THE TORSION INDEX OF A $p$-COMPACT GROUP

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Abstract. We extend the theory of torsion indices of compact connected Lie groups to $p$-compact groups and compute these indices in all cases.

1. Introduction and statement of results

The torsion index of a compact connected Lie group was defined by Grothendieck in 1958 ([13]) and has been investigated by several authors ([14], [6], [15], etc.). Recently, the computation of the torsion indices of all simply connected compact Lie groups has been completed (see [16]). Since we are going to work at a single prime $p$, instead of the torsion index of a Lie group $G$, we want to consider its $p$-primary part $t_p(G)$. We summarize the properties of $t_p(G)$ which are relevant to the present work in the following proposition ($\mathbb{Z}_p$ denotes the ring of $p$-adic integers).

Theorem 1.1. Let $p$ be a prime and let $G$ be a compact connected Lie group with a maximal torus $T$ and corresponding Weyl group $W$. The positive integer $t_p(G)$ has the following properties:

1. **(TI1)** If $A$ is a finite abelian $p$-subgroup of $G$, then $A$ has a subgroup of index dividing $t_p(G)$ which is contained in a conjugate of $T$.

2. **(TI2)** $t_p(G)$ kills the kernel and the cokernel of the homomorphism $H^*(BG; \mathbb{Z}_p) \to H^*(BT; \mathbb{Z}_p)^W$.

3. **(TI3)** $H^*(G/T; \mathbb{Z}_p)$ is torsion free and concentrated in even degrees $\leq N = \dim(G) - \text{rank}(G)$, with $H^N(G/T; \mathbb{Z}_p) \cong \mathbb{Z}_p$. Then, $t_p(G)$ is the order of the cokernel of $H^N(BT; \mathbb{Z}_p) \to H^N(G/T; \mathbb{Z}_p)$.

4. **(TI4)** If $p$ is not a torsion prime for $G$, then $t_p(G) = 1$.

Notice that the property (TI3) can be taken as a definition of the $(p$-primary) torsion index $t_p(G)$. The other properties are well known and can be found in [15], which provides proofs or references for all of them. Actually, the properties above are usually stated using $H^*(-; \mathbb{Z})$ and $t(G) = \prod_p t_p(G)$ instead of $H^*(-; \mathbb{Z}_p)$ and $t_p(G)$, but it is easy to see that both formulations are indeed equivalent. For property (TI2) one should notice that $H^*(BT; \mathbb{Z}_p)^W = H^*(BT; \mathbb{Z})^W \otimes \mathbb{Z}_p$. This follows from exactness of $- \otimes \mathbb{Z}_p$ and the fact that the elements invariant under $W$ can be viewed as the kernel of the homomorphism $\bigoplus_{g \in W}(1 - g)$.
The purpose of this paper is to extend the theorem above to connected \( p \)-compact groups ([8]) and to compute the torsion indices in all cases. We prove:

**Theorem 1.2.** Let \( p \) be a prime and let \( X \) be a connected \( p \)-compact group with maximal torus \( T \) and corresponding Weyl group \( W \). There is an integer \( t_p(X) \) such that:

1. The properties (TI1), (TI2), (TI3), (TI4) in Theorem 1.1 hold after replacing \( G \) with \( X \).
2. If \( X \) is exotic, then \( t_p(X) = 1 \) for \( p \) odd and \( t_2(X) = 2 \).

Here we use the work exotic with the same meaning as in [1]: A \( p \)-compact group \( X \) is exotic if the associated pseudoreflection representation of the Weyl group of \( X \) over the \( p \)-adic field is irreducible and does not come from a reflection group over \( \mathbb{Z} \).

Section 2 deals with the (easier) odd prime case, and we show that if we define \( t_p(X) = 1 \) for any exotic \( X \), then properties (TI1), (TI2), (TI3), (TI4) hold true. The hardest part consists of computing the torsion index of the only exotic 2-compact group, which we (following [12]) denote \( G_3 \) (other authors denote it as \( DI(4) \)). We need a comprehensive review of the cohomology of \( G_3 \) and \( BG_3 \) (section 3) and some computations on the cohomology of the exotic homogeneous space \( G_3/\text{Spin}(7) \) (section 4) before we can prove that \( t_2(G_3) = 2 \). Finally, we prove Theorem 1.2 in section 6.

### 2. The odd prime case

The classification theorem for \( p \)-compact groups ([2]) tells us that any connected \( p \)-compact group \( X \) splits uniquely as a product \( X \cong G_p^0 \times X_1 \), where \( G \) is a compact connected Lie group and \( X_1 \) is a product of exotic \( p \)-compact groups. Notice that the splitting is as \( p \)-compact groups and not just as spaces. This splitting implies that it is enough to prove Theorem 1.2 for each exotic \( p \)-compact group, since it is already known to be true for the \( (p) \)-completions of compact connected Lie groups. Let us discuss this in some more detail. If Theorem 1.2 holds for the \( p \)-compact groups \( X_1 \) and \( X_2 \), let \( X = X_1 \times X_2 \) and let us define \( t_p(X) = t_p(X_1) t_p(X_2) \). We need to check that properties (TI1) to (TI4) hold for \( X \) if they hold for \( X_1 \) and \( X_2 \).

We prove (TI1) trivial and (TI3) is straightforward. To prove (TI2) let us observe that the kernel of \( \gamma : H^*(BX; \mathbb{Z}_p) \to H^*(BT; \mathbb{Z}_p)^W \) is equal to the torsion elements in \( H^*(BX; \mathbb{Z}_p) \). If \( X \) is of Lie type, this is well known (cf. [9]). If \( X \) is exotic and \( p = 2 \) (i.e. \( X = G_3 \)), then this is assertion 4 in [12]; and if \( p \) is odd, this is proven in [4]. Then, it is clear that \( t_p(X_1) t_p(X_2) \) kills the kernel of \( \gamma \). It is obvious that \( t_p(X_1) t_p(X_2) \) kills the cokernel of \( \gamma \) as well. Finally, (TI1) follows easily since we can use the theory of kernels of homomorphisms between \( p \)-compact groups which is developed in [8], section 7.

Let us assume now that \( p \) is odd and let \( X \) be an exotic \( p \)-compact group. These objects are very well understood. In particular, they satisfy the following properties (see [1]). Let \( T \) and \( W \) denote a maximal torus of \( X \) and the corresponding Weyl group, respectively. Then:

1. \( X \) is simply connected and center free and \( H^*(X; \mathbb{Z}_p) \) is torsion free.
2. The natural map \( BT \to BX \) induces an isomorphism \( H^*(BX; \mathbb{Z}_p) \cong H^*(BT; \mathbb{Z}_p)^W \).
In particular, $H^*(BX;\mathbb{Z}_p)$ is concentrated in even degrees.

(3) $H^*(X/T;\mathbb{Z}_p)$ is a free $\mathbb{Z}_p$-module concentrated in even degrees. Moreover (see [13], th. 7.5.1) $H^*(X/T;\mathbb{Z}_p) \otimes \mathbb{Q}$ is a Poincaré duality algebra with fundamental class in degree $\text{dim}(X) - \text{rank}(X)$. Actually, as a $W$-module, $H^*(X/T;\mathbb{Z}_p) \otimes \mathbb{Q}$ coincides with the regular representation of $W$.

We also need another property of $p$-compact groups (which holds also for $p = 2$) that follows from the work in [3].

(4) If $X$ is any $p$-compact group such that $H^*(BX;\mathbb{F}_p)$ is concentrated in even degrees, then any finite abelian $p$-subgroup of $X$ is conjugated to a subgroup of the maximal torus of $X$. In particular, this holds for any product of exotic $p$-compact groups for $p$ odd.

Theorem [12] for $p$ odd follows immediately from all these properties of $p$-compact groups.

\[ \Box \]

3. The 2-Compact Group $G_3$ and its Maximal Torus

In this section we recollect several properties of $G_3$ that we need in the forthcoming sections. We state these properties without proof because either they can be found in the papers [7], [12], [4], [11] or they follow from straightforward computations that are left to the reader.

As is well known, $G_3$ is an exotic connected 2-compact group of rank three whose Weyl group $W$ is the reflection group number 24 in the Shephard-Todd list of finite complex reflection groups. Its existence was established by Dwyer and Wilkerson in [4]. We remind the reader that some authors call this 2-compact group $DI(4)$, but we follow the notation used in [12]. As an abstract group, $W$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times GL_3(\mathbb{F}_2)$ and for a maximal torus $T$ of $G_3$, there is a basis $\{e_1, e_2, e_3\}$ of $H^2(BT;\mathbb{Z}_2)$ such that the action of $W$ on $H^*(BT;\mathbb{Z}_2)$ is given by the pseudoreflections

$$s_1 = \begin{pmatrix} -1 & \alpha & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 1 & 0 & 0 \\ -\alpha & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad s_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix},$$

where $\alpha, \bar{\alpha} \in \mathbb{Z}_2$ are the roots of $x^2 - x + 2$ chosen in such a way that $\alpha$ is odd and $\bar{\alpha}$ is even.

$G_3$ has $\text{Spin}(7)$ as a 2-compact subgroup of maximal rank. This means that there is a map $\phi : B\text{Spin}(7) \to BG_3$ whose homotopical fibre is $\mathbb{F}_2$-finite. It is natural to denote this fibre by $G_3/\text{Spin}(7)$. The restriction of $\phi$ to a maximal torus of $\text{Spin}(7)$ is a maximal torus of $G_3$.

There is a subgroup $V \subset \text{Spin}(7)$ (explicitly described in [7]) which is an elementary abelian 2-group of rank four and such that the homomorphisms

$$H^*(BG_3;\mathbb{F}_2) \xrightarrow{\phi^*} H^*(B\text{Spin}(7);\mathbb{F}_2) \xrightarrow{k^*} H^*(BV;\mathbb{F}_2) \cong \mathbb{F}_2[V^*]$$

are monomorphisms ($k^*$ is induced by the inclusion $V \subset \text{Spin}(7)$). Moreover, the image of $(\phi k)^*$ coincides with the rank four Dickson algebra which is the algebra of invariants of $H^*(BV;\mathbb{F}_2)$ under the action of the full linear group $GL(V^*)$, and the image of $k^*$ coincides with the algebra of invariants $H^*(BV;\mathbb{F}_2)^H$ where $H \subset GL(V^*)$ can be described, in some appropriate basis of $V^*$, as the set of matrices with first row equal to $(1, 0, 0, 0)$. These algebras of invariants are well
known (also as algebras over the Steenrod algebra) and we have isomorphisms (subscripts denote degrees)
\[ H^*(BG; \mathbb{F}_2) \cong \mathbb{F}_2[c_8, c_{12}, c_{14}, c_{15}], \]
\[ H^*(B\text{Spin}(7); \mathbb{F}_2) \cong \mathbb{F}_2[d_4, d_6, d_7, d_8], \]
where the generators \( c_i \) and \( d_i \) can be explicitly described. In particular, we can see that \( \phi^* \) is given by \( \phi^*(c_8) = d_6^2 + d_8, \phi^*(c_{12}) = d_6^2 + d_4d_8, \phi^*(c_{14}) = d_6^2 + d_6d_8, \phi^*(c_{15}) = d_7d_8 \). \( Sq^1 \) vanishes on \( d_4, d_7, d_8 \), while \( Sq^1(d_6) = d_7 \).

As was said before, a maximal torus \( T \) of \( \text{Spin}(7) \) is also a maximal torus of \( G_3 \). We have maps
\[ BT_2^n \rightarrow B\text{Spin}(7)_2^n \rightarrow BG_3 \]
and we can view the Weyl group \( W_1 \) of \( \text{Spin}(7) \) as a subgroup of \( W \), namely \( W_1 = \langle s_1, s_2, s_1s_3s_2s_1s_2s_3s_1 \rangle \). It is known that the homomorphism
\[ i^* : H^*(B\text{Spin}(7); \mathbb{Z}_2) \rightarrow H^*(BT; \mathbb{Z}_2)^{W_1} \]
is surjective and its kernel coincides with the ideal of torsion elements. The integral invariants of \( W_1 \) are computed in [4]. They turn out to form a polynomial algebra on generators of degrees 4, 8, 12:
\[ H^*(BT; \mathbb{Z}_2)^{W_1} \cong \mathbb{Z}_2[u_4, u_8, u_{12}]. \]
Choosing an appropriate basis \( \{x_1, x_2, A\} \) of \( H^3(BT; \mathbb{Z}_2) \), these generators are
\[ u_4 = (1/2)(x_1^2 + x_2^2 + x_3^2), \]
\[ u_8 = (1/16)(x_1^4 + x_2^4 + x_3^4 - 2x_1^2x_2^2 - 2x_1^2x_3^2 - 2x_2^2x_3^2), \]
\[ u_{12} = x_1^2x_2^2x_3^2, \]
where we have used the notation \( x_3 = 2A - x_1 - x_2 \), and one can check that in spite of the denominators, these polynomials belong to \( \mathbb{Z}_2[x_1, x_2, A] \).

The generators \( u_4, u_8 \) and \( u_{12} \) have a rather simple form as polynomials on \( x_1, x_2, A \), but this basis of \( H^2(BT; \mathbb{Z}_2) \) does not coincide with the basis \( \{\epsilon_1, \epsilon_2, \epsilon_3\} \) that we have used to describe the action of \( W \) on \( H^*(BT; \mathbb{Z}_2) \). The matrix that expresses \( \{\epsilon_1, \epsilon_2, \epsilon_3\} \) in terms of \( \{x_1, x_2, A\} \) is
\[
\begin{pmatrix}
0 & -\alpha / 2 & -(1 + \alpha) / 2 \\
1 & 0 & -(1 + \alpha) / 2 \\
\bar{\alpha} & \alpha & \bar{\alpha}
\end{pmatrix} \in GL_3(\mathbb{Z}_2).
\]
Using this matrix we can express the generators \( u_4, u_8, u_{12} \) as polynomials in \( \epsilon_1, \epsilon_2, \epsilon_3 \) and so we have an explicit description of the homomorphism
\[ \mathbb{Z}_2[u_4, u_8, u_{12}] = H^*(B\text{Spin}(7); \mathbb{Z}_2)_{/\text{Torsion}} \rightarrow H^*(BT; \mathbb{Z}_2) = \mathbb{Z}_2[\epsilon_1, \epsilon_2, \epsilon_3]. \]

Finally, we want to use this to describe the homomorphism
\[ \mathbb{F}_2[d_4, d_6, d_7, d_8] = H^*(B\text{Spin}(7); \mathbb{F}_2) \xrightarrow{\iota} H^*(BT; \mathbb{F}_2) = \mathbb{F}_2[\epsilon_1, \epsilon_2, \epsilon_3]. \]
In the Bockstein spectral sequence for \( B\text{Spin}(7) \) we have \( E_2 = E_\infty = \mathbb{F}_2[d_4, d_6, d_7, d_8] \), and the surjection
\[ j : H^*(B\text{Spin}(7); \mathbb{Z}_2)_{/\text{Torsion}} \rightarrow E_\infty \]
is given by \( j(u_4) = \bar{d}_4, j(u_8) = \bar{d}_8, j(u_{12}) = \bar{d}_6^2 \). From this it is straightforward to perform the computations that yield

\[
\begin{align*}
    i^*(d_4) &= \epsilon_1^2 + \epsilon_1 \epsilon_2 + \epsilon_2^2, \\
    i^*(d_6) &= Sq^2 i^*(d_4) = \epsilon_1^2 \epsilon_2 + \epsilon_1 \epsilon_2^2, \\
    i^*(d_7) &= 0, \\
    i^*(d_8) &= \epsilon_1 \epsilon_2 \epsilon_3 (\epsilon_1 + \epsilon_2 + \epsilon_3) + \epsilon_2^2 (\epsilon_1 + \epsilon_2 + \epsilon_3)^2.
\end{align*}
\]

4. The Exotic Homogeneous Space \( G_3/\Spin(7) \)

In this section we want to investigate the cohomology of the exotic homogeneous space \( G_3/\Spin(7) \). The computations presented here are probably known to experts, but it may be worthwhile to work them out here in some detail.

Let us consider the fibration \( G_3/\Spin(7) \to B\Spin(7) \to BG_3 \) and let \( V \subset \Spin(7) \) denote the elementary abelian 2-group of rank 4 considered in the preceding section. To simplify the notation, let us write \( B\Spin(7) \) to denote the elementary abelian 2-group of rank 4.

We obtain the following:

\( \text{Tor} \quad \text{BG}_3(\mathbb{F}_2, H^* (\Spin(7); \mathbb{F}_2)) \Rightarrow H^* (\Spin(7); \mathbb{F}_2) \). Here the key observation is that \( H^* (\Spin(7); \mathbb{F}_2) \) is a free module over \( H^* (BG_3; \mathbb{F}_2) = S^G \) because of the following classic argument. \( S \) is an integral extension of \( S^G \); hence \( S^H \) is also an integral extension of \( S^G \) and, since \( S^H \) is a finitely generated algebra, we obtain that \( S^H \) is a finitely generated \( S^G \)-module. But both \( S^H \) and \( S^G \) are polynomial algebras, and we can apply \[, Chap. V, 5.5, or \[, 6.7.1, to conclude that \( S^H \) is \( S^G \)-free.

Hence the Eilenberg-Moore spectral sequence collapses to an isomorphism

\[
H^* (G_3/\Spin(7); \mathbb{F}_2) \cong \mathbb{F}_2 [\bar{d}_4, \bar{d}_6, \bar{d}_7]/(\bar{d}_6^2 + \bar{d}_6^3 \bar{d}_7^2 + \bar{d}_4^2 \bar{d}_6, \bar{d}_4^2 \bar{d}_7),
\]

where \( \bar{d}_4, \bar{d}_6, \bar{d}_7 \) are the images of \( d_4, d_6, d_7 \in H^* (\Spin(7); \mathbb{F}_2) \), respectively. It is rather easy to completely work out the algebra structure of \( H^* (G_3/\Spin(7); \mathbb{F}_2) \). We obtain the following:

1. The Poincaré series of \( H^* (G_3/\Spin(7); \mathbb{F}_2) \) is
   \[1 + t^4 + t^6 + t^7 + t^8 + t^{10} + t^{11} + t^{12} + t^{13} + t^{14} + t^{16} + t^{17} + t^{20} + t^{24}\]
   and the Euler characteristic is \( 7 = [W:H] \).

2. An additive basis for \( H^* (G_3/\Spin(7); \mathbb{F}_2) \) is given by
   \[\{\bar{d}_i^i, i = 0, \ldots, 6, \bar{d}_6, \bar{d}_7, \bar{d}_4 \bar{d}_6, \bar{d}_4 \bar{d}_7, \bar{d}_6 \bar{d}_7, \bar{d}_4^2 \bar{d}_6, \bar{d}_4 \bar{d}_6 \bar{d}_7, \bar{d}_4^3 \bar{d}_6\}\].

3. \( H^* (G_3/\Spin(7); \mathbb{F}_2) \) is a Poincaré duality algebra with top class \( \bar{d}_4^6 \) (see \[, 6.5).

4. The Bockstein spectral sequence of \( H^* (G_3/\Spin(7); \mathbb{F}_2) \) collapses after the second term; i.e. \( H^* (G_3/\Spin(7); \mathbb{Z}_2) \) has only torsion of order 2. We have

\[
H^* (G_3/\Spin(7); \mathbb{Z}_2)/\text{Torsion} \cong \mathbb{Z}_2 [\bar{a}] / (\bar{a}^7, \bar{a}^3, \bar{a}^2 \bar{c}, 2\bar{c})
\]

and

\[
H^* (G_3/\Spin(7); \mathbb{Z}_2) \cong \mathbb{Z}_2 [\bar{a}, \bar{c}] / (\bar{a}^7, \bar{b}^3, \bar{a}^2 \bar{c}, 2\bar{c}).
\]
In particular, the top class in \( H^*(G_3/\text{Spin}(7);\mathbb{Z}_2) \) is \( \tilde{d}_4^6 \) in dimension 24, and it is in the image of
\[
\phi^* : H^*(B\text{Spin}(7);\mathbb{Z}_2) \to H^*(G_3/\text{Spin}(7);\mathbb{Z}_2).
\]

5. The torsion index of \( G_3 \)

To compute the torsion index of the 2-compact group \( G_3 \) we need a lemma on Poincaré duality in fibrations. I’m grateful to Aniceto Murillo for some helpful conversations on this subject. For this lemma we use the following notation. Let \( \mathcal{O} \) denote the ring of integers or the ring of \( p \)-adic integers. Cohomology is taken with coefficients in \( \mathcal{O} \), and we assume that all spaces are of finite type over \( \mathcal{O} \). We say that \( \eta \in H^n(X) \) is an orientation class if \( H^i(X) = 0 \) for \( i > n \), \( H^n(X) \cong \mathcal{O} \), and \( \eta \) is a generator of \( H^n(X) \).

**Lemma 5.1.** Let \( F \xrightarrow{j} E \xrightarrow{\pi} B \) be a fibration of 1-connected spaces and assume that \( \eta^F \in H^n(F) \) and \( \eta^B \in H^n(B) \) are orientation classes. Assume \( \alpha \in H^m(E) \) is such that \( j^*(\alpha) = \lambda \eta^F \) for some \( \lambda \neq 0 \). Then there is an orientation class \( \eta^E \) for \( E \) such that \( \alpha \cdot \pi^*(\eta^B) = \lambda \eta^E \).

**Proof.** This follows easily from the cohomology spectral sequence of the fibration \( F \xrightarrow{j} E \xrightarrow{\pi} B \). First of all, it is clear that \( H^i(E) = 0 \) for \( i > n + m \) while \( H^{n+m}(E) = E_{\infty}^{n,m} = E_{2}^{n,m} \cong \mathcal{O} \). Recall that the cohomology spectral sequence is multiplicative in the sense that (up to some signs which would not play any role here) the product in \( E_2 \) induced by the products in \( H^*(B) \) and \( H^*(F) \) yields a product in each \( E_r \), \( 2 \leq r \leq \infty \), in such a way that the product in \( E_\infty \) is compatible with the product in \( H^*(E) \).

At the \( E_2 \) level we have that \( \eta^E := \eta^F \cdot \eta^B \) is a generator of \( E_2^{n,m} = E_{\infty}^{n,m} = H^{n+m}(E) \). The hypothesis \( j^*(\alpha) = \lambda \eta^F \), \( \lambda \neq 0 \) implies that \( \alpha \) has filtration zero in \( H^m(E) \) and its image in \( E_{\infty}^{0,m} \) is \( \lambda \eta^E \). Then, \( \lambda \eta^E = (\lambda \eta^F) \cdot \eta^B \) holds in \( E_\infty \) where \([\eta^B]\) denotes the image of \( \eta^B \) in \( E_{\infty}^{0,0} \). Since \( E_{\infty}^{n+m-n,i} = 0 \) for \( i \neq n \), we deduce \( \lambda \eta^E = \alpha \cdot \pi^*(\eta^B) \), as desired. \( \square \)

Now we can proceed to the computation of the torsion index of \( G_3 \), or, to be more precise, to the computation of the order of the cokernel of \( k^* : H^{42}(BT;\mathbb{Z}_2) \to H^{42}(G_3/T;\mathbb{Z}_2) \). We consider the diagram

\[
(\text{Spin}(7)/T)^2 \xrightarrow{j} G_3/T \xrightarrow{\pi} G_3/\text{Spin}(7) \xrightarrow{\phi} (B\text{Spin}(7))^2
\]

where \( \omega \in H^*(BT;\mathbb{Z}) \) such that \( f^*(\omega) = 2\eta \) for the natural map \( f : \text{Spin}(7)/T \to BT \).

The computations in the preceding section show that there is an orientation class \( \rho \in H^{24}(G_3/\text{Spin}(7);\mathbb{Z}_2) \) which is in the image of \( \phi^* \). Let \( \rho = \phi^*(\gamma) \). We can now apply the lemma above to the fibration \( \text{Spin}(7)/T \to G_3/T \to G_3/\text{Spin}(7) \) with \( \alpha = k^*(\omega) \) and deduce that there is an orientation class \( \theta \in H^{42}(G_3/T;\mathbb{Z}_2) \) such that \( k^*(\omega \cdot i^*(\gamma)) = 2\theta \). This implies that the torsion index of \( G_3 \) divides 2.
Next, we prove that the torsion index of $G_3$ cannot be equal to 1. It is enough to prove that the homomorphism $H^{42}(BT; F_2) \to H^{42}(G_3/T; F_2)$ is equal to zero. Let us consider the $F_2$-spectral sequence of the fibration $G_3 \to G_3/T \to BT_2^\wedge$. We have that

$$H^*(G_3; F_2) \cong F_2[x_7]/x_7^2 \otimes E(x_{11}, x_{13}),$$

$$Sq^4(x_7) = x_{11}, \quad Sq^2(x_{11}) = x_{13}, \quad Sq^1(y_{13}) = x_7^2.$$  

Hence, the generators $x_7, x_{11}, x_{13}, x_7^2$ are transgressive to $c_8, c_{12}, c_{14}, 0$, respectively. Here we denote by $c_8, c_{12}, c_{14}$ the images in $H^*(BT; F_2)$ of the generators $c_8, c_{12}, c_{14} \in H^*(BG_3; F_2)$. Recall that in section 3 we have computed these elements as explicit polynomials in some basis $\{\epsilon_1, \epsilon_2, \epsilon_3\}$ of $H^2(BT; F_2)$.

In the $E_2$-term of the spectral sequence of $G_3 \to G_3/T \to BT_2^\wedge$, let us consider the row containing $x_7^2$. All elements in this row are permanent cycles, and the only boundaries are the elements of the form $x_7^2q$ with $q$ in the ideal of $F_2[\epsilon_1, \epsilon_2, \epsilon_3]$ generated by $c_8, c_{12}, c_{14}$. If we compute the quotient algebra $F_2[\epsilon_1, \epsilon_2, \epsilon_3]/(c_8, c_{12}, c_{14})$ (using any choice of computer algebra software), we see that it is a graded algebra with Poincaré series equal to

$$1 + 3t^2 + 6t^4 + 10t^6 + 14t^8 + 18t^{10} + 21t^{12} + 22t^{14} + 21t^{16} + 18t^{18} + 14t^{20} + 10t^{22} + 6t^{24} + 3t^{26} + t^{28},$$

and so in particular there is an element $q \in H^{28}(BT; F_2)$ which does not belong to the ideal $(c_8, c_{12}, c_{14})$. Hence, the element $x_7^2q$ in the $E_2$-term of the spectral sequence survives to a nontrivial element in $H^{42}(G_3/T; F_2)$ which cannot be in the image of $H^*(BT; F_2)$. This finishes the proof of

**Theorem 5.2.** The cokernel of $H^{42}(BT; Z_2) \to H^{42}(G_3/T; Z_2)$ has order two.

**6. Proof of Theorem 1.2**

In section 2 we saw that it is enough to prove Theorem 1.2 for each exotic $p$-compact group, and we also saw that Theorem 1.2 is true for all odd primes. Since it is known (2) that the only exotic 2-compact group is $G_3$, the only thing that remains to be proved is that $G_3$ satisfies the properties (TI1) to (TI4) with $t_2(G_3) = 2$.

(TI4) is void, and (TI3) is just Theorem 5.2 plus some facts about $G_3/T$ which were proven in [2]. In [12] it is proven that the torsion elements in $H^*(BG_3; Z_2)$ are of order two and the homomorphism $H^*(BG_3; Z_2) \to H^*(BT; Z_2)^W$ is surjective. This implies immediately that (TI2) holds. Let $A$ be a nontrivial finite abelian 2-subgroup of $G_3$ and let $E$ be a subgroup of $A$ of order 2. Then, $A$ factors through the centralizer of $E$ in $G_3$ which is $\text{Spin}(7)$. Since $\text{Spin}(7)$ has 2-torsion index equal to 2, we deduce that $A$ has a subgroup of index at most 2 which is included in a maximal torus of $G_3$. So, we have (TI1), and the proof is complete.

**References**


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