Exotic normal fusion subsystems of the General Linear group

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Groups Geometry Topology Málaga 06

Contents

Introduction

p-local finite groups arose in the work of Broto-Levi-Oliver as a formalization of the fusion in a finite group for a fixed prime p.

A p-local finite group consists on a triple $(S, \mathcal{F}, \mathcal{L})$ where:

S is a p-group and plays the role of a Sylow p-subgroup.	$\mathcal{F}$ is a category and models a fusion over S.	$\mathcal{L}$ is a category with extra information in such a way that there is a classifying space.

Theorem (BLO)

If G is a finite group and S is a Sylow p-subgroup, we can construct a p-local finite group $(S, \mathcal{F}_S(G), \mathcal{L}_S^c(G))$ and $|\mathcal{L}_S^c(G)|_p^\wedge \simeq BG_p^\wedge$.

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There are p-local finite groups which cannot be constructed from a finite group: exotic examples.

Construction of exotic examples:

- Combinatorial. Broto-Levi-Oliver, R-Viruel i Díaz-R-Viruel.
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Saturated fusion subsystems (BCGLO)

Broto-Castellana-Grodal-Levi-Oliver study the saturated fusion subsystems (and the extensions) of a p-local finite group $(S, \mathcal{F}, \mathcal{L})$, and they give a description in the following cases:

- p-power index subsystems: these are the subsystems containing all the $\mathcal{F}$-automorphisms of P of order prime to p, for all $P \leq S$.
- Subsystems of index prime to p: these are the subsystems containing all the $\mathcal{F}$-automorphisms of P of p-power order, for all $P \leq S$.

Saturated fusion subsystems of index prime to p

Notation:

- $O_*^{p'}(\mathcal{F})$ smallest subcategory in $\mathcal{F}$ with the same objects and containing all the restrictions of automorphisms of p-power order.
- $$\mathrm{Out}^0_{\mathcal{F}}(S) \stackrel{\text{def}}{=} \langle \alpha \in \mathrm{Out}_{\mathcal{F}}(S) \mid \alpha|_P \in \mathrm{Mor}_{O_*^{p'}(\mathcal{F})}(P,S), \text{ some } \mathcal{F}\text{-centric } P \leq S \rangle$$

Theorem (BCGLO)

There is a bijection between the saturated fusion subsystems of index prime to p of $\mathcal{F}$ and the subgroups of $\Gamma_{p'}(\mathcal{F}) \stackrel{\text{def}}{=} \mathrm{Out}_{\mathcal{F}}(S)/\mathrm{Out}^0_{\mathcal{F}}(S)$.

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$$\mathrm{Out}^0_{\mathcal{F}}(S) = \langle \alpha \in \mathrm{Out}_{\mathcal{F}}(S) \mid \alpha|_P \in \mathrm{Mor}_{O_*^{p'}(\mathcal{F})}(P,S), \text{ some } \mathcal{F}\text{-centric, } \mathcal{F}\text{-radical } P \leq S \rangle$$

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$P \leq S$ is $\mathcal{F}$-centric if P and all its $\mathcal{F}$-conjugated subgroups contain their S-centralizer.

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There is a bijection between the saturated fusion subsystems of index prime to p

$\Gamma_{p'}(\mathcal{F}) \stackrel{def}{=} \mathrm{Out}_{\mathcal{F}}($

$P \leq S$ is $\mathcal{F}$-radical if $\mathrm{Out}_{\mathcal{F}}(P)$ does not contain any nontrivial normal p-subgroup.

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p-radical subgroups in $\mathrm{GL}_n(q)$ (AF)

Remark 1

If $(S, \mathcal{F}, \mathcal{L})$ is the fusion system of a finite group G over a Sylow p-subgroup S, being $\mathcal{F}$-radical and being p-radical in G are independent definitions.

Remark 2

If $P \leq S$ is a $\mathcal{F}$-radical, $\mathcal{F}$-centric subgroup, then P is p-radical in G.

Alperin-Fong give a list of the p-radical subgroups in $\mathrm{GL}_n(q)$, where q is a prime power prime to p, so we can use it as a list of the possible $\mathcal{F}$-centric, $\mathcal{F}$-radical subgroups.

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Example 1

Notation

- q a prime power,
- p a prime such that $p|(q-1)$.
- $l = \nu_p(q-1)$ i.e. $p^l|(q-1)$ and $p^{l+1} \not|(q-1)$.

Possible $\mathcal{F}$-centric, $\mathcal{F}$-radical subgroups in $\mathrm{GL}_p(q)$

- A cyclic subgroup of order p^l in the center of $\mathrm{GL}_p(q)$.
- The *maximal torus*, generated by the diagonal matrices of p-power order.
- The *Sylow p-subgroup*, generated by the maximal torus and the permutation $(1, 2, \ldots, p)$.
- A subgroup isomorphic to an extension of a cyclic subgroup of order p^l and a extraspecial subgroup p_+^{1+2}.
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Possible $\mathcal{F}$-centric, $\mathcal{F}$-radical subgroups in $\mathrm{GL}_{ep}(q)$

- The *maximal torus*, generated by p-power order $e \times e$ boxes in the diagonal.
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Comparing examples 1 and e

Similarities

- $\mathrm{GL}_p(q^e)$ and $\mathrm{GL}_{ep}(q)$ have isomorphic Sylow p-subgroups.
- There is an inclusion $\mathrm{GL}_p(q^e) \leq \mathrm{GL}_{ep}(q)$, so the fusion of $\mathrm{GL}_p(q^e)$ is contained in the fusion of $\mathrm{GL}_{ep}(q)$.

Differences

- For any p-subgroup P, the difference between $\mathrm{Aut}_{\mathrm{GL}_p(q^e)}(P)$ and $\mathrm{Aut}_{\mathrm{GL}_{ep}(q)}(P)$ are copies of $\mathbb{Z}/e$.
- The Galois group of the extension $\mathbb{F}_q \leq \mathbb{F}_{q^e}$ is a cyclic subgroup of order e which acts over the $\mathcal{F}$-centric, $\mathcal{F}$-radical subgroups in $\mathrm{GL}_{ep}(q)$.

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Fusion subsystems

As before, q a prime power prime to p, and e the order of q modulo p.
$(S_{n,q}, \mathcal{F}_{n,q}, \mathcal{L}_{n,q})$ the p-local finite group induced by $\mathrm{GL}_n(q)$ over $S_{n,q}$ (a fixed Sylow p-subgroup).

Theorem

If $n \geq ep$, then $\Gamma_{p'}(\mathcal{F}_{n,q}) \cong \mathbb{Z}/e$.

Corollary

For each divisor r of e, there exists a p-local finite group $(S_{n,q}, \mathcal{F}_{n,q,r}, \mathcal{L}_{n,q,r})$ such that $\mathcal{F}_{n,q,r}$ is a saturated fusion subsystem in $\mathcal{F}_{n,q}$ of index prime to p.

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Sketch of the proofs

We must compute $\Gamma_{p'}(\mathcal{F}) \overset{\text{def}}{=} \mathrm{Out}_{\mathcal{F}}(S)/\,\mathrm{Out}^0_{\mathcal{F}}(S)$, where:

- S is the Sylow p-subgroup of $\mathrm{GL}_n(q)$,
- $\mathcal{F}$ is the induced fusion by $\mathrm{GL}_n(q)$ over S,
- $O^{p'}_*(\mathcal{F})$ is the smallest subcategory in $\mathcal{F}$ with the same objects and the restrictions of all the automorphisms of p-power order.
- $$\mathrm{Out}^0_{\mathcal{F}}(S) \overset{\text{def}}{=} \langle \alpha \in \mathrm{Out}_{\mathcal{F}}(S) \mid \alpha|_P \in \mathrm{Mor}_{O^{p'}_*(\mathcal{F})}(P,S), \text{ some } \mathcal{F}\text{-centric and } \mathcal{F}\text{-radical } P \leq S\rangle$$

Sketch of the proofs

Proof of the Theorem

1. $\mathrm{Out}_{\mathcal{F}}(S)$ is described in (AF): a product of semidirect products which involves copies of $\mathbb{Z}/e$, $\mathbb{Z}/(p-1)$ and symmetric groups Σ_a, with $a < p$.
2. First, consider $e = 1$ and prove $\mathrm{Out}^0_{\mathcal{F}}(S) = \mathrm{Out}_{\mathcal{F}}(S)$.
3. Then compare $\mathrm{GL}_m(q^e)$ and $\mathrm{GL}_{em}(q)$.
4. Prove that just a copy of $\mathbb{Z}/e$ survives (the one induced by the Galois group).

Proof of the corollary

Follows from BCGLO's Theorem and the calculus of $\mathrm{Out}_{\mathcal{F}}(S)/\mathrm{Out}^0_{\mathcal{F}}(S)$.

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Two questions

1. Are there exotic examples of type $(S_{n,q}, \mathcal{F}_{n,q,r}, \mathcal{L}_{n,q,r})$?
2. Can we identify them?

Relation with p-compact groups

Let p be a prime number, $r \geq 1$, $e \geq 1$ natural numbers such that $r|e|(p-1)$.

Let $G(e,r,m) \leq \mathrm{GL}_m(\mathbb{Z}_p^\wedge)$ be the subgroup generated by:

$$A(e,r,m) \stackrel{\text{def}}{=} \{\mathrm{diag}(a_1,\ldots,a_m) \mid a_i^e = 1 \text{ i } (a_1 \cdots a_m)^{e/r} = 1\}$$

and the permutation matrices.

Consider $BX(e,r,m)$ the p-compact group realizing the pseudoreflexion group $G(e,r,m)$.

Finally consider the pullback:

$$\begin{tikzcd}
BX(e,r,m)(q) \arrow[r] \arrow[d] & BX(e,r,m) \arrow[d, "\Delta"] \\
BX(e,r,m) \arrow[r, "1\times\varphi^q"] & BX(e,r,m) \times BX(e,r,m)
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where Δ the diagonal map and φ^q is an unstable Adams map of exponent q, a p-adic unit.

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Identifying $(S_{n,q}, \mathcal{F}_{n,q,r}, \mathcal{L}_{n,q,r})$

Theorem

$|\mathcal{L}_{n,q,r}| \simeq BX(e, r, [n/e])(q^e)$ up to p-completion, where $BX(e, r, [n/e])$ is a generalized Grassmannian.

Corollary

If $r > 2$, $(S_{n,q}, \mathcal{F}_{n,q,r}, \mathcal{L}_{n,q,r})$ is an exotic p-local finite group.

Corollary

There is a fibration:

$$|\mathcal{L}_{n,q,r}| \to |\mathcal{L}_{n,q}| \to B(\mathbb{Z}/r)$$

where the basis and the total space correspond to finite groups and the fibre is an exotic p-local finite group.

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If $r > 2$, $(S_{n,q}, \mathcal{F}_{n,q,r}, \mathcal{L}_{n,q,r})$ is an exotic p-local finite group.

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There is a fibration:

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where the basis and the total space correspond to finite groups and the fibre is an exotic p-local finite group.

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Sketch of the proofs

Proof of the Theorem

1. Check the case $(S_{ep,q}, \mathcal{F}_{ep,q,r}, \mathcal{L}_{ep,q,r})$ from the description of the $\mathcal{F}$-centric, $\mathcal{F}$-radical subgroups and their $\mathcal{F}$-automorphisms.
2. Induction using the centralizers decomposition.

Proof of the first Corollary

Broto-Møller prove that $BX(e, r, m)(q)$ is exotic for $r \geq 2$.

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Broto-Castellana-Grodal-Levi-Oliver describe this fibration for any saturated fusion subsystem of index prime to p.

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Application: Oliver-Ventura's examples

Example [OV] Fix a finite group G with $S \in \mathrm{Syl}_p(G)$, and a finite p-group A. Fix

$$\chi\colon \pi_1(|\mathcal{L}_S^c(G)|) \longrightarrow \mathrm{Aut}(A),$$

such that $|\mathrm{Im}(\chi)|$ has order prime to p. Let $\mathcal{T}_1 \subseteq \mathcal{T}$ be the subcategory with the same objects, and with morphisms those $\varphi \in \mathrm{Mor}(\mathcal{T})$ with $\chi(\varphi) = \mathrm{Id}_A$. Let $\mathcal{F}_1^c \subseteq \mathcal{F}^c$ be the image of $\mathcal{T}_1$ in $\mathcal{F}$. Then $\mathcal{F}_1^c$ is a full subcategory of a saturated fusion system $\mathcal{F}_1 \subseteq \mathcal{F}$. If $\mathcal{F}_1$ is exotic, then for any extension

$$1 \longrightarrow A \longrightarrow \tilde{\mathcal{T}} \longrightarrow \mathcal{L}_S^c(G) \longrightarrow 1$$

where $\mathcal{L}_S^c(G)$ acts on A via χ, $\tilde{\mathcal{T}}$ is a centric linking system associated to an exotic saturated fusion system $\tilde{\mathcal{F}}$.

Application: Oliver-Ventura's examples

To construct new exotic examples they need:

p a prime, q a p-power such that e, the order of q modulo p, is bigger than 2: for example $p = 5$ and $q = 2$. Then $e = 4$.

G a finite group: $G = \mathrm{GL}_{20}(2)$.

A a p-group, and a map $\chi\colon \pi_1(|\mathcal{L}_S^c(G)|) \to \mathrm{Aut}(A)$ such that $p \not| |\mathrm{Im}(\chi)|$.

Moreover we need the "kernel" of χ to be exotic.

So consider $A = \mathbb{Z}/5$ ($\mathrm{Aut}(A) = \mathbb{Z}/4$) and use the fibration:

$$|\mathcal{L}_{20,2,4}| \to |\mathcal{L}_{20,2}| \to B(\mathbb{Z}/4)\,.$$

Then we can construct a new exotic example as an extension:

$$1 \to \mathbb{Z}/5 \to \mathcal{T} \to \mathcal{L}_S^c(\mathrm{GL}_{20}(2)) \to 1$$

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