

(b) If $f \in VMO(\mathbb{C})$, then

$$|\langle \overline{\partial} f - \mu\, \partial f, \varphi \rangle| \leq C\ (1 + \|\mu\|_\infty + \|\partial \mu\|_p)\ (\|\varphi\|_\infty + \|D\varphi\|_\infty)\ \|f\|_*\ \left(\mathcal{M}^1_*(E) + \frac{\mathcal{M}^1(E)^{\frac{2}{q}}}{\|f\|_*^{\frac{2}{q}}} \right)$$

(c) If $f \in \mathrm{Lip}_\alpha(\mathbb{C})$, then

$$|\langle \overline{\partial} f - \mu\, \partial f, \varphi \rangle| \leq C\ (1 + \|\mu\|_\infty + \|\partial \mu\|_p)\ (\|\varphi\|_\infty + \|D\varphi\|_\infty)\ \|f\|_\alpha\ \left(\mathcal{M}^{1+\alpha}(E) + \frac{\mathcal{M}^{1+\alpha}(E)^{\frac{2}{q}}}{\|f\|_\alpha^{\frac{2}{q}}} \right)$$

Proof. We repeat the argument in Lemma 8, and consider the function $\delta = \delta(t)$ defined for $0 \leq t \leq 1$ by

$$\delta(t) = \sup_{\mathrm{diam}(D) \leq 2t} \left(\frac{1}{|D|} \int_D |f - f_D|^q \right)^{\frac{1}{q}}$$

and

$$\delta(t) = \sup_{D \subset \mathbb{C}} \left(\frac{1}{|D|} \int_D |f - f_D|^q \right)^{\frac{1}{q}}$$

if $t \geq 1$. Here $q = \frac{p}{p-1}$. By construction, for each disk $D \subset \mathbb{C}$ we have

$$\left(\frac{1}{|D|} \int_D |f - f_D|^q \right)^{\frac{1}{q}} \leq \delta \left(\frac{\mathrm{diam}(D)}{2} \right)$$

Now consider the measure function $h(t) = t\, \delta(t)$. Let D_j be a covering of E by disks, such that

$$\sum_j h(\mathrm{diam}(D_j)) \leq \mathcal{M}^h(E) + \varepsilon.$$

Again, as in the proof of Lemma 8, we have $\mathcal{C}^\infty$ functions ψ_j compactly supported in $2D_j$ satisfying $|D\psi_j(z)| \leq \frac{C}{\mathrm{diam}(2D_j)}$ and $0 \leq \sum_j \psi_j \leq 1$ on $\mathbb{C}$. In particular, $\sum_j \psi_j = 1$ on $\cup_j D_j$. For every test function $\varphi \in \mathcal{D}$,

$$-\langle \overline{\partial} f - \mu\, \partial f, \varphi \rangle = \sum_{j=1}^n \langle f - c_j, \overline{\partial}(\varphi\, \psi_j) \rangle - \sum_{j=1}^n \langle f - c_j, \partial(\mu\, \varphi\, \psi_j) \rangle. \tag{12}$$

An analogous procedure to that in Lemma 8 gives

$$\left|\sum_{j=1}^{n}\langle f-c_j,\overline{\partial}(\varphi\,\psi_j)\rangle\right| \lesssim \left(\mathcal{M}^h(E)+\varepsilon\right)\left(\|\varphi\|_\infty+\|D\varphi\|_\infty\right).$$

The other sum in (12) is again divided into two parts,

$$\left|\sum_{j=1}^{n}\langle f-c_j,\partial(\mu\,\varphi\,\psi_j)\rangle\right| \leq \sum_j\int_{2D_j}|f-c_j|\,|\partial\mu|\,|\varphi\,\psi_j|+\sum_j\int_{2D_j}|f-c_j|\,|\mu|\,|\partial(\varphi\,\psi_j)|.$$

The second one, as before,

$$\sum_j\int_{2D_j}|f-c_j|\,|\mu|\,|\partial(\varphi\,\psi_j)| \lesssim \|\mu\|_\infty\,\left(\mathcal{M}^h(E)+\varepsilon\right)\left(\|\varphi\|_\infty+\|D\varphi\|_\infty\right).$$

For the first term,

$$\begin{aligned}\sum_j\int|f-c_j|\,|\partial\mu|\,|\varphi\,\psi_j| &\leq \|\varphi\|_\infty\sum_j\left(\int|f-c_j|^q\,|\psi_j|\right)^{\frac{1}{q}}\left(\int|\partial\mu|^p\,|\psi_j|\right)^{\frac{1}{p}}\\ &\leq \|\varphi\|_\infty\left(\sum_j|2D_j|\,\delta(\operatorname{diam}(D_j))^q\right)^{\frac{1}{q}}\left(\sum_j\int|\partial\mu|^p\,\psi_j\right)^{\frac{1}{p}}\\ &\lesssim \|\varphi\|_\infty\left(\sum_j\operatorname{diam}(D_j)^2\,\delta(\operatorname{diam}(D_j))^q\right)^{\frac{1}{q}}\left(\int_{\mathbb{D}}|\partial\mu|^p\right)^{\frac{1}{p}}.\end{aligned}$$

Now assume that $f\in BMO(\mathbb{C})$. We then have $\delta(t)\leq C\,\|f\|_*$ for some constant $C>0$ and all t. Thus, since $q>2$ then $\left(\frac{\delta(t)}{C\,\|f\|_*}\right)^q\leq\left(\frac{\delta(t)}{C\,\|f\|_*}\right)^2$, and we obtain

$$\begin{aligned}\left(\sum_j\operatorname{diam}(D_j)^2\,\delta(\operatorname{diam}(D_j))^q\right)^{\frac{1}{q}} &\lesssim \|f\|_*^{1-\frac{2}{q}}\left(\sum_j\operatorname{diam}(D_j)^2\,\delta(\operatorname{diam}(D_j))^2\right)^{\frac{1}{q}}\\ &\lesssim \|f\|_*^{1-\frac{2}{q}}\left(\sum_j\operatorname{diam}(D_j)\,\delta(\operatorname{diam}(D_j))\right)^{\frac{2}{q}}\\ &\lesssim \|f\|_*^{1-\frac{2}{q}}\left(\mathcal{M}^h(E)+\varepsilon\right)^{\frac{2}{q}}.\end{aligned}$$

Analogous estimates are found when $f\in VMO(\mathbb{C})$ of $f\in\operatorname{Lip}_\alpha(\mathbb{C})$. The rest of the proof follows as in Lemma 8. □

As in the case $\mu\in W^{1,2}$, from these lemmas one obtains precise results on how removable sets for VMO, BMO and Hölder continuous functions are preserved by μ-quasiconformal mappings with $\mu\in W^{1,p}$. This implies the following results on distortions of different measure functions.

Proposition 20. *Let $\frac{2K}{K+1} < p \leq 2$, and let $\mu \in W^{1,p}$ be a compactly supported Beltrami coefficient. If ϕ is μ-quasiconformal, then*

a)

$$\dim(E) \leq 1 \qquad \Rightarrow \qquad \dim(\phi(E)) \leq 1$$

b)

$$\mathcal{M}^1(E) = 0 \qquad \Rightarrow \qquad \mathcal{M}^1(\phi(E)) = 0$$

c)

$$\mathcal{M}^1_*(E) = 0 \qquad \Rightarrow \qquad \mathcal{M}^1_*(\phi(E)) = 0$$

Proof. The proofs for (b) and (c) are similar to those in Section 4. We just prove (a). If $\dim(E) \leq 1$, then $\mathcal{H}^{1+\alpha}(E) = 0$ for all $\alpha > 0$, and hence E is μ-removable for α-Hölder continuous functions, for every $\alpha > 0$. Now let $\beta > 0$, and let $h : \mathbb{C} \to \mathbb{C}$ be a Lip_β function, holomorphic on $\mathbb{C} \setminus \phi(E)$. Then, $h \circ \phi$ is a $\mathrm{Lip}_{\beta/K}$ function, μ-quasiregular on $\mathbb{C} \setminus E$ and hence has a μ-quasiregular extension to the whole of $\mathbb{C}$. Then, h extends holomorphically. This means that $\phi(E)$ is removable for holomorphic β-Hölder continuous functions, so that $\mathcal{H}^{1+\beta}(\phi(E)) = 0$. Since this holds for any $\beta > 0$, then we get $\dim(\phi(E)) \leq 1$. □

The above results are not if and only if conditions. The reason is that if $\mu \in W^{1,p}$ and $p > \frac{2K}{K+1}$, then the Beltrami coefficient ν of the inverse mapping ϕ^{-1} is just known to be in $W^{1,1}$ (see Remark 1). However if we restrict ourselves to Beltrami coefficients $\mu \in W^{1,p}$ with $p > \frac{2K^2}{K^2+1}$, the inverse mapping ϕ^{-1} has a Beltrami coefficient ν within the Sobolev range $(\frac{2K}{K+1}, 2)$. In this situation we can strength Proposition 20.

Proposition 21. *Let $\frac{2K^2}{K^2+1} < p < 2$. Let $\mu \in W^{1,p}$ be a compactly supported Beltrami coefficient, and let ϕ be μ-quasiconformal. Then,*

a)

$$\dim(E) \leq 1 \qquad \Leftrightarrow \qquad \dim(\phi(E)) \leq 1$$

b)

$$\mathcal{M}^1(E) = 0 \qquad \Leftrightarrow \qquad \mathcal{M}^1(\phi(E)) = 0$$

c)

$$\mathcal{M}^1_*(E) = 0 \qquad \Leftrightarrow \qquad \mathcal{M}^1_*(\phi(E)) = 0$$

Finally we address the question of rectifiability in this weaker setting. If $p > \frac{2K}{K+1}$, then by Lemma 14 we get that ϕ rectifiable sets to rectifiable set. Again, as above, this need not be an equivalence in general. Moreover, it could be that $\mathcal{H}^1(E) > 0$ and $\mathcal{H}^1(\phi(E)) = 0$. Thus to control how analytic capacity is distorted we need the extra regularity $\frac{2K^2}{K^2+1} < p < 2$. Arguing as in Section 5, we obtain the following unexpected result.

Theorem 22. *Let $\mu \in W^{1,p}$ be a compactly supported Beltrami coefficient, $\|\mu\|_\infty \leq \frac{K-1}{K+1}$ and $\frac{2K^2}{K^2+1} < p < 2$, and let $\phi : \mathbb{C} \to \mathbb{C}$ be a μ-quasiconformal mapping. Then,*

$$\gamma(E) = 0 \qquad \Leftrightarrow \qquad \gamma(\phi(E)) = 0$$

for any compact set E with σ-finite $\mathcal{H}^1(E)$.

Note added in proof. Very recently, the first author and X. Tolsa [10] have shown that the σ-finiteness assumption in Theorem 16 can be removed.

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