Exotic normal fusion subsystems of the General Linear group

Albert Ruiz

Universitat Autònoma de Barcelona

Groups Geometry Topology
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$p$-local finite groups arose in the work of Broto-Levi-Oliver as a formalization of the fusion in a finite group for a fixed prime $p$.

A $p$-local finite group consists on a triple $(S, \mathcal{F}, \mathcal{L})$ where:

<table>
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<tr>
<th>$S$ is a $p$-group</th>
<th>$\mathcal{F}$ is a category</th>
<th>$\mathcal{L}$ is a category with extra information in such a way that there is a classifying space.</th>
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<td>and plays the role of a Sylow $p$-subgroup.</td>
<td>and models a fusion over $S$.</td>
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**Theorem (BLO)**

If $G$ is a finite group and $S$ is a Sylow $p$-subgroup, we can construct a $p$-local finite group $(S, \mathcal{F}_S(G), \mathcal{L}_S^c(G))$ and $|\mathcal{L}_S^c(G)|_p^\wedge \simeq BG^\wedge_p$. 
$p$-local finite groups arose in the work of Broto-Levi-Oliver as a formalization of the fusion in a finite group for a fixed prime $p$.

A $p$-local finite group consists on a triple $(S, F, L)$ where:

| $S$ is a $p$-group and plays the role of a Sylow $p$-subgroup. | $F$ is a category and models a fusion over $S$. | $L$ is a category with extra information in such a way that there is a classifying space. |

**Theorem (BLO)**

If $G$ is a finite group and $S$ is a Sylow $p$-subgroup, we can construct a $p$-local finite group $(S, \mathcal{F}_S(G), \mathcal{L}^c_S(G))$ and $|\mathcal{L}^c_S(G)|_p \wedge p \simeq B G^\wedge_p$. 

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\( p \)-local finite groups arose in the work of Broto-Levi-Oliver as a formalization of the fusion in a finite group for a fixed prime \( p \).

A \( p \)-local finite group consists on a triple \((S, \mathcal{F}, \mathcal{L})\) where:

| S is a \( p \)-group and plays the role of a Sylow \( p \)-subgroup. | \( \mathcal{F} \) is a category and models a fusion over \( S \). | \( \mathcal{L} \) is a category with extra information in such a way that there is a classifying space. |

\[ |\mathcal{L}^c_S(G)|_p^\wedge \cong BG_p^\wedge. \]

**Theorem (BLO)**

If \( G \) is a finite group and \( S \) is a Sylow \( p \)-subgroup, we can construct a \( p \)-local finite group \((S, \mathcal{F}_S(G), \mathcal{L}^c_S(G))\) and \( |\mathcal{L}^c_S(G)|_p^\wedge \cong BG_p^\wedge. \)
There are $p$-local finite groups which cannot be constructed from a finite group: exotic examples.

Construction of exotic examples:
- Solomon’s group. Levi-Oliver.
- Homotopic fixed points. Broto-Møller.
- Fusion subsystems. R.
There are $p$-local finite groups which cannot be constructed from a finite group: **exotic examples**.

**Construction of exotic examples:**

- **Combinatorial.** Broto-Levi-Oliver, R-Viruel i Díaz-R-Viruel.
- **Solomon’s group.** Levi-Oliver.
- **Homotopic fixed points.** Broto-Møller.
- **Fusion subsystems.** R.
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   - Application: Oliver-Ventura’s examples
Broto-Castellana-Grodal-Levi-Oliver study the saturated fusion subsystems (and the extensions) of a $p$-local finite group $(S, \mathcal{F}, \mathcal{L})$, and they give a description in the following cases:

- **$p$-power index subsystems**: these are the subsystems containing all the $\mathcal{F}$-automorphisms of $P$ of order prime to $p$, for all $P \leq S$.
- **Subsystems of index prime to $p$**: these are the subsystems containing all the $\mathcal{F}$-automorphisms of $P$ of $p$-power order, for all $P \leq S$. 

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Exotic normal fusion subsystems of the General Linear group
Saturated fusion subsystems of index prime to $p$

Notation:
- $O_{p'}^*(\mathcal{F})$ smallest subcategory in $\mathcal{F}$ with the same objects and containing all the restrictions of automorphisms of $p$-power order.
- $\text{Out}^0_{\mathcal{F}}(S) \overset{\text{def}}{=} \langle \alpha \in \text{Out}_\mathcal{F}(S) \mid \alpha|_P \in \text{Mor}_{O_{p'}^*(\mathcal{F})}(P, S), \text{some } \mathcal{F}\text{-centric } P \leq S \rangle$

Theorem (BCGLO)
There is a bijection between the saturated fusion subsystems of index prime to $p$ of $\mathcal{F}$ and the subgroups of $\Gamma_{p'}(\mathcal{F}) \overset{\text{def}}{=} \text{Out}_\mathcal{F}(S)/\text{Out}^0_{\mathcal{F}}(S)$.

Lemma
$\text{Out}^0_{\mathcal{F}}(S) = \langle \alpha \in \text{Out}_\mathcal{F}(S) \mid \alpha|_P \in \text{Mor}_{O_{p'}^*(\mathcal{F})}(P, S), \text{some } \mathcal{F}\text{-centric, } \mathcal{F}\text{-radical } P \leq S \rangle$

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Theorem (BCGLO)
There is a bijection between the saturated fusion subsystems of index prime to $p$ of $\mathcal{F}$ and the subgroups of $\Gamma_{p'}(\mathcal{F}) \overset{\text{def}}{=} \text{Out}_\mathcal{F}(S)/\text{Out}^0_\mathcal{F}(S)$.

Lemma
$\text{Out}^0_\mathcal{F}(S) = \langle \alpha \in \text{Out}_\mathcal{F}(S) \mid \alpha|_P \in \text{Mor}_{O^p_*(\mathcal{F})}(P, S), \text{some } S\text{-centralizer} P \leq S \rangle$

$P \leq S$ is $\mathcal{F}$-centric if $P$ and all its $\mathcal{F}$-conjugated subgroups contain their $S$-centralizer.
Saturated fusion subsystems of index prime to $p$

Notation:
- $O_{\ast}^p(\mathcal{F})$ smallest subcategory in $\mathcal{F}$ with the same objects and containing all the restrictions of automorphisms of $p$-power order.
- $\text{Out}_0^\mathcal{F}(S) \overset{\text{def}}{=} \langle \alpha \in \text{Out}_\mathcal{F}(S) \mid \alpha|_P \in \text{Mor}_{O_{\ast}^p(\mathcal{F})}(P, S), \text{some } \mathcal{F}\text{-centric } P \leq S \rangle$

Theorem (BCGLO)
There is a bijection between the saturated fusion subsystems of index prime to $p$ of $\mathcal{F}$ and the subgroups of $\Gamma_{p'}(\mathcal{F}) \overset{\text{def}}{=} \text{Out}_\mathcal{F}(S)/\text{Out}_0^\mathcal{F}(S)$.

Lemma
$\text{Out}_0^\mathcal{F}(S) = \langle \alpha \in \text{Out}_\mathcal{F}(S) \mid \alpha|_P \in \text{Mor}_{O_{\ast}^p(\mathcal{F})}(P, S), \text{some } \mathcal{F}\text{-centric, } \mathcal{F}\text{-radical } P \leq S \rangle$
Saturated fusion subsystems of index prime to $p$

Notation:
- $O_{p'}^*(\mathcal{F})$ smallest subcategory in $\mathcal{F}$ with the same objects and containing all the restrictions of automorphisms of $p$-power order.
  
  \[
  \text{Out}_0^0 F(S) \overset{\text{def}}{=} \langle \alpha \in \text{Out}_F(S) \mid \alpha|_P \in \text{Mor}_{O_{p'}^*(\mathcal{F})}(P, S), \text{some } \mathcal{F}\text{-centric } P \leq S \rangle 
  \]

Theorem (BCGLO)

There is a bijection between the saturated fusion subsystems of index prime to $p$ of $\mathcal{F}$ and the subgroups of

\[
\Gamma_{p'}(\mathcal{F}) \overset{\text{def}}{=} \text{Out}_F(S)/\text{Out}_0 F(S).
\]

Lemma

\[
\text{Out}_0^0 F(S) = \langle \alpha \in \text{Out}_F(S) \mid \alpha|_P \in \text{Mor}_{O_{p'}^*(\mathcal{F})}(P, S), \text{some } \mathcal{F}\text{-centric, } \mathcal{F}\text{-radical } P \leq S \rangle
\]
Saturated fusion subsystems of index prime to $p$

Notation:
- $O_{*}^{p'}(\mathcal{F})$ smallest subcategory in $\mathcal{F}$ with the same objects and containing all the restrictions of automorphisms of $p$-power order.
- $\text{Out}_{\mathcal{F}}^{0}(S) \overset{\text{def}}{=} \langle \alpha \in \text{Out}_{\mathcal{F}}(S) \mid \alpha|_{P} \in \text{Mor}_{O_{*}^{p'}(\mathcal{F})}(P, S), \text{ some } \mathcal{F}\text{-centric } P \leq S \rangle$

Theorem (BCGLO)

There is a bijection between the saturated fusion subsystems of index prime to $p$ of $\mathcal{F}$ and the subgroups of $\Gamma_{p'}(\mathcal{F}) \overset{\text{def}}{=} \text{Out}_{\mathcal{F}}(S)/\text{Out}_{\mathcal{F}}^{0}(S)$.

Lemma

$\text{Out}_{\mathcal{F}}^{0}(S) = \langle \alpha \in \text{Out}_{\mathcal{F}}(S) \mid \alpha|_{P} \in \text{Mor}_{O_{*}^{p'}(\mathcal{F})}(P, S), \text{ some } \mathcal{F}\text{-centric, } \mathcal{F}\text{-radical } P \leq S \rangle$
Saturated fusion subsystems of index prime to $p$

Notation:
- $O^p_*(\mathcal{F})$ smallest subcategory in $\mathcal{F}$ with the same objects and containing all the restrictions of automorphisms of $p$-power order.
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Theorem (BCGLO)

There is a bijection between the saturated fusion subsystems of index prime to $p$ and the subgroups of $\Gamma_{p^*}(\mathcal{F}) \overset{\text{def}}{=} \text{Out}_{\mathcal{F}}(S) / \text{Out}^0_{\mathcal{F}}(S)$.

Lemma

$P \leq S$ is $\mathcal{F}$-radical if $\text{Out}_{\mathcal{F}}(P)$ does not contain any nontrivial normal $p$-subgroup.
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**Remark 1**

If \((S, \mathcal{F}, \mathcal{L})\) is the fusion system of a finite group \(G\) over a Sylow \(p\)-subgroup \(S\), being \(\mathcal{F}\)-radical and being \(p\)-radical in \(G\) are independent definitions.

**Remark 2**

If \(P \leq S\) is a \(\mathcal{F}\)-radical, \(\mathcal{F}\)-centric subgroup, then \(P\) is \(p\)-radical in \(G\).

Alperin-Fong give a list of the \(p\)-radical subgroups in \(\text{GL}_n(q)\), where \(q\) is a prime power prime to \(p\), so we can use it as a list of the possible \(\mathcal{F}\)-centric, \(\mathcal{F}\)-radical subgroups.
Remark 1
If \((S, \mathcal{F}, \mathcal{L})\) is the fusion system of a finite group \(G\) over a Sylow \(p\)-subgroup \(S\), being \(\mathcal{F}\)-radical and being \(p\)-radical in \(G\) are independent definitions.

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If \(P \leq S\) is a \(\mathcal{F}\)-radical, \(\mathcal{F}\)-centric subgroup, then \(P\) is \(p\)-radical in \(G\).

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**p-radical subgroups in $GL_n(q)$ (AF)**

**Remark 1**

If $(S, F, L)$ is the fusion system of a finite group $G$ over a Sylow $p$-subgroup $S$, being $F$-radical and being $p$-radical in $G$ are independent definitions.

**Remark 2**

If $P \leq S$ is a $F$-radical, $F$-centric subgroup, then $P$ is $p$-radical in $G$.

Alperin-Fong give a list of the $p$-radical subgroups in $GL_n(q)$, where $q$ is a prime power prime to $p$, so we can use it as a list of the possible $F$-centric, $F$-radical subgroups.
Example 1

**Notation**

- $q$ a prime power,
- $p$ a prime such that $p \mid (q - 1)$.
- $l = \nu_p(q - 1)$ i.e. $p^l \mid (q - 1)$ and $p^{l+1} \nmid (q - 1)$.

**Possible $F$-centric, $F$-radical subgroups in $GL_p(q)$**

- A cyclic subgroup of order $p^l$ in the center of $GL_p(q)$.
- The *maximal torus*, generated by the diagonal matrices of $p$-power order.
- The *Sylow $p$-subgroup*, generated by the maximal torus and the permutation $(1, 2, \ldots, p)$.
- A subgroup isomorphic to an extension of a cyclic subgroup of order $p^l$ and a extraspecial subgroup $p^{1+2}$.
- A cyclic subgroup of order $p^{l+1}$. 
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Possible \( \mathcal{F} \)-centric, \( \mathcal{F} \)-radical subgroups in \( \text{GL}_p(q) \)

- A cyclic subgroup of order \( p^l \) in the center of \( \text{GL}_p(q) \).
- The \textit{maximal torus}, generated by the diagonal matrices of \( p \)-power order.
- The \textit{Sylow \( p \)-subgroup}, generated by the maximal torus and the permutation \((1, 2, \ldots, p)\).
- A subgroup isomorphic to an extension of a cyclic subgroup of order \( p^l \) and a extraspecial subgroup \( p^{1+2}_+ \).
- A cyclic subgroup of order \( p^{l+1} \).
Example e

Notation

- $q$ a prime power,
- $p$ a prime such that $p \nmid q$.
- $e$ the order of $q$ modulo $p$, so $p \mid (q^e - 1)$.
- $l = \nu_p(q^e - 1)$ i.e. $p^l \mid (q^e - 1)$ and $p^{l+1} \nmid (q^e - 1)$.

Possible $F$-centric, $F$-radical subgroups in $GL_{ep}(q)$

- The maximal torus, generated by $p$-power order $e \times e$ boxes in the diagonal.
- The Sylow $p$-subgroup, generated by the maximal torus and the permutation $(1, 2, \ldots, p)$.
- A subgroup isomorphic to a extension of a cyclic subgroup of order $p^l$ and a extraspecial $p^{1+2}$.
Example $e$

**Notation**

- $q$ a prime power,
- $p$ a prime such that $p \nmid q$.
- $e$ the order of $q$ modulo $p$, so $p \mid (q^e - 1)$.
- $l = \nu_p(q^e - 1)$ i.e. $p^l \mid (q^e - 1)$ and $p^{l+1} \nmid (q^e - 1)$.

**Possible $F$-centric, $F$-radical subgroups in $GL_{ep}(q)$**

- The *maximal torus*, generated by $p$-power order $e \times e$ boxes in the diagonal.
- The *Sylow $p$-subgroup*, generated by the maximal torus and the permutation $(1, 2, \ldots, p)$.
- A subgroup isomorphic to a extension of a cyclic subgroup of order $p^l$ and an extraspecial $p_{1+2}^{1+2}$.
Comparing examples 1 and $e$

**Similarities**
- $GL_p(q^e)$ and $GL_{ep}(q)$ have isomorphic Sylow $p$-subgroups.
- There is an inclusion $GL_p(q^e) \leq GL_{ep}(q)$, so the fusion of $GL_p(q^e)$ is contained in the fusion of $GL_{ep}(q)$.

**Differences**
- For any $p$-subgroup $P$, the difference between $Aut_{GL_p(q^e)}(P)$ and $Aut_{GL_{ep}(q)}(P)$ are copies of $\mathbb{Z}/e$.
- The Galois group of the extension $\mathbb{F}_q \leq \mathbb{F}_{q^e}$ is a cyclic subgroup of order $e$ which acts over the $\mathcal{F}$-centric, $\mathcal{F}$-radical subgroups in $GL_{ep}(q)$.
Comparing examples 1 and e

**Similarities**

- $\text{GL}_p(q^e)$ and $\text{GL}_{ep}(q)$ have isomorphic Sylow $p$-subgroups.
- There is an inclusion $\text{GL}_p(q^e) \leq \text{GL}_{ep}(q)$, so the fusion of $\text{GL}_p(q^e)$ is contained in the fusion of $\text{GL}_{ep}(q)$.

**Differences**

- For any $p$-subgroup $P$, the difference between $\text{Aut}_{\text{GL}_p(q^e)}(P)$ and $\text{Aut}_{\text{GL}_{ep}(q)}(P)$ are copies of $\mathbb{Z}/e$.
- The Galois group of the extension $\mathbb{F}_q \leq \mathbb{F}_{q^e}$ is a cyclic subgroup of order $e$ which acts over the $\mathcal{F}$-centric, $\mathcal{F}$-radical subgroups in $\text{GL}_{ep}(q)$. 
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   - Application: Oliver-Ventura’s examples
As before, \( q \) a prime power prime to \( p \), and \( e \) the order of \( q \) modulo \( p \).

\((S_n, q, \mathcal{F}_n, q, \mathcal{L}_n, q)\) the \( p \)-local finite group induced by \( GL_n(q) \) over \( S_n, q \) (a fixed Sylow \( p \)-subgroup).

**Theorem**

If \( n \geq ep \), then \( \Gamma_p'(\mathcal{F}_n, q) \cong \mathbb{Z}/e \).

**Corollary**

For each divisor \( r \) of \( e \), there exists a \( p \)-local finite group \((S_n, q, \mathcal{F}_n, q, r, \mathcal{L}_n, q, r)\) such that \( \mathcal{F}_n, q, r \) is a saturated fusion subsystem in \( \mathcal{F}_n, q \) of index prime to \( p \).
As before, \( q \) a prime power prime to \( p \), and \( e \) the order of \( q \) modulo \( p \).

\((S_n,q, \mathcal{F}_{n,q}, \mathcal{L}_{n,q})\) the \( p \)-local finite group induced by \( \text{GL}_n(q) \) over \( S_n,q \) (a fixed Sylow \( p \)-subgroup).

**Theorem**

If \( n \geq ep \), then \( \Gamma^p_2(\mathcal{F}_{n,q}) \cong \mathbb{Z}/e \).

**Corollary**

For each divisor \( r \) of \( e \), there exists a \( p \)-local finite group \((S_n,q, \mathcal{F}_{n,q},r, \mathcal{L}_{n,q},r)\) such that \( \mathcal{F}_{n,q},r \) is a saturated fusion subsystem in \( \mathcal{F}_{n,q} \) of index prime to \( p \).
Fusion subsystems

As before, $q$ a prime power prime to $p$, and $e$ the order of $q$ modulo $p$.

$(S_{n,q}, \mathcal{F}_{n,q}, \mathcal{L}_{n,q})$ the $p$-local finite group induced by $\text{GL}_n(q)$ over $S_{n,q}$ (a fixed Sylow $p$-subgroup).

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If $n \geq ep$, then $\Gamma_{p'}(\mathcal{F}_{n,q}) \cong \mathbb{Z}/e$.

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For each divisor $r$ of $e$, there exists a $p$-local finite group $(S_{n,q}, \mathcal{F}_{n,q,r}, \mathcal{L}_{n,q,r})$ such that $\mathcal{F}_{n,q,r}$ is a saturated fusion subsystem in $\mathcal{F}_{n,q}$ of index prime to $p$. 
Sketch of the proofs

We must compute $\Gamma_{p'}(\mathcal{F}) \overset{\text{def}}{=} \text{Out}_\mathcal{F}(S)/\text{Out}^0_\mathcal{F}(S)$, where:

- $S$ is the Sylow $p$-subgroup of $\text{GL}_n(q)$,
- $\mathcal{F}$ is the induced fusion by $\text{GL}_n(q)$ over $S$,
- $O_{p'}^*(\mathcal{F})$ is the smallest subcategory in $\mathcal{F}$ with the same objects and the restrictions of all the automorphisms of $p$-power order.

$$\text{Out}^0_\mathcal{F}(S) \overset{\text{def}}{=} \langle \alpha \in \text{Out}_\mathcal{F}(S) \mid \alpha|_P \in \text{Mor}_{O_{p'}^*(\mathcal{F})}(P, S), \text{ some } \mathcal{F}\text{-centric and } \mathcal{F}\text{-radical } P \leq S \rangle$$
Sketch of the proofs

Proof of the Theorem

1. Out$_F(S)$ is described in (AF): a product of semidirect products which involves copies of $\mathbb{Z}/e$, $\mathbb{Z}/(p-1)$ and symmetric groups $\Sigma_a$, with $a < p$.

2. First, consider $e = 1$ and prove Out$_0^F(S) = \text{Out}_F(S)$.

3. Then compare GL$_m(q^e)$ and GL$_{em}(q)$.

4. Prove that just a copy of $\mathbb{Z}/e$ survives (the one induced by the Galois group).

Proof of the corollary

Follows from BCGLO’s Theorem and the calculus of Out$_F(S)/\text{Out}_0^F(S)$.
### Sketch of the proofs

#### Proof of the Theorem

1. \( \text{Out}_F(S) \) is described in (AF): a product of semidirect products which involves copies of \( \mathbb{Z}/e, \mathbb{Z}/(p - 1) \) and symmetric groups \( \Sigma_a \), with \( a < p \).
2. First, consider \( e = 1 \) and prove \( \text{Out}^0_F(S) = \text{Out}_F(S) \).
3. Then compare \( \text{GL}_m(q^e) \) and \( \text{GL}_{em}(q) \).
4. Prove that just a copy of \( \mathbb{Z}/e \) survives (the one induced by the Galois group).

#### Proof of the corollary

Follows from BCGLO’s Theorem and the calculus of \( \text{Out}_F(S)/\text{Out}^0_F(S) \).
Two questions

1. Are there exotic examples of type \((S_n,q, F_{n,q,r}, \mathcal{L}_{n,q,r})\)?
2. Can we identify them?
Relation with $p$-compact groups

Let $p$ be a prime number, $r \geq 1$, $e \geq 1$ natural numbers such that $r|e|(p - 1)$. Let $G(e, r, m) \leq \text{GL}_m(\mathbb{Z}_p^\wedge)$ be the subgroup generated by:

$$A(e, r, m) \overset{\text{def}}{=} \{\text{diag}(a_1, \ldots, a_m) | a_i^e = 1 \text{ i } (a_1 \cdots a_m)^{e/r} = 1\}$$

and the permutation matrices.

Consider $BX(e, r, m)$ the $p$-compact group realizing the pseudoreflexion group $G(e, r, m)$. Finally consider the pullback:

$$BX(e, r, m)(q) \xrightarrow{\phi^q} BX(e, r, m)$$

$$BX(e, r, m) \xrightarrow{1 \times \phi^q} BX(e, r, m) \times BX(e, r, m)$$

where $\Delta$ the diagonal map and $\phi^q$ is an unstable Adams map of exponent $q$, a $p$-adic unit.
Relation with $p$-compact groups

Let $p$ be a prime number, $r \geq 1$, $e \geq 1$ natural numbers such that $r|e|(p-1)$.

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Finally consider the pullback:

$$\begin{array}{ccc}
BX(e, r, m)(q) & \overset{\rightarrow}{\longrightarrow} & BX(e, r, m) \\
\downarrow & & \downarrow \Delta \\
BX(e, r, m) & \overset{1 \times \varphi^q}{\longrightarrow} & BX(e, r, m) \times BX(e, r, m)
\end{array}$$

where $\Delta$ the diagonal map and $\varphi^q$ is an unstable Adams map of exponent $q$, a $p$-adic unit.
Relation with $p$-compact groups

Let $p$ be a prime number, $r \geq 1$, $e \geq 1$ natural numbers such that $r | e | (p - 1)$.

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and the permutation matrices.

Consider $BX(e, r, m)$ the $p$-compact group realizing the pseudoreflexion group $G(e, r, m)$.

Finally consider the pullback:

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Relation with $p$-compact groups

Let $p$ be a prime number, $r \geq 1$, $e \geq 1$ natural numbers such that $r|e|(p−1)$.

Let $G(e, r, m) \leq \text{GL}_m(\mathbb{Z}_p)$ be the subgroup generated by:

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and the permutation matrices.

Consider $BX(e, r, m)$ the $p$-compact group realizing the pseudoreflexion group $G(e, r, m)$.

Finally consider the pullback:

$$BX(e, r, m)(q) \to BX(e, r, m)$$

$$BX(e, r, m) \xrightarrow{1 \times \varphi^q} BX(e, r, m) \times BX(e, r, m)$$

where $\Delta$ the diagonal map and $\varphi^q$ is an unstable Adams map of exponent $q$, a $p$-adic unit.
Introduction

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GL

Subsystems in GL

Subsystems

Identifying the examples

Appl: OV’s examples

Relation with $p$-compact groups

Let $p$ be a prime number, $r \geq 1$, $e \geq 1$ natural numbers such that $r|e|(p-1)$.

Let $G(e, r, m) \leq \text{GL}_m(\mathbb{Z}_p^\wedge)$ be the subgroup generated by:

$$A(e, r, m) \overset{\text{def}}{=} \{ \text{diag}(a_1, \ldots, a_m) \mid a_1^{e_i} = 1 \ (i) \ (a_1 \cdot \ldots \cdot a_m)^{e/r} = 1 \}$$

and the permutation matrices.

Consider $BX(e, r, m)$ the pseudoreflexion group $G(e, r, m)$.

Finally consider the pullback:

$$BX(e, r, m)(q) \rightarrow BX(e, r, m)$$

where $\Delta$ the diagonal map and $\varphi^q$ is an unstable Adams map of exponent $q$, a $p$-adic unit.

Broto-Møller proved that this construction gives a $p$-local finite group.
Identifying \((S_{n,q}, F_{n,q,r}, L_{n,q,r})\)

**Theorem**

\[|L_{n,q,r}| \simeq BX(e, r, [n/e])(q^e) \text{ up to } p\text{-completion, where} BX(e, r, [n/e]) \text{ is a generalized Grassmannian.}\]

**Corollary**

If \(r > 2\), \((S_{n,q}, F_{n,q,r}, L_{n,q,r})\) is an exotic p-local finite group.

**Corollary**

There is a fibration:

\[|L_{n,q,r}| \to |L_{n,q}| \to B(\mathbb{Z}/r)\]

where the basis and the total space correspond to finite groups and the fibre is an exotic p-local finite group.
Identifying \((S_{n,q}, \mathcal{F}_{n,q,r}, \mathcal{L}_{n,q,r})\)

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Sketch of the proofs

Proof of the Theorem

1. Check the case \((S_{ep,q}, \mathcal{F}_{ep,q,r}, \mathcal{L}_{ep,q,r})\) from the description of the \(\mathcal{F}\)-centric, \(\mathcal{F}\)-radical subgroups and their \(\mathcal{F}\)-automorphisms.

2. Induction using the centralizers decomposition.

Proof of the first Corollary

Broto-Møller prove that \(BX(e, r, m)(q)\) is exotic for \(r \geq 2\).

Proof of the second Corollary

Broto-Castellana-Grodal-Levi-Oliver describe this fibration for any saturated fusion subsystem of index prime to \(p\).
Sketch of the proofs

**Proof of the Theorem**
1. Check the case \((S_{e\text{p},q}, \mathcal{F}_{e\text{p},q,r}, \mathcal{L}_{e\text{p},q,r})\) from the description of the \(\mathcal{F}\)-centric, \(\mathcal{F}\)-radical subgroups and their \(\mathcal{F}\)-automorphisms.
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### Sketch of the proofs

#### Proof of the Theorem

1. Check the case \((S_{ep,q}, F_{ep,q,r}, L_{ep,q,r})\) from the description of the \(F\)-centric, \(F\)-radical subgroups and their \(F\)-automorphisms.
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#### Proof of the second Corollary

Broto-Castellana-Grodal-Levi-Oliver describe this fibration for any saturated fusion subsystem of index prime to \(p\).
**Application: Oliver-Ventura’s examples**

**Example [OV]** Fix a finite group $G$ with $S \in \text{Syl}_p(G)$, and a finite $p$-group $A$. Fix

$$\chi: \pi_1(|\mathcal{L}_S^c(G)|) \longrightarrow \text{Aut}(A),$$

such that $|\text{Im}(\chi)|$ has order prime to $p$. Let $\mathcal{T}_1 \subseteq \mathcal{T}$ be the subcategory with the same objects, and with morphisms those $\varphi \in \text{Mor}(\mathcal{T})$ with $\chi(\varphi) = \text{Id}_A$. Let $\mathcal{F}_1^c \subseteq \mathcal{F}^c$ be the image of $\mathcal{T}_1$ in $\mathcal{F}$. Then $\mathcal{F}_1^c$ is a full subcategory of a saturated fusion system $\mathcal{F}_1 \subseteq \mathcal{F}$. If $\mathcal{F}_1$ is exotic, then for any extension

$$1 \longrightarrow A \longrightarrow \tilde{\mathcal{T}} \longrightarrow \mathcal{L}_S^c(G) \longrightarrow 1$$

where $\mathcal{L}_S^c(G)$ acts on $A$ via $\chi$, $\tilde{\mathcal{T}}$ is a centric linking system associated to an exotic saturated fusion system $\tilde{\mathcal{F}}$. 
Application: Oliver-Ventura’s examples

To construct new exotic examples they need:

- $p$ a prime, $q$ a $p$-power such that $e$, the order of $q$ modulo $p$, is bigger than 2: for example $p = 5$ and $q = 2$. Then $e = 4$.
- $G$ a finite group: $G = \text{GL}_{20}(2)$.
- $A$ a $p$-group, and a map $\chi: \pi_1(|L_{S}(G)|) \to \text{Aut}(A)$ such that $p \nmid |\text{Im}(\chi)|$.
- Moreover we need the “kernel” of $\chi$ to be exotic.

So consider $A = \mathbb{Z}/5$ ($\text{Aut}(A) = \mathbb{Z}/4$) and use the fibration:

$$|L_{20,2,4}| \to |L_{20,2}| \to B(\mathbb{Z}/4).$$

Then we can construct a new exotic example as an extension:

$$1 \to \mathbb{Z}/5 \to T \to L_{S}^{E}(\text{GL}_{20}(2)) \to 1$$

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