TRANSVERSE STRUCTURE OF LIE FOLIATIONS

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Abstract. We study if two non isomorphic Lie algebras can appear as transverse algebras to the same Lie foliation. In particular we prove a quite surprising result: every Lie $\mathcal{G}$-flow of codimension 3 on a compact manifold, with basic dimension 1, is transversely modeled on one, two or countable many Lie algebras. We also solve an open question stated in [GR91] about the realization of the three dimensional Lie algebras as transverse algebras to Lie flows on a compact manifold.

0. Introduction

This paper deals with the problem of the realization of a given Lie algebra as transverse algebra to a Lie foliation on a compact manifold.

Lie foliations have been studied by several authors (cf. [KAH86], [KAN91], [Fed71], [Mas], [Ton88]). The importance of this study was increased by the fact that they arise naturally in Molino's classification of Riemannian foliations (cf. [Mol82]).

To each Lie foliation are associated two Lie algebras, the Lie algebra $\mathcal{G}$ of the Lie group on which the foliation is modeled and the structural Lie algebra $\mathcal{H}$. The latter algebra is the Lie algebra of the Lie foliation $\mathcal{F}$ restricted to the clousure of any one of its leaves. In particular, it is a subalgebra of $\mathcal{G}$. We remark that although $\mathcal{H}$ is canonically associated to $\mathcal{F}$, $\mathcal{G}$ is not.

Thus two interesting problems are naturally posed: the *realization problem* and the *change problem*.

The *realization problem* is to know which pairs of Lie algebras $(\mathcal{G}, \mathcal{H})$, with $\mathcal{H}$ subalgebra of $\mathcal{G}$, can arise as transverse and structural Lie algebras, respectively, of a Lie foliation $\mathcal{F}$ on a compact manifold M.

This problem is closely related to the following Haefliger's problem (see [Hae84]): given a subgroup Γ of a Lie group G, is there a Lie G-foliation on a compact manifoild M with holonomy group Γ?

The present formulation of the *realization problem* in terms of Lie algebras was first considered in [Lla88], and [GR91] made a very detailed study of Lie flows of codimension 3. But a completre classification was not obtained because of the following open questions:

i) Let $\mathcal{G}_7^k$ be the family of Lie algebras for which there is a basis $\{e_1, e_2, e_3\}$ such that

$$[e_1, e_2] = 0, \qquad [e_1, e_3] = e_1, \qquad [e_2, e_3] = ke_2, \quad k \in [-1, 0) \cup (0, 1].$$

For which k is there a Lie $\mathcal{G}_7^k$-flow on a compact manifold with basic dimension 1 ?

ii) Let $\mathcal{G}_8^h$ be the family of Lie algebras for which there is a basis $\{e_1, e_2, e_3\}$ such that

$$[e_1, e_2] = 0, \qquad [e_1, e_3] = e_2, \qquad [e_2, e_3] = -e_1 + he_2, \quad h \in (0, 2).$$

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For which h is there a Lie $\mathcal{G}_8^h$-flow on a compact manifold with basic dimension 1 or basic dimension 2 ?

We solve these problems here, for basic dimension 1, and give a complete characterization in terms of k and h for a given Lie algebra of the families $\mathcal{G}_7^k$ and $\mathcal{G}_8^h$ to be realizable in the above conditions.

The *change problem* is to know if a given Lie $\mathcal{G}$-foliation can be at the same time a Lie $\mathcal{G}'$-foliation, where $\mathcal{G}$ and $\mathcal{G}'$ are two non isomorphic Lie algebras. An exemple of this situation, given by P. Molino, can be found in [GR91]. As far as we know this problem has no been treated for non-trivial basic dimension. The only *a priori* restriction is that the structural Lie algebra $\mathcal{H}$ must be a Lie subalgebra of $\mathcal{G}$ and $\mathcal{G}'$.

As a first step in the study of this two problems we consider the case of codimension 3 (the cases of codimension 1 and 2 are trivial). We expect that this study becomes useful in order to attack the general case.

We begin this paper with some results on abelian and nilpotent Lie foliations, that we shall use later. We first prove:

Proposition 2.2. *Every no dense Lie abelian foliation of codimension 3 on a compact manifold M is also a Lie $\mathcal{G}_8^{h=0}$-foliation.*

We also give an example to show that the converse is not true. But the converse is true for Lie flows of basic dimension 1 or 2:

Corollary 2.4. *Let $\mathcal{F}$ be a Lie $\mathcal{G}_8^{h=0}$-flow on a compact manifold M with basic dimension 1, then $\mathcal{F}$ is also a Lie abelian flow.*

Corollary 2.7. *Let $\mathcal{F}$ be a Lie $\mathcal{G}_8^{h=0}$-foliation on a compact manifold M with basic dimension 2, then $\mathcal{F}$ is also a Lie abelian foliation.* This corollary enable

us to construct a pair of Lie groups (G, Γ), with Γ a finitely generated subgroup of G, $\bar{\Gamma}$ uniform, such that there are no Lie G-flows on a compact manifold with holonomy group Γ. But Γ is realizable as the holonomy group of a Lie G-foliation of a compact manifold.

We also prove (Proposition 2.9) that if a given Lie foliation is transversely modeled on two nilpotent Lie algebras, then these algebras are isomorphic.

In §4 we study the *realization problem* and we obtain:

Theorem 4.1. *There is a compact manifold M endowed with a Lie $\mathcal{G}_8^h$-flow $\mathcal{F}$, $h \neq 0$, of basic dimension 1 if and only if*

$$h = \frac{2\ln\lambda}{\sqrt{4\omega^2 + \ln^2\lambda}}$$

where λ and ω are two real numbers, with $\lambda > 1$ and $\omega \neq k\pi$ $(k \in \mathbf{Z})$, such that λ, $\dfrac{1}{\sqrt{\lambda}}(\cos\omega \pm \mathbf{i}\sin\omega)$ are the roots of a monic polynomial of degree 3 with integer coefficients.

Theorem 4.4. *There is a compact manifold M endowed with a $\mathcal{G}_7^k$-flow $\mathcal{F}$ of basic dimension 1 if and only if*

$$k = \frac{\ln b}{\ln a} \qquad \text{and } k \notin \mathbf{Q}$$

where a, b, $\frac{1}{ab}$ are positive real roots of a monic polynomial of degree 3 with integer coefficients.

We devote §5 to the *change problem*. We obtain a quite surprising result, essentially that each Lie flow of codimension 3 on a compact manifold with basic dimension 1 is transversely modeled on one, two or countable many Lie algebras. Concretely, we prove (see §1 for the description of the algebras $\mathcal{G}_i$):

Theorem 5.4. *Let $\mathcal{F}$ be a Lie flow of codimension 3 on a compact manifold M with basic dimension 1. Then only three cases are possible:*

i) *$\mathcal{F}$ is transversely modeled exactly on one Lie algebra. This occurs if and only if the transverse Lie algebra is $\mathcal{G}_5$ or $\mathcal{G}_7^k$.*
ii) *$\mathcal{F}$ is transversely modeled exactly on two Lie algebras. This occurs if and only if these two transverse Lie algebras are $\mathcal{G}_1$ and $\mathcal{G}_8^{h=0}$.*
iii) *$\mathcal{F}$ is transversely modeled on countable many Lie algebras. This occurs if and only if $\mathcal{F}$ is transversely modeled on $\mathcal{G}_8^{h\neq 0}$.*
   In this case
   a) *There exist two real numbers $\lambda > 1$ and ω such that $h = \dfrac{2\ln\lambda}{\sqrt{4\omega^2+\ln^2\lambda}}$.*
   b) *$\mathcal{F}$ is also transversely modeled on $\mathcal{G}_8^{h'}$ for each*
   $$h' = \frac{2\ln\lambda}{\sqrt{4(\omega+2k\pi)^2+\ln^2\lambda}} \qquad (\forall k \in \mathcal{Z}).$$
   c) *If $\mathcal{F}$ is also transversely modeled on $\mathcal{G}$ then $\mathcal{G} = \mathcal{G}_8^{h'}$ for some of the above h'.*

1. Preliminaries

Let $\mathcal{F}$ be a smooth foliation of codimension n on a differentiable manifold M given by an integrable subbundle $L \subset TM$. We denote by $T\mathcal{F}$ the Lie algebra of the vector fields tangents to the foliation, i.e. the sections of L. A vector field $Y \in \mathcal{X}(M)$ is said to be *$\mathcal{F}$-foliated* (or simply *foliated*) if and only if $[X, Y] \in T\mathcal{F}$ for all $X \in T\mathcal{F}$. The Lie algebra of foliated vector fields is denoted by $\mathcal{L}(M, \mathcal{F})$. Clearly, $T\mathcal{F}$ is an ideal of $\mathcal{L}(M, \mathcal{F})$ and the elements of $\mathcal{X}(M/\mathcal{F}) = \mathcal{L}(M, \mathcal{F})/T\mathcal{F}$ are called *transverse (or basic) vector fields.*

If there is a family $\{X_1, \ldots, X_n\}$ of foliated vector fields on M such that the corresponding family $\{\overline{X}_1, \ldots, \overline{X}_n\}$ of basic vector fields has rank n everywhere the foliation is called transversely parallelizable and $\{\overline{X}_1, \ldots, \overline{X}_n\}$ is a transverse parallelism. If the vector subspace $\mathcal{G}$ of $\mathcal{X}(M/\mathcal{F})$ generated by $\{\overline{X}_1, \ldots, \overline{X}_n\}$ is a Lie subalgebra, the foliation is called *Lie $\mathcal{G}$-foliation* and we say that $\mathcal{F}$ is *transversely modeled* on the Lie algebra $\mathcal{G}$.

We shall use the following structure theorems:

Theorem 1.1. [Mol82] *Let $\mathcal{F}$ be a transversely parallelizable foliation on a compact manifold M, of codimension n. Then*

a) *There is a Lie algebra $\mathcal{H}$ of dimension $g \leq n$.*
b) *There is a locally trivial fibration $\pi : M \to W$ with compact fibre F and*
$$dim\ W = n - g = m.$$
c) *There is a dense Lie $\mathcal{H}$-foliation on F such that :*
   i) *The fibres of π are the adherences of the leaves of $\mathcal{F}$.*
   ii) *The foliation induced by $\mathcal{F}$ on each fibre of $\pi : M \to W$ is isomorphic to the $\mathcal{H}$-foliation on F.*

$\mathcal{H}$ is called the *structural Lie algebra* of $(M, \mathcal{F})$, π the *basic fibration* and W the *basic manifold.* The foliation given by the fibres of π is denoted by $\overline{\mathcal{F}}$.

Note that the *basic dimension* (i.e. the dimension of W) is

$$\dim W = \text{codim}\ \overline{\mathcal{F}} = \text{codim}\ \mathcal{F} - \dim \mathcal{H}.$$

Theorem 1.2. [Fed71] *Let G be the connected and simply connected Lie group with Lie algebra $\mathcal{G}$. $\mathcal{F}$ is a Lie $\mathcal{G}$-foliation on a compact manifold M if and only if there exists a homomorphism $\Phi : \pi_1(M) \longrightarrow G$ and a covering $p : \widetilde{M} \longrightarrow M$ such that*
i) *There is a locally trivial fibration $D : \widetilde{M} \longrightarrow G$ equivariant under the action of $\pi_1(M)$, where $\mathbf{Aut}(p) \cong Im\ \Phi$.*
ii) *The fibres of D are the leaves of the lifted foliation $\widetilde{\mathcal{F}} = p^*\mathcal{F}$ of $\mathcal{F}$.*

Condition $i)$ means that if $g_\gamma = \Phi([\gamma])$ and $\widetilde{\gamma}$ is the corresponding element to $[\gamma]$ in $\mathbf{Aut}(p)$, then

$$D(\widetilde{\gamma}(x)) = g_\gamma \cdot D(x) \qquad \forall x \in \widetilde{M}.$$

The subgroup $\Gamma = \text{Im } \Phi$ is called the *holonomy group* of the foliation.

For a Lie $\mathcal{G}$-foliation the structural Lie algebra $\mathcal{H}$ is always a subalgebra of $\mathcal{G}$.

A geometrical characterization of the fact that $\mathcal{H}$ is an ideal of $\mathcal{G}$ is the following

Lemma 1.3. [Lla88] *Let $\mathcal{F}$ be a Lie $\mathcal{G}$-foliation of codimension n on a compact manifold M. The structural algebra $\mathcal{H}$ is an ideal of $\mathcal{G}$ if and only if there exist a Lie $\mathcal{G}$ parallelism $\{\overline{Y}_1, \ldots, \overline{Y}_n\}$ such that the foliated vector fields $Y_1, \ldots, Y_t$ ($t = dim\ \mathcal{H}$) are tangent to $\overline{\mathcal{F}}$ at each point.*

The basic cohomology $H^*(M/\mathcal{F})$ of a foliation $\mathcal{F}$ on a manifold M is the cohomology of the complex of basic forms, i.e., the subcomplex $\Omega^*(M/\mathcal{F}) \subset \Omega(M)$ of the De Rham complex given by the forms α satisfiying $i_X\alpha = 0$ and $L_X\alpha = 0$ for all vector field $X \in T\mathcal{F}$.

For a Riemannian foliation it is well known (cf. [KAHS85]) that $H^n(M/\mathcal{F}) = 0$ or $\mathbf{R}$, where n is the codimension of the foliation. We have the following result

Theorem 1.4. [LR88] *Let $\mathcal{F}$ be a Lie $\mathcal{G}$-foliation of codimension n on a compact manifold M.*
i) If $H^n(M/\mathcal{F}) = \mathbf{R}$ then $H^n(\mathcal{G}) = \mathbf{R}$ and $H^p(\mathcal{G}) \subset H^p(M/\mathcal{F})$. In this case $\mathcal{F}$ will be called unimodular.
ii) If $H^n(\mathcal{G}) = \mathbf{R}$ and the structural Lie algebra is an ideal of $\mathcal{G}$ then $H^n(M/\mathcal{F}) = \mathbf{R}$.

Given a basis $\{e_1, \ldots, e_n\}$ of $\mathcal{G}$ with structure constants $\{c_{ij}^k\}$, we say that the foliated vector fields $Y_1, \ldots, Y_n \in \mathcal{L}(M, \mathcal{F})$ are a *foliated realization* of the basis $\{e_1, \ldots, e_n\}$ if the Lie brackets of these vector fields are

$$[Y_i, Y_j] = \sum_{k=1}^{n} c_{ij}^k Y_k + T\mathcal{F} \mod T\mathcal{F}.$$

We shall use the following classification of the 3 dimensional Lie algebras:

- $\mathcal{G}_1$ (Abelian):
$$[e_1, e_2] = [e_1, e_3] = [e_2, e_3] = 0.$$
- $\mathcal{G}_2$ (Heisenberg):
$$[e_1, e_2] = [e_1, e_3] = 0, \quad [e_2, e_3] = e_1.$$
- $\mathcal{G}_3$ ($so(3)$):
$$[e_1, e_2] = e_3, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = e_2.$$
- $\mathcal{G}_4$ (sl(2)):
$$[e_1, e_2] = e_3, \quad [e_2, e_3] = -e_1, \quad [e_3, e_1] = e_2.$$
- $\mathcal{G}_5$ (Affine):
$$[e_1, e_2] = e_1, \quad [e_1, e_3] = [e_2, e_3] = 0.$$

- $\mathcal{G}_6$:
$$[e_1,e_2]=0, \quad [e_1,e_3]=e_1 \quad [e_2,e_3]=e_1+e_2.$$
- The family $\mathcal{G}_7^k$:
$$[e_1,e_2]=0, \quad [e_1,e_3]=e_1, \quad [e_2,e_3]=ke_2 \qquad k\neq 0.$$
The algebras $\mathcal{G}_7^k$ and $\mathcal{G}_7^{k'}$ are isomorphic if and only if $k=k'$ or $k=\frac{1}{k'}$. From now on we consider that the family is parametrized by $k\in[-1,0)\cup(0,1]$.
- The family $\mathcal{G}_8^h$:
$$[e_1,e_2]=0, \quad [e_1,e_3]=e_2, \quad [e_2,e_3]=-e_1+he_2 \qquad h^2<4.$$
The algebras $\mathcal{G}_8^h$ and $\mathcal{G}_8^{h'}$ are isomorphic if and only if $h=h'$ or $h=-h'$. From now on we consider that the family is parametrized by $h\in[0,2)$. Notice that for $h^2\geq 4$ we obtain an algebra isomorphic to $\mathcal{G}_6$.

The Lie algebras $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$, $\mathcal{G}_4$ are unimodular. The Lie algebras $\mathcal{G}_5$, $\mathcal{G}_6$ are not unimodular. The only unimodular Lie algebra of the family $\mathcal{G}_7$ is $\mathcal{G}_7^{k=-1}$ and the only unimodular Lie algebra of the family $\mathcal{G}_8$ is $\mathcal{G}_8^{h=0}$.

In §3 we shall need an explicit description of the connected simply connected Lie groups corresponding to $\mathcal{G}_5$, $\mathcal{G}_7^k$, $\mathcal{G}_8^h$. These groups are given by

$$G_5=\left\{\begin{pmatrix}\mathrm{e}^t & 0 & x\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix}\ ;\ x,y,t\in\mathbb{R}\right\}$$

$$G_7^k=\left\{\begin{pmatrix}\mathrm{e}^{-t} & 0 & x\\ 0 & \mathrm{e}^{-kt} & y\\ 0 & 0 & 1\end{pmatrix}\ ;\ x,y,t\in\mathbb{R}\right\}$$

$$G_8^h=\left\{\begin{pmatrix}c(t)\cos(\varphi+t) & -c(t)\sin t & x\\ c(t)\sin t & c(t)\cos(\varphi-t) & y\\ 0 & 0 & 1\end{pmatrix}\ ;\ x,y,t\in\mathbb{R}\right\}$$

where $c(t)=\dfrac{2\mathrm{e}^{\beta t}}{\alpha}$, $\alpha=\sqrt{4-h^2}$ and $\beta=\tan\varphi=\dfrac{h}{\alpha}$.

These groups can also be thought as $\mathbb{R}^3=\mathbb{R}^2\times\mathbb{R}$ with the product
$$(p,t)\cdot(p',t')=(p+\mathrm{e}^{-\Lambda t}p',t+t')$$
where Λ depends on the algebra:

For $\mathcal{G}_5$,
$$\Lambda=\begin{pmatrix}-1 & 0\\ 0 & 0\end{pmatrix}, \qquad \mathrm{e}^{-\Lambda t}=\begin{pmatrix}\mathrm{e}^t & 0\\ 0 & 1\end{pmatrix}.$$

For $\mathcal{G}_7^k$,
$$\Lambda=\begin{pmatrix}1 & 0\\ 0 & k\end{pmatrix}, \qquad \mathrm{e}^{-\Lambda t}=\begin{pmatrix}\mathrm{e}^{-t} & 0\\ 0 & \mathrm{e}^{-kt}\end{pmatrix}.$$

For $\mathcal{G}_8^h$,
$$\Lambda=\begin{pmatrix}0 & 1\\ -1 & h\end{pmatrix}, \qquad \mathrm{e}^{-\Lambda t}=c(t)\begin{pmatrix}\cos(\varphi+t) & -\sin t\\ \sin t & \cos(\varphi-t)\end{pmatrix}.$$

The above basis of these Lie algebras are given by:

$$\text{for } \mathcal{G}_5 \qquad \begin{cases} e_1 &= \mathrm{e}^t \dfrac{\partial}{\partial x} \\[2ex] e_2 &= \dfrac{\partial}{\partial y} \\[2ex] e_3 &= -\dfrac{\partial}{\partial t} \end{cases}$$

$$\text{for } \mathcal{G}_7^k \qquad \begin{cases} e_1 &= \mathrm{e}^{-t} \dfrac{\partial}{\partial x} \\[2ex] e_2 &= \mathrm{e}^{-kt} \dfrac{\partial}{\partial y} \\[2ex] e_3 &= \dfrac{\partial}{\partial t} \end{cases}$$

$$\text{for } \mathcal{G}_8^h \qquad \begin{cases} e_1 &= \dfrac{2}{\alpha}\mathrm{e}^{-\beta t}(\cos(\varphi + t)\dfrac{\partial}{\partial x} + \sin t \dfrac{\partial}{\partial y}) \\[2ex] e_2 &= \dfrac{2}{\alpha}\mathrm{e}^{-\beta t}(-\sin t \dfrac{\partial}{\partial x} + \cos(\varphi - t)\dfrac{\partial}{\partial y}) \\[2ex] e_3 &= -\dfrac{\alpha}{2}\dfrac{\partial}{\partial t} \end{cases}$$

2. Abelian and nilpotent Lie foliations

Many of the problems about Lie foliations are, in fact, problems on Lie groups. For instance, as a consequence of the following proposition, every Lie abelian foliation without dense leaves is also a Lie $\mathcal{G}_8^{h=0}$-foliation:

Proposition 2.1. *Let H be a proper closed uniform subgroup of the abelian group $(\mathbf{R}^3, +)$. Then there is a product $\odot$ on $\mathbf{R}^3$ such that the Lie group $(\mathbf{R}^3, \odot)$ is isomorphic to $G_8^{h=0}$ and*

$$h + g = h \odot g \qquad \forall h \in H,\ \forall g \in \mathbf{R}^3.$$

Proof. The proper closed uniform subgroups of $(\mathbb{R}^3, +)$ are isomorphic to

$$\mathbb{R}^2 \times \mathbb{Z}, \quad \mathbb{R} \times \mathbb{Z}^2 \quad \text{or} \quad \mathbb{Z}^3.$$

Thus there are vectors $v_1, \dots, v_a, w_1, \dots, w_b$ linearly independent in $\mathbb{R}^3$ with $a+b = 3$ and $a \neq 3$, such that H is exactly the set of vectors that can be written in the form

$$x_1 v_1 + \cdots + x_a v_a + y_1 w_1 + \cdots + y_b w_b$$

where $x_1, \dots, x_a \in \mathbb{R}$ and $y_1, \dots, y_b \in \mathbb{Z}$. With respect to this new basis we define

$$(x, y, z) \odot (x', y', z') = ((x, y) + R_{2\pi z}(x', y'), z + z')$$

$\forall (x, y, z), (x', y', z') \in \mathbb{R}^3$, where $R_{2\pi z}$ is the rotation of angle $2\pi z$.

Then $(\mathbb{R}^3, \odot)$ is clearly isomorphic to $G_8^{h=0}$.

Moreover, since $z \in \mathbb{Z}$ for each $(x, y, z) \in H$, we have

$$(x, y, z) \odot (x', y', z') = (x, y, z) + (x', y', z')$$

$\forall (x, y, z) \in H, \forall (x', y', z') \in \mathbb{R}^3$.$\square$

Proposition 2.2. *Every no dense Lie abelian foliation of codimension 3 on a compact manifold M is also a Lie $\mathcal{G}_8^{h=0}$-foliation.*

Proof. Since $\mathcal{F}$ is a Lie abelian foliation we have, by Theorem 1.2, a homomorphism
$\Phi : \pi_1(M) \longrightarrow (\mathbb{R}^3, +)$ and a locally trivial fibration $D : \widetilde{M} \longrightarrow \mathbb{R}^3$ equivariant under the action $\pi_1(M)$. Put $\Gamma = \mathrm{Im}\ \Phi$.

Then $H = \overline{\Gamma}$ is a proper closed uniform subgroup of the abelian group $(\mathbf{R}^3, +)$.

Proposition 2.1 implies that Φ also defines a homomorphism $\pi_1(M) \longrightarrow (\mathbb{R}^3, \odot)$ with respect to which D is again equivariant.

Thus $\mathcal{F}$ is a Lie $\mathcal{G}_8^{h=0}$-foliation. □

This proposition is a generalization of the following example given by P. Molino (cf. [GR91]) in which the change of the parallelism was explicitely given:

Let us consider the flow given by the fibres of the trivial bundle

$$\mathbb{T}^1 \times \mathbb{T}^3 \longrightarrow \mathbb{T}^3$$

Let $\theta^0, \theta^1, \theta^2, \theta^3$ denote the canonical coordinates in $\mathbb{T}^1 \times \mathbb{T}^3$. The parallelism given by $\dfrac{\partial}{\partial\theta^1}, \dfrac{\partial}{\partial\theta^2}, \dfrac{\partial}{\partial\theta^3}$ shows that the fibres of this bundle are the leaves of a Lie abelian foliation. But the parallelism given by the vector fields

$$\begin{aligned} e_1 &= \cos\theta^1 \frac{\partial}{\partial\theta^2} + \sin\theta^1 \frac{\partial}{\partial\theta^3} \\ e_2 &= -\sin\theta^1 \frac{\partial}{\partial\theta^2} + \cos\theta^1 \frac{\partial}{\partial\theta^3} \\ e_3 &= -\frac{\partial}{\partial\theta^1} \end{aligned}$$

is a Lie parallelism with $[e_1, e_2] = 0$, $[e_1, e_3] = e_2$, $[e_2, e_3] = -e_1$, that is, the flow is also transversely modeled on $\mathcal{G}_8^{h=0}$.

The converse of Proposition 2.2 is not true in general. That is, there are Lie $\mathcal{G}_8^{h=0}$-foliations which are not abelian foliations.

For instance, take the uniform discrete subgroup

$$\Gamma = \{(m, n, \pi t) |\ m, n, t \in \mathbf{Z}\} \subset G_8^{h=0}$$

Let M be the compact manifold $\Gamma \backslash G_8^{h=0}$. Then the trivial fibration $M \times S^1 \longrightarrow M$ is a Lie $\mathcal{G}_8^{h=0}$-flow with basic dimension 3 that can not be abelian. In fact any abelian parallelism of the above flow would induce 3 linearly independent vector fields on M which pairwise commute, i.e. M would be a 3-dimensional torus $\mathbb{T}^3$. But this is impossible because the fundamental group of M is the non abelian group Γ.

The converse of Proposition 2.2 is, however, true for Lie flows of basic dimension 1 and for Lie foliations of basic dimension 2.

For basic dimension 1 it is a corollary of the following:

Proposition 2.3. *Let $\mathcal{G}$ be an unimodular Lie algebra of dimension n. If $\mathcal{F}$ is a Lie $\mathcal{G}$-flow on a compact manifold with basic dimension 1, then $\mathcal{F}$ is also a Lie abelian flow.*

Proof. It is well known that the structural Lie algebra $\mathcal{H}$ of a Lie flow is abelian (cf. [Car84]). In this case it is a subalgebra of dimension $n-1$ of an unimodular Lie algebra $\mathcal{G}$ of dimension n. Then $\mathcal{H}$ is an ideal of $\mathcal{G}$. By Lemma 1.3, one can find a $\mathcal{G}$-parallelism $Y_1, \ldots, Y_n$ such that $Y_i \in T\overline{\mathcal{F}}$ for $i = 1, \ldots, n-1$ and, hence, $Y_n \notin T\overline{\mathcal{F}}$ at any point.

In this situation we have that $\mathcal{F}$ is an unimodular Lie flow, that is $H^n(M/\mathcal{F}) \neq 0$ (cf. [Lla88]). By [MS85] there exist $X_1, \ldots, X_{n-1}$ foliated vector fields such that

$[X_i, Y] \in T\mathcal{F}$ for all foliated vector field Y (i.e. the transverse central sheaf has a global trivialization). Thus, $X_1, \dots, X_{n-1}, Y_n$ is a Lie abelian parallelism. □

Corollary 2.4. *Let $\mathcal{F}$ be a Lie $\mathcal{G}_8^{h=0}$-flow on a compact manifold M with basic dimension 1, then $\mathcal{F}$ is also a Lie abelian flow.*

In the case of basic dimension 2, we have

Proposition 2.5. *Let $\mathcal{F}$ be a Lie $\mathcal{G}_8^{h=0}$-foliation on a compact manifold M with basic dimension 2, then $\mathcal{F}$ is also a Lie abelian foliation.*

The proof is based on the following lemma:

Lemma 2.6. *Let $\mathcal{F}$ be a Lie $\mathcal{G}_8^{h=0}$-foliation on a compact manifold M with basic dimension 2 and let $\overline{Y}_1, \overline{Y}_2, \overline{Y}_3$ be the Lie $\mathcal{G}_8^{h=0}$-parallelism corresponding to the basis $\{e_1, e_2, e_3\}$ given in §1. Then the foliated vector field Y_3 is not tangent to $\overline{\mathcal{F}}$ at any point and there exists a global foliated vector field Y tangent to $\overline{\mathcal{F}}$ which commutes (modulo $T\mathcal{F}$) with every foliated vector field (i.e., the commuting sheaf is globally trivial).*

Proof. For each point $x \in M$ there exists a foliated vector field Z_U in a neighbourhood U of x, such that Z_U is tangent to $\overline{\mathcal{F}}$, no tangent to $\mathcal{F}$, and commutes (modulo $T\mathcal{F}$) with every global foliated vector field. Here we are considering a local section of the commuting sheaf (cf. [God91], [Mol82]). Moreover if Z_V is another vector field in a neighbourhood V of x with the same property then $Z_V = \alpha Z_U$ (modulo $T\mathcal{F}$) where α is a locally constant function.

We can assume that the vector field Z_U can be written as

$$Z_U = a_U Y_1 + b_U Y_2 + c_U Y_3$$

where a_U, b_U, c_U are basic functions on U.

Since $[Y_i, Z_U] \in T\mathcal{F}$ we obtain the equations:

$$\begin{array}{ccc} Y_1(a_U) = 0 & Y_2(a_U) = c_U & Y_3(a_U) = -b_U \\ Y_1(b_U) = -c_U & Y_2(b_U) = 0 & Y_3(b_U) = a_U \\ Y_1(c_U) = 0 & Y_2(c_U) = 0 & Y_3(c_U) = 0. \end{array}$$

We deduce from these equations that c_U is constant on U.

Since $Z_V = \alpha Z_U$ (modulo $T\mathcal{F}$), with α a locally constant function, if $c_U = 0$ then $c_V = 0$. Then there are only two possibilities:

i) for any point $y \in M$ and any neighbourhood W of y we have $c_W = 0$ or
ii) for any point $y \in M$ and any neighbourhood W of y we have $c_W \neq 0$.

Let us prove that ii) is not possible:

In this case Y_1, Y_2 are not tangents to $\overline{\mathcal{F}}$ at any point. We take a riemannian metric and we define Y_i^N as the normal component to $\overline{\mathcal{F}}$ of Y_i. Then Y_3^N is a combination of Y_1^N and Y_2^N at each point, i.e., there are basic functions f, g such that

$$(Y_3^N)_p = f(p)(Y_1^N)_p + g(p)(Y_2^N)_p \qquad \forall p \in M.$$

In this case we obtain

$$Y_2^N = [Y_1, Y_3]^N = Y_1(f)Y_1^N + Y_1(g)Y_2^N$$

$$-Y_1^N = [Y_2, Y_3]^N = Y_2(f)Y_1^N + Y_2(g)Y_2^N$$

then $Y_1(g)$ is the constant 1 and $Y_2(f)$ is the constant -1. This is no possible because f, g are continuous functions on a compact manifold.

Thus $c_W = 0$ in each neighbourhood W and this means that Y_3 is not tangent to $\overline{\mathcal{F}}$ at any point. This proves the first part of the lemma.

Now we consider $\alpha = i_{Y_3}\pi * \omega$, where π is the basic projection and ω is a volume form on the basic manifold. We take the global basic functions

$$A = i_{Y_2}\alpha, \quad B = -i_{Y_1}\alpha.$$

As $A^2 + B^2 \neq 0$, we can consider the foliated vector field $Y = aY_1 + bY_2$ where

$$a = \frac{A}{\sqrt{A^2+B^2}}, \quad b = \frac{B}{\sqrt{A^2+B^2}}.$$

Using that $\alpha(Y) = 0$ it is easy to see that Y is tangent to $\bar{\mathcal{F}}$ everywhere and not tangent to $\mathcal{F}$ at any point.

It remains to prove that Y commutes (modulo $T\mathcal{F}$) with every global foliated vector field. Since $Y \in T\overline{\mathcal{F}}$ it suffices to prove that $[Y_i, Y] \in T\mathcal{F}$.

From the fact that $[Y_i, Y] \in T\overline{\mathcal{F}}$ we have $[Y_i, Y] = \lambda_i Y + T\mathcal{F}$ and we obtain the equations

$$\begin{array}{lll} Y_1(a) = \lambda_1 a & Y_2(a) = \lambda_2 a & Y_3(a) + b = \lambda_3 a \\ Y_1(b) = \lambda_1 b & Y_2(b) = \lambda_2 b & Y_3(b) - a = \lambda_3 b \end{array}$$

From these equations we deduce

$$\lambda_i = \lambda_i(a^2+b^2) = aY_i(a) + bY_i(b) = \frac{1}{2}Y_i(a^2+b^2) = \frac{1}{2}Y_i(1) = 0.$$

Hence the result. $\square$

Proof of Proposition 2.5. By the above lemma there exists a foliated vector field $Y = aY_1 + bY_2$ tangent to $\overline{\mathcal{F}}$ at any point with a, b basic functions such that $a^2 + b^2 = 1$ and Y commutes (modulo $T\mathcal{F}$) with every global foliated vector field.

Let us see that $X_1 = Y$, $X_2 = -bY_1 + aY_2$, $X_3 = Y_3$ give a Lie abelian parallelism. Clearly these vector fields are transversely independent. Using now

$$Y_3(a) + b = \lambda_3 a = 0, \qquad Y_3(b) - a = \lambda_3 b = 0 \qquad \text{and} \qquad [\overline{Y}, \overline{Y}_i] = \lambda_i \overline{Y} = 0$$

we have

$$\begin{array}{rcl} [\overline{X_1}, \overline{X_2}] & = & [\overline{Y}, -b\overline{Y_1} + a\overline{Y_2}] = -b[\overline{Y}, \overline{Y_1}] + a[\overline{Y}, \overline{Y_2}] = (-b\lambda_1 + a\lambda_2)\overline{Y} = 0 \\ [\overline{X_2}, \overline{X_3}] & = & [-b\overline{Y_1} + a\overline{Y_2}, \overline{Y_3}] = -(b + Y_3(a))\overline{Y_2} + (Y_3(b) - a)\overline{Y_1} = 0 \\ [\overline{X_1}, \overline{X_3}] & = & [\overline{Y}, \overline{Y_3}] = 0. \end{array}$$

This ends the proof. $\square$

Corollary 2.7. *Let $\mathcal{F}$ be a Lie $\mathcal{G}_8^{h=0}$-flow on a compact manifold M with basic dimension 2, then $\mathcal{F}$ is also a Lie abelian flow.*

Proof. It follows from the above demonstration that it suffices to show that the commuting sheaf admits a global trivialization. In the case of flows this condition is true if and only if the flow is isometric (cf. [MS85]).

If the basic manifold W is not orientable we take a double covering D over the orientation covering of W. Here, by the above proposition, the commuting sheaf is trivial and, hence, the flow on D is isometric. This implies that the initial flow is isometric. $\square$

Next corollary is closely related to the Haefliger's problem.

Corollary 2.8. *There is a pair of Lie groups (G, Γ), with Γ a finitely generated subgroup of G, $\bar{\Gamma}$ uniform, such that there are no Lie G-flows on a compact manifold with holonomy group Γ.*

Proof. Let G be the Lie group $G_8^{h=0}$ and let Γ be the Lie subgroup

$$\Gamma = \langle (1,0,0), (\xi,0,0), (0,1,0), (0,0,\pi) \rangle \qquad \xi \notin \mathbf{Q}.$$

We have $\overline{\Gamma} \overset{diff}{\cong} \mathbb{R} \times \mathbb{Z}^2$ because the elements of $\overline{\Gamma}$ are of the form $(a, m, n\pi)$ with $a \in \mathbb{R}$ and $m, n \in \mathbb{Z}$. Moreover $\overline{\Gamma}$ is an uniform not abelian subgroup of $G_8^{h=0}$.

Assume that there is a Lie $\mathcal{G}_8^{h=0}$-flow $\mathcal{F}$ on a compact manifold with holonomy group Γ (this implies that $\mathcal{F}$ has basic dimension 2). Then, by Corollary 2.7, $\mathcal{F}$ would be also a Lie abelian flow and then Γ would be abelian, which is not possible. $\square$ Note that the pair (G, Γ) is realizable, i.e., there is a lie G-foliation of acompact manifold with the holonomy group Γ. The reason is because $G_8^{h=0}$ is solvable and Γ is polycyclic (cf. [Mei95]).

In order to generalize this kind of results to Lie foliations of arbitrary codimension we consider the case of Lie foliations with nilpotent transverse Lie algebra. In this category the transverse Lie algebra is canonically associated to the foliation:

Proposition 2.9. *Let $\mathcal{F}$ be a Lie foliation on a compact manifold transversely modeled on two nilpotent Lie algebras $\mathcal{G}$ and $\mathcal{H}$. Then $\mathcal{G}$ and $\mathcal{H}$ are isomorphic.*

Proof. Again, this is a consequence of a result on Lie groups.

In fact, if $p : \widetilde{M} \longrightarrow M$ is the universal covering of M and we fix points $x_0 \in M$ and $\widetilde{x}_0 \in \widetilde{M}$ with $p(\widetilde{x}_0) = x_0$, the developing diagrams are given by the canonical projection

$$D : \widetilde{M} \longrightarrow \widetilde{M}/p^{-1}(\mathcal{F})$$

and the holonomy morphisms

$$\Phi_1 : \pi_1(M, x_0) \longrightarrow G \qquad \Phi_2 : \pi_1(M, x_0) \longrightarrow H$$

where G and H are the connected simply connected groups corresponding to $\mathcal{G}$ and $\mathcal{H}$ respectively. Note that as differentiable manifolds we have:

$$G \cong \widetilde{M}/p^{-1}(\mathcal{F}) \cong H.$$

Since

$$D(\widetilde{\gamma}(\widetilde{x})) = \Phi_1([\gamma]) \cdot D(\widetilde{x}) = \Phi_2([\gamma]) * D(\widetilde{x})$$

the holonomy groups $\Gamma_1 = \Phi_1(\pi_1(M, x_0))$ and $\Gamma_2 = \Phi_2(\pi_1(M, x_0))$ are related by

$$\Gamma_1 = (r_H)_e \Gamma_2$$

where $(r_H)_e$ is the right translation in H with respect to the unit element e of G. In fact, $(r_H)_e$ restricted to Γ_2 is a morphism of Lie groups. This implies that the closures $\overline{\Gamma}_1$ and $\overline{\Gamma}_2$ are isomorphic.

As they are also uniforme subgroups of the nilpotent Lie groups G and H respectively, this two groups are isomorphic as Lie groups (cf. [Rag72]). $\square$

The foliation by points of the quotien of the Heisenberg group $\mathcal{N}/\Gamma$ (cf. [GR91]) is a good example of this situation.

3. Lie flows of basic dimension one

Let $\mathcal{F}$ be a Lie $\mathcal{G}$-flow of codimension 3 and basic dimension 1 on a compact manifold M. There is a matrix $A \in SL(3, \mathbf{Z})$ with an eigenvalue $\lambda > 0$ and an eigenvector v, whose components are rationally independents, such that M is the manifold $\mathbf{T}^3 \times_A \mathbf{R}$ (usually called $\mathbf{T}^4_A$) described as the quotient of $\mathbf{T}^3 \times \mathbf{R}$ by the equivalence relation $(p, t) \sim (Ap, t+1)$. The flow $\mathcal{F}$ is the induced by the linear flow $\mathcal{F}_v$ (cf. [AM86], [Car84]).

The basic fibration corresponding to this Lie flow is a $\mathbb{T}^3$ bundle over $\mathbb{T}^1$:

$$\mathbb{T}^3 \longrightarrow M \longrightarrow \mathbb{T}^1$$

The homotopy sequence of the basic fibration gives the exact sequence of fundamental groups

$$0 \longrightarrow Z^3 \longrightarrow \pi_1(M) \longrightarrow \mathbb{Z} \longrightarrow 0$$

from which we have that $\pi_1(M) \cong \mathbb{Z}^3 \times_A \mathbb{Z}$, i.e., $\pi_1(M)$ is $\mathbb{Z}^3 \times \mathbb{Z}$ with the product

$$(x, t)(y, s) = (x + A^t y, t + s)$$

This implies, in particular, a certain kind of uniqueness for the matrix A: If A and B are two elements of $SL(3; \mathbb{Z})$ such that the fibre bundle $\mathbb{T}^4_A$ is diffeomorphic to the fibre bundle $\mathbb{T}^4_B$ then A is conjugated to B or to B^{-1}.

On the other hand, the only possibilities for the Lie algebra $\mathcal{G}$ are

$$\mathcal{G} = \mathcal{G}_1, \quad \mathcal{G} = \mathcal{G}_5, \quad \mathcal{G} = \mathcal{G}_7^k (k \notin \mathbf{Q}) \quad \text{or} \quad \mathcal{G} = \mathcal{G}_8^h.$$

In fact it is evident that the algebras $\mathcal{G}_3$, $\mathcal{G}_4$ are not possible because they do not have any abelian subalgebra of dimension 2. That the algebras $\mathcal{G}_2$, $\mathcal{G}_6$ and $\mathcal{G}_7^k$ ($k \in \mathbf{Q}$) are not realizable as transverse algebras of a Lie flow of basic dimension 1 is proved in [GR91].

When $\mathcal{F}$ is an unimodular Lie $\mathcal{G}$-flow then the Lie algebra $\mathcal{G}$ is also unimodular. Hence the only possibilities for an unimodular Lie $\mathcal{G}$-flow of basic dimension 1 are $\mathcal{G} = \mathcal{G}_1$ or $\mathcal{G} = \mathcal{G}_8^{h=0}$, and it follows from Proposition 2.2 and Corollary 2.4 that $\mathcal{F}$ is transversely modeled on both algebras.

For a non unimodular Lie $\mathcal{G}$-flow $\mathcal{F}$ of basic dimension one the Lie algebra $\mathcal{G}$ is not unimodular (Theorem 1.4), hence the only possibilities are $\mathcal{G} = \mathcal{G}_5$, $\mathcal{G} = \mathcal{G}_7^k$ or $\mathcal{G} = \mathcal{G}_8^{h \neq 0}$. Notice that in all this cases the connected simply connected groups associated to these algebras are diffeomorphic to $\mathbb{R}^3$. Moreover the matrix A used in the construction of $\mathbf{T}^4_A$ is not the identity matrix because the flow is unimodular if and only if $A =$Id (cf. [AM86]).

The following theorem proves that the matrix A is closely related to the matrix defining the group structure.

Theorem 3.1. *Let $\mathcal{F}$ be a non unimodular Lie $\mathcal{G}$-flow of codimension 3 and basic dimension 1 on a compact manifold M. Then the matrix $A \in SL(3; \mathbb{Z})$ defining M is conjugated to:*

$$B = \begin{pmatrix} \lambda^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix} \qquad \textit{for } \mathcal{G} = \mathcal{G}_5,$$

$$B = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda^k & 0 \\ 0 & 0 & \lambda^{-1-k} \end{pmatrix} \qquad \textit{for } \mathcal{G} = \mathcal{G}_7^k,$$

and

$$B = c(\lambda) \begin{pmatrix} \cos(\varphi + \lambda) & -\sin\lambda & 0 \\ \sin\lambda & \cos(\varphi - \lambda) & 0 \\ 0 & 0 & c(\lambda)^{-3} \cos^{-2}\varphi \end{pmatrix}, \quad \lambda \in \mathbb{R}, \qquad \textit{for } \mathcal{G} = \mathcal{G}_8^h$$

Proof. Recall that $\pi_1(M)$ is $\mathbb{Z}^3 \times \mathbb{Z}$ with the product

$$(x, t)(y, s) = (x + A^t y, t + s)$$

Thus the holonomy representation of $\mathcal{F}$ is a morphism $\Phi : \mathbb{Z}^3 \times_A \mathbb{Z} \longrightarrow G$, where G is G_5, $G_7^{k \neq 1}$ or $G_8^{h \neq 0}$ and, hence, it is diffeomorphic to $\mathbb{R}^3$.

The image of Φ, $\Gamma = \Phi(\mathbb{Z}^3 \times_A \mathbb{Z})$, is the holonomy group of $\mathcal{F}$ and, since the basic manifold $\mathbb{T}^1$ is diffeomorphic to $\overline{\Gamma} \setminus G$, we have dim $\overline{\Gamma} = 2$.

In spite of the fact that the identity component, $\overline{\Gamma}_e$, of $\overline{\Gamma}$ is abelian, $\overline{\Gamma}$ is not abelian because $A \neq$Id and Φ is injective.

For G_7^k and G_8^h it is easy to prove (cf., for example, [GR91]) that for every abelian subgroup H of the holonomy group Γ, either H is contained in $\mathbb{R}2 \times \{0\}$ (the only subalgebra of dimension two is, in both cases, the subalgebra generated by e_1, e_2) or there is an element $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $\alpha_3 \neq 0$, such that $H = \{\alpha^n |\, n \in \mathbb{Z}\}$.

For G_5 every abelian subgroup H of Γ is contained in $\mathbb{R}^2 \times \{0\}$ or there is an element $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $\alpha_3 \neq 0$, such that

$$H \subset \left\{ (C(1 - \mathrm{e}^t), y, t) |\, C = \frac{\alpha_1}{1 - \mathrm{e}^{\alpha_3}} \right\}.$$

Using now that the subgroup $\Phi(0 \times \mathbb{Z})$ can not be contained in $\Phi(\mathbb{Z}^3 \times 0)$ (because Γ is not abelian) and that $\Phi(\mathbb{Z}^3 \times 0)$ is an abelian normal subgroup of Γ, an easy computation shows that in the three cases (G_5, $G_7^{k\neq 1}$, $G_8^{h\neq 0}$) we have

$$\Phi(\mathbb{Z}^3 \times 0) \subset \mathbb{R}^2 \times \{0\}.$$

Let $\Phi(\mathbb{Z}^3 \times 0) = \langle (p_1, 0), (p_2, 0), (p_3, 0) \rangle$ and $\Phi(\mathbb{Z}) = \langle \Phi(0, 1) \rangle = \langle (p, \vartheta) \rangle$, $\vartheta > 0$. Then the normality condition is

$$(p, \vartheta)(p_i, 0)(p, \vartheta)^{-1} = (p, \vartheta)(p_i, 0)(-\mathrm{e}^{\Lambda\vartheta} p, -\vartheta) \subset \Phi(\mathbb{Z}^2 \times 0)$$

Thus

$$\mathrm{e}^{-\Lambda\vartheta} p_i = \sum \lambda_i^j p_j, \qquad \lambda_i^j \in \mathbb{Z} \qquad (1)$$

Let us consider $q_i \in \mathbb{Z}^3$ such that $\Phi((q_i, 0)) = (p_i, 0)$. Then

$$(p, \vartheta)(p_i, 0)(-\mathrm{e}^{\Lambda\vartheta} p, -\vartheta) = \Phi((0, 1)(q_i, 0)(0, 1)^{-1}) =$$
$$= \Phi((0, 1)(q_i, -1)) = \Phi(A(q_i), 0) = \Phi((\sum \lambda_i^j q_j, 0))$$

i.e., the matrix (λ_i^j) is the matrix A in the basis q_i.

Next we consider the vectors $v_1 = (a_1, a_2, a_3)$ and $v_2 = (b_1, b_2, b_3)$ where $p_1 = (a_1, b_1)$, $p_2 = (a_2, b_2)$, $p_3 = (a_3, b_3)$. The equation (1) can be written now as:

i) For $G = G_5$

$$\begin{pmatrix} \mathrm{e}^{\vartheta} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_i \\ b_i \end{pmatrix} = \lambda_i^1 \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + \lambda_i^2 \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} + \lambda_i^3 \begin{pmatrix} a_3 \\ b_3 \end{pmatrix}$$

that is $Av_1 = \mathrm{e}^{\vartheta} v_1$ and $Av_2 = v_2$.

ii) For $G = G_7^k$

$$\begin{pmatrix} \mathrm{e}^{-\vartheta} & 0 \\ 0 & \mathrm{e}^{-k\vartheta} \end{pmatrix} \begin{pmatrix} a_i \\ b_i \end{pmatrix} = \lambda_i^1 \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + \lambda_i^2 \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} + \lambda_i^3 \begin{pmatrix} a_3 \\ b_3 \end{pmatrix}$$

that is $Av_1 = \mathrm{e}^{-\vartheta} v_1$ and $Av_2 = \mathrm{e}^{-k\vartheta} v_2$.

iii) For $G = G_8^{h\neq 0}$

$$c(\vartheta) \begin{pmatrix} \cos(\varphi + \vartheta) & -\sin\vartheta \\ \sin\vartheta & \cos(\varphi - \vartheta) \end{pmatrix} \begin{pmatrix} a_i \\ b_i \end{pmatrix} = \lambda_i^1 \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + \lambda_i^2 \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} + \lambda_i^3 \begin{pmatrix} a_3 \\ b_3 \end{pmatrix}$$

that is $Av_1 = c(\vartheta)(\cos(\varphi + \vartheta) v_1 - \sin\vartheta v_2)$ and $Av_2 = c(\vartheta)(\sin\vartheta v_1 + \cos(\varphi - \vartheta) v_2)$.

Set $\lambda = \mathrm{e}^{-\vartheta}$. Completing v_1, v_2 to a basis $\{v_1, v_2, v_3\}$ the matrix A is equivalent to

$$B = \begin{pmatrix} \lambda^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix} \qquad \text{for } \mathcal{G} = \mathcal{G}_5$$

$$B = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda^k & 0 \\ 0 & 0 & \lambda^{-1-k} \end{pmatrix} \qquad \text{for } \mathcal{G} = \mathcal{G}_7^k$$

and

$$B = c(\lambda)\begin{pmatrix} \cos(\varphi+\lambda) & \sin\lambda & 0 \\ -\sin\lambda & \cos(\varphi-\lambda) & 0 \\ 0 & 0 & c(\lambda)^{-3}\cos^{-2}\varphi \end{pmatrix} \qquad \text{for } \mathcal{G} = \mathcal{G}_8^h$$

Hence the result. □

Corollary 3.2. *Let $\mathcal{F}$ be a Lie $\mathcal{G}$-flow of basic dimension 1. Assume that $\mathcal{F}$ is also a Lie $\mathcal{G}'$-flow.*

i) *If $\mathcal{G} = \mathcal{G}_5$ then $\mathcal{G}' = \mathcal{G}_5$.*
ii) *If $\mathcal{G} = \mathcal{G}_7^k$ then $\mathcal{G}' = \mathcal{G}_7^{k'}$.*
iii) *If $\mathcal{G} = \mathcal{G}_8^h$ then $\mathcal{G}' = \mathcal{G}_8^{h'}$.*

Proof. The manifold M is diffeomeorphic to $\mathbb{T}_A^4$ by the existence of a Lie $\mathcal{G}$-flow, and is also diffeomorphic to $\mathbb{T}_B^4$ by the existence of a $\mathcal{G}'$-flow. Then A and B or B^{-1} are conjugated matrices of $SL(3,\mathbb{Z})$. By the above theorem we have the following 3 cases:

i) The matrix A has three real eigenvalues $\lambda,\ 1, \lambda^{-1}$. Then 1 is also an eigenvalue of the matrix B. The only possibility is $\mathcal{G}' = \mathcal{G}_5$.
ii) The matrix A has three real eigenvalues $\lambda,\ \lambda^k,\ \lambda^{-(k+1)}$ with $\lambda \neq 1$. Then the matrix B has also three real eigenvalues not equal to 1. The only possibility is $\mathcal{G}' = \mathcal{G}_7^{k'}$.
iii) The matrix A has two complex eigenvalues. Then the matrix B has also two complex eigenvalues. The only possibility is $\mathcal{G}' = \mathcal{G}_8^{h'}$. □

We improve this result in §5.

4. The realization problem with basic dimension one

The realization problem that we consider is the following:

Given a Lie algebra $\mathcal{G}$ of codimension 3 and an integer q, $0 \le q \le 3$, is there a compact manifold endowed with a Lie $\mathcal{G}$-flow of basic dimension q ?

This problem has been studied in [Lla88] and [GR91], but two cases remains still open: namely the cases corresponding to the family of Lie algebras $\mathcal{G}_8^h$, $h \neq 0$, for $q = 1, 2$, and to the family $\mathcal{G}_7^k$ for $q = 1$.

In this paper we solve the case $q = 1$. In fact we give, for $q = 1$, a necessary and sufficient condition, in terms of h (rep. k), for an algebra of the $\mathcal{G}_8^h$-family (resp. $\mathcal{G}_7^k$) to be realizable.

Theorem 4.1. *There is a compact manifold M endowed with a Lie $\mathcal{G}_8^h$-flow $\mathcal{F}$, $h \neq 0$, of basic dimension 1 if and only if*

$$h = \frac{2\ln\lambda}{\sqrt{4\omega^2 + \ln^2\lambda}}$$

where λ and ω are two real numbers, with $\lambda > 1$ and $\omega \neq k\pi$ ($k \in \mathbf{Z}$), such that λ, $\dfrac{1}{\sqrt{\lambda}}(\cos\omega \pm \mathbf{i}\sin\omega)$ are the roots of a monic polynomial of degree 3 with integer coefficients.

Observe that $\pm\omega$ is determined except an additive multiple integer of 2π.

Proof. First assume that $\mathcal{F}$ is a Lie $\mathcal{G}_8^h$-flow, $h \neq 0$, on a compact manifold M with $\dim W = \dim M/\overline{\mathcal{F}} = 1$.

Let $\{e_1, e_2, e_3\}$ be the basis of $\mathcal{G}_8^h$ described in §1. Since $\mathcal{F}$ is a riemannian flow its structural Lie algebra $\mathcal{H}$ must be abelian (cf. [Car84]) and it is a subalgebra of $\mathcal{G}_8^h$. The only abelian subalgebra of $\mathcal{G}_8^h$ is the generated by e_1, e_2, which is an ideal. Then, by Lemma 1.3, we can find a transverse Lie parallelism $\{\overline{Y}_1, \overline{Y}_2, \overline{Y}_3\}$

associated to the basis $\{e_1, e_2, e_3\}$ such that Y_1, Y_2 are tangent to $\overline{\mathcal{F}}$ at any point. Hence Y_3 is not tangent to $\overline{\mathcal{F}}$ at any point.

Recall that there exists a matrix $A \in SL(3, \mathbf{Z})$ with an eigenvalue $\lambda > 0$ and a eigenvector v, whose components are rationally independents, such that M is the manifold $\mathbf{T}^4_A$ and the flow $\mathcal{F}$ is the induced by the linear flow $\mathcal{F}_v$. If necessary, we take the matrix A^{-1} for λ to be > 1.

It follows from Theorem 3.1 that the roots of the characteristic polynomial $p(x)$ of A are λ, $z = \dfrac{1}{\sqrt{\lambda}}(\cos\omega + \mathbf{i}\sin\omega)$ and $\bar{z} = \dfrac{1}{\sqrt{\lambda}}(\cos\omega - \mathbf{i}\sin\omega)$.

We can find a basis $\{v, u_1, u_2\}$ of $\mathbf{R}^3$ such that $Au_1 = u_2$ and $Au_2 = -\dfrac{1}{\lambda}u_1 + \alpha u_2$, where $\alpha = \frac{2}{\sqrt{\lambda}}\cos\omega$.

We define the vector fields of $\mathbf{T}^3 \times \mathbf{R}$

$$X_{1(p,t)} = (u_1, 0),\ X_{2(p,t)} = (u_2, 0) \in T_p\mathbf{T}^3 \oplus T_t\mathbf{R}.$$

The transverse vector fields $\overline{Y}_1, \overline{Y}_2, \overline{Y}_3$ are the projection of the transverse vector fields of $\mathbf{T}^3 \times \mathbf{R}$ given by:

$$\begin{array}{rcl}
\overline{Z}_{1(p,t)} & = & a(t)\overline{X}_{1(p,t)} + b(t)\overline{X}_{2(p,t)} \\
\overline{Z}_{2(p,t)} & = & c(t)\overline{X}_{1(p,t)} + d(t)\overline{X}_{2(p,t)} \\
\overline{Z}_{3(p,t)} & = & (0, k\overline{\frac{\partial}{\partial t}}) \qquad (k \in \mathbf{R},\ \text{constant})
\end{array}$$

The reason is as follows.

Observe that each transverse vector field $\overline{Y}_i$ is the projection of some transverse vector field $\overline{Z}_i$ of $\mathbf{T}^3 \times \mathbf{R}$, and the these transverse vector fields are of the form:

$$\overline{Z}_{i(p,t)} = F_i(p,t)\overline{X}_{1(p,t)} + G_i(p,t)\overline{X}_{2(p,t)} + f_i(p,t)\overline{\frac{\partial}{\partial t}}$$

The fact that Y_i is a foliated vector field implies that the functions F_i, G_i, f_i does not depend on p. Since Y_1, Y_2 are tangent to $\overline{\mathcal{F}}$, the corresponding vector fields Z_1, Z_2 must be tangent to $\mathbf{T}^3$ in $\mathbf{T}^3 \times \mathbf{R}$, this means that $f_1 = f_2 = 0$.

Therefore the transverse vector fields $\overline{Z}_i$ are of the form

$$\begin{array}{rcl}
\overline{Z}_{1(p,t)} & = & a(t)\overline{X}_{1(p,t)} + b(t)\overline{X}_{2(p,t)} \\
\overline{Z}_{2(p,t)} & = & c(t)\overline{X}_{1(p,t)} + d(t)\overline{X}_{2(p,t)} \\
\overline{Z}_{3(p,t)} & = & F_3(t)\overline{X}_{1(p,t)} + G_3(t)\overline{X}_{2(p,t)} + f(t)\overline{\frac{\partial}{\partial t}}
\end{array}$$

with $f(t+1) = f(t)$ (because $\overline{Z}_3$ is projectable) and $f(t) \neq 0$ (since Y_3 is not tangent to $\overline{\mathcal{F}}$ at any point).

Observe that

$$[\overline{Z}_i, \overline{Z}_3] = [\overline{Z}_i, f(t)\overline{\frac{\partial}{\partial t}}] \qquad (i = 1, 2)$$

Then $\overline{Y}_1 = \pi_*(\overline{Z}_1)$, $\overline{Y}_2 = \pi_*(\overline{Z}_2)$, $\pi_*(f(t)\overline{\frac{\partial}{\partial t}})$ are also a realization of the basis $\{e_1, e_2, e_3\}$.

Finally, if f is not constant, we can take the reparametrization of $\mathbf{R}$ given by:

$$s(t) = \frac{\int_0^t \frac{1}{f(x)}\,\mathrm{d}x}{k} \qquad \text{where } k = \int_0^1 \frac{1}{f(x)}\,\mathrm{d}x$$

to obtain $f(t)\overline{\dfrac{\partial}{\partial t}} = k\overline{\dfrac{\partial}{\partial s}}$ and $s(t+1) = s(t) + 1$.

The identification $(x, t) \sim (Ax, t+1)$ is equivalent to $(x, s(t)) \sim (Ax, s(t)+1)$ and the projection of the transverse vector fields $\overline{Z}_{1(p,s)} = \tilde{a}(s)\overline{X}_{1(p,s)} + \tilde{b}(s)\overline{X}_{2(p,s)}$, $\overline{Z}_{2(p,s)} = \tilde{c}(s)\overline{X}_{1(p,s)} + \tilde{d}(s)\overline{X}_{2(p,s)}$ and $\overline{Z}_{3(p,s)} = (0, k\overline{\frac{\partial}{\partial s}})$ give $\overline{Y}_1, \overline{Y}_2, \overline{Y}_3$.

It can be shown that the constant k is always $\neq 1$ (cf. Remark 4.3).

Thus we can assume that the transverse vector fields $\overline{Z}_i$ are of the form:

$$\overline{Z}_1 = a(t)\overline{X}_1 + b(t)\overline{X}_2, \quad \overline{Z}_2 = c(t)\overline{X}_1 + d(t)\overline{X}_2, \quad \overline{Z}_3 = k\overline{\frac{\partial}{\partial t}}$$

The condition that the $\overline{Z}_i$ are projectable is equivalent to

$$\left.\begin{array}{ll} a(t+1) = -\dfrac{1}{\lambda}b(t) & b(t+1) = a(t) + \alpha b(t) \\ c(t+1) = -\dfrac{1}{\lambda}d(t) & d(t+1) = c(t) + \alpha d(t) \end{array}\right\} (1)$$

On the other hand,

$$c(t)\overline{X}_1 + d(t)\overline{X}_2 = \overline{Z}_2 = [\overline{Z}_1, \overline{Z}_3] = -ka'(t)\overline{X}_1 - kb'(t)\overline{X}_2$$

$$hc(t)\overline{X}_1 + hd(t)\overline{X}_2 - a(t)\overline{X}_1 - b(t)\overline{X}_2 = h\overline{Z}_2 - \overline{Z}_1 = [\overline{Z}_2, \overline{Z}_3]$$
$$= -kc'(t)\overline{X}_1 - kd'(t)\overline{X}_2$$

Then $a(t), c(t)$ and $b(t), d(t)$ are two independent solutions of the system of differential equations:

$$\left.\begin{array}{r} y(t) = -kx'(t) \\ hy(t) - x(t) = -ky'(t) \end{array}\right\}$$

Therefore, $a(t), b(t)$ are solutions of the differential equation

$$k^2x''(t) + khx'(t) + x = 0$$

then

$$(2)\quad \left.\begin{array}{r} a(t) = \mathrm{e}^{-\frac{h}{2k}t}(A_1\cos\theta t + A_2\sin\theta t) \\ b(t) = \mathrm{e}^{-\frac{h}{2k}t}(B_1\cos\theta t + B_2\sin\theta t) \\ c(t) = -ka'(t) = \dfrac{h}{2}a(t) - k\mathrm{e}^{-\frac{h}{2k}t}(-\theta A_1\sin\theta t + \theta A_2\cos\theta t) \\ d(t) = -kb'(t) = \dfrac{h}{2}b(t) - k\mathrm{e}^{-\frac{h}{2k}t}(-\theta B_1\sin\theta t + \theta B_2\cos\theta t) \end{array}\right\}$$

where A_1, A_2, B_1, B_2 are constants, $\theta = \dfrac{\sqrt{4-h^2}}{2k}$ and $A_1B_2 - A_2B_1 \neq 0$.

By substituing this values in (1) we obtain:

$$\left.\begin{array}{r} \mathrm{e}^{-\frac{h}{2k}}(A_2\cos\theta - A_1\sin\theta) = -\dfrac{1}{\lambda}B_2 \\ \mathrm{e}^{-\frac{h}{2k}}(A_1\cos\theta + A_2\sin\theta) = -\dfrac{1}{\lambda}B_1 \\ \mathrm{e}^{-\frac{h}{2k}}(B_1\cos\theta + B_2\sin\theta) = A_1 + B_1\alpha \\ \mathrm{e}^{-\frac{h}{2k}}(B_2\cos\theta - B_1\sin\theta) = A_2 + B_2\alpha \end{array}\right\} (3)$$

Then (A_1, A_2, B_1, B_2) is a non trivial solution of the homogeneous system equations whose matrix of coefficients is:

$$\begin{pmatrix} -\mathrm{e}^{-\frac{h}{2k}}\sin\theta & \mathrm{e}^{-\frac{h}{2k}}\cos\theta & 0 & \dfrac{1}{\lambda} \\ \mathrm{e}^{-\frac{h}{2k}}\cos\theta & \mathrm{e}^{-\frac{h}{2k}}\sin\theta & \dfrac{1}{\lambda} & 0 \\ 1 & 0 & \alpha - \mathrm{e}^{-\frac{h}{2k}}\cos\theta & -\mathrm{e}^{-\frac{h}{2k}}\sin\theta \\ 0 & 1 & \mathrm{e}^{-\frac{h}{2k}}\sin\theta & \alpha - \mathrm{e}^{-\frac{h}{2k}}\cos\theta \end{pmatrix} (4)$$

The determinant of this matrix must be zero.

Therefore

$$\mathrm{e}^{-\frac{h}{k}}\alpha^2 - 2\mathrm{e}^{-\frac{h}{2k}}\cos\theta(\mathrm{e}^{-\frac{h}{k}} + \frac{1}{\lambda})\alpha + (\mathrm{e}^{-\frac{h}{k}} - \frac{1}{\lambda})^2 + 4\frac{1}{\lambda}\mathrm{e}^{-\frac{h}{k}}\cos^2\theta = 0$$

Then α is a real root of the polynomial:

$$q(x) = \mathrm{e}^{-\frac{h}{k}}x^2 - 2\mathrm{e}^{-\frac{h}{2k}}\cos\theta(\mathrm{e}^{-\frac{h}{k}} + \frac{1}{\lambda})x + (\mathrm{e}^{-\frac{h}{k}} - \frac{1}{\lambda})^2 + 4\frac{1}{\lambda}\mathrm{e}^{-\frac{h}{k}}\cos^2\theta$$

whose discriminant is $-\sin^2\theta e^{-\frac{h}{k}}(e^{-\frac{h}{k}}-\frac{1}{\lambda})^2 \leq 0$.

Since $\sin\theta \neq 0$ (otherwise $A_1B_2 - A_2B_1 = 0$), the only possibility is $\lambda = e^{\frac{h}{k}}$ and, hence, $\alpha = 2e^{-\frac{h}{2k}}\cos\theta$.

Then we have

$$2e^{-\frac{h}{2k}}\cos\theta = \alpha = \frac{2}{\sqrt{\lambda}}\cos\omega = 2e^{-\frac{h}{2k}}\cos\omega$$

hence $\cos\omega = \cos\theta$. Since $\lambda = e^{\frac{h}{k}}$ and $\theta = \frac{\sqrt{4-h^2}}{2k}$, taking the argument of the complex roots $\omega = \theta$ we have

$$h = \frac{2\ln\lambda}{\sqrt{4\omega^2 + \ln^2\lambda}}$$

as stated in theorem.

Finally, we prove the converse:

Let $p(x)$ be the polynomial $x^3 - mx^2 + nx - 1$ ($m, n \in \mathbf{Z}$) with only one real root λ ($\lambda > 1$). Recall that $p(x)$ has only one real root if and only if

$$\frac{m^3+n^3}{27} - \frac{m^2n^2}{108} - \frac{nm}{6} + \frac{1}{4} > 0,$$

and this root is $\neq 1$ if $(m,n) \neq (0,0), (1,1), (2,2)$.

One can consider the matrix

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -n \\ 0 & 1 & m \end{pmatrix}$$

whose characteristic polynomial is $p(x)$. A computation shows that the root λ is irrational and the components of an eigenvector v are rationally independent.

If we define $k = \frac{h}{\ln\lambda}$, $\theta = \omega$ and $\alpha = 2e^{-\frac{h}{2k}}\cos\theta$, then the determinant of the matrix (4) is zero and the system (3) has non trivial solutions. Let A_1, A_2, B_1, B_2 be one of these solutions. Observe that $A_1B_2 - A_2B_1 \neq 0$ (because $\sin\theta \neq 0$).

We consider the functions $a(t), b(t), c(t), d(t)$ as in (2). Note that

$$a(t)\cdot d(t) - b(t)\cdot c(t) = -k\theta e^{-\frac{h}{2k}t}(A_1B_2 - A_2B_1) \neq 0.$$

Let $\{v, u_1, u_2\}$ be the basis of $\mathbf{R}^3$ such that

$$Av = \lambda v, \quad Au_1 = u_2, \quad Au_2 = -\frac{1}{\lambda}u_1 + \frac{2}{\sqrt{\lambda}}\cos\omega\alpha.$$

The vector fields

$$Z_{1(p,t)} = a(t)u_1 + b(t)u_2, \quad Z_{2(p,t)} = c(t)u_1 + d(t)u_2, \quad Z_{3(p,t)} = k\frac{\partial}{\partial t}$$

are projectable by construction and their projection gives a foliated realization of the basis $\{e_1, e_2, e_3\}$ of $\mathcal{G}_8^h$.

Observe that if we make the same construction taking $\theta = \omega + 2k\pi$ then the same flow is also a Lie $\mathcal{G}_8^{h'}$-flow with

$$h' = \frac{2\ln\lambda}{\sqrt{4(\omega+2k\pi)^2 + \ln^2\lambda}} \neq h. \quad \square$$

Corollary 4.2. *Only a numerable set of algebras of $\mathcal{G}_8^h$ are realizable on a compact manifold as transverse algebras to a basic dimension 1 Lie flow.*

Remark 4.3. Let $\mathcal{F}$ be a Lie $\mathcal{G}_8^h$-flow, $h \neq 0$, of basic dimension 1. We have seen in the proof of Theorem 4.1 that for a foliated realization Y_1, Y_2, Y_3 of the basis e_1, e_2, e_3 the vector field Y_3 is the projection of the vector field $(0, k\frac{\partial}{\partial t})$ of $\mathbf{T}^3 \times \mathbf{R}$. The constant k is always $\neq 1$.

The reason is as follows.

Assume that $k = 1$. Following the proof of Theorem 4.1 and using the same notation we have that there is a basis $\{v, u_1, u_2\}$ of $\mathbf{R}^3$ such that

$$Av = \lambda v, \qquad Au_1 = u_2, \qquad Au_2 = -\frac{1}{\lambda}u_1 + \alpha u_2$$

with $\lambda = \mathrm{e}^h$ and $\alpha = 2\mathrm{e}^{-\frac{h}{2}}\cos\theta$. Therefore the characteristic polynomial of A is equal to

$$p(x) = x^3-(\lambda+\alpha)x^2+(\alpha\lambda+\frac{1}{\lambda})x-1 = x^3-(\mathrm{e}^h+2\mathrm{e}^{-\frac{h}{2}}\cos\theta)x^2+(2\mathrm{e}^{\frac{h}{2}}\cos\theta+\mathrm{e}^{-h})x-1$$

Hence e^h $(h \in (0,2))$ is a solution of the system

$$\begin{cases} m = \mathrm{e}^h + 2\mathrm{e}^{-\frac{h}{2}}\cos\left(\dfrac{\sqrt{4-h^2}}{2}\right) \\ n = 2\mathrm{e}^{\frac{h}{2}}\cos\left(\dfrac{\sqrt{4-h^2}}{2}\right) + \mathrm{e}^{-h} \end{cases} \qquad m, n \in \mathbf{Z}$$

But it can be shown, after a quite long computation, that this system has no solutions for $h \in [0, 2]$.

Theorem 4.4. *There is a compact manifold M endowed with a $\mathcal{G}_7^k$-flow $\mathcal{F}$ of basic dimension 1 if and only if*

$$k = \frac{\ln b}{\ln a}, \qquad \text{and } k \notin \mathbf{Q}$$

where a, b, $\frac{1}{ab}$ are positive real roots of a monic polynomial of degree 3 with integer coefficients.

Proof.

Let $\mathcal{F}$ be a Lie $\mathcal{G}_7^k$-flow of dimension 1 on a compact manifold M. Then $M = \mathbb{T}3 \times_A \mathbb{R}$. By theorem 3.1, the matrix $A \in SL(3,\mathbb{Z})$ has the three real roots

$$a = e^{-\vartheta}, b = e^{-\vartheta k}, c = e^{\vartheta(k+1)}, \vartheta > 0$$

and $k \notin \mathbf{Q}$ (cf. [GR91]).

Conversely:

Let $x^3 - mx^2 + nx - 1$ a polynomial with real roots a, b, c such that

$$k = \frac{\ln b}{\ln a} \notin \mathbf{Q}.$$

Let us consider the matrix

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -n \\ 0 & 1 & m \end{pmatrix} \in SL(3,\mathbb{Z})$$

As the eigenvalues of A are not in $\mathbf{Q}$, the components of the eigenvectors of A are rationally independent.

If we define $\vartheta = -\ln a$, then we have that the roots of the polynomial $p(x)$ are

$$e^{-\vartheta}, e^{-\vartheta k}, e^{\vartheta(k+1)}.$$

Let $\{v_1, v_2, v_3\}$ be a basis of eigenvectors of A:

$$A(v_1) = e^{-\vartheta}v_1, A(v_2) = e^{-\vartheta k}v_2, A(v_3) = e^{\vartheta(k+1)}v_3.$$

Then $\mathcal{F}$ the linear flow on $\mathbb{T}^3$ given for the v_3 vector induces a flow $\overline{\mathcal{F}}$ on $M = \mathbb{T}^4_A$.

The vector fields on $\mathbb{T}^3 \times \mathbb{R}$

$$\begin{cases} \widetilde{X}_1(p,t) &= e^{-t\vartheta}(v_1, 0) \\ \widetilde{X}_2(p,t) &= e^{-tk\vartheta}(v_2, 0) \\ \widetilde{X}_3(p,t) &= \dfrac{1}{\vartheta}\dfrac{\partial}{\partial t} \end{cases}$$

are projectable on M because $\widetilde{X}_i(p, t+1) = A(\widetilde{X}_i(p,t))$. The projections X_1, X_2, X_3 of these vector fields are foliated vector fields transverse to the flow such that

$$\begin{aligned} [X_1, X_2]_{(p,t)} &= [e^{-t\vartheta}(v_1,0), e^{-tk\vartheta}(v_2,0)] = 0 \\ [X_1, X_3]_{(p,t)} &= [e^{-t\vartheta}(v_1,0), \frac{1}{\vartheta}\frac{\partial}{\partial t}] = e^{-t\vartheta}(v_1,0) = X_{1(p,t)} \\ [X_2, X_3]_{(p,t)} &= [e^{-tk\vartheta}(v_2,0), \frac{1}{\vartheta}\frac{\partial}{\partial t}] = kX_{2(p,t)} \quad \square \end{aligned}$$

Corollary 4.5. *Only a numerable set of algebras of $\mathcal{G}_7^k$ are realizable on a compact manifold as transverse algebras to a Lie flow with basic dimension 1.*

5. The change problem

The aim of this section is to prove that a Lie $\mathcal{G}$-flow of codimension 3 on a compact manifold with basic dimension 1 can be transversely modeled on one, two or countable many Lie algebras.

Proposition 5.1. *Let $\mathcal{F}$ be a Lie $\mathcal{G}_7^k$-flow of basic dimension 1 on a compact manifold M transversely modeled on another Lie algebra $\mathcal{G}$. Then $\mathcal{G} = \mathcal{G}_7^{k'}$ and k' is one of the values*

$$k' = k, \quad \frac{1}{k}; \quad \frac{1}{-1-k}, \quad -1-k; \quad -\frac{k}{1+k}, \quad -\frac{1+k}{k}.$$

Proof. By Corollary 3.2 the Lie algebra $\mathcal{G}$ is $\mathcal{G}_7^{k'}$. Then the matrix $A \in SL(3;\mathbb{Z})$ defining M has the eingenvalues a, a^k, a^{-1-k} and also $b, b^{k'}, b^{-1-k'}$. As they are the same, we have $3! = 6$ possibilities that imply directely the result. $\square$

Remark 5.2. If $\mathcal{F}$ is a Lie $\mathcal{G}_7^k$-flow on a compact manifold M, then we also have on M a Lie $\mathcal{G}_7^{-1-k}$-flow and a Lie $\mathcal{G}_7^{-k/1+k}$-flow.

To see this, let u_1, u_2, u_3 be the eigenvectors of A with eigenvalues a, a^k, a^{-1-k} respectively.

The vector fields on $\mathbb{T}^3 \times \mathbb{T}^1$

$$\widetilde{X}_1 = a^t u_1, \qquad \widetilde{X}_2 = a^{kt}u_2, \qquad \widetilde{X}_3 = a^{-(1+k)t}u_3, \qquad \widetilde{X}_0 = -\frac{1}{\ln a}\frac{\mathrm{d}}{\mathrm{d}t}$$

are invariant for A, so they induce vector fields on $M = \mathbb{T}^3 \times_A \mathbb{T}^1$, that we denote by X_1, X_2, X_3, X_0, respectively.

The foliation induced by X_1 admits $X_2, X_3, -\dfrac{1}{1+k}X_0$ as transverse parallelism and

$$[X_2, X_3] = 0 \qquad [X_3, -\frac{1}{1+k}X_0] = X_2 \qquad [X_2, -\frac{1}{1+k}X_0] = -\frac{1}{1+k}X_2,$$

i.e., it is a Lie $\mathcal{G}_7^{-k/(1+k)}$-flow.

The foliation induced by X_2 admits X_1, X_3, X_0 as transverse parallelism and

$$[X_1, X_3] = 0 \qquad [X_1, X_0] = X_1 \qquad [X_3, X_0] = -(1+k)X_3,$$

i.e., it is a Lie $\mathcal{G}_7^{-1-k}$-flow.

The foliation induced by X_3 admits X_1, X_2, X_0 as transverse parallelism and

$$[X_1, X_2] = 0 \qquad [X_1, X_0] = X_1 \qquad [X_2, X_0] = kX_2,$$

i.e., it is a Lie $\mathcal{G}_7^k$-flow.

It follows from this remark that it is not possible to deduce from Theorem 3.1 if a Lie $\mathcal{G}_7^k$-flow of basic dimension 1 is at the same time a Lie $\mathcal{G}_7^{k'}$-flow with $k \neq k'$.

A priori the six cases of Proposition 5.1 are possible. Nevertheless we have

Proposition 5.3. *Let $\mathcal{F}$ be a Lie flow of basic dimension 1 on a compact manifold M transversely modeled on two Lie algebras, $\mathcal{G}_7^k$ and $\mathcal{G}_7^{k'}$, of the family $\mathcal{G}_7$. Then $\mathcal{G}_7^k \cong \mathcal{G}_7^{k'}$.*

Proof. Assume that the flow is transversely modeled on two Lie algebras, $\mathcal{G}_7^k$ and $\mathcal{G}_7^{k'}$, of the family $\mathcal{G}_7$. Since the structural Lie algebra is abelian and the only abelian subalgebra of $\mathcal{G}_7$ is the ideal generated by e_1, e_2 we have two Lie parallelisms $\overline{Y}_1, \overline{Y}_2, \overline{Y}_3$ and $\overline{Z}_1, \overline{Z}_2, \overline{Z}_3$ with products

$$(1) \qquad [\overline{Y}_1, \overline{Y}_2] = 0, \qquad [\overline{Y}_1, \overline{Y}_3] = \overline{Y}_1, \qquad [\overline{Y}_2, \overline{Y}_3] = k\overline{Y}_2.$$

$$(2) \qquad [\overline{Z}_1, \overline{Z}_2] = 0, \qquad [\overline{Z}_1, \overline{Z}_3] = \overline{Z}_1, \qquad [\overline{Z}_2, \overline{Z}_3] = k'\overline{Z}_2$$

such that the foliated vector fields Y_1, Y_2 and Z_1, Z_2 are tangent to $\overline{\mathcal{F}}$ everywhere.

The transverse vector fields $\overline{Z}_i$ are

$$\begin{aligned} \overline{Z}_1 &= f_1^1\overline{Y}_1 + f_1^2\overline{Y}_2 \\ \overline{Z}_2 &= f_2^1\overline{Y}_1 + f_2^2\overline{Y}_2 \\ \overline{Z}_3 &= f_3^1\overline{Y}_1 + f_3^2\overline{Y}_2 + f_3^3\overline{Y}_3 \end{aligned}$$

with f_i^j basic functions.

By substitution of this values in the equations (2) we have the equations:

$$\begin{cases} f_1^1 = f_1^1 f_3^3 - f_3^3 Y_3(f_1^1) \\ f_1^2 = k f_1^2 f_3^3 - f_3^3 Y_3(f_1^2) \\ k' f_2^1 = f_2^1 f_3^3 - f_3^3 Y_3(f_2^1) \\ k' f_2^2 = k f_2^2 f_3^3 - f_3^3 Y_3(f_2^2) \end{cases}$$

(observe that as Y_1, Y_2 are tangent to $\overline{\mathcal{F}}$, then $Y_1(f_i^j) = Y_2(f_i^j) = 0$.)

Note that f_3^3 is not zero at any point. Then all these equations are of the form $y' = g(x)y$.

By interpreting this system of differential equations on the basic manifold S^1, we have the solutions:

$$\begin{aligned} f_1^1 &= C_1 \mathrm{e}^x \mathrm{e}^{-\int_{t_0}^x \frac{1}{f_3^3}\mathrm{d}t} \\ f_1^2 &= C_2 \mathrm{e}^{kx} \mathrm{e}^{-\int_{t_0}^x \frac{1}{f_3^3}\mathrm{d}t} \\ f_2^1 &= C_3 \mathrm{e}^x \mathrm{e}^{-\int_{t_0}^x \frac{k'}{f_3^3}\mathrm{d}t} \\ f_2^2 &= C_4 \mathrm{e}^{kx} \mathrm{e}^{-\int_{t_0}^x \frac{k'}{f_3^3}\mathrm{d}t} \end{aligned}$$

Then there exist periodic solutions with $f_1^1 f_2^2 - f_1^2 f_2^1 \neq 0$ if and only if $k = k'$ or $k = \dfrac{1}{k'}$. $\square$

Now we are in the position to prove the main result of this section.

Theorem 5.4. *Let $\mathcal{F}$ be a Lie flow of codimension 3 on a compact manifold M with basic dimension 1. Then only three cases are possible:*

i) $\mathcal{F}$ is transversely modeled exactly on one Lie algebra. This occurs if and only if the transverse Lie algebra is $\mathcal{G}_5$ or $\mathcal{G}_7^k$.
ii) $\mathcal{F}$ is transversely modeled exactly on two Lie algebras. This occurs if and only if these two transverse Lie algebras are $\mathcal{G}_1$ and $\mathcal{G}_8^{h=0}$.
iii) $\mathcal{F}$ is transversely modeled on countable many Lie algebras. This occurs if and only if $\mathcal{F}$ is transversely modeled on $\mathcal{G}_8^{h\neq 0}$.
In this case

a) *There exist two real numbers $\lambda > 1$ and ω such that $h = \dfrac{2\ln\lambda}{\sqrt{4\omega^2 + \ln^2\lambda}}$.*

b) *$\mathcal{F}$ is also transversely modeled on $\mathcal{G}_8^{h'}$ for each*
$$h' = \frac{2\ln\lambda}{\sqrt{4(\omega + 2k\pi)^2 + \ln^2\lambda}} \qquad (\forall k \in \mathbb{Z}).$$

c) *If $\mathcal{F}$ is also transversely modeled on $\mathcal{G}$ then $\mathcal{G} = \mathcal{G}_8^{h'}$ for some of the above h'.*

Proof. Assume that $\mathcal{F}$ is a Lie $\mathcal{G}$-flow of codimension 3 on the compact manifold M with basic dimension 1. Then $\mathcal{G}$ is one of the following Lie algebras: $\mathcal{G}_1$, $\mathcal{G}_5$, $\mathcal{G}_7^k$ or $\mathcal{G}_8^h$.

i) If $\mathcal{G} = \mathcal{G}_5$ then by Corollary 3.2 $\mathcal{F}$ is not transversely modeled on any other Lie algebra.
If $\mathcal{G} = \mathcal{G}_7^k$ then by Proposition 5.3, $\mathcal{F}$ is not transversely modeled on any other Lie algebra.

ii) If $\mathcal{G} = \mathcal{G}_1$ then by Proposition 2.2, $\mathcal{F}$ is also transversely modeled on $\mathcal{G}' = \mathcal{G}_8^{h=0}$.
Conversely if $\mathcal{G} = \mathcal{G}_8^{h=0}$, Corollary 2.4 proves that $\mathcal{F}$ is also a Lie abelian flow.

iii) It only remains the case that $\mathcal{G} = \mathcal{G}_8^{h\neq 0}$.

a) We have, by Theorem 4.1, that
$$h = \frac{2\ln\lambda}{\sqrt{4\omega^2 + \ln^2\lambda}}$$
where λ, $\dfrac{1}{\sqrt{\lambda}}(\cos\omega \pm \mathbf{i}\sin\omega)$, $\lambda > 1$, are the roots of a polynomial of integer coefficients.

b) Theorem 4.1 also proves that $\mathcal{F}$ is also a Lie $\mathcal{G}_8^{h'}$-flow with
$$h' = \frac{2\ln\lambda}{\sqrt{4(\omega + 2k\pi)^2 + \ln^2\lambda}} \qquad \text{for any } k \in \mathbb{Z}.$$

c) If $\mathcal{F}$ is also a Lie $\mathcal{G}'$-flow, it follows from Corollary 3.2 that $\mathcal{G} = \mathcal{G}_8^{h'}$. Again, by Theorem 4.1 we have that
$$h' = \frac{2\ln\lambda'}{\sqrt{4(\omega')^2 + \ln^2\lambda'}}.$$
But using Theorem 3.1 is easy to see that $\lambda' = \lambda'$ and $\omega' = \omega + 2k\pi$. □

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