Gauss and geometry

Geodesy and non-euclidean geometry

AGUSTÍ REVENTÓS
may 29, 2006
Universität Stuttgart

Geometry

- I. Euclidean geometry.
- II. Non-euclidean geometry.
- III. Geodesy.
- IV. Differential geometry (Surfaces).
- V. DG-NEG.

Geometry

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- II. Non-euclidean geometry. Bolyai
- III. Geodesy.
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Geometry

- I. Euclidean geometry.
- II. Non-euclidean geometry. Bolyai
- III. Geodesy.
- IV. Differential geometry (Surfaces).
- V. DG-NEG. Bolyai

I. Euclidean geometry

1796

- March 29, 1796. Seventeen sides.
- Letter to Gerling 1819.

Das Geschichtliche jener Entdeckung ist bisher nirgends von mir öffentl erwähnt; ich kann es aber sehr genau angeben. Der Tag war der 29 *März* 1796, *und der Zufall hatte gar keinen Antheil daran.*

The history of that discovery is not so far anywhere mentioned. I can indicate very exactly. The day was March 29, 1796, and the coincidence has nothing to do with it.

1796

- March 29, 1796. Seventeen sides.
- Letter to Gerling 1819.

Schon früher war alles was auf die Zertheilung der Wurzeln der Gleichung

$$\frac{x^p - 1}{x - 1} = 0$$

in zwei Gruppen [...]

Everything was to divide the roots of the equation

1796

- March 29, 1796. Seventeen sides.
- Letter to Gerling 1819.

[...] glückte es mir bei einem Ferenaufenthalt in Braunschweig, am Morgen des gedachten Tages (ehe ich aus dem Bette aufgestanden war) diesen Zusammenhang auf das klarste anzuschauen, so dass ich die specielle Anwendung auf das 17-Eck und die numerische Bestätigung auf der Stelle machen konnte.

During a holidays in B. one morning (before I had risen from the bed) I saw clearly all the correlations and applied to the 17-gon the corresponding numerical confirmation.

The Diary

- The day after, March 30, 1796, he begins the Diary, one month before turning 19 years.

  The items [1],[3],[55],[65],[66],[116], talk about polygons.

Braunschweig

Göttingen

II. Non-euclidean geometry

1792

- Letter to Schumaker (09-28-1846)

Ein gewisser Schweikart nannte eine solche Geometrie Astralgeometrie, Lobatchevski imaginäire Geometrie. Sie wissen, dass ich schon seit 54 *Jahren (seit* 1792*) dieselbe Überzeugung habe.*

.. I have the same opinion already for 54 years (since 1792)

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.. I have the same opinion already for 54 years (since 1792)

He was 15!

1794

- Letter to Gerling (10-10-1846)

  Der Satz, den Ihnen Hr. Schweikart erwähnt hat, dass in jeder Geometrie die Summe aller äussern Polygonwinkel von $360°$ um eine Grösse verschieden ist, [....] welche dem Flächeninhalt proportional ist, ist der erste gleichsam an der Schwelle liegende Satz der Theorie, den ich schon im Jahr 1794 als nothwending erkannte.

  The theorem that Mr. S. mention to you, that in each geometry the sum .. is the first theorem on the threshold of this theory, which I already recognized in the year 1794.

The Diary

- July 28, 1797.

[72] *Plani possibilitatem demonstravi.*

The Diary

- July 28, 1797.

[72] *I have demonstrated the possibility of the plane.*

The Diary

- September 1799.

[99] *In principiis Geometriae egregios progressus fecimus.*

The Diary

- September 1799.

[99] *We have made exceptional progress in the principles of Geometry.*

Parallelentheorie

- Notes found among Gauss's papers 1831.
- We know what he thought on the subject from a few letters.
- Nevertheless, all the results in *Astralgeometry* that appear in these letters can be deduced directly from *Lambert analogy*.

Lambert (1728-1777)

- Lambert suggests that the *geometry of the acute angle* corresponds to the geometry on a sphere of imaginary radius.
- Gauss consults Lambert's work in the library of Göttingen on October 24, 1795 and on January 2, 1797.
- The *analogy* was developed by Taurinus (1794-1874).

Analogy

$\cos\frac{a}{R} = \cos\frac{b}{R} \cdot \cos\frac{c}{R}$	$\cosh\frac{a}{R} = \cosh\frac{b}{R} \cdot \cosh\frac{c}{R}$
A $= R^2(\alpha + \beta + \gamma - \pi)$	A$= R^2(\pi - (\alpha + \beta + \gamma))$
L$= 2\pi R \sin\frac{r}{R}$	L$= 2\pi R \sinh\frac{r}{R}.$

Default

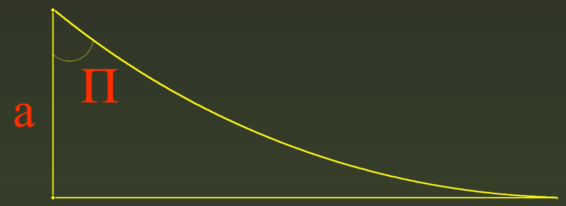
The analogy in trigonometry:

$$\cos\alpha = \frac{\cosh\frac{a}{R}}{1+\cosh\frac{a}{R}}$$

Hence $\alpha < \pi/3$.

- $\alpha + \alpha + \alpha < \pi$

The angle of parallelism



$$1 = \sin \Pi(a) \cosh \frac{a}{R}$$

$$\boxed{\Pi(a) = 2 \arctan e^{-a/R}}$$

Some letters

Letter to **Farkas Bolyai** (12-16-1799)

- *Wenn man beweisen könnte, dass ein geradliniges Dreieck möglich sei, dessen Inhalt grösser wäre als eine jede gegeben Fläche, so bin ich im Stande die ganze Geometrie völlig strenge zu beweisen. Die meisten würden nun whol jenes als ein Axiom gelten lassen; ich nicht;*

If one could prove that there exists a triangle of area as great as you want, then I am in conditions to prove rigorously the whole geometry. Many people could take this as an axiom, but I don't.

Letter to Gerling (11-04-1816)

- *Es wäre sogar wünschenswerth, dass die Geometrie Euklids nicht wahr wäre, weil wir dann ein allgemeines Mass a priori hätten, z. B. könnte man als Raumeinheit die Seite desjenigen gleichseitigen Dreiecks annehmen, dessen Winkel* $= 59°59'59''.99999$.

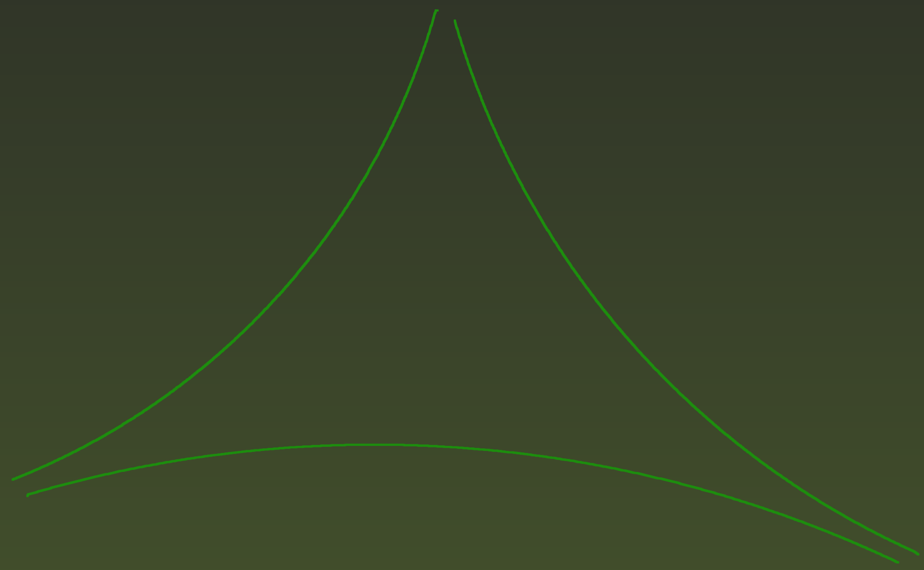
It would be even desirable that EG would not be true, because we would have a unit of measure a priori. For instance, the side of an equilateral triangle...

Letter to Gerling (16-03-1819)

- *Der Defect der Winkelsumme im ebenen Dreieck gegen* 180° *ist z. B. nicht bloss desto grösser, je grösser der Flächeninhalt ist, sondern ihm genau proportional, so dass der Flächeninhalt eine Grenze hat, die er nie erreichen kann, und welche Grenze selbst dem Inhalt der zwischen drei sich asymptotisch berührenden geraden Linien enthalten Fläche gleich ist, die Formel für diese Grenze ist*

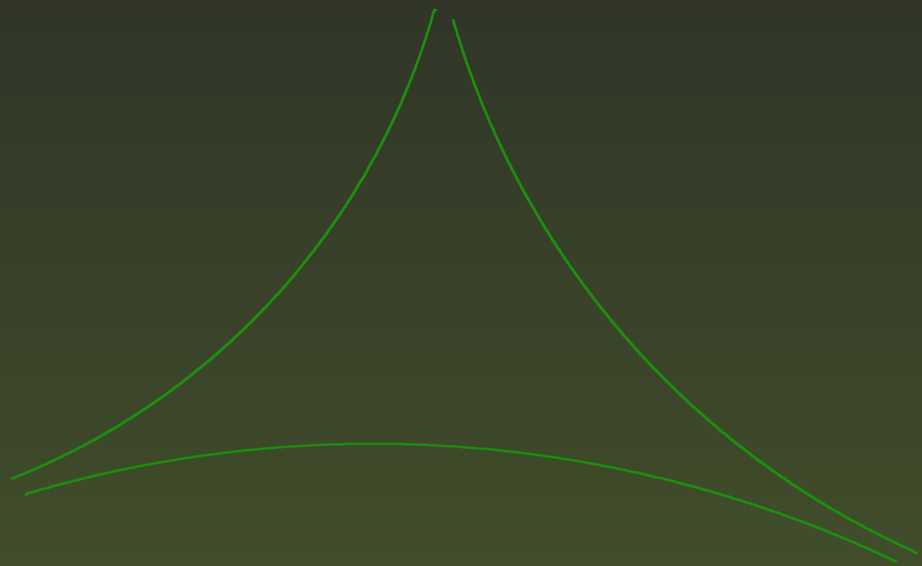
  The Defect is not only larger when the area is larger, but it is exactly proportional to it, in such a way that the area has a bound which it can never reach, and this bound is equal to the area enclosed by 3 asymptotic lines. The formula for this bound is

Same letter



$$\textit{Limes areae trianguli plani} = \frac{\pi CC}{(\log \text{hyp}(1+\sqrt{2}))^2}$$

Same letter



$$\text{\textit{Limes areae trianguli plani}} = \frac{\pi CC}{(\log\,\mathrm{hyp}(1+\sqrt{2}))^2}$$

$$\Pi(1) = \frac{\pi}{4}$$

Letter to **Schumaker** (05-17-1831)

- *Von meinen eigenen Meditationen, die zum Theil schon gegen 40 Jahr alt sind, wovon ich aber nie etwas aufgeschrieben habe, und daher manches 3 oder 4 mal von neuem auszusinnen genöthigt gewesen bin, habe ich vor einigen Wochen doch einiges aufzuschreiben angefangen. Ich wünschte doch, dass es nicht mit mir unterginge.*

In the last few weeks I have begun to put down a few of my own meditations, which are already to some extent nearly 40 years old. These I have never put in writing, so that I have been compelled 3 or 4 times to go over the whole matter afresh in my head. I wished that it should not perish with me.

Letter to Schumaker (07-12-1831)

- In der That ist in der Nicht-Euklidischen Geometrie der halbe Umfang eines Kreises, dessen Halbmesser $= r$:

$$L = \pi k (e^{r/k} - e^{-r/k}),$$

wo k eine Constante ist, von der wir durch Erfahrung wissen, dass sie gegen alles durch uns messbare ungeheuer gross sein muss. In Euklids Geometrie wird sie unendlich.

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- Gauss interrupts the writing in 1832, when he knew the work of János Bolyai.

Letter to Gerling (14-02-1832)

- *Noch bemerke ich, dass ich dieser Tage eine Schrift aus Ungarn über die Nicht-Euklidische Geometrie erhalten habe, worin ich alle meine eigenen Ideen und Resultate wiederfinde, mit grosser Eleganz entwicklet,*

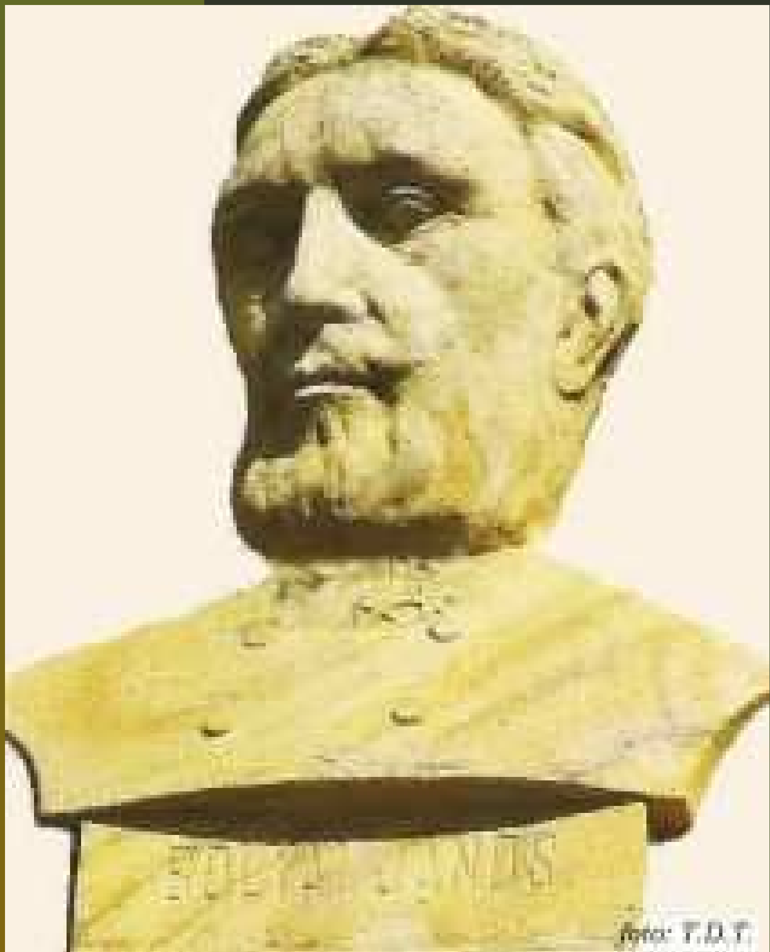
In addition I note that in recent days I received a small work from Hungary on NEG in which I find all of my ideas and results developed with great elegance

Letter to **Gerling** (14-02-1832)

- *Der Verfasser ist ein sehr junger ósterreichischer Officier, Sohn eines Jugendfreundes von mir, mit dem ich 1798 mich oft über die Sache unterhalten hatte, wiewohl damals meine Ideen noch viel weiter von der Ausbildung und Reife entfernt waren [...] Ich halte diesen jungen Geometer v. Bolyai für ein Genie erster Grösse...*

  The author is a very young Austrian officer, the son of a friend of my youth with whom I had often discussed the subject in 1798, although my ideas at that time were much less developed [...] I consider this young geometer, v, Bolyai, to be a genius of the first class.

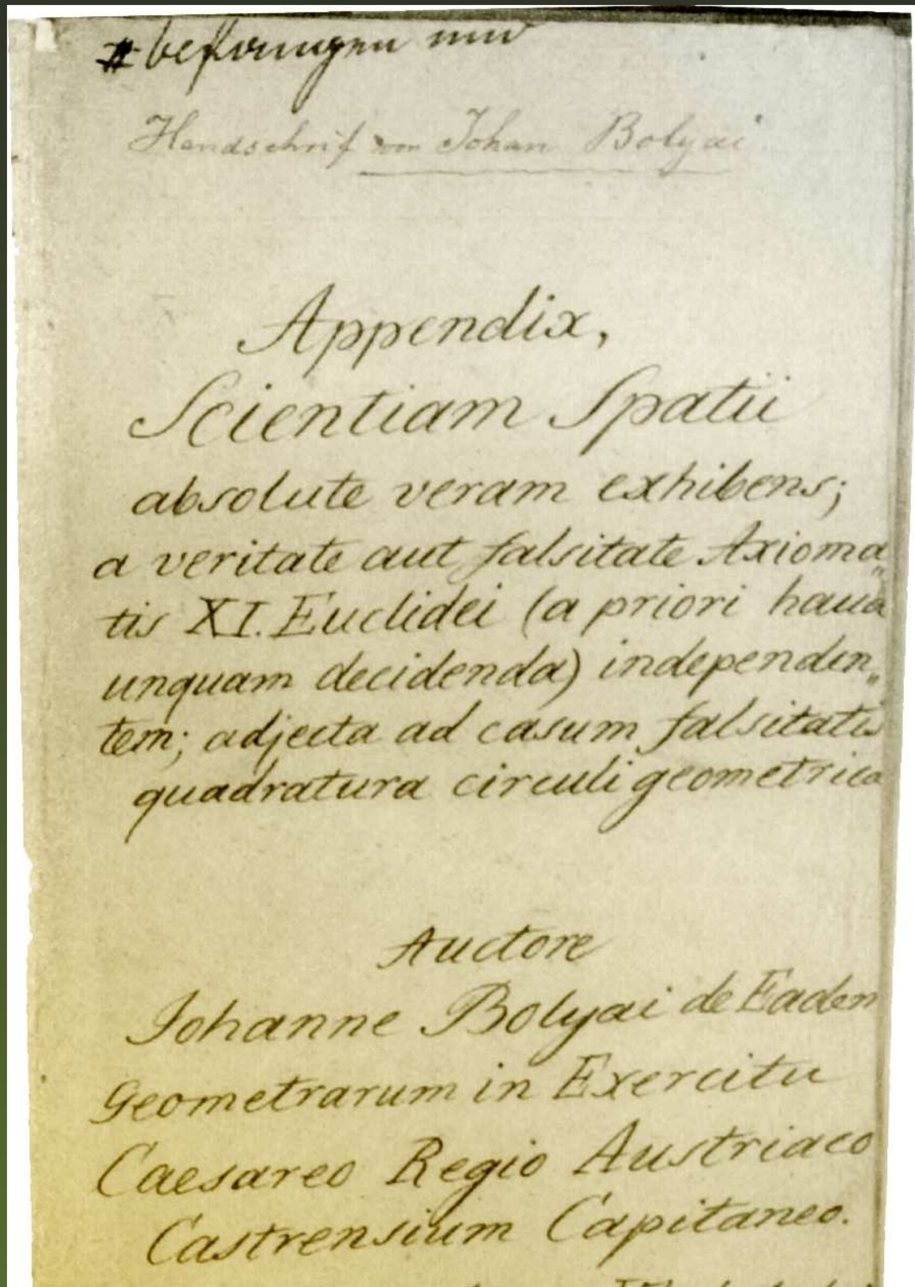
János Bolyai (1802-1860)



Farkas to János. April 1820

For God's sake! Leave parallels alone, abjure them like indecent talk, they may deprive you (just like me) from your time, health, tranquility and the happiness of your life.

Tentamen


Handschrif von Johan Bolyai

Appendix,
Scientiam Spatii
absolute veram exhibens;
a veritate aut falsitate Axioma-
tis XI. Euclidei (a priori haud
unquam decidenda) independen-
tem; adjecta ad casum falsitatis
quadratura circuli geometrica

Auctore
Johanne Bolyai de Eadem
Geometrarum in Exercitu
Caesareo Regio Austriaco
Castrensium Capitaneo.

Letter to Farkas Bolyai (6-03-1832)

- *Und höchst erfreulich ist es mir, dass gerade der Sohn meines alten Freundes es ist, der mir auf eine so merkwürdige Art zuvorgekommen ist.*

  And it is the greatest joy for me that precisely the son of my old friend is the one who preceded me in such a remarkable manner.

Letter to Farkas Bolyai (6-03-1832)

- *Und höchst erfreulich ist es mir, dass gerade der Sohn meines alten Freundes es ist, der mir auf eine so merkwürdige Art zuvorgekommen ist.*

  And it is the greatest joy for me that precisely the son of my old friend is the one who preceded me in such a remarkable manner.

- History could have changed if Gauss had made public his good opinion of the work of János Bolyai!

III. Geodesy

Geodesy

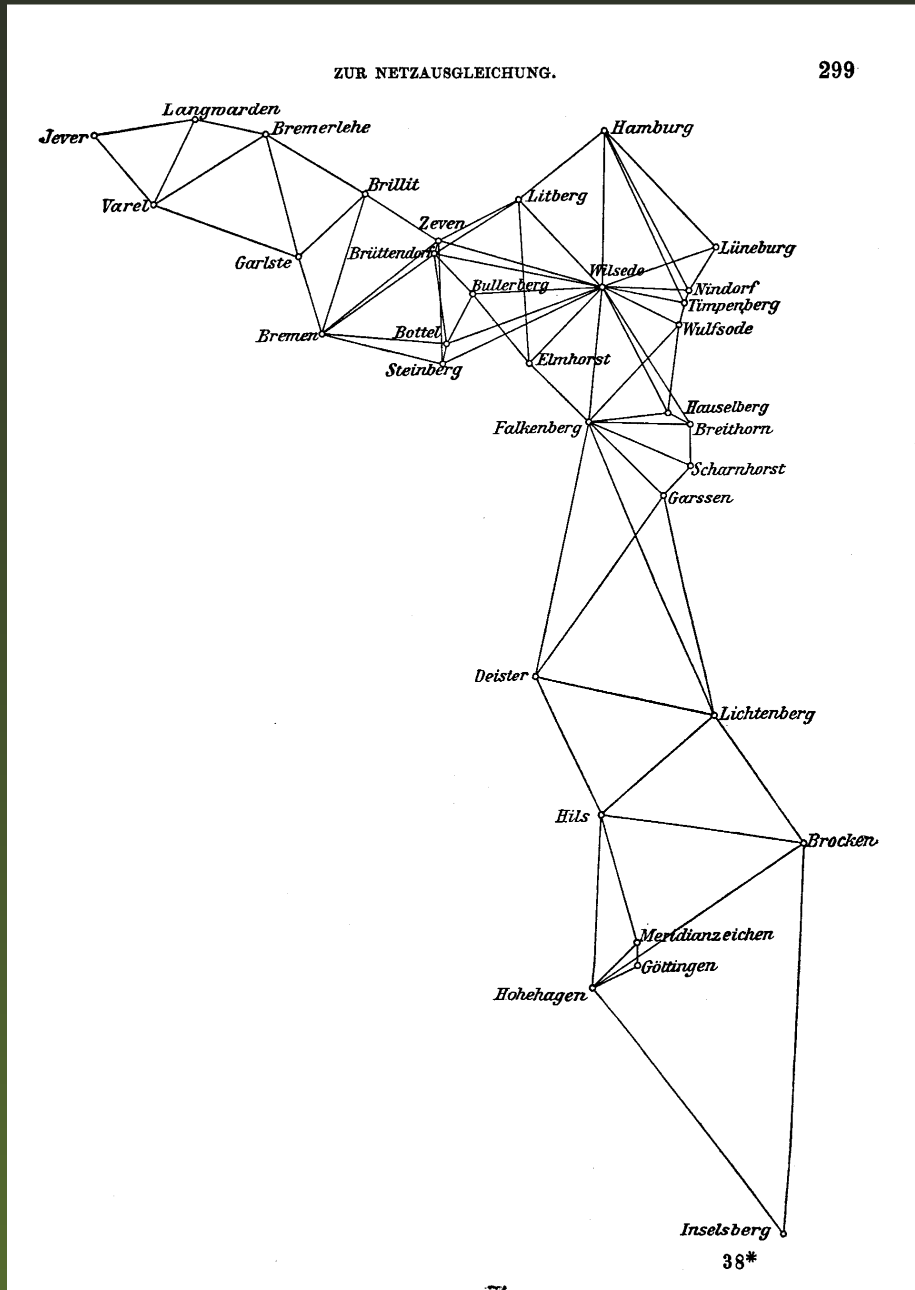
[Braunschweig museum]*The most refined geometer and the perfect astronomer-these are two separate titles which I highly esteem with all my heart.*

Hannover

Gauss was hired by King George III of Britain, Elector of Hannover, to survey Hannover to within an accuracy sufficient to produce useful maps. 1818.

- Devotes to this about 8 years.
- Use the method of least squares.
- Invents the heliotrope.
- The results were not sufficiently satisfactory (basic line).
- Bessel thought that Gauss's geodetic work should be carried on by one of lower mathematical stature .

Hannover


ZUR NETZAUSGLEICHUNG.
299
Langwarden
Jever
Bremerlehe
Varel
Brillit
Litberg
Hamburg
Zeven
Garlste
Brüttendorf
Lüneburg
Wilsede
Bullerberg
Nindorf
Timpenberg
Bremen
Bottel
Wulfsode
Elmhorst
Steinberg
Hauselberg
Falkenberg
Breithorn
Scharnhorst
Garssen
Deister
Lichtenberg
Hils
Brocken
Meridianzeichen
Göttingen
Hohehagen
Inselsberg
38*

Hannover

[3.]

[Zusammenstellung der beobachteten Dreiecke und ihrer Widersprüche.]

Nr.	Eck-punkt	Winkel	Excess	Nr.	Eck-punkt	Winkel	Excess	Nr.	Eck-punkt	Winkel	Excess
	1	115° 58′ 47,″435			8	23° 11′ 52,″206			12	27° 27′ 12,″046	
1	2	48 19 36, 048		10.	9	99 14 52, 452		19	13	148 10 28, 108	
	3	15 41 35, 239			10	57 33 19, 258			15	4 22 19, 354	
		179 59 58, 722	0,″158			180 0 3, 916	4,″245			179 59 59, 508	0,″321
		−1, 436				−0, 329				−0, 813	
	2	119 37 29, 268			9	87 32 16, 986			13	34 25 46, 752	
2	3	42 31 25, 667		11	10	21 0 11, 004		20	14	109 38 36, 566	
	4	17 51 7, 707			11	71 27 33, 968			15	35 55 37, 227	
		180 0 2, 642	1, 348			180 0 1, 958	0, 758			180 0 0, 545	1, 295
		+1, 294				+1, 200				−0, 750	
	3	52 29 10, 876			10	22 10 9, 986			14	80 10 54, 559	
3	4	84 40 26, 895		12	11	64 11 24, 606		21	15	15 24 48, 626	
	5	42 50 30, 659			12	93 38 25, 839			16	84 24 15, 820	
		180 0 8, 430	6, 568			180 0 0, 431	0, 759			179 59 59, 005	0, 349
		+1, 862				−0, 328				−1, 344	
	3	86 13 58, 366			10	8 0 47, 395			15	7 35 56, 089	
4	5	53 6 45, 642		13	12	28 17 42, 299		22	16	96 37 6, 464	
	6	40 39 30, 165			13	143 41 29, 140			17	75 46 59, 128	
		180 0 14, 173	14, 853			179 59 58, 834	0, 202			180 0 1, 681	0, 176
		−0, 680				−1, 368				+1, 505	

Hannover

1. $0 = -0{,}718 + 3(1) - (2)$
2. $0 = +0{,}647 - (1) + 3(2) + (3)$
3. $0 = +0{,}931 + (2) + 3(3) - (4) - (5)$
4. $0 = -0{,}340 - (3) + 3(4)$
5. $0 = -0{,}331 - (3) + 3(5) - (6)$
6. $0 = -0{,}032 - (5) + 3(6) - (7) - (8)$

(7. $0 = +0{,}233.5 - (6) + 3(7) + (8) + (9) - (10)$)

8. $0 = +0{,}510.5 - (6) + (7) + 3(8) - (9) + (10)$
9. $0 = -0{,}442 + (7) - (8) + 3(9) + (10) - (11)$
10. $0 = -0{,}164.5 - (7) + (8) + (9) + 3(10) - (11)$
11. $0 = +0{,}600 - (9) - (10) + 3(11) - (12)$
12. $0 = -0{,}164 - (11) + 3(12) - (13) - (14)$

(13. $0 = -0{,}684 - (12) + 3(13) + (14) - (15) - (16) - (19)$)

14. $0 = -0{,}569.5 - (12) + (13) + 3(14) + (16) + (17) - (18) + (19)$

(15. $0 = +0{,}886.5 - (13) + 3(15) + (16) - (17) + (20)$)

16. $0 = +0{,}521 - (13) + (14) + (15) + 3(16) + (17) - (18) - (19) - (20)$
17. $0 = -0{,}740.5 + (14) - (15) + (16) + 3(17) - (18) + (20) - (21)$
18. $0 = -0{,}918.5 - (14) - (16) - (17) + 3(18) - (27) - (28)$
19. $0 = -0{,}406.5 - (13) + (14) - (16) + 3(19) + (20)$
20. $0 = -0{,}375 + (15) - (16) + (17) + (19) + 3(20) - (21)$
21. $0 = -0{,}672 - (17) - (20) + 3(21) - (22) - (23)$
22. $0 = +0{,}752.5 - (21) + 3(22) + (23) - (24) - (25) + (38)$
23. $0 = -0{,}228 - (21) + (22) + 3(23) + (25) + (26) - (29) - (38)$
24. $0 = +0{,}660 - (22) + 3(24) + (25) - (26) + (39)$
25. $0 = +0{,}108 - (22) + (23) + (24) + 3(25) + (26) - (29) + (38) - (39)$
26. $0 = -0{,}688.5 + (23) - (24) + (25) + 3(26) - (29) + (39)$
27. $0 = -1{,}577.5 - (18) + 3(27) + (28) - (29) + (30) + (31)$
28. $0 = -0{,}914 - (18) + (27) + 3(28) - (32) + (33)$
29. $0 = -0{,}600.5 - (23) - (25) - (26) - (27) + 3(29) - (30) - (31)$
30. $0 = -0{,}378 + (27) - (29) + 3(30) + (31) - (32) - (34) + (35) - (40)$
31. $0 = -0{,}900 + (27) - (29) + (30) + 3(31) - (35) - (37) + (40)$
32. $0 = +0{,}290 - (28) - (30) + 3(32)$ [illegible]

Hannover

[8.]

Ausgleichungswerthe.

Nr.	Eck-punkt	Ausgeglichene Winkel	Log. der Seiten	Nr.	Eck-punkt	Ausgeglichene Winkel	Log. der Seiten
	1	115° 58′ 47″,885	4,221 7939		7	66° 1′ 19″,251	4,780 5184
1	2	48 19 36, 540	4,141 3507	7	8	66 39 58, 719	4,782 6578
	3	15 41 35, 731	3,700 2059		9	47 18 48, 840	4,686 0435
	2	119 37 28, 959	4,674 4426		7	55 34 15, 737	4,848 5425
2	3	42 31 25, 063	4,565 1592	8	8	89 51 50, 793	4,932 1822
	4	17 51 7, 328	4,221 7939		10	34 34 2, 142	4,686 0435
	3	52 29 10, 229	4,741 3374		7	10 27 3, 514	4,449 6103
3	4	84 40 26, 275	4,840 0752	9	9	146 33 41, 664	4,932 1822
	5	42 50 30, 066	4,674 4426		10	22 59 17, 205	4,782 6579
	3	86 13 58, 691	5,025 2012		8	23 11 52, 074	4,449 6103
4	5	53 6 45, 967	4,929 1248	10	9	99 14 52, 824	4,848 5425
	6	40 39 30, 195	4,840 0752		10	57 33 19, 347	4,780 5184
	4	49 57 23, 197	4,627 7548		9	87 32 16, 652	4,472 3562
5	5	46 6 58, 284	4,601 5606	11	10	21 0 10, 563	4,027 1430
	7	83 55 42, 791	4,741 3374		11	71 27 33, 545	4,449 6103

Hannover

Hannover

Konforme Abbildung

Konforme Abbildung

- Letter to Schumaker (5-07-1816).

Mir war eine interessante Aufgabe eingefallen, nemlich:

> *allgemein eine gegebene Fläche so auf einer andern (gegebenen) zu projiciren (abzubilden), dass das Bild dem Original in den kleinsten Theilen ähnlich werde.*

Ein specialler Fall ist, wenn die erste Fläche eine Kugel, die zweite eine Ebene ist. Hier sind die stereographische und die merkatorische Projectionen particuläre Auflösungen.

I have thought of an interesting problem: In the general case ...

Konforme Abbildung

- This question was published by the Copenhagen Scientific Society in 1821.
- The answer was given by Gauss himself on 11-12-1822.

Isotherme Flächenkoordinaten

- The day after, 12-12-1822, he writes in his private notes *Stand meiner untersuchung über die umformung der Flächen*:

$$k = -\frac{1}{2m}\left(\frac{\partial^2 \log m}{\partial u^2} + \frac{\partial^2 \log m}{\partial v^2}\right)$$

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$$k = -\frac{1}{2m}\left(\frac{\partial^2 \log m}{\partial u^2} + \frac{\partial^2 \log m}{\partial v^2}\right)$$

- $ds^2 = m(du^2 + dv^2)$. This formula does not appear (explicitly) in *Disquisitiones*.

Isotherme Flächenkoordinaten

- The day after, 12-12-1822, he writes in his private notes *Stand meiner untersuchung über die umformung der Flächen*:

$$k = -\frac{1}{2m}\left(\frac{\partial^2 \log m}{\partial u^2} + \frac{\partial^2 \log m}{\partial v^2}\right)$$

- *Das Krümmungsmass behält denselben Werth bei allen Umformungen der Fläche, die deren Linienelement* $m(du^2 + dv^2)$ *unverändert lassen.*

  The curvature keeps the same value under all transformations of the surface which leave the line element unchanged.

Konforme Abbildung

- The answer to the 1821 question, was given on 1822, but not published till 1825 in Astronomische Abhandlungen, Altona:

  Allgemeine Auflösung der Aufgabe die Theile einer gegebnen Fläche auf einer andern gegebnen Fläche so abzubilden, dass die Abbildung dem Abgebildeten in den Kleinsten Theilen ähnlich wird.

A general solution to the problem of mapping the parts of a given surface onto another surface such that the image and the mapped part are similar in the smallest parts.

Konforme Abbildung

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  Allgemeine Auflösung der Aufgabe die Theile einer gegebnen Fläche auf einer andern gegebnen Fläche so abzubilden, dass die Abbildung dem Abgebildeten in den Kleinsten Theilen ähnlich wird.

Ab his via sternitur ad maiora.

Following Newton in ’De quadratura curvarum’, prelude of the calculus of fluxions.

Ab his via sternitur ad maiora

- Kugel $\longrightarrow$ Ebene
- $ds^2 = a^2 \sin^2 u \, dt^2 + a^2 du^2$

Ab his via sternitur ad maiora

- Kugel $\longrightarrow$ Ebene
- $ds^2 = a^2 \sin^2 u \, dt^2 + a^2 du^2$
- Put $ds^2 = 0$ and isolate dt.
- $dt = \pm i \frac{du}{\sin u}$ $\quad (dt + i \frac{du}{\sin u})(dt - i \frac{du}{\sin u}) = \frac{1}{a^2 \sin^2 u} ds^2$

Ab his via sternitur ad maiora

- Kugel $\longrightarrow$ Ebene
- $ds^2 = a^2 \sin^2 u \, dt^2 + a^2 du^2$
- Put $ds^2 = 0$ and isolate dt.
- $dt = \pm i \frac{du}{\sin u}$ $\quad (dt + i\frac{du}{\sin u})(dt - i\frac{du}{\sin u}) = \frac{1}{a^2 \sin^2 u} ds^2$
- Integrating
- $t \pm i \log \cot \frac{u}{2} = constant$

Ab his via sternitur ad maiora

- Kugel $\longrightarrow$ Ebene
- $ds^2 = a^2 \sin^2 u \, dt^2 + a^2 du^2$
- Put $ds^2 = 0$ and isolate dt.
- $dt = \pm i \frac{du}{\sin u}$ $\qquad (dt + i\frac{du}{\sin u})(dt - i\frac{du}{\sin u}) = \frac{1}{a^2 \sin^2 u} ds^2$
- Integrating
- $t \pm i \log \cot \frac{u}{2} = constant$
- $p = t, \; q = \log \cot \frac{u}{2}$
- $ds^2 = a^2 \sin^2 u (dp^2 + dq^2)$

Ab his via sternitur ad maiora

$$P+iQ=f(p+iq)$$

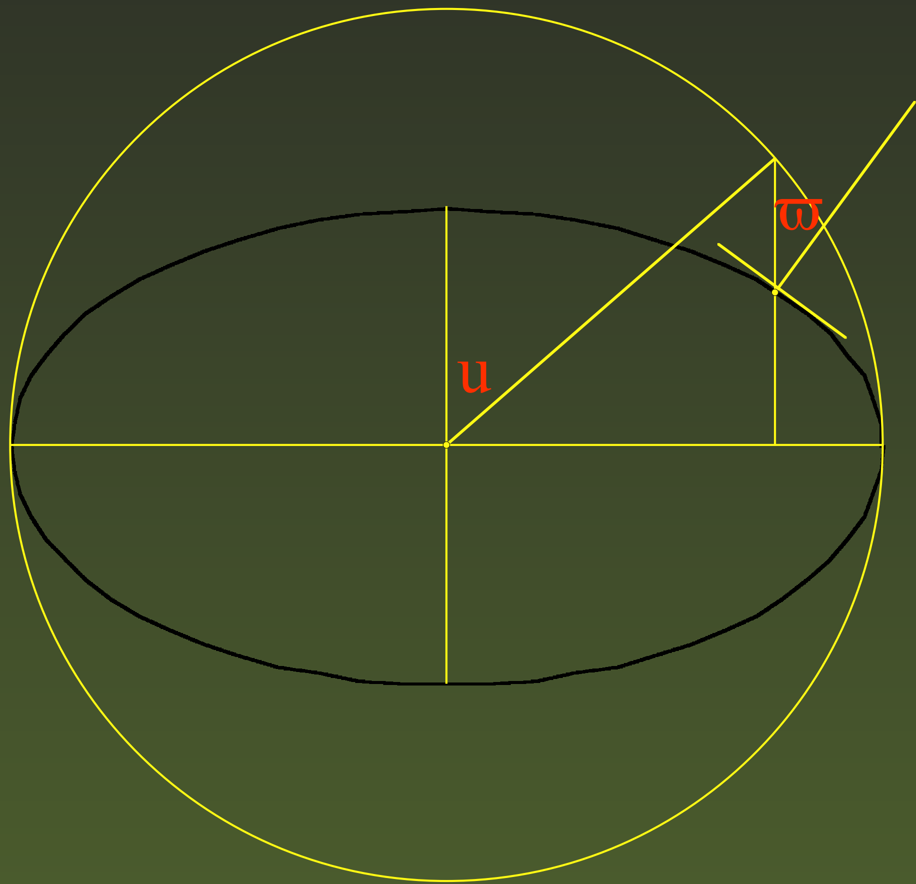
Ab his via sternitur ad maiora

- Ellipsoid $\longrightarrow$ Kugel

$$T + i \log\cot\frac{U}{2} = f(t + i\log(\cot\frac{\omega}{2}\cdot(\frac{1-e\cos\omega}{1+e\cos\omega})^{\frac{e}{2}}))$$

- T und $90 - U$ die Länge und Breite auf dem Kugelfläche.
- t und $90 - \omega$ die Länge und Breite auf dem Ellipsoid.

Ab his via sternitur ad maiora



$$\tan\omega = \frac{b}{a}\tan u$$

Höhern Geodaesie

Untersuchungen über Gegenstände der Höhern Geodaesie. I, II.

1844 and 1847 in *Abhanlungen der Königl. Gesellschaft der Wissenschaften zu Göttingen.*

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- I. Ellipsoid $\longrightarrow$ Kugel.
- II. Ellipsoidische Trigonometrie.

Höhern Geodaesie I

- He takes as holomorphic map $f(z) = \alpha z - i \log k$, with α, k and A (radius of the sphere) determined in order to have isometry ($m = 1$) on the mean parallel of Hannover, $Q = 52^\circ\ 40'\ 0''$.

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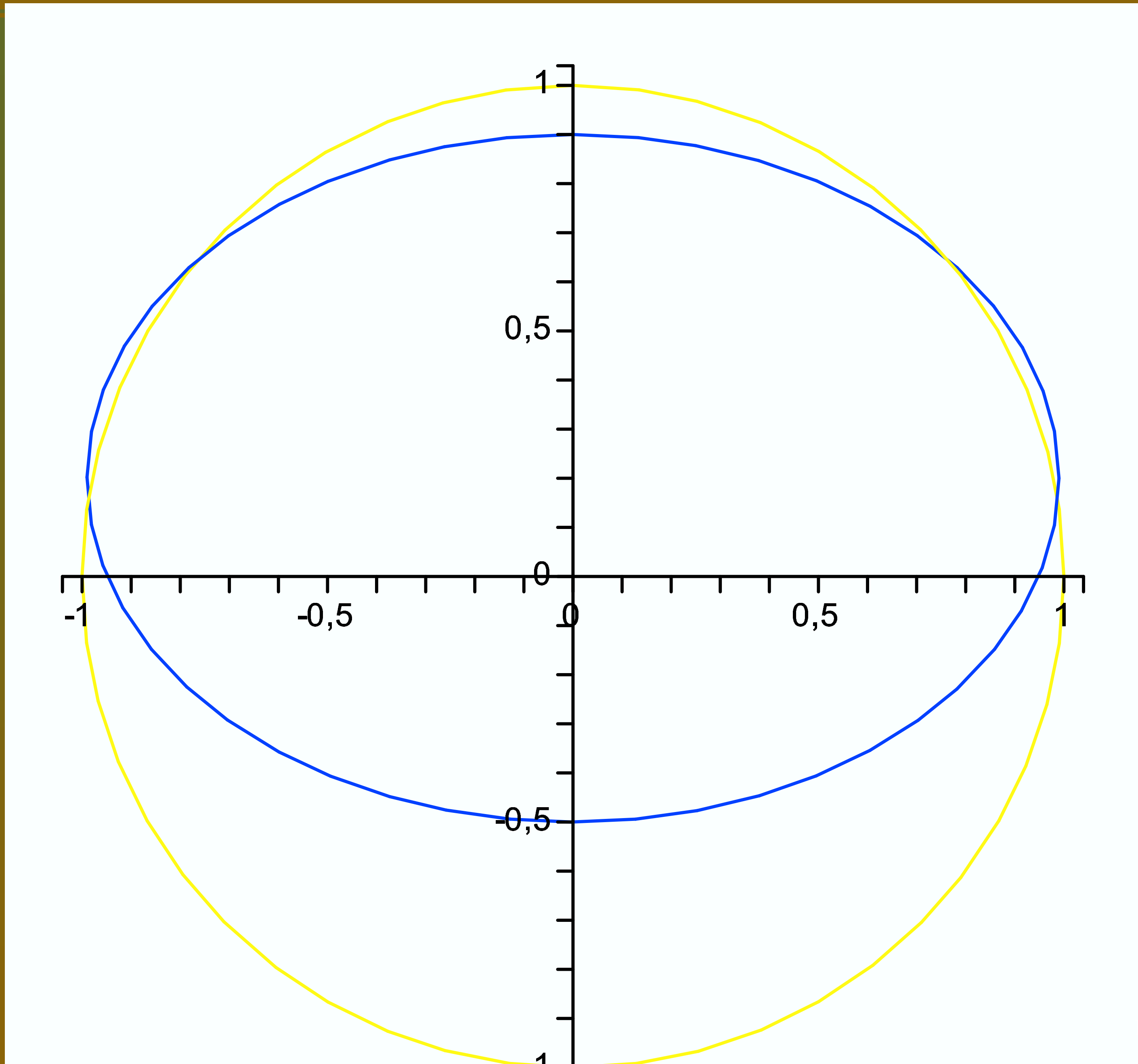
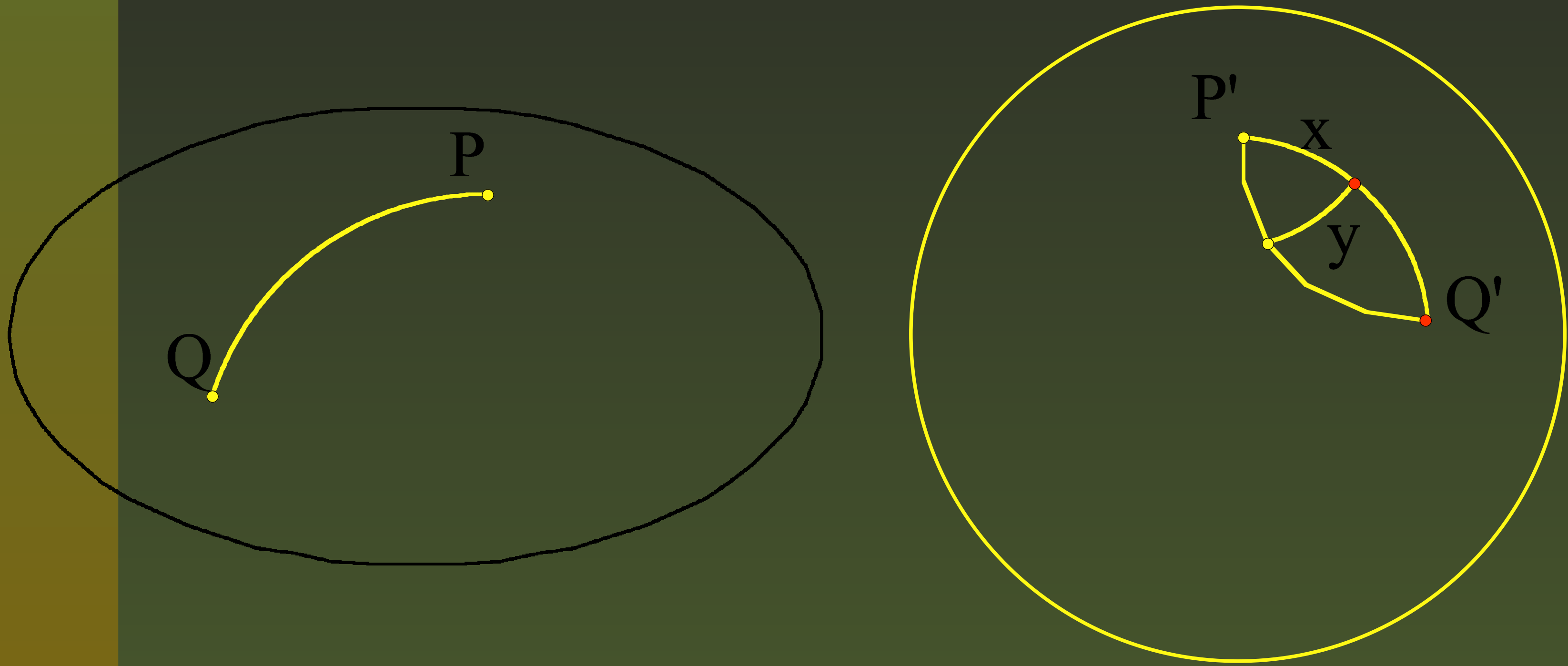
$$
\begin{aligned}
\log a &= 6.5148235337 \quad \text{Toise} \\
\log e &= 8.9122052079 \\
P &= 52^o\ 42'\ 2.53251'' \\
\log \frac{1}{k} &= 0.0016708804 \\
\log \alpha &= 0.0001966553 \\
\log A &= 6.5152074703 \quad \text{Toise}
\end{aligned}
$$

Höhern Geodaesie I



Höhern Geodaesie I



$$d(P,Q) = \frac{d(P',Q')}{\sqrt{m_p m_Q}}$$

IV. Differential geometry

Conjecture

- Gauss could have extended the Lambert *analogy* to differential geometry, in order to find a surface representing the imaginary sphere. In particular, of negative curvature.

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  > [...] *aber die Entwickelung ist auf anderm Wege gemacht*

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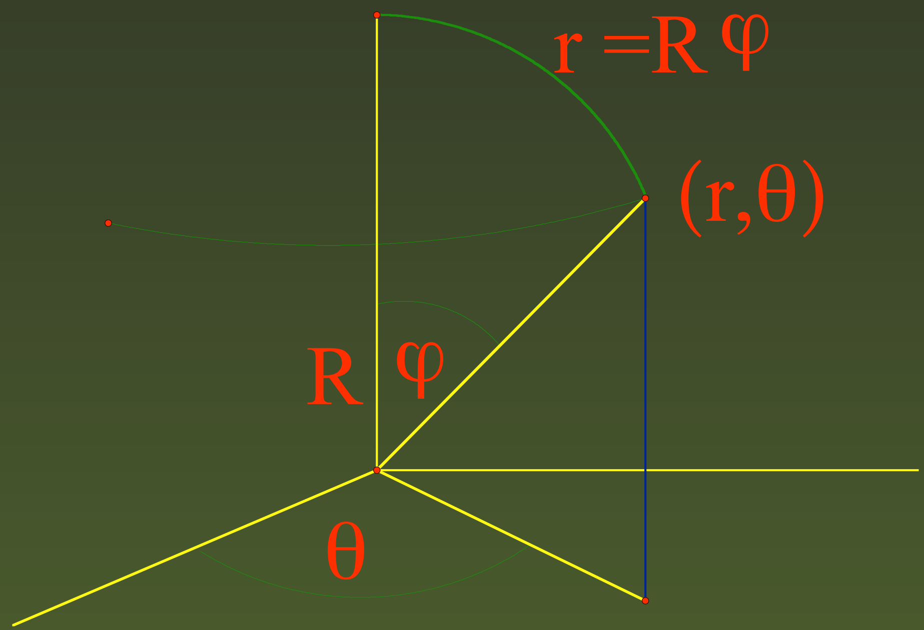
- Was the Disquisitiones written [partially] with this idea?

The length element

- $ds^2 = dr^2 + R^2 \sin^2(\frac{r}{R})\, d\theta^2$



Objective

- To find a surface with

$$ds^2 = dr^2 + R^2 \sinh^2(\frac{r}{R})\, d\theta^2$$

Disquisitiones Generales Circa Superficies Curvas

October 7, 1827

Letter to Schumaker (21-11-1825)

- *Ich habe seit einiger Zeit angefangen, einen Theil der allgemeinen Untersuchungen über die krummen Flächen wieder vorzunehmen, die die Grundlage meines projectirten Werks über Höhere Geodäsie werden sollen.*

Recently I have taken up again the study of curved surfaces as a basis of my projected essay on advanced geodesy.

Letter to **Schumaker** (21-11-1825)

- *Ich finde leider, dass ich dabei sehr weit werde ausholen müssen, da auch das Bekannte in einer andern, den neuen Untersuchungen anpassenden Form entwickelt werden muss.*

Unfortunately I have to go far back in the exposition because even what is known must be developed in another different way adapted to the new investigations.

Disquisitiones Generales

- 40 pages; 29 sections.
- 5 new (?) concepts; 10 theorems.
- He mentions Euler (§8), and Legendre (§27).
- The only surface that appears is the sphere.

Unfinished project (?)

- *We must reserve for another occasion the more extended exposition of the the theory of these figures...* §6
- *The study of which opens to geometry a new a fertile field...* §13
- *The consideration of the rectilinear triangle whose sides are equal to* a, b, c *is of great advantage...* §26

Unfinished project (?)

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- *The study of which opens to geometry a new a fertile field...* §13
- *The consideration of the rectilinear triangle whose sides are equal to* a, b, c *is of great advantage...* §26
- He does not find the imaginary sphere.

New concepts

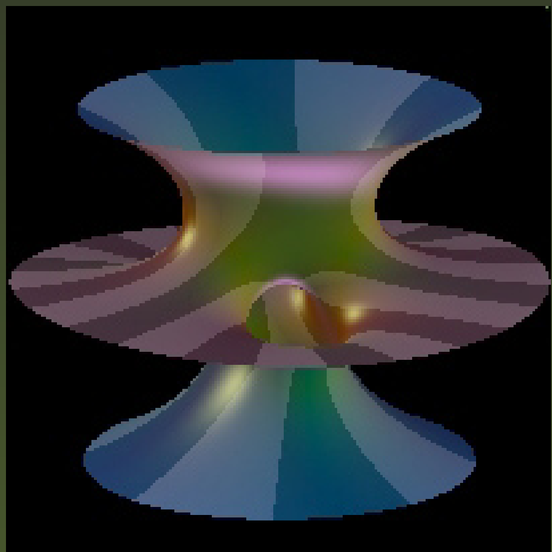
- Gauss map. §6
- Gauss curvature. §6
- Total curvature. §6
- Angular variation. §17
- Geodesic-abscissa orthogonal chart. §19

New theorems

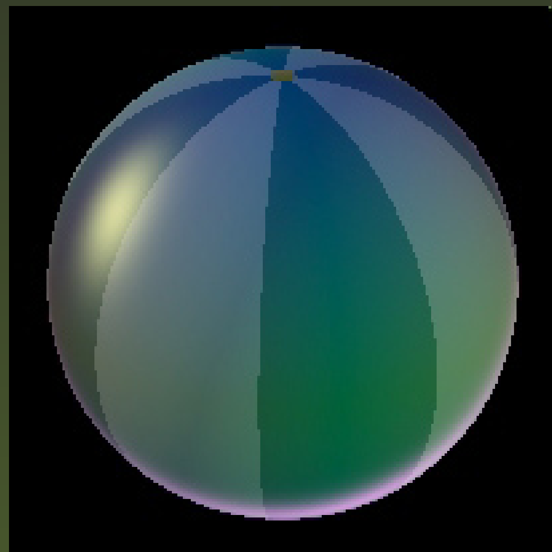
- $k = \det \Phi_2 / \det T_2$. §7
- $k = k_1 \cdot k_2$. §8
- The egregium theorem. §12
- Gauss lemma. §16
- $k = -\frac{1}{\sqrt{G}}(\sqrt{G})_{rr}, \qquad \frac{d\gamma}{d\theta} = -(\sqrt{G})_r$. §19
- Default theorem. §20
- $A^* = A - \frac{1}{12}\sigma(2k(A) + k(B) + k(C))$. §27

Curvature. §6

$\gamma : S \longrightarrow S^2$ Gauss map.



$\longrightarrow$



$$k(P) = \lim_{S \to P} \frac{\text{Area of } \gamma(S)}{\text{Àrea de } S}$$

Euler curvature. §8

THEOREM. *The measure of curvature at any point whatever of the surface is equal to a fraction whose numerator is unity, and whose denominator is the product of the two extreme radii of curvature of the sections by normal planes.*

$$k = k_1 \cdot k_2$$

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THEOREM. *The measure of curvature at any point whatever of the surface is equal to a fraction whose numerator is unity, and whose denominator is the product of the two extreme radii of curvature of the sections by normal planes.*

$$k = k_1 \cdot k_2$$

Just before he says: *These conclusions contain almost all that the illustrious Euler was the first to prove on the curvature of curved surfaces.*

Olinde Rodrigues (1794-1851)

Recherches sur la théorie analytique des lignes et des rayons de courbure des surfaces, et sur la transformation d'une class d'intégrales doubles, qui ont un rapport direct avec les formules de cette théorie, Correspondance sur l'Ecole Polytechnique, Vol 3, pag.$162-182$, 1815.

Olinde Rodrigues (1794-1851)

- Gauss map.
- Gauss curvature.
- $k = k_1 \cdot k_2$.
- $N'(t) = \lambda x'(t)$.

Olinde Rodrigues (1794-1851)

- Gauss map.
- Gauss curvature.
- $k = k_1 \cdot k_2$.
- $N'(t) = \lambda x'(t)$.

- Gauss knew the 3 first points before 1813 (not published).

The egregium theorem. §11

$$
\begin{aligned}
4\quad & (EG - FF)^2 \, k = E\left(\frac{dE}{dq}\cdot\frac{dG}{dq} - 2\frac{dF}{dp}\frac{dG}{dq} + \left(\frac{dG}{dp}\right)^2\right) \\
+\quad & F\left(\frac{dE}{dp}\frac{dG}{dq} - \frac{dE}{dq}\frac{dG}{dp} - 2\frac{dE}{dq}\frac{dF}{dq} + 4\frac{dF}{dp}\frac{dF}{dq} - 2\frac{dF}{dp}\frac{dG}{dp}\right) \\
+\quad & G\left(\frac{dE}{dp}\frac{dG}{dp} - 2\frac{dE}{dp}\frac{dF}{dq} + \left(\frac{dE}{dq}\right)^2\right) \\
-\quad & 2(EG - FF)\left(\frac{ddE}{dq^2} - 2\frac{ddF}{dp\cdot dq} + \frac{ddG}{dp^2}\right).
\end{aligned}
$$

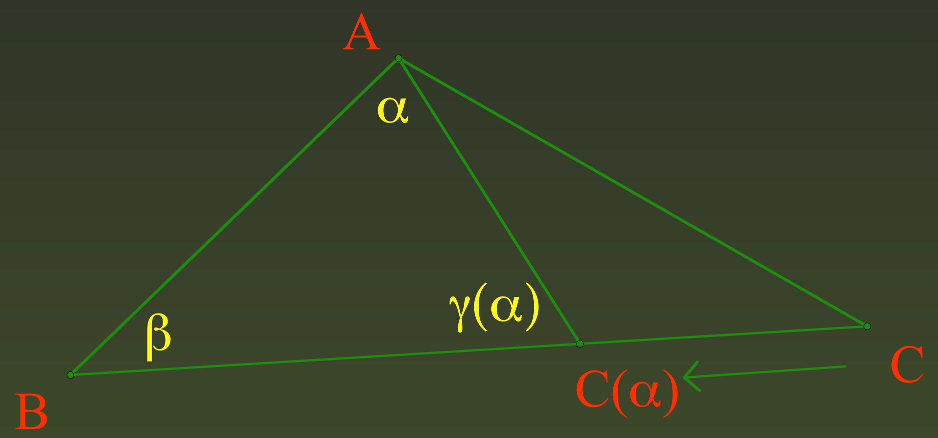
The egregium theorem. §12

Formula itaque art. prae. sponte perducit ad egregium

THEOREMA *Si superficies curva in quamcunque aliam superficiem explicatur, mensura curvaturae in singulis punctis invariate manet.*

Angular variation

Angular variation in the plane



α+β+γ=π;   1+γ' =0

$$\boxed{\frac{d\gamma}{d\alpha} = -1}$$

- Observe $\gamma(0) = \pi - \beta$.

Angular variation in the sphere

$$\frac{d\gamma}{d\alpha} = -\cos\frac{b(\alpha)}{R}$$

Area of a triangle in $S^2(R)$

$$\text{Area} = \int_0^{\alpha}\int_0^{r(\theta)} R\sin\frac{r}{R}\,dr\,d\theta$$

$$= R^2\alpha - \int_0^{\alpha} R^2\cos\frac{r(\theta)}{R}\,d\theta$$

Area of a triangle in $S^2(R)$

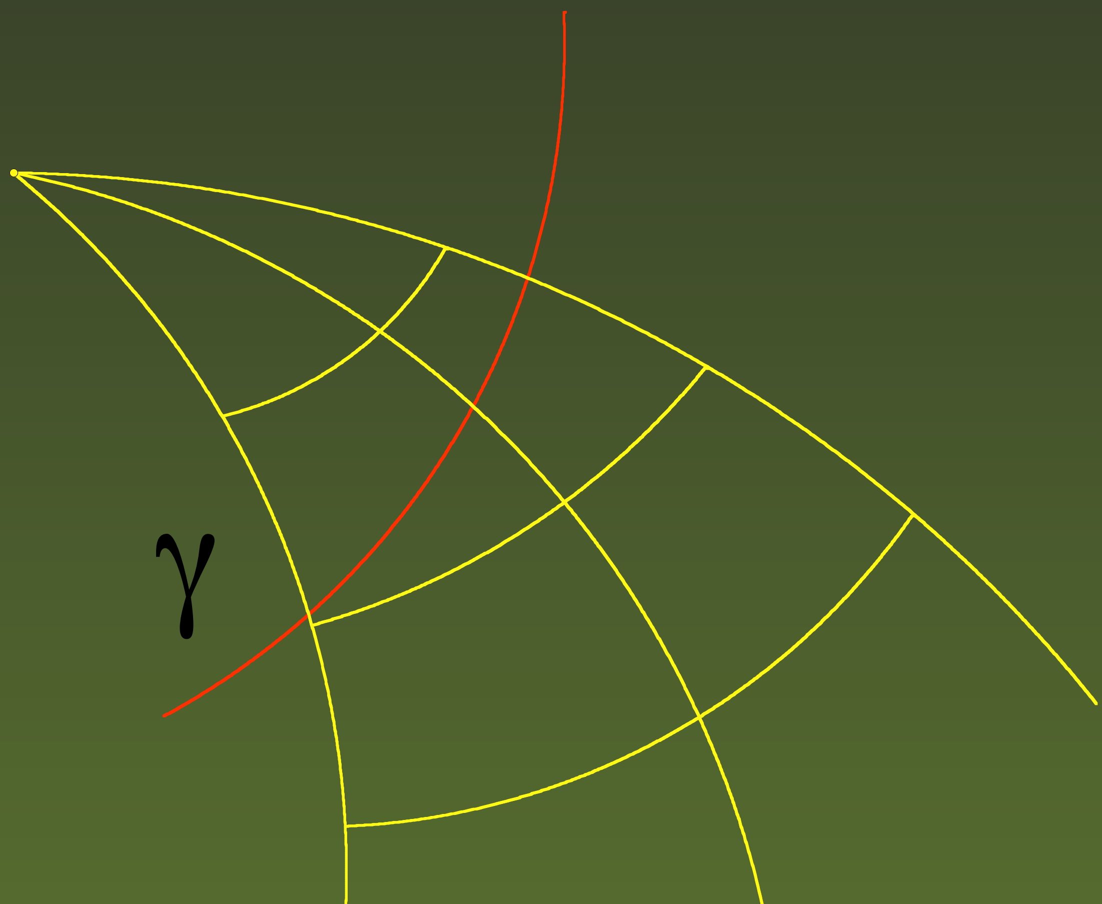
$$\begin{aligned}
\text{Area} &= \int_0^{\alpha}\int_0^{r(\theta)} R\sin\frac{r}{R}\,dr\,d\theta \\
&= R^2\alpha - \int_0^{\alpha} R^2\cos\frac{r(\theta)}{R}\,d\theta \\
&= R^2\alpha + R^2(\gamma(\alpha)-\gamma(0)) \\
&= R^2(\alpha+\beta+\gamma-\pi) \\
&= R^2\cdot\text{Excess}.
\end{aligned}$$

Disquisitiones (continuation)

Angular variation. §19

Length element in a geodesic-abscissa orthogonal chart:

$$ds^2 = dr^2 + G(r, \theta) d\theta^2$$



Angular variation. §19

$$\frac{d\gamma}{d\theta} = -\frac{\partial}{\partial r}\sqrt{G}$$

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- Polar coordinates in the plane:

$G = r^2;$ $\quad \dfrac{d\gamma}{d\theta} = -1$

- Polar coordinates in the sphere:

$G = R^2 \sin^2 \dfrac{r}{R};$ $\quad \dfrac{d\gamma}{d\theta} = -\cos \dfrac{r}{R}$

Curvature. §19

$$k = -\frac{1}{\sqrt{G}} \cdot \frac{\partial^2 \sqrt{G}}{\partial r^2}$$

- Implies the egregium theorem.

Default theorem. §20

- From

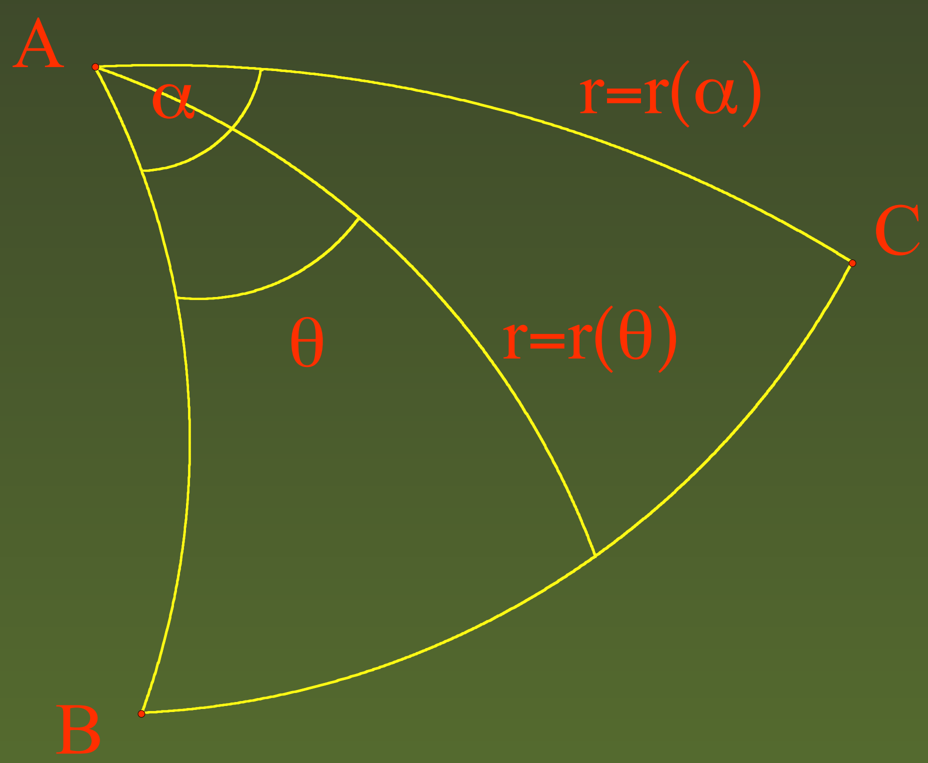
$$k\sqrt{G} = -\frac{\partial^2\sqrt{G}}{\partial r^2}$$

Default theorem. §20

- From

$$k\sqrt{G} = -\frac{\partial^2 \sqrt{G}}{\partial r^2}$$

- and integrating on the triangle



Default theorem. §20

- $\int_0^{r(\theta)} k\sqrt{G}dr = 1 - \frac{d}{dr}\sqrt{G}$

Default theorem. §20

- $\int_0^{r(\theta)} k\sqrt{G} dr = 1 - \frac{d}{dr}\sqrt{G}$
- $\int_0^{\alpha} \int_0^{r(\theta)} k\sqrt{G} dr\, d\theta = \alpha - \int_0^{\alpha} \frac{d}{dr}\sqrt{G}\, d\theta$

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$$\int_0^{\alpha} \int_0^{r(\theta)} k\sqrt{G}dr\, d\theta = (\alpha + \beta + \gamma) - \pi$$

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$$\int_T kdA = (\alpha + \beta + \gamma) - \pi$$

- Total curvature = Area spherical image = Default

The version of 1825.

- He proves $\text{Area}(\nu(T)) = \text{Default}(T)$ but he says:

  Der Beweis wird in der Form einiger Modification und Erläuterung bedürfen, wenn der Punkt (3) innerhalb des Dreiecks fällt.

  The proof will require some modification and explanation, when the point (3) is interior to the triangle.

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- Default $\longleftrightarrow$ Egregium

Last sections

Comparison theorems. §26

To quantities of the fourth order:

- $A^* = A - \frac{\sigma}{12}(2k(A) + k(B) + k(C))$

Comparison theorems. §26

To quantities of the fourth order:

- $A^* = A - \frac{\sigma}{12}(2k(A) + k(B) + k(C))$

  - $\sigma =$ Area ABC.
  - $k(A) =$ curvature at A.
  - $A^* =$ angle of the euclidian triangle whose sides are equal to the sides of the triangle on the surface.

The sphere §27

- The above formulas were first established by Legendre on the sphere.

$$A^* = A - \frac{\sigma}{3R^2}$$

The sphere §27

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  $$A^* = A - \frac{\sigma}{3R^2}$$

- Adding, we obtain the default theorem.

  $$\pi = A + B + C - \frac{\sigma}{R^2}$$

BHI §28

If BHI was spherical

- $B^* = B - \frac{\sigma}{3R^2} = B - \frac{14.85348''}{3} = B - 4''.95116$

BHI §28

If BHI was spherical

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On the earth ellipsoid *the calculation gave*

- Hohehagen $-4''.95113$
- Brocken $-4''.95104$
- Inselsberg $-4''.95131$

BHI §28

Letter to Olbers (March 1827).

In praktischer Rücksicht ist dies zwar ganz unwichtig, weil in der That bei den grössten Dreieecken, die sich auf der Erde messen lassen, diese Ungleichheit in der Vertheilung unmerklich wird; aber die Würde der Wissenchaft erfordert doch, dass man die Natur dieser Ungleichheit klar begreife.

In practical considerations this is not important, because even in the largest triangles that can be measured on earth, becomes imperceptible. But the dignity of science required that one understand the nature of this inequality clearly.

BHI. p. 314, Vol IX.

The rectilinear triangle with these sides

$$\begin{aligned} HI &= 84942.45328 \\ IB &= 105974.4570 \\ BH &= 69195.07749 \end{aligned}$$

has angles

$$\begin{aligned} B^* &= 53^o\ \ 6'\ 41.009760'' \\ H^* &= 86^o\ 13'\ 53.763480'' \\ I^* &= 40^o\ 39'\ 25.227360'' \end{aligned}$$

BHI. p. 314, Vol IX.

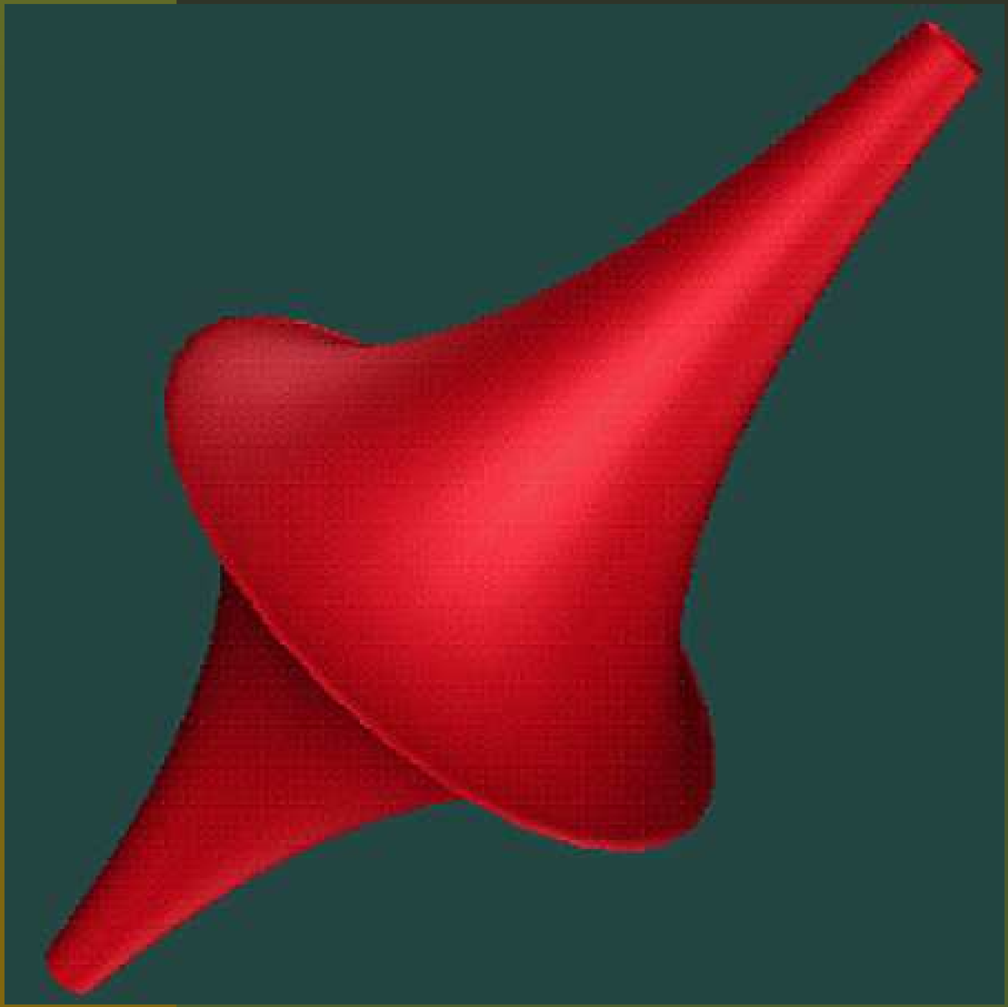
The excess $14.8523''$ of BHI is distributed:

$$\begin{aligned} H - H^* &= 4.9275 \\ B - B^* &= 4.9572 \\ I - I^* &= 4.9676 \end{aligned}$$

V. DG-NEG.

Let us find the imaginary sphere

Pseudosphere. F. Minding (1840)

Tractrice

- Curve with subtangent 1.
- $y' = -\frac{y}{\sqrt{1-y^2}}$

Pseudosphere

$$ds^2 = \frac{1}{y^2}(dx^2 + dy^2)$$

$x =$ angle of rotation; $y = e^\tau$ where $\tau =$ distance on the tractrice.

In geodesic polar coordinates

$$ds^2 = dr^2 + R^2 \sinh^2 \frac{r}{R} d\alpha^2$$

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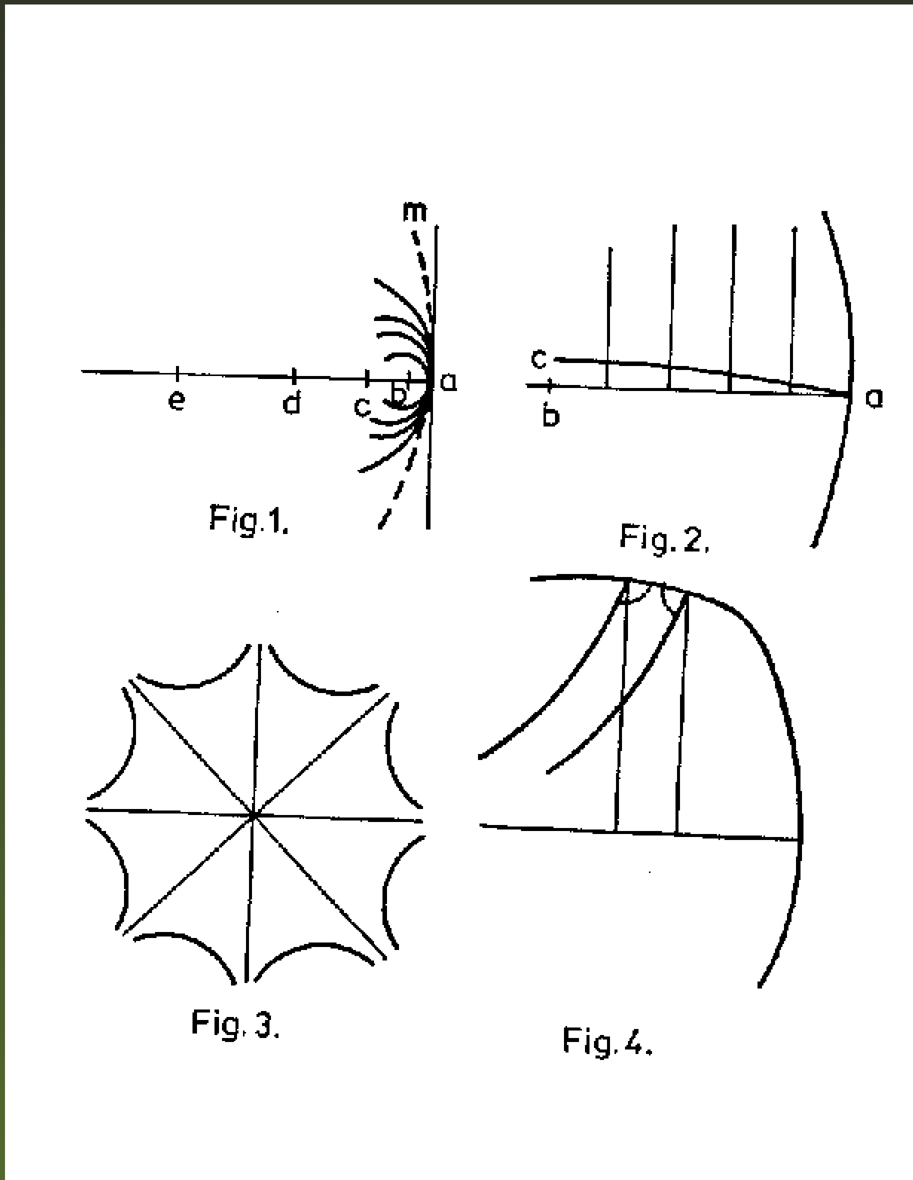
In geodesic polar coordinates

$$ds^2 = dr^2 + R^2 \sinh^2 \frac{r}{R} d\alpha^2$$

- Local

Ein Genie erster Grösse

János Bolyai



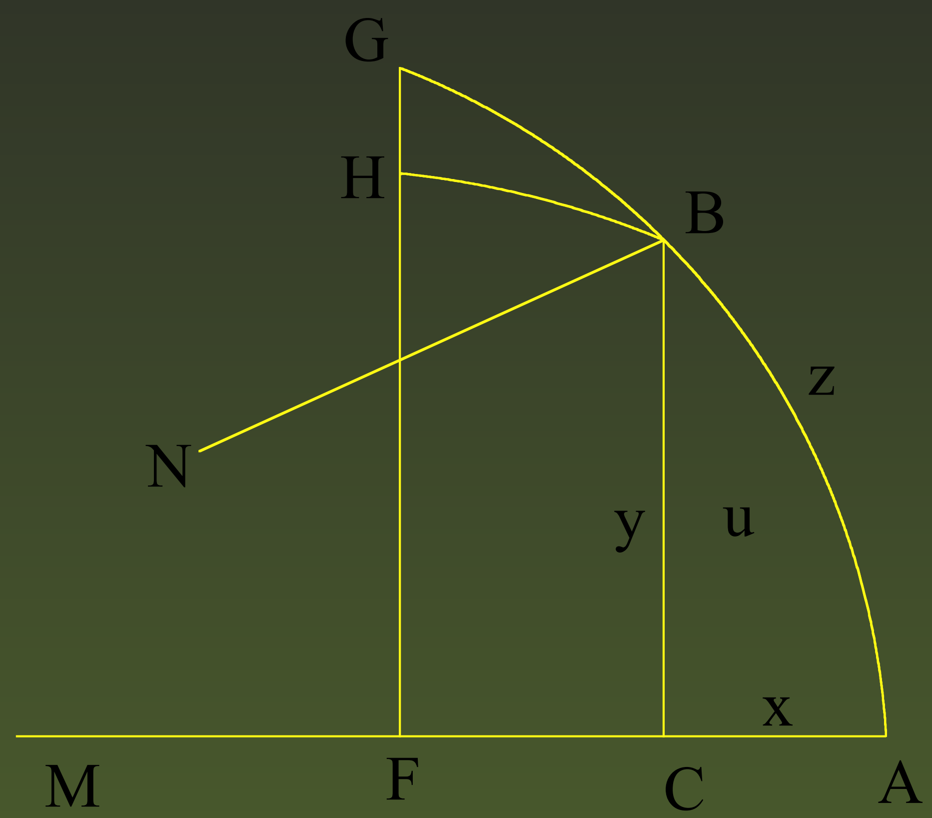
János Bolyai

gatur): innotescet exinde etiam limes ipsius $\frac{\quad}{bh}$, adeoque tang *hba*; eritque (cum *hbc* manifesto nec $>$ nec $<$ adeoque $= R$ sit), *tangens* in *b* ipsius *bg* per *y* determinata.

II. Demonstrari potest, esse $\frac{dz^2}{dy^2 + bh^2} \sim 1$;

Hinc *limes* ipsius $\frac{dz}{dx}$, et inde *z* integratione (per *x* expressum) reperitur. Et potest lineae cuiusvis *in concreto datae* aequatio in *S* inveniri, e. g. ipsius

János Bolyai



$$\frac{dz^2}{dy^2 + \overline{BH}^2} \doteq 1$$

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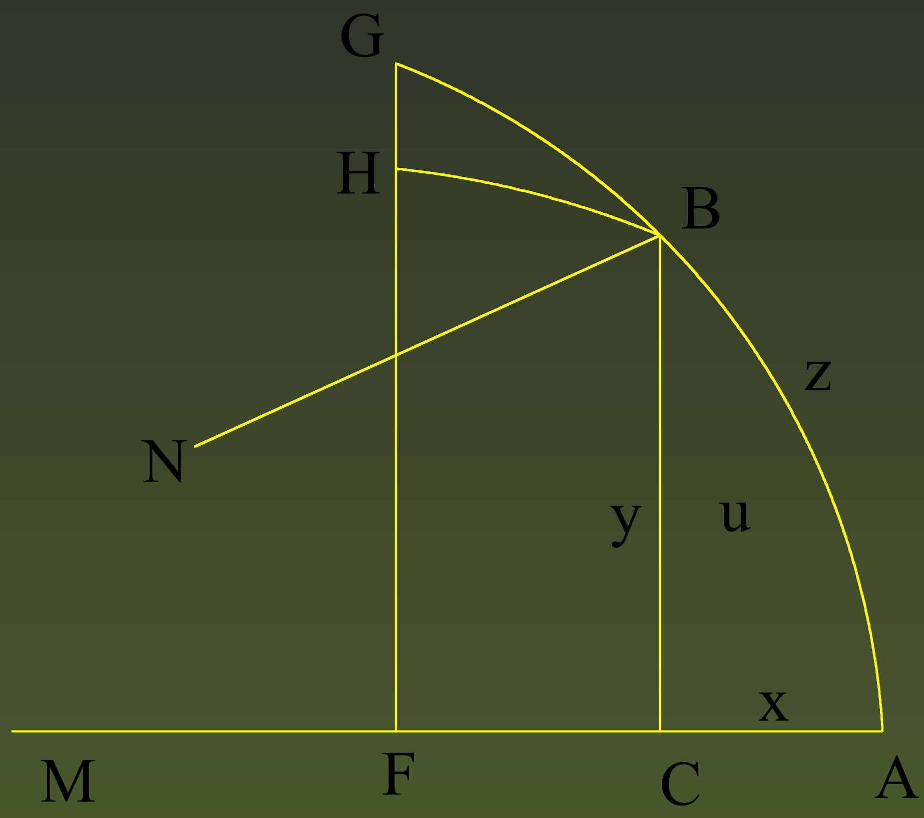
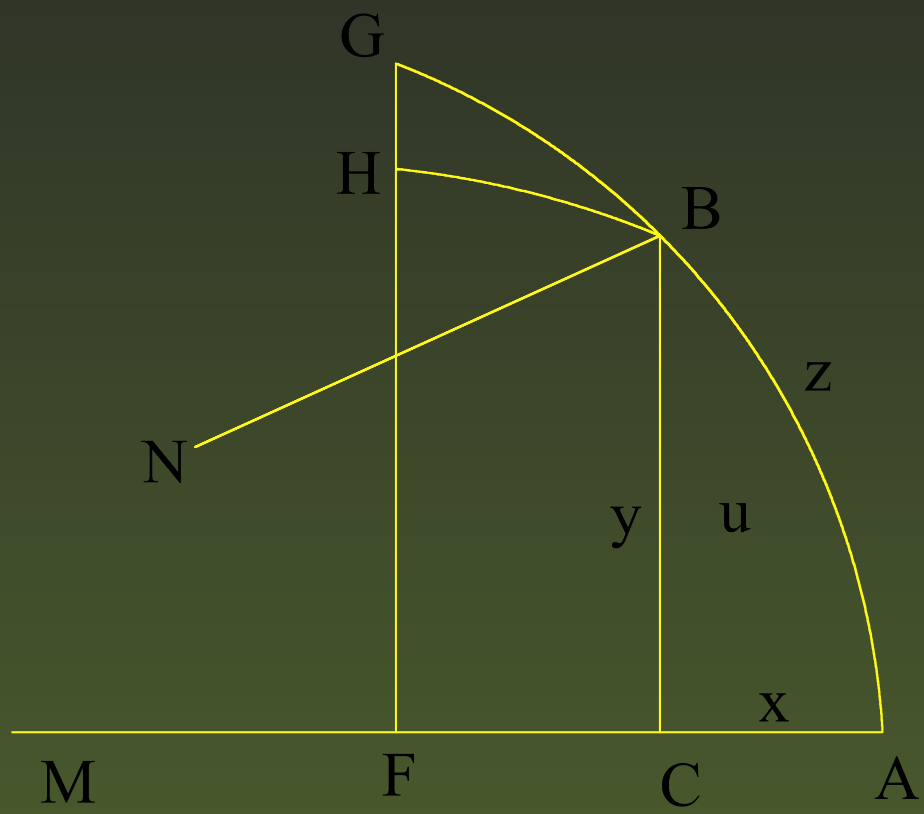


$$dz^2 \doteq \cosh^2 \frac{y}{R} dx^2 + dy^2$$

János Bolyai



$$ds^2 = dr^2 + R^2 \sinh^2 \frac{r}{R} d\alpha^2$$

A new world created from nothing

Marosvásárhely

Braunschweig