WEIGHTED NORM INEQUALITIES
FOR THE BILINEAR MAXIMAL OPERATOR
ON VARIABLE LEBESGUE SPACES

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Abstract: We extend the theory of weighted norm inequalities on variable Lebesgue spaces to the case of bilinear operators. We introduce a bilinear version of the variable $A_{p(\cdot)}$ condition and show that it is necessary and sufficient for the bilinear maximal operator to satisfy a weighted norm inequality. Our work generalizes the linear results of the first author, Fiorenza, and Neugebauer [7] in the variable Lebesgue spaces and the bilinear results of Lerner et al. [22] in the classical Lebesgue spaces. As an application we prove weighted norm inequalities for bilinear singular integral operators in the variable Lebesgue spaces.

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1. Introduction

In this paper we develop the theory of bilinear weighted norm inequalities in the variable Lebesgue spaces. To put our results in context we will first describe some previous results; for brevity, we will defer the majority of definitions until below. The Hardy–Littlewood maximal operator is defined by

$$Mf(x) = \sup_Q \int_Q |f(y)| \, dy \cdot \chi_Q(x),$$

where the supremum is taken over all cubes in $\mathbb{R}^n$ with sides parallel to the coordinate axes. The now classical result of Muckenhoupt ([23]) is
that a necessary and sufficient condition for $M$ to be bounded on the weighted Lebesgue space $L^p(w)$, $1 < p < \infty$, i.e., that

$$\int_{\mathbb{R}^n} (Mf)^p w \, dx \lesssim \int_{\mathbb{R}^n} |f|^p w \, dx,$$

is that $w \in A_p$:

$$\sup_Q \int_Q w \, dx \left( \int_Q w^{1-p'} \, dx \right)^{p-1} < \infty,$$

where again the supremum is taken over all cubes in $\mathbb{R}^n$ with sides parallel to the coordinate axes.

This result has been generalized in two directions. First, Lerner et al. [22], as part of the theory of weighted norm inequalities for bilinear Calderón–Zygmund singular integrals, introduced the bilinear (more properly, “bisublinear”) maximal operator:

$$M(f_1, f_2)(x) = \sup_Q \int_Q |f_1(y)| \, dy \int_Q |f_2(y)| \, dy \cdot \chi_Q(x).$$

It is immediate that $M(f_1, f_2)(x) \leq Mf_1(x)Mf_2(x)$, and so by Hölder’s inequality,

(1.1) $$M: L^{p_1}(w_1) \times L^{p_2}(w_2) \to L^p(w),$$

where $1 < p_1, p_2 < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $w_j \in A_{p_j}$, $j = 1, 2$, and $w = w_1^{\frac{p}{p_1}} w_2^{\frac{p}{p_2}}$.

However, while this condition is sufficient, it is not necessary. In [22] they introduced the class $A_{\vec{p}}$ of vector weights defined as follows. With the previous definitions, let $\vec{p} = (p_1, p_2, p)$ and let $\vec{w} = (w_1, w_2, w)$.

Then $\vec{w} \in A_{\vec{p}}$ if

$$\sup_Q \left( \int_Q w \, dx \right)^{\frac{1}{p}} \left( \int_Q w_1^{1-p_1'} \, dx \right)^{\frac{1}{p_1'}} \left( \int_Q w_2^{1-p_2'} \, dx \right)^{\frac{1}{p_2'}} < \infty.$$

They proved that a necessary and sufficient condition for inequality (1.1) to hold is that $\vec{w} \in A_{\vec{p}}$. If $w_j \in A_{p_j}$, then $\vec{w} \in A_{\vec{p}}$, but they gave examples to show that the class $A_{\vec{p}}$ is strictly larger than the weights gotten from $A_{p_1} \times A_{p_2}$.

A second generalization of Muckenhoupt’s result is to the setting of the variable Lebesgue spaces. The first author, Fiorenza, and Neugebauer ([6]) proved that given an exponent function $p(\cdot): \mathbb{R}^n \to [1, \infty)$ such that $1 < p_- \leq p_+ < \infty$ and $p(\cdot)$ is log-Hölder continuous, then the maximal operator is bounded on $L^{p(\cdot)}$. Given $p(\cdot)$ a log-Hölder continuous function, in [7] (see also [4]) they proved the corresponding weighted
norm inequality: a necessary and sufficient condition for the maximal operator to be bounded on $L^{p(\cdot)}(w)$, i.e., that $\|(Mf)w\|_{p(\cdot)} \lesssim \|fw\|_{p(\cdot)}$, is that $w \in A_{p(\cdot)}$,

$$\sup_{Q} |Q|^{-1} \|w \chi_{Q}\|_{p(\cdot)} \|w^{-1} \chi_{Q}\|_{p'(\cdot)} < \infty.$$ 

When $p(\cdot) = p$ is a constant function, then this reduces to the classical result of Muckenhoupt, since $L^{p(\cdot)}(w) = L^{p}(w^{p})$ and $w \in A_{p(\cdot)}$ is equivalent to $w^{p} \in A_{p}$.

The purpose of this paper is to extend both of these results and characterize the class of weights necessary and sufficient for the bilinear maximal operator to satisfy bilinear weighted norm inequalities over the variable Lebesgue spaces. The remainder of this paper is organized as follows. In Section 2 we make the necessary definitions to state our two main results. In particular, we introduce the class of vector weights $A_{\vec{p}(\cdot)}$. Our first result, Theorem 2.4, is for the bilinear maximal operator. Our second, Theorem 2.8, shows that the weight condition $A_{\vec{p}(\cdot)}$ is sufficient for bilinear Calderón–Zygmund singular integral operators to satisfy weighted norm inequalities over the variable Lebesgue spaces. This generalizes the main result of [22].

In Section 3 we gather some basic results about weights and the variable Lebesgue spaces that we need in our proof, and in Section 4 we prove some properties of $A_{p(\cdot)}$ and $A_{\vec{p}(\cdot)}$ weights. In Section 5 we give a characterization of vector weights $A_{\vec{p}(\cdot)}$ in terms of averaging operators. In Section 6 we prove Theorem 2.4. The proof is broadly similar to the proof in the linear case given in [7], but there are many additional technical obstacles. Finally, in Section 7 we prove Theorem 2.8. The proof relies on an extrapolation theorem in the scale of weighted variable Lebesgue spaces proved in [13].

Remark 1.1 (Added in proof). One of the anonymous referees for this paper asked whether a shorter proof of Theorem 2.4 could be gotten by adapting the ideas of [4] to the bilinear case. We originally tried this approach, but were unsuccessful. This remains an open problem.

Throughout this paper, $n$ will denote the dimension of the underlying space $\mathbb{R}^{n}$. A cube $Q \subset \mathbb{R}^{n}$ will always have its sides parallel to the coordinate axes. Let $\ell(Q)$ denote the side-length of $Q$. Given a cube $Q$ and a function $f$, we will denote averages as follows:

$$\frac{1}{|Q|} \int_{Q} f \, dx = \hat{f}_{Q}, \quad \int_{Q} f \, dx = \langle f \rangle_{Q}.$$
Similarly, if $\sigma$ is a non-negative measure, we denote weighted averages by

$$\frac{1}{\sigma(Q)} \int_Q f \sigma \, dx = (f)_{\sigma,Q}.$$  

Constants will be denoted by $C$, $c$, etc. and their value may change from line to line, even in the same computation. If we need to emphasize the dependence of a constant on some parameter we will write, for instance, $C(n)$. Given two positive quantities $A$ and $B$, we will write $A \lesssim B$ if there is a constant $c$ such that $A \leq cB$. If $A \lesssim B$ and $B \lesssim A$, then we write $A \approx B$.

2. Main results

We first recall the definition of variable Lebesgue spaces. For more information, see [5]. Let $\mathcal{P}$ denote the collection of measurable functions $p(\cdot) : \mathbb{R}^n \to [1, \infty]$ and $\mathcal{P}_0$ the collection of measurable functions $p(\cdot) : \mathbb{R}^n \to (0, \infty]$. Given $p(\cdot) \in \mathcal{P}_0$ and a set $E \subset \mathbb{R}^n$, define

$$p_-(E) = \text{ess inf}_{x \in E} p(x), \quad p_+(E) = \text{ess sup}_{x \in E} p(x).$$

For simplicity we will write $p_- = p_-(\mathbb{R}^n)$ and $p_+ = p_+(\mathbb{R}^n)$. Given $p(\cdot) \in \mathcal{P}$ we define the dual exponent $p'(\cdot)$ pointwise a.e. by

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1,$$

with the convention that $\frac{1}{\infty} = 0$.

The space $L^{p(\cdot)}$ consists of all complex-valued, measurable functions $f$ such that for some $\lambda > 0$,

$$\rho_{p(\cdot)}(f/\lambda) = \int_{\mathbb{R}^n \setminus \Omega_\infty} \left| \frac{f(x)}{\lambda} \right|^{p(x)} dx + \lambda^{-1}\|f\|_{L^\infty(\Omega_\infty)} < \infty,$$

where $\Omega_\infty = \{x \in \mathbb{R}^n : p(x) = \infty\}$. This becomes a quasi-Banach function space when equipped with the quasi-norm

$$\|f\|_{L^{p(\cdot)}} = \|f\|_{p(\cdot)} = \inf\{\lambda > 0 : \rho_{p(\cdot)}(f/\lambda) \leq 1\};$$

when $p_- \geq 1$ it is a Banach space. When $p(\cdot) = p$, $0 < p < \infty$, then $L^{p(\cdot)} = L^p$ with equality of quasi-norms.

An exponent $p(\cdot) \in \mathcal{P}_0$ is said to be locally log-Hölder continuous, denoted by $p(\cdot) \in LH$, if there exists a constant $C_0$ such that

$$\left| \frac{1}{p(x)} - \frac{1}{p(y)} \right| \leq \frac{C_0}{-\log(|x-y|)}, \quad |x-y| < \frac{1}{2};$$
\( p(\cdot) \) is said to be log-Hölder continuous at infinity, denoted by \( p(\cdot) \in LH_\infty \), if there exist \( C_\infty, p_\infty > 0 \) such that

\[
\left| \frac{1}{p(x)} - \frac{1}{p_\infty} \right| \leq \frac{C_\infty}{\log(e + |x|)}.
\]

If \( p(\cdot) \in LH = LH_0 \cap LH_\infty \), we simply say that it is log-Hölder continuous.

**Remark 2.1.** For our main results we will assume \( p_+ < \infty \). In this case \( \Omega_\infty \) has measure zero and the definition of the norm is simpler. Moreover, in the definition of log-Hölder continuity, we can replace the left-hand sides by \( |p(x) - p(y)| \) and \( |p(x) - p_\infty| \), respectively, requiring only new constants that depend on \( p_+ \).

By a weight \( w \) we mean a non-negative function such that \( 0 < w(x) < \infty \) a.e. Given a weight \( w \) and \( p(\cdot) \in \mathcal{P}_0 \), we say \( f \in L^{p(\cdot)}(w) \) if \( f w \in L^{p(\cdot)} \).

**Definition 2.2.** Given exponents \( p_1(\cdot), p_2(\cdot) \in \mathcal{P}(\mathbb{R}^n) \), define \( p(\cdot) \) for a.e. \( x \) by

\[
\frac{1}{p(x)} = \frac{1}{p_1(x)} + \frac{1}{p_2(x)},
\]

and let \( \bar{p}(\cdot) = (p_1(\cdot), p_2(\cdot), p(\cdot)) \). Given weights \( w_1, w_2 \), let \( w = w_1 w_2 \) and define \( \bar{w} = (w_1, w_2, w) \). We say that \( \bar{w} \in \mathcal{A}_{\bar{p}(\cdot)} \) if

\[
\sup_Q |Q|^{-2} \|w \chi_Q\|_{p(\cdot)} \|w_1^{-1} \chi_Q\|_{p_1(\cdot)} \|w_2^{-1} \chi_Q\|_{p_2(\cdot)} < \infty.
\]

**Remark 2.3.** If \( p_1(\cdot) \) and \( p_2(\cdot) \) are constant, then this condition reduces to the \( A_{\bar{p}} \) condition for the triple \( (w_1^{p_1}, w_2^{p_2}, (w_1 w_2)^p) \).

We can now state our first main result.

**Theorem 2.4.** Given \( p_1(\cdot), p_2(\cdot) \in \mathcal{P} \), suppose \( 1 < (p_j)_{-} \leq (p_j)_{+} < \infty \) and \( p_j(\cdot) \in LH \), \( j = 1,2 \). Define \( p(\cdot) \) by (2.1). Let \( w_1, w_2 \) be weights and define \( w = w_1 w_2 \). Then the bilinear maximal operator satisfies

\[
\mathcal{M} : L^{p_1(\cdot)}(w_1) \times L^{p_2(\cdot)}(w_2) \to L^{p(\cdot)}(w)
\]

if and only if \( \bar{w} \in \mathcal{A}_{\bar{p}(\cdot)} \).

**Remark 2.5.** We do not believe that the assumption \( p(\cdot) \in LH \) is necessary in Theorem 2.4, but some additional hypothesis is. In the linear, unweighted case, while it is sufficient to assume that the exponent \( p(\cdot) \) is log-Hölder continuous for the maximal operator to be bounded on \( L^{p(\cdot)} \), it is not necessary: see [5, Section 4.4] for examples. Diening and Hästö ([17]) conjectured that in the weighted case, a
necessary and sufficient condition for $M$ to be bounded on $L^{p(\cdot)}(w)$ is
that the maximal operator is bounded on $L^{p(\cdot)}$ and $w \in A_{p(\cdot)}$. (The latter condition is given in Definition 4.1.) Unlike in the constant exponent case, when $w = 1$ these conditions are not the same since $1 \in A_{p(\cdot)}$ is a necessary but not sufficient condition for $M$ to be bounded on $L^{p(\cdot)}$ [5, Corollary 4.50, Example 4.51]. We conjecture that the analogous result holds in the bilinear case: $M$ satisfies (2.2) if and only if $M$ satisfies an unweighted bilinear estimate and $\vec{w} \in A_{\vec{p}(\cdot)}$.

Remark 2.6. In the linear case, the maximal operator is bounded on $L^{p(\cdot)}$ if $p_+ > 1$ and $1/p(\cdot) \in LH$: we can allow $p_+ = \infty$. (See [5] for details and references.) In [7] it was conjectured that $M$ is bounded on $L^{p(\cdot)}(w)$ with the same hypotheses if $w \in A_{p(\cdot)}$. This condition is well defined even if $p_- = 1$ and $p_+ = \infty$. Moreover, this conjecture is true if $p(\cdot) = \infty$ a.e. This is equivalent to a classical but often overlooked result of Muckenhoupt [23], that if $w^{-1} \in A_1$ and $fw \in L^\infty$, then $(Mf)w \in L^\infty$.

Here we conjecture that we can remove the hypothesis $p_+ < \infty$ from Theorem 2.4. However, as in the linear case we believe that this will require a very different argument, as the fact that $p_+, (p_j)_+ < \infty$ plays an important role in our proof.

Remark 2.7. Though we have only proved our result in the bilinear case, an $m$-linear version of Theorem 2.4, $m \geq 3$, should be true with the obvious changes in the definition of $A_{\vec{p}(\cdot)}$ and the statement of the theorem. But even in the bilinear case the proof is quite technical, and so we decided to avoid making our proof even more obscure by trying to prove the general result.

Our second main result is for bilinear Calderón–Zygmund singular integrals. These operators have been considered by a number of authors, and we refer the reader to [22] for details and further references.

Let $K(x, y, z)$ be a complex-valued, locally integrable function on $\mathbb{R}^{3n} \setminus \Delta$, where $\Delta = \{(x, x, x) : x \in \mathbb{R}^n\}$. $K$ is a Calderón–Zygmund kernel if there exist $A > 0$ and $\delta > 0$ such that for all $(x, y, z) \in \mathbb{R}^{3n} \setminus \Delta$,

$$|K(x, y, z)| \leq \frac{A}{(|x - y| + |x - z| + |y - z|)^{2n}}$$

and

$$|K(x, y, z) - K(\tilde{x}, y, z)| \leq \frac{A|x - \tilde{x}|^\delta}{(|x - y| + |x - z| + |y - z|)^{2n+\delta}},$$

whenever $|x - \tilde{x}| \leq \frac{1}{2} \max(|x - z|, |x - y|)$. We also assume that the two analogous difference estimates with respect to the variables $y$ and $z$ hold.
An operator \( T : S \times S \to S' \) is a bilinear Calderón–Zygmund singular integral if:

1. there exists a bilinear Calderón–Zygmund kernel \( K \) such that
   \[
   T(f_1, f_2)(x) = \int_{\mathbb{R}^{2n}} K(x, y, z) f_1(y) f_2(z) \, dy \, dz
   \]
   for all \( f_1, f_2 \in C_c(\mathbb{R}^n) \) and all \( x \notin \text{supp}(f_1) \cap \text{supp}(f_2) \);
2. there exist \( 1 \leq p, q < \infty \) and \( r \) such that \( \frac{1}{r} = \frac{1}{p} + \frac{1}{q} \) and \( T \) can be extended to a bounded operator from \( L^p \times L^q \) into \( L^r \).

**Theorem 2.8.** Given \( p_1(\cdot), p_2(\cdot) \in \mathcal{P} \), suppose \( 1 < (p_j)_- \leq (p_j)_+ < \infty \) and \( p_j(\cdot) \in LH \), \( j = 1, 2 \). Define \( p(\cdot) \) by (2.1). Let \( w_1, w_2 \) be weights, define \( w = w_1 w_2 \), and assume \( \vec{w} \in A_{p(\cdot)} \). If \( T \) is a bilinear Calderón–Zygmund singular integral, then

\[
T : L^{p_1(\cdot)}(w_1) \times L^{p_2(\cdot)}(w_2) \to L^{p(\cdot)}(w).
\]

**Remark 2.9.** As for the bilinear maximal operator, we do not believe that the assumption that \( p_1(\cdot), p_2(\cdot) \in LH \) is necessary for the conclusion in Theorem 2.8 to hold. In [11], the authors proved that in the unweighted case it was sufficient to assume that the (linear) maximal operator is bounded on \( L^p_1(\cdot) \) and \( L^p_2(\cdot) \). We conjecture that with this hypothesis, or even the weaker assumption that \( \mathcal{M} \) satisfies the associated unweighted bilinear inequality, and \( \vec{w} \in A_{p(\cdot)} \), then (2.3) holds.

**Remark 2.10.** Alongside the variable Lebesgue spaces there is a theory of variable Hardy spaces; see [12]. Very recently, the first author, Moen, and Nguyen ([10]) proved unweighted estimates on variable Hardy spaces for bilinear Calderón–Zygmund singular integrals. It would be interesting to extend these results to weighted variable Hardy spaces using the \( A_{p(\cdot)} \) weights.

### 3. Preliminaries

In this section we gather some basic results about weights and about variable Lebesgue spaces that we will need in the subsequent sections.

**Weights.** First, we recall the definition of the class \( A_\infty \):

\[
A_\infty = \bigcup_{p > 1} A_p.
\]

We will need the following property of \( A_\infty \) weights. For a proof, see [18].

**Lemma 3.1.** Let \( w \in A_\infty \). Then for each \( 0 < \alpha < 1 \) there exists \( 0 < \beta < 1 \) such that if \( Q \) is any cube and \( E \subset Q \) is such that \( \alpha |Q| \leq |E| \), then \( \beta w(Q) \leq w(E) \). Similarly, for each \( 0 < \gamma < 1 \) there exists \( 0 < \delta < 1 \) such that if \( |E| \leq \gamma |Q| \), then \( w(E) \leq \delta w(Q) \).
To state our next result, we introduce the weighted dyadic maximal operator. Given a weight $\sigma$,

$$M_{\sigma}^{D_0} f(x) = \sup_{Q \in D_0} \frac{1}{\sigma(Q)} \int_Q |f| \sigma \cdot \chi_Q(x) = \sup_{Q \in D_0} \langle |f| \sigma \rangle Q \chi_Q(x),$$

where the supremum is taken over all cubes in the collection $D_0$ of dyadic cubes:

$$D_0 = \{2^{-k}([0,1)^n + j) : k \in \mathbb{Z}, j \in \mathbb{Z}^n\}.$$

The following result is well-known but an explicit proof does not seem to have appeared in the literature. The proof is essentially the same as for the classical dyadic maximal operator; see Grafakos [19].

**Lemma 3.2.** Given a weight $\sigma$, then for $1 < p < \infty$,

$$\int_{\mathbb{R}^n} (M_{\sigma}^{D_0} f)^p \sigma \, dx \lesssim \int_{\mathbb{R}^n} |f|^p \sigma \, dx$$

and the implicit constant depends only on $p$.

**Variable Lebesgue spaces.** Here we gather some basic results about variable Lebesgue spaces. All of these are found in the literature (with some minor variations). In some cases they were only proved for exponents $p(\cdot) \in \mathcal{P}$, but essentially the same proof works for $p(\cdot) \in \mathcal{P}_0$.

**Lemma 3.3** ([5, Proposition 2.18]). Given $p(\cdot) \in \mathcal{P}_0$, suppose $|\Omega_\infty| = 0$. If $s > 0$, then

$$\| \|f\|^s\|_{p(\cdot)} = \|f\|_{s p(\cdot)}^s.$$

**Lemma 3.4** ([5, Theorem 2.61]). Given $p(\cdot) \in \mathcal{P}_0$, if $f \in L^{p(\cdot)}$ is such that $\{f_k\}$ converges to $f$ pointwise a.e., then

$$\|f\|_{p(\cdot)} \leq \liminf_{k \to \infty} \|f_k\|_{p(\cdot)}.$$

**Lemma 3.5** ([5, Corollary 2.23]). Given $p(\cdot) \in \mathcal{P}_0$, suppose $0 < p_- \leq p_+ < \infty$. If $\|f\|_{p(\cdot)} > 1$, then

$$\rho_{p(\cdot)}(f)^{\frac{1}{p_-}} \leq \|f\|_{p(\cdot)} \leq \rho_{p(\cdot)}(f)^{\frac{1}{p_+}}.$$

If $\|f\|_{p(\cdot)} \leq 1$, then

$$\rho_{p(\cdot)}(f)^{\frac{1}{p_-}} \leq \|f\|_{p(\cdot)} \leq \rho_{p(\cdot)}(f)^{\frac{1}{p_+}}.$$

Consequently, $\|f\|_{p(\cdot)} \lesssim 1$ if and only if $\rho_{p(\cdot)}(f) \lesssim 1$. 
Lemma 3.6 ([5, Corollary 2.30]). Fix $k \geq 2$ and let $p_j(\cdot) \in \mathcal{P}$ satisfy for a.e. $x$,
\[ \sum_{j=1}^{k} \frac{1}{p_j(x)} = 1. \]
Then, for all $f_j \in L^{p_j(\cdot)}$, $1 \leq j \leq k$,
\[ \int_{\mathbb{R}^n} |f_1 \cdots f_k| \, dx \lesssim \prod_{j=1}^{k} \|f_j\|_{p_j(\cdot)}. \]
The implicit constant depends only on the $p_j(\cdot)$.

Lemma 3.7 ([5, Corollary 2.28]). Given $p_1(\cdot), p_2(\cdot) \in \mathcal{P}_0$, define $p(\cdot) \in \mathcal{P}_0$ by (2.1). Then
\[ \|fg\|_{p(\cdot)} \lesssim \|f\|_{p_1(\cdot)} \|g\|_{p_2(\cdot)}. \]
The implicit constant depends only on $p_1(\cdot)$ and $p_2(\cdot)$.

Lemma 3.8 ([5, Theorem 2.34]). Given $p(\cdot) \in \mathcal{P}$, then for every $f \in L^{p(\cdot)}$,
\[ \|f\|_{p(\cdot)} \approx \sup_{\|g\|_{p'(\cdot)} \leq 1} \int_{\mathbb{R}^n} |fg| \, dx. \]
The implicit constants depend only on $p(\cdot)$.

Remark 3.9. It is immediate that in the weighted space $L^{p(\cdot)}(w)$, the same result is true if we take the supremum over all $g \in L^{p'(\cdot)}(w^{-1})$ with $\|gw^{-1}\|_{p'(\cdot)} \leq 1$.

Lemma 3.10 ([16, Corollary 4.5.9]). Let $p(\cdot) \in \mathcal{P}$ be such that $p(\cdot) \in LH$. Then for every cube $Q$,
\[ \|\chi_Q\|_{p(\cdot)} \approx |Q|^{\frac{1}{p_Q}}, \]
where $p_Q$ is the harmonic mean of $p(\cdot)$ on $Q$:
\[ \frac{1}{p_Q} = \int_Q \frac{dx}{p(x)}. \]
The implicit constants depend only on $p(\cdot)$.

Lemma 3.11 ([5, Lemma 3.24]). Given $p(\cdot) \in \mathcal{P}_0$, suppose $p(\cdot) \in LH_0$ and $0 < p_- \leq p_+ < \infty$. Then for every cube $Q$,
\[ |Q|^{p_-(Q) - p_+(Q)} \lesssim 1, \]
and the implicit constant depends only on $p(\cdot)$ and $n$. The same inequality holds if we replace one of $p_+(Q)$ or $p_-(Q)$ by $p(x)$ for any $x \in Q$. 
Remark 3.12. Lemma 3.11 is sometimes referred to as Diening’s condition, and it is the principal way in which we will apply the \( \mathcal{L}H_0 \) condition.

Lemma 3.13 ([2, Lemmas 2.7, 2.8]). Given two exponents \( r(\cdot), s(\cdot) \in \mathcal{P}_0 \), suppose

\[
|s(y) - r(y)| \leq \frac{C_0}{\log(e + |y|)}.
\]

Then, given any set \( G \) and any non-negative measure \( \mu \), for every \( t \geq 1 \) there exists a constant \( C = C(t, C_0) \) such that for all functions \( f \) such that \( |f(y)| \leq 1 \),

\[
\int_Q |f(y)|^{s(y)} d\mu(y) \leq C \int_G |f(y)|^{r(y)} d\mu(y) + \int_G \frac{1}{(e + |y|)^{tns_+(G)}} d\mu(y).
\]

If we instead assume that

\[
0 \leq r(y) - s(y) \leq \frac{C_0}{\log(e + |y|)},
\]

then the same inequality holds for any function \( f \).

Remark 3.14. Lemma 3.13 is the principal way in which we will apply the \( \mathcal{L}H_\infty \) condition.

4. Properties of \( \mathcal{A}_{p(\cdot)} \) and \( \mathcal{A}_{\bar{p}(\cdot)} \) weights

In this section we give some properties of the \( \mathcal{A}_{p(\cdot)} \) and \( \mathcal{A}_{\bar{p}(\cdot)} \) weights that will be used in the proof of Theorem 2.4. For simplicity, throughout this section assume that \( w_1, w_2 \) are weights and let \( w = w_1 w_2 \) and \( \tilde{w} = (w_1, w_2, w) \). Similarly, whenever we are given \( p_1(\cdot), p_2(\cdot) \in \mathcal{P} \), define \( p(\cdot) \) by (2.1) and let \( \bar{p}(\cdot) = (p_1(\cdot), p_2(\cdot), p(\cdot)) \). Note that in this case we always have that \( p_- \geq \frac{1}{2} \).

We begin by recalling the definition of \( \mathcal{A}_{p(\cdot)} \) weights and then state several results from [7] on their properties.

Definition 4.1. Given \( p(\cdot) \in \mathcal{P} \) and a weight \( w \), we say \( w \in \mathcal{A}_{p(\cdot)} \) if

\[
\sup_Q |Q|^{-1} \| w^Q \|_{p(\cdot)} \| w^{-1}^Q \|_{p'(\cdot)} < \infty.
\]

The next two lemmas show the relationship between \( \mathcal{A}_{p(\cdot)} \) and \( A_\infty \) weights.

Lemma 4.2 ([7, Lemma 3.4]). Given \( p(\cdot) \in \mathcal{P} \), suppose \( p(\cdot) \in LH \) and \( p_+ < \infty \). If \( w \in \mathcal{A}_{p(\cdot)} \), then \( u(\cdot) = w(\cdot)^{p(\cdot)} \in A_\infty \).
Lemma 4.3 ([7, Lemmas 3.5, 3.6]). Given \( p(\cdot) \in \mathcal{P} \), suppose \( p(\cdot) \in LH \) and \( p_+ < \infty \). Let \( w \in \mathcal{A}_{p(\cdot)} \) and let \( u(x) = w(x)^{p(x)} \). Then, given any cube \( Q \) such that \( \|w\chi_Q\|_{p(\cdot)} \geq 1 \), we have \( \|w\chi_Q\|_{p(\cdot)} \approx u(Q)^{\frac{1}{p_\infty}} \). Moreover, given any \( E \subset Q \),

\[
\frac{|E|}{|Q|} \lesssim \left( \frac{u(E)}{u(Q)} \right)^{\frac{1}{p_\infty}}.
\]

The implicit constants depend only on \( w \) and \( p(\cdot) \).

Remark 4.4. To apply Lemma 4.3, note that by Lemma 3.5, \( \|w\chi_Q\|_{p(\cdot)} \geq 1 \) if and only if \( u(Q) \geq 1 \).

The next result is a weighted version of the Diening condition in Lemma 3.11 and will be used to apply the \( LH_0 \) condition.

Lemma 4.5 ([7, Lemma 3.3]). Given \( p(\cdot) \in \mathcal{P} \) such that \( p(\cdot) \in LH \), if \( w \in \mathcal{A}_{p(\cdot)} \), then for all cubes \( Q \),

\[
\|w\chi_Q\|_{p(\cdot)}^{p^-(Q) - p^+(Q)} \lesssim 1.
\]

The implicit constant depends only on \( p(\cdot) \) and \( w \).

The final lemma is an integral estimate that, in conjunction with Lemma 3.13, will be used to apply the \( LH_\infty \) condition.

Lemma 4.6 ([7, Inequality (3.3)]). Given \( p(\cdot) \in \mathcal{P} \), suppose \( p(\cdot) \in LH \). If \( w \in \mathcal{A}_{p(\cdot)} \), then there exists a constant \( t > 1 \), depending only on \( w \), \( p(\cdot) \), and \( n \), such that

\[
\int_{\mathbb{R}^n} \frac{w(x)^{p(x)}}{(e + |x|)^{tnp_-}} \, dx \leq 1.
\]

We now turn to the \( \mathcal{A}_{p(\cdot)} \) condition. If \( w_1 \in \mathcal{A}_{p_1(\cdot)} \) and \( w_2 \in \mathcal{A}_{p_2(\cdot)} \), then by Lemma 3.6 we have that \( w \in \mathcal{A}_{\tilde{p}(\cdot)} \). However, this inclusion is proper, since it is in the constant exponent case. Nevertheless, we can characterize the bilinear \( \mathcal{A}_{\tilde{p}(\cdot)} \) weights in terms of the \( \mathcal{A}_{p(\cdot)} \) condition. In the constant exponent case this is proved in [22], and our argument is adapted from theirs.

Proposition 4.7. Given \( \tilde{w} \) and \( \tilde{p}(\cdot) \), \( \tilde{w} \in \mathcal{A}_{\tilde{p}(\cdot)} \) if and only if

\[
\begin{cases}
    w_j^{-\frac{1}{2}} \in \mathcal{A}_{2p_j(\cdot)}, & j = 1, 2, \\
    w^{\frac{1}{2}} \in \mathcal{A}_{2p(\cdot)}.
\end{cases}
\]

Remark 4.8. Note that since \( p_- \geq \frac{1}{2} \), we have \( 2p(\cdot) \in \mathcal{P} \) and \( \mathcal{A}_{2p(\cdot)} \) is well defined.
Proof: First assume that \( \vec{w} \in A_{\vec{p}(\cdot)} \). Then for a.e. \( x \),
\[
\frac{1}{2p(x)} + \frac{1}{2p'(x)} = 1 - \frac{1}{(2p)'(x)} + \frac{1}{2p'(x)} \\
= 1 - \left( \frac{1}{2p'(x)} + \frac{1}{2p'(x)} \right) + \frac{1}{2p'(x)} \\
= 1 - \frac{1}{2p'(x)} = \frac{1}{(2p)'(x)}.
\]
Therefore, by Lemmas 3.7 and 3.3, and by the definition of \( A_{\vec{p}(\cdot)} \),
\[
\| w^{-\frac{1}{2}} \chi_Q \|_{2p'(\cdot)} \| w^{\frac{1}{2}} \chi_Q \|_{(2p')'(\cdot)} = \| w^{-\frac{1}{2}} \chi_Q \|_{2p'(\cdot)} \| w^{\frac{1}{2}} \frac{1}{2} w^{\frac{1}{2}} \chi_Q \|_{(2p')'(\cdot)} \\
\lesssim \| w^{-\frac{1}{2}} \chi_Q \|_{2p'(\cdot)} \| w^{\frac{1}{2}} \chi_Q \|_{2p(\cdot)} \| w^{-\frac{1}{2}} \chi_Q \|_{2p'(\cdot)} \\
= \left( \| w \chi_Q \|_{p(\cdot)} \| w^{-1} \chi_Q \|_{p'(\cdot)} \right)^{\frac{1}{2}} \\
\lesssim |Q|.
\]
Hence, \( w^{-\frac{1}{2}} \in A_{2p'(\cdot)} \). The same argument shows that \( w^{\frac{1}{2}} \in A_{2p'(\cdot)} \).

Finally, we have that
\[
\| w^{\frac{1}{2}} \chi_Q \|_{2p(\cdot)} \| w^{-\frac{1}{2}} \chi_Q \|_{(2p)'(\cdot)} \lesssim \| w^{\frac{1}{2}} \chi_Q \|_{2p(\cdot)} \prod_{j=1}^{2} \| w^{\frac{1}{2}} \chi_Q \|_{2p'(\cdot)} \\
= \left( \| w \chi_Q \|_{p(\cdot)} \prod_{j=1}^{2} \| w^{-1} \chi_Q \|_{p'(\cdot)} \right)^{\frac{1}{2}} \lesssim |Q|.
\]

Thus (4.1) holds.

Conversely, now suppose that (4.1) holds. Then for a.e. \( x \),
\[
\frac{1}{2(2p)'(x)} + \frac{1}{2(2p'_1)'(x)} + \frac{1}{2(2p'_2)'(x)} = 1,
\]
so by Lemmas 3.6 and 3.3, for any cube \( Q \),
\[
1 = \langle w^{-\frac{1}{2}} w^{\frac{1}{2}} w^{-\frac{1}{2}} \rangle_Q \lesssim |Q|^{-2} \| w^{-\frac{1}{2}} \chi_Q \|_{(2p)'(\cdot)} \prod_{j=1}^{2} \| w^{\frac{1}{2}} \chi_Q \|_{2(2p'_j)'(\cdot)} \\
= |Q|^{-2} \| w^{-\frac{1}{2}} \chi_Q \|_{(2p)'(\cdot)} \prod_{j=1}^{2} \| w^{\frac{1}{2}} \chi_Q \|_{2(2p'_j)'(\cdot)}.
\]
Therefore,
\[
\|w \chi_Q\|_{\vec{p}(\cdot)}^2 \prod_{j=1}^2 \|w_j^{-1} \chi_Q\|_{\vec{p}_j'(\cdot)}^2 = \|w^{\frac{1}{2}} \chi_Q\|_{2\vec{p}(\cdot)}^2 \prod_{j=1}^2 \|w_j^{-\frac{1}{2}} \chi_Q\|_{2\vec{p}_j'(\cdot)}^2 \\
\lesssim |Q|^{-2} \|w^{\frac{1}{2}} \chi_Q\|_{2\vec{p}(\cdot)} \|w^{-\frac{1}{2}} \chi_Q\|_{(2\vec{p}_j')'(\cdot)} \prod_{j=1}^2 \|w_j^{\frac{1}{2}} \chi_Q\|_{(2\vec{p}_j')'(\cdot)} \|w_j^{-\frac{1}{2}} \chi_Q\|_{2\vec{p}_j'(\cdot)} \\
\lesssim |Q|,
\]
and so \(w \in A_{\vec{p}(\cdot)}\).

Proposition 4.7 has the following corollary which will be used in our proof of Theorem 2.4.

**Corollary 4.9.** Given \(p_1(\cdot), p_2(\cdot) \in \mathcal{P}\), suppose \(p_j(\cdot) \in LH\) and \((p_j)_+ < \infty\), \(j = 1, 2\). If \(w \in A_{\vec{p}(\cdot)}\), then \(u(\cdot) = w(\cdot)^{p(\cdot)}\) and \(\sigma_j(\cdot) = w_j(\cdot)^{-p'_j(\cdot)}\), \(j = 1, 2\), are in \(A_\infty\).

**Proof:** This follows immediately from Lemma 4.2 and Proposition 4.7.

\(\square\)

Our next lemma is a variant of Lemma 4.5 for \(A_{\vec{p}(\cdot)}\) weights. The proof is adapted from the proof in [7].

**Proposition 4.10.** Given \(p_1(\cdot), p_2(\cdot) \in \mathcal{P}\), suppose \(p_j(\cdot) \in LH\), \(j = 1, 2\). Define \(p(\cdot)\) by (2.1) and suppose \(p_+ < \infty\). For every cube \(Q\), define \(q(Q)\) by
\[
\frac{1}{q(Q)} = \frac{1}{(p_1)_-(Q)} + \frac{1}{(p_2)_-(Q)}.
\]
Then, given \(v \in A_{p(\cdot)}\), for a.e. \(x \in Q\),
\[
(4.2) \quad \|v^{-1} \chi_Q\|_{p'(\cdot)}^{q(Q)-p(x)} \lesssim 1.
\]
The implicit constant depends on \(p_1(\cdot), p_2(\cdot), n\), and \(v\), but is independent of \(Q\).

**Remark 4.11.** When we apply Proposition 4.10 below, we will do so in conjunction with Proposition 4.7 to \(w^{\frac{1}{2}} \in A_{2p(\cdot)}\), so we will let \(v^{-1} = w^{-\frac{1}{2}}\) and replace \(p(\cdot)\) by \(2p(\cdot)\) and \(q\) by \(2q\). We could have stated this result in these terms, but for the purposes of the proof, it seemed easier to suppress the factor of 2. Recall that \(\ell(Q)\) denotes the side-length of the cube \(Q\) (see Section 1).
Proof: Fix a cube \( Q \subset \mathbb{R}^n \). It follows from the definition that for a.e. \( x \in Q \), \( q(Q) \leq p(x) \leq p_+ \), so if \( \|v^{-1}\chi_Q\|_{p'(\cdot)} > 1 \), (4.2) holds immediately. Therefore, we may assume without loss of generality that \( \|v^{-1}\chi_Q\|_{p'(\cdot)} \leq 1 \).

Let \( Q_0 \) be the cube centered at the origin with \( |Q_0| = 1 \). Then either \( |Q| \leq |Q_0| \) or \( |Q| > |Q_0| \). We will prove (4.2) in the first case; the proof of the second case is the same, exchanging the roles of \( Q \) and \( Q_0 \). Suppose that \( \text{dist}(Q, Q_0) \leq \ell(Q_0) \). Then \( Q \subset 5Q_0 \). Define

\[
q_\ast = \inf_Q q(Q) \leq \sup_Q q(Q) \leq p_+ < \infty.
\]

Then

\[
\frac{1}{q(Q)} - \frac{1}{p(x)} \leq \left( \frac{1}{(p_1)_-(Q)} - \frac{1}{(p_1)_+(Q)} \right) + \left( \frac{1}{(p_2)_-(Q)} - \frac{1}{(p_2)_+(Q)} \right),
\]

so there exists a constant \( C = C(p_1(\cdot), p_2(\cdot)) \) such that

\[
p(x) - q(Q) \leq C[(p_1)_+(Q) - (p_1)_-(Q)] + C[(p_2)_+(Q) - (p_2)_-(Q)].
\]

Therefore, by Lemma 3.6 and the \( \mathcal{A}_{p'(\cdot)} \) condition we have that

\[
|Q| = \int_Q v^{-1}v \, dx \lesssim \|v^{-1}\chi_Q\|_{p'(\cdot)} \|v\chi_Q\|_{p(\cdot)}
\]

\[
\leq 5^n \|v^{-1}\chi_Q\|_{p'(\cdot)} |5Q_0|^{-1} \|v\chi_{5Q_0}\|_{p(\cdot)}
\]

\[
\lesssim \|v^{-1}\chi_Q\|_{p'(\cdot)} \|v^{-1}\chi_{5Q_0}\|_{p'(\cdot)}^{-1}.
\]

Hence, by (4.4) and Lemma 3.11,

\[
\|v^{-1}\chi_Q\|_{p'(\cdot)}^{q(Q)-p(x)} \lesssim \|v^{-1}\chi_{5Q_0}\|_{p'(\cdot)}^{q(Q)-p(x)} |Q|^{q(Q)-p(x)}
\]

\[
\leq \left( 1 + \|v^{-1}\chi_{5Q_0}\|_{p'(\cdot)}^{-1} \right)^{p_+ - q_-} |Q|^{q(Q)-p(x)} \lesssim 1.
\]

Now assume that \( \text{dist}(Q, Q_0) \geq \ell(Q_0) \). Then there exists a cube \( \hat{Q} \) such that \( Q, Q_0 \subset \hat{Q} \) and \( \ell(Q) \approx \text{dist}(Q, Q_0) \approx \text{dist}(Q, 0) = d_Q \). Therefore, arguing as we did in inequality (4.5), replacing \( 5Q_0 \) by \( \hat{Q} \), we get

\[
|Q| \lesssim |\hat{Q}| \|v^{-1}\chi_Q\|_{p'(\cdot)} \|v^{-1}\chi_{\hat{Q}}\|_{p'(\cdot)}^{-1}.
\]

If we continue the above argument and use the fact that \( \|v^{-1}\chi_Q\|_{p'(\cdot)} \leq \|v^{-1}\chi_{\hat{Q}}\|_{p'(\cdot)} \), we get

\[
\|v^{-1}\chi_Q\|_{p'(\cdot)}^{q(Q)-p(x)} \lesssim |\hat{Q}|^{p(x)-q(Q)}.
\]
To estimate this final term, note that since $p_j(\cdot) \in LH$, there exist $x_1, x_2 \in Q$ such that $(p_1)_{\cdot}(Q) = p_1(x_1)$ and $(p_2)_{\cdot}(Q) = p_2(x_2)$. Moreover, $|x_1|, |x_2| \approx d_Q$. Therefore, again by log-Hölder continuity, and using that $\frac{1}{p_{\infty}} = \frac{1}{(p_1)_{\infty}} + \frac{1}{(p_2)_{\infty}}$,
\[
\left| \frac{1}{q(Q)} - \frac{1}{p_{\infty}} \right| \leq \left| \frac{1}{p_1(x_1)} - \frac{1}{(p_1)_{\infty}} \right| + \left| \frac{1}{p_2(x_2)} - \frac{1}{(p_2)_{\infty}} \right| \lesssim \frac{1}{\log(e + d_Q)}.
\]
Therefore, for $x \in Q$, since $|x| \approx d_Q$,
\[
\left| \frac{1}{q(Q)} - \frac{1}{p(x)} \right| \leq \left| \frac{1}{q(Q)} - \frac{1}{p_{\infty}} \right| + \left| \frac{1}{p_{\infty}} - \frac{1}{p(x)} \right| \lesssim \frac{1}{\log(e + d_Q)}.
\]
Given this, and since $|Q| \lesssim (e + d_Q)^n$, we thus have that
\[
\hat{Q}^{p(x) - q(Q)} \lesssim 1,
\]
and so $\|v^{-1}\chi_Q\|_{p'(\cdot)}^{q(Q)-p(x)} \lesssim 1$. \hfill \Box

5. Characterization of $A_{\vec{p}(\cdot)}$

In this section we give two characterizations of the $A_{\vec{p}(\cdot)}$ condition in terms of averaging operators. The first is a very general condition that does not require assuming that the exponent functions are log-Hölder continuous. The second requires the additional assumption that $p_1(\cdot), p_2(\cdot)$ are log-Hölder continuous.

Given a cube $Q$, define the multilinear averaging operator $A_Q$ by
\[
A_Q(f_1, f_2)(x) := \langle f_1 \rangle_Q \langle f_2 \rangle_Q \chi_Q(x).
\]
More generally, given a family $Q = \{Q\}$ of disjoint cubes, we define
\[
T_Q(f_1, f_2)(x) = \sum_{Q \in Q} A_Q(f_1, f_2)(x) \chi_Q(x).
\]

**Theorem 5.1.** Given $p_1(\cdot), p_2(\cdot) \in P$ and $\vec{w}$, then $\vec{w} \in A_{\vec{p}(\cdot)}$ if and only if
\[
\sup_Q \|A_Q(f_1, f_2)w\|_{p(\cdot)} \lesssim \|f_1 w_1\|_{p_1(\cdot)} \|f_2 w_2\|_{p_2(\cdot)},
\]
where the supremum is taken over all cubes $Q$. If we assume further that $p_1(\cdot), p_2(\cdot) \in LH$, then $\vec{w} \in A_{\vec{p}(\cdot)}$ if and only if
\[
\sup_Q \|T_Q(f_1, f_2)w\|_{p(\cdot)} \lesssim \|f_1 w_1\|_{p_1(\cdot)} \|f_2 w_2\|_{p_2(\cdot)},
\]
where the supremum is taken over all collections $Q$ of disjoint cubes.
Remark 5.2. When \( p_+ \geq 1 \) (i.e., when \( L^{p(\cdot)} \) is a Banach space) the characterization in terms of the operators \( T_Q \) is a consequence of a general result in the setting of Banach lattices due to Kokilashvili et al. [21]. However, even in this special case we would be required to show that condition \( \mathcal{G} \) defined below holds in order to apply their result. In our case we can use the rescaling properties of variable Lebesgue spaces to prove it directly.

Remark 5.3. A very deep result in the theory of variable Lebesgue spaces is that the uniform boundedness of the linear version of the averaging operators \( T_Q \) is equivalent to the boundedness of the Hardy–Littlewood maximal operator, but the uniform boundedness of the (linear) operators \( A_Q \) is not. See [5, Section 4.4], [15], and [16, Section 5.2] for details and further references. We conjecture that the corresponding result holds in the bilinear case.

The proof of Theorem 5.1 is straightforward for \( A_Q \), and so we give this proof separately.

Proof of Theorem 5.1 for \( A_Q \): Let be \( w \in \mathcal{A}_{\vec{p}(\cdot)} \). Then, given any cube \( Q \), by Lemma 3.6 and the definition of \( \mathcal{A}_{\vec{p}(\cdot)} \) we get

\[
\| A_Q(f_1, f_2)w \|_{p(\cdot)} = |Q|^{-2} \int_Q |f_1| w_1^{-1} \chi_Q^2 \int_Q |f_2| w_2^{-1} \chi_Q^2 dy \| w \chi_Q \|_{p(\cdot)} \\
\lesssim |Q|^{-2} \| w_1^{-1} \chi_Q \|_{p_1(\cdot)} \| w_2^{-1} \chi_Q \|_{p_2(\cdot)} \| w \chi_Q \|_{p(\cdot)} \| f_1 w_1 \|_{p_1(\cdot)} \| f_2 w_2 \|_{p_2(\cdot)} \\
\lesssim \| f_1 w_1 \|_{p_1(\cdot)} \| f_2 w_2 \|_{p_2(\cdot)}.
\]

Since the implicit constant depends only on the \( \mathcal{A}_{\vec{p}(\cdot)} \) condition and is independent of \( Q \), we get (5.1).

Now assume that (5.1) holds. By Lemma 3.8, there exist \( h_j w_j \in L^{p_j(\cdot)}, \| h_j w_j \|_{p_j(\cdot)} \leq 1, j = 1, 2 \), such that

\[
\| w \chi_Q \|_{p(\cdot)} \prod_{j=1}^2 \| w_j^{-1} \chi_Q \|_{p_j(\cdot)} \lesssim \| w \chi_Q \|_{p(\cdot)} \int_Q h_1 dy \int_Q h_2 dy = \| A_Q(h_1, h_2)w \|_{p(\cdot)} |Q|^2 \\
\lesssim \| h_1 w_1 \|_{p_1(\cdot)} \| h_2 w_2 \|_{p_2(\cdot)} |Q|^2 \\
\lesssim |Q|^2.
\]

Again, the constant is independent of \( Q \), so \( \vec{w} \in \mathcal{A}_{\vec{p}(\cdot)} \). \( \square \)
The proof of Theorem 5.1 for $T_Q$ requires two ancillary tools. The first is a bilinear averaging operator that generalizes a linear operator introduced in [16]. Given $p(\cdot) \in \mathcal{P}$, define the $p(\cdot)$-average

$$\langle h \rangle_{p(\cdot), Q} := \frac{\| h \chi_Q \|_{p(\cdot)}}{\| \chi_Q \|_{p(\cdot)}},$$

and given a disjoint family of cubes $Q$ define the $p(\cdot)$-averaging operator

$$T_{p(\cdot), Q} f(x) = \sum_{Q \in Q} \langle h \rangle_{p(\cdot), Q} \cdot \chi_Q(x).$$

In [16, Corollary 7.3.21] the authors showed that if $p(\cdot) \in LH$, then

$$\| T_{p(\cdot), Q} f \|_{p(\cdot)} \lesssim \| f \|_{p(\cdot)}. \tag{5.3}$$

We define the bilinear $p(\cdot)$-average operator analogously: given $p_1(\cdot), p_2(\cdot)$, and a family of disjoint cubes $Q$, let

$$\tilde{T}_{p(\cdot), Q} (f_1, f_2)(x) = \sum_{Q \in Q} \frac{\| f_1 \chi_Q \|_{p_1(\cdot)} \| g_1 \chi_Q \|_{p_2(\cdot)}}{\| \chi_Q \|_{p(\cdot)}} \cdot \chi_Q(x).$$

**Lemma 5.4.** Given $p_1(\cdot), p_2(\cdot) \in \mathcal{P}$, suppose $p_j(\cdot) \in LH$, $j = 1, 2$. Then

$$\sup_Q \| \tilde{T}_{p(\cdot), Q} (f_1, f_2) \|_{p(\cdot)} \lesssim \| f_1 \|_{p_1(\cdot)} \| f_2 \|_{p_2(\cdot)},$$

where the supremum is taken over all collections $Q$ of disjoint cubes.

**Proof:** Since $p_1(\cdot), p_2(\cdot) \in LH$, $p(\cdot) \in LH$, and so by Lemma 3.10,

$$\| \chi_Q \|_{p(\cdot)} \approx |Q|^{\frac{1}{p(\cdot)}} |Q|^{\frac{1}{p_1(\cdot)}} |Q|^{\frac{1}{p_2(\cdot)}} \approx \| \chi_Q \|_{p_1(\cdot)} \| \chi_Q \|_{p_2(\cdot)}.$$ 

Therefore,

$$\tilde{T}_{p(\cdot), Q} (f_1, f_2)(x) \approx \sum_{Q \in Q} \langle f_1 \rangle_{p_1(\cdot), Q} \langle f_2 \rangle_{p_2(\cdot), Q} \cdot \chi_Q,$$

and so by Lemma 3.6 and (5.3),

$$\| \tilde{T}_{p(\cdot), Q} (f_1, f_2) \|_{p(\cdot)} \lesssim \left\| \sum_{Q \in Q} \langle f_1 \rangle_{p_1(\cdot), Q} \langle f_2 \rangle_{p_2(\cdot), Q} \cdot \chi_Q \right\|_{p(\cdot)} \lesssim \left\| \sum_{Q \in Q} \langle f_1 \rangle_{p_1(\cdot), Q} \cdot \chi_Q \sum_{Q \in Q} \langle f_2 \rangle_{p_2(\cdot), Q} \cdot \chi_Q \right\|_{p(\cdot)} \lesssim \| T_{p_1(\cdot), Q} f_1 \|_{p_1(\cdot)} \| T_{p_2(\cdot), Q} f_2 \|_{p_2(\cdot)} \lesssim \| f_1 \|_{p_1(\cdot)} \| f_2 \|_{p_2(\cdot)}. \qed
The second tool is a summation property. Given $p_1(\cdot), p_2(\cdot) \in \mathcal{P}$, suppose $p(\cdot)$ is such that $p_- \geq 1$. Then we say that $\tilde{p}(\cdot) \in \breve{G}$ if for every family of disjoint cubes $Q$,
\[ \sum_{Q \in \mathcal{Q}} \| f_1 \chi_Q \|_{p_1(\cdot)} \| f_2 \chi_Q \|_{p_2(\cdot)} \| h \chi_Q \|_{p'(\cdot)} \lesssim \| f_1 \|_{p_1(\cdot)} \| f_2 \|_{p_2(\cdot)} \| h \|_{p'(\cdot)}, \]
where the implicit constant is independent of $Q$.

**Remark 5.5.** The linear version of property $G$ is due to Berezhnoi [1] in the setting of Banach function spaces. See also [16] where it is used to prove (5.3).

**Lemma 5.6.** Given $p_1(\cdot), p_2(\cdot) \in \mathcal{P}$, suppose $p_j(\cdot) \in LH$, $j = 1, 2$. Then $\tilde{p}(\cdot) \in \breve{G}$.

**Proof:** Since both $p(\cdot), p'(\cdot) \in LH$, by Lemma 3.10, for any cube $Q$,
\[ \| \chi_Q \|_{p(\cdot)} \| \chi_Q \|_{p'(\cdot)} \approx |Q|^{\frac{1}{p}} |Q|^{\frac{1}{p'}} = |Q|. \]
Hence, by Lemma 3.6, (5.3), and Lemma 5.4,
\[ \sum_{Q \in \mathcal{Q}} \| f_1 \chi_Q \|_{p_1(\cdot)} \| f_2 \chi_Q \|_{p_2(\cdot)} \| h \chi_Q \|_{p'(\cdot)} \]
\[ \approx \sum_{Q \in \mathcal{Q}} \int_{\mathbb{R}^n} \frac{\| f_1 \chi_Q \|_{p_1(\cdot)} \| f_2 \chi_Q \|_{p_2(\cdot)} \| h \chi_Q \|_{p'(\cdot)}}{\| \chi_Q \|_{p(\cdot)} \| \chi_Q \|_{p'(\cdot)}} \cdot \chi_Q \, dx \]
\[ \lesssim \int_{\mathbb{R}^n} \sum_{Q \in \mathcal{Q}} \frac{\| f_1 \chi_Q \|_{p_1(\cdot)} \| f_2 \chi_Q \|_{p_2(\cdot)}}{\| \chi_Q \|_{p(\cdot)}} \cdot \chi_Q \sum_{Q \in \mathcal{Q}} \frac{\| h \chi_Q \|_{p'(\cdot)}}{\| \chi_Q \|_{p'(\cdot)}} \cdot \chi_Q \, dx \]
\[ = \int_{\mathbb{R}^n} \tilde{T}_{\tilde{p}(\cdot), Q}(f_1, f_2) T_{p'(\cdot), Q} h \, dx \]
\[ \lesssim \| \tilde{T}_{\tilde{p}(\cdot), Q}(f_1, f_2) \|_{p(\cdot)} \| T_{p'(\cdot), Q} h \|_{p'(\cdot)} \]
\[ \lesssim \| f_1 \|_{p_1(\cdot)} \| f_2 \|_{p_2(\cdot)} \| h \|_{p'(\cdot)}. \]

**Proof of Theorem 5.1 for $T_Q$:** We first prove that the $A_{\tilde{p}(\cdot)}$ condition is sufficient. Since $|T_Q(f_1, f_2)(x)| \leq T_Q(|f_1|, |f_2|)(x)$, we may assume without loss of generality that $f_1, f_2$ are non-negative. Because $(p_j)_- \geq 1,$
j = 1, 2, we have 2p_ - ≥ 1, and so 2p(·) ∈ P. Therefore, by Lemmas 3.3 and 3.8, there exists hw_ - \frac{1}{2} ∈ L((2p)'(·)), with \|hw_ - \frac{1}{2}\|_{(2p)'(·)} ≤ 1, such that

\|T_Q(f_1, f_2)w\|_{p(·)}^{\frac{1}{2}} = \|T_Q(f_1, f_2)\frac{1}{2}w_\frac{1}{2}\|_{2p(·)}

≈ \int_{\mathbb{R}^n} T_Q(f_1, f_2)\frac{1}{2}w_\frac{1}{2}hw_ - \frac{1}{2} dx

≤ \sum_{Q ∈ Q} \langle f_1 \rangle_Q^{\frac{1}{2}} \langle f_2 \rangle_Q^{\frac{1}{2}} \int_{Q} hw_\frac{1}{2}w_ - \frac{1}{2} dx

= \sum_{Q ∈ Q} \|f_1^{\frac{1}{2}}w_1^{\frac{1}{2}}w_{1 -}^{\frac{1}{2}}\chi_Q\|_2\|f_2^{\frac{1}{2}}w_2^{\frac{1}{2}}w_{2 -}^{\frac{1}{2}}\chi_Q\|_2\|hw_\frac{1}{2}w_ - \frac{1}{2}\chi_Q\|_{1|Q|^{-1}};

by Lemmas 3.6 and 3.7,

≤ \sum_{Q ∈ Q} \left[ \|f_1^{\frac{1}{2}}w_1^{\frac{1}{2}}\chi_Q\|_{2p_1(·)}\|w_{1 -}^{\frac{1}{2}}\chi_Q\|_{2p_1'(·)} \right.

× \|f_2^{\frac{1}{2}}w_2^{\frac{1}{2}}\chi_Q\|_{2p_2(·)}\|w_{2 -}^{\frac{1}{2}}\chi_Q\|_{2p_2'(·)}\|h w_ - \frac{1}{2}\chi_Q\|_{(2p)'(·)}\|w_\frac{1}{2}\chi_Q\|_{2p(·)}|Q|^{-1} \left. \right] \]

≤ \sum_{Q ∈ Q} \left[ \|f_1^{\frac{1}{2}}w_1^{\frac{1}{2}}\chi_Q\|_{2p_1(·)}\|f_2^{\frac{1}{2}}w_2^{\frac{1}{2}}\chi_Q\|_{2p_2(·)}\|hw_ - \frac{1}{2}\chi_Q\|_{(2p)'(·)} \right.

× \|w_{1 -}^{\frac{1}{2}}\chi_Q\|_{2p_1'(·)}\|w_{2 -}^{\frac{1}{2}}\chi_Q\|_{2p_2'(·)}\|w_\frac{1}{2}\chi_Q\|_{2p_1(·)}\|w_\frac{1}{2}\chi_Q\|_{2p_2(·)}|Q|^{-1} \left. \right] \]

by Proposition 4.7, Lemma 5.6 applied to the exponents 2p_1(·), 2p_2(·), and Lemma 3.3,

≤ \sum_{Q ∈ Q} \|f_1^{\frac{1}{2}}w_1^{\frac{1}{2}}\chi_Q\|_{2p_1(·)}\|f_2^{\frac{1}{2}}w_2^{\frac{1}{2}}\chi_Q\|_{2p_2(·)}\|hw_ - \frac{1}{2}\chi_Q\|_{(2p)'(·)}

≤ \|f_1^{\frac{1}{2}}w_1^{\frac{1}{2}}\|_{2p_1(·)}\|g_1 w_2^{\frac{1}{2}}\|_{2p_2(·)}\|hw_ - \frac{1}{2}\|_{(2p)'(·)}

≤ \|f_1 w_1^{\frac{1}{2}}\|_{p_1(·)}\|g_1 w_2^{\frac{1}{2}}\|_{p_2(·)}.

Since the implicit constants are independent of our choice of Q, we conclude that \vec{w} ∈ A_{\vec{p}(·)} implies (5.2).

The converse, that the A_{\vec{p}(·)} condition is necessary, follows from the corresponding implication for A_Q proved above.

6. Proof of Theorem 2.4

In this section we prove Theorem 2.4. As before, given weights w_1 and w_2 we define w = w_1 w_2 and let \vec{w} = (w_1, w_2, w). Given exponents p_1(·), p_2(·) ∈ P, we define p(·) by (2.1) and let \vec{p}(·) = (p_1(·), p_2(·), p(·)).
We first prove the necessity of the $\mathcal{A}_{\vec{p}(\cdot)}$ condition. This is an immediate consequence of Theorem 5.1. Given any cube $Q$, we have

$$|A_Q(f_1, f_2)(x)| \leq \mathcal{M}(f_1, f_2)(x).$$

Therefore, given weights $w_1, w_2$ such that the boundedness condition (2.2) holds, we immediately have that (5.1) holds, and so $\vec{w} \in \mathcal{A}_{\vec{p}(\cdot)}$.

**Remark 6.1.** The proof that the $\mathcal{A}_{\vec{p}(\cdot)}$ condition is necessary does not require us to assume that the exponents are log-Hölder continuous.

The proof of the sufficiency of the $\mathcal{A}_{\vec{p}(\cdot)}$ condition is considerably more complicated. Fix $p_1(\cdot), p_2(\cdot) \in \mathcal{P}$ such that $(p_j)_- > 1$ and $p_j(\cdot) \in LH$, $j = 1, 2$. Let $w_1, w_2$ be such that $\vec{w} \in \mathcal{A}_{\vec{p}(\cdot)}$.

We begin with a series of reductions. First, for $t \in \{0, 1/3\}^n$, define

$$D_t = \{2^{-k}([0, 1)^n + j + (-1)^k t) : k \in \mathbb{Z}, j \in \mathbb{Z}^n\}.$$

Each $D_t$ is a “1/3” translate of the standard dyadic grid, and has exactly the same properties as $D_0$ defined above. (Note that the two definitions agree when $t = 0$.) Define the dyadic bilinear maximal operator

$$\mathcal{M}^{D_t}(f_1, f_2)(x) = \sup_{Q \in D_t} \int_Q |f_1(y)| dy \int_Q |f_2(y)| dy \chi_Q(x).$$

Then we have the following remarkable inequality: there exists a constant $C(n)$ such that

$$\mathcal{M}(f_1, f_2)(x) \leq C(n) \sum_{t \in \{0, 1/3\}^n} \mathcal{M}^{D_t}(f_1, f_2)(x).$$

This was first proved in [14]. (For the linear case, see also [3].)

Therefore, to prove that inequality (2.2) holds, it suffices to prove it with $\mathcal{M}$ replaced by $\mathcal{M}^{D_t}$, and in fact it suffices to prove it for $\mathcal{M}^d = \mathcal{M}^{D_0}$, since the same proof holds for any dyadic grid $D_t$ with different constants, where the difference only depends on $t$. (Below we will describe where this difference arises.)

Second, we may assume that $f, g$ are non-negative, bounded functions with compact support. It is clear from the definition of $\mathcal{M}^d$ that we may take them non-negative. To show the approximation, it suffices to note that given $f_1, f_2$, there exists a sequence of non-negative, bounded functions of compact support, $g_k, h_k$, that increase pointwise to $f$ and $g$ and such that

$$\lim_{k \to \infty} \mathcal{M}^d(g_k, h_k)(x) = \mathcal{M}^d(f_1, f_2)(x).$$

In the linear case this is proved in [5, Lemma 3.30] and the same proof (with the obvious changes) works in the bilinear case. The desired result then follows by Lemma 3.4.
Third, we restate the desired inequality in an equivalent fashion. Given an exponent \( p(\cdot) \in P_0 \) and a weight \( v \), define \( L_v^{p(\cdot)} \) to be the quasi-Banach function space with norm
\[
\| g \|_{L_v^{p(\cdot)}} := \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \left( \frac{|g(x)|}{\lambda} \right)^{p(x)} v(x) \, dx \leq 1 \right\}.
\]
In other words, \( L_v^{p(\cdot)} \) is defined exactly as \( L^{p(\cdot)} \) with Lebesgue measure replaced by the measure \( v \, dx \). This norm has many of the same basic properties as the \( L^{p(\cdot)} \) norm.

Let \( u(\cdot) = w(\cdot)^{p(\cdot)} \) and \( \sigma_1(\cdot) = w_l(\cdot)^{-p_l'(\cdot)} \), \( l = 1, 2 \). Then
\[
(\sigma_1(x)w_l(x))^{p_l(x)} = (w_l(x)^{1-p_l'(x)})^{p_l(x)} = w_l(x)^{-p_l'(x)} = \sigma_1(x).
\]
Therefore,
\[
\| M^d(f_1 \sigma_1, f_2 \sigma_2) \|_{L_u^{p(\cdot)}} = \| M^d(f_1 \sigma_1, f_2 \sigma_2) w \|_{L_v^{p(\cdot)}},
\]
and for \( l = 1, 2 \),
\[
\| f_l \|_{L_{p_l(\cdot)^1}} \leq \| f_1 \sigma_1 w_l \|_{L_{p_l(\cdot)}}.
\]
Hence, it will suffice to prove that
\[
\| M^d(f_1 \sigma_1, f_2 \sigma_2) \|_{L_u^{p(\cdot)}} \lesssim \| f_1 \|_{L_{p_l(\cdot)^1}} \| f_2 \|_{L_{p_2(\cdot)}},
\]
since if we replace \( f_l \) by \( f_l/\sigma_1 \), \( l = 1, 2 \), we get (2.2).

Finally, by homogeneity we may assume without loss of generality that \( \| f_l \|_{L_{p_l(\cdot)^1}} = 1 \), \( l = 1, 2 \), which by Lemma 3.5 (which holds in this setting) implies that
\[
\int_{\mathbb{R}^n} |f_l(x)|^{p_l(x)} \sigma_1(x) \, dx \leq 1.
\]
Thus it will suffice to prove that
\[
\| M^d(f_1 \sigma_1, f_2 \sigma_2) \|_{L_u^{p(\cdot)}} \lesssim 1,
\]
which, again by Lemma 3.5, is equivalent to proving that
\[
(6.1) \quad \int_{\mathbb{R}^n} M^d(f_1 \sigma_1, f_2 \sigma_2)^{p(x)} u(x) \, dx \lesssim 1
\]
with a constant independent of \( f_l \), \( l = 1, 2 \).
We now begin our main estimate, which is to prove that (6.1) holds. Define the functions

\[ h_1 = f_1 \chi_{\{f_1 > 1\}}, \quad h_2 = f_1 \chi_{\{f_1 \leq 1\}}, \]
\[ h_3 = f_2 \chi_{\{f_2 > 1\}}, \quad h_4 = f_2 \chi_{\{f_2 \leq 1\}}, \]

and for brevity define

\[ \rho(1) = 1, \quad \rho(2) = 1, \quad \rho(3) = 2, \quad \rho(4) = 2. \]

Then we can write

\[
\int_{\mathbb{R}^n} \mathcal{M}^d(f_1 \sigma_1, f_2 \sigma_2)(x)^{\rho(x)} u(x) \, dx \leq \int_{\mathbb{R}^n} \mathcal{M}^d(h_1 \sigma_1, h_3 \sigma_2)(x)^{\rho(x)} u(x) \, dx
\]
\[
+ \int_{\mathbb{R}^n} \mathcal{M}^d(h_1 \sigma_1, h_4 \sigma_2)(x)^{\rho(x)} u(x) \, dx
\]
\[
+ \int_{\mathbb{R}^n} \mathcal{M}^d(h_2 \sigma_1, h_3 \sigma_2)(x)^{\rho(x)} u(x) \, dx
\]
\[
+ \int_{\mathbb{R}^n} \mathcal{M}^d(h_2 \sigma_1, h_4 \sigma_2)(x)^{\rho(x)} u(x) \, dx
\]
\[
= I_1 + I_2 + I_3 + I_4. \]

We will estimate each term on the right separately. The integral \( I_1 \) is the “local” term and the estimate will use the \( LH_0 \) condition. The integral \( I_4 \) is the “global” term and the estimate will use the \( LH_\infty \) condition. The estimates of \( I_2 \) and \( I_3 \) involve both local and global estimates and are the most complicated: this is where our proof diverges most significantly from the linear case. Note, however, that the estimates for these integrals are the same (making the obvious changes) so we will only estimate \( I_2 \).

**The estimate for \( I_1 \).** We begin by forming the bilinear Calderón–Zygmund cubes associated with \( \mathcal{M}^d(h_1 \sigma_1, h_3 \sigma_2) \). For the details of this decomposition, see [22]. Fix \( a > 2^{2n} \) and for each \( k \in \mathbb{Z} \) define

\[ \Omega_k = \{ x \in \mathbb{R}^n : \mathcal{M}^d(h_1 \sigma_1, h_3 \sigma_2)(x) > a^k \} \]

Then \( \Omega_k = \bigcup_j Q_j^k \) where \( \{Q_j^k\}_{k,j} \) is a family of maximal dyadic cubes contained in \( \Omega_k \) with the property that

\[ a^k < \langle h_1 \rangle_{Q_j^k} \langle h_3 \rangle_{Q_j^k} \leq a^{k+1}. \]

Moreover, since \( \Omega_{k+1} \subset \Omega_k \), the sets \( E_j^k = Q_j^k \setminus \Omega_{k+1} \) are pairwise disjoint and there exists \( 0 < \alpha < 1 \) such that

\[ \alpha |Q_j^k| < |E_j^k|. \]
By Corollary 4.9, $u$, and $\sigma_l$, $l = 1, 2$, are $A_\infty$ weights, so by Lemma 3.1 there exists $0 < \beta < 1$ such that

$$\beta u(Q_j^k) \leq u(E_j^k), \quad \beta \sigma_l(Q_j^k) \leq \sigma_l(E_j^k).$$

We will use this fact repeatedly throughout the proof without further comment.

We can now estimate $I_1$ as follows:

$$I_1 = \int_{\mathbb{R}^n} \mathcal{M}^d(h_1\sigma_1, h_3\sigma_2)(x)^{p(x)} u(x) \, dx$$

$$\leq \sum_{k=0}^{\infty} \int_{\Omega_k \setminus \Omega_{k+1}} a^{(k+1)p(x)} u(x) \, dx$$

$$\lesssim \sum_{k,j} \int_{E_j^k} \prod_{l=1,3} (h_l^{\sigma_1(y)})^{p(x)} u(x) \, dx$$

$$= \sum_{k,j} \int_{E_j^k} \prod_{l=1,3} \left( \int_{Q_j^k} h_l^{\sigma_1(y)} \, dy \right)^{p(x)} |Q_j^k|^{-2p(x)} u(x) \, dx.$$

Since $h_1(x) \geq 1$ or $h_1(x) = 0$, we have that

$$\int_{Q_j^k} h_1(y) \sigma_1(y) \, dy \leq \int_{Q_j^k} h_1(y)^{p_1(y)} \sigma_1(y) \, dy$$

$$\leq \int_{\mathbb{R}^n} f_1(y)^{p_1(y)} \sigma_1(y) \, dy \leq 1. \tag{6.2}$$

The same estimate holds for $h_3$. For each $j$, $k$ define

$$1 \over q(Q_j^k) = 1 \over (p_1) - (Q_j^k) + 1 \over (p_2) - (Q_j^k),$$

and note that for $x \in Q_j^k$, $q(Q_j^k) \leq p_-(Q_j^k) \leq p(x)$. Thus,

$$I_1 \leq \sum_{k,j} \int_{E_j^k} \prod_{l=1,3} \left( \int_{Q_j^k} h_l(y)\sigma_1(y) \, dy \right)^{q(Q_j^k)} |Q_j^k|^{-2p(x)} u(x) \, dx$$

$$\leq \sum_{k,j} \int_{E_j^k} \prod_{l=1,3} \left( \frac{1}{\sigma_1(y)} \int_{Q_j^k} h_l(y) \frac{1}{(Q_j^k)^p(y)} \sigma_1(y) \, dy \right)^{q(Q_j^k)}$$

$$\times \sigma_1(y) |Q_j^k|^{-2p(x)} u(x) \, dx.$$
By Hölder’s inequality with measure $\sigma_l \, dx$, for $l = 1, 3$,

$$
\left( \frac{1}{\sigma_{\rho(l)}(Q_j^k)} \int_{Q_j^k} h_l(y) \frac{p_l(y)}{(p_l)_+ - (Q_j^k)_+} \sigma_{\rho(l)}(y) \, dy \right)^{q(Q_j^k)}
\leq \left( \frac{1}{\sigma_{\rho(l)}(Q_j^k)} \int_{Q_j^k} h_l(y) \frac{p_l(y)}{(p_l)_+ - (Q_j^k)_+} \sigma_{\rho(l)}(y) \, dy \right)^{(p_l)_- - \frac{q(Q_j^k)}{(p_l)_+ - (Q_j^k)_+}}
= \langle h_l \rangle_{\sigma_{\rho(l)}, Q}^{(p_l)_- - \frac{q(Q_j^k)}{(p_l)_+ - (Q_j^k)_+}}.
$$

(6.3)

Further, we claim that

$$
\hat{E}_{k,j} \prod_{l=1,3} \sigma_{\rho(l)}(Q_j^k) |Q_j^k|^{-2p(x)} u(x) \, dx
\lesssim \sigma_1(Q_j^k)^{(p_1)_+ - (Q_j^k)_+} \sigma_2(Q_j^k)^{(p_2)_+ - (Q_j^k)_+}.
$$

(6.4)

If we assume this for the moment, then we can argue as follows: since

$$
1 = \frac{q(Q_j^k)}{(p_1)_+ - (Q_j^k)_+} + \frac{q(Q_j^k)}{(p_2)_+ - (Q_j^k)_+},
$$

by (6.3) and Young’s inequality,

$$
I_1 \lesssim \sum_{k,j} \prod_{l=1,3} \langle h_l \rangle_{\sigma_{\rho(l)}, Q}^{(p_l)_- - \frac{q(Q_j^k)}{(p_l)_+ - (Q_j^k)_+}} \sigma_{\rho(l)}(Q_j^k) \sigma_{\rho(l)}(Q_j^k)^{\frac{q(Q_j^k)}{(p_l)_+ - (Q_j^k)_+}}
\leq \sum_{k,j} \sum_{l=1,3} \langle h_l \rangle_{\sigma_{\rho(l)}, Q}^{(p_l)_- - \frac{q(Q_j^k)}{(p_l)_+ - (Q_j^k)_+}} \sigma_{\rho(l)}(Q_j^k)
\leq \sum_{k,j} \sum_{l=1,3} \langle h_l \rangle_{\sigma_{\rho(l)}, Q}^{(p_l)_- - \frac{q(Q_j^k)}{(p_l)_+ - (Q_j^k)_+}} \sigma_{\rho(l)}(E_j^k).
$$

(6.5)

By Lemma 3.2, since $(p_l)_+ > 1$,

$$
\leq \sum_{l=1,3} \int_{\mathbb{R}^n} \frac{M_{\sigma_{\rho(l)}}(h_l)}{\langle h_l \rangle_{(p_l)_- - \sigma_{\rho(l)}}(x)} (x) \sigma_{\rho(l)}(x) \, dx
\lesssim \sum_{l=1,3} \int_{\mathbb{R}^n} h_l(x)(p_l(x)) \sigma_{\rho(l)}(x) \, dx
\lesssim 1.
$$
Therefore, to complete the estimate of $I_1$ we will prove (6.4). First, rewrite the left-hand side as follows:

$$
\int \prod_{l=1,3} \sigma_{\rho(l)}(Q^k_j) q(Q^k_j) |Q^k_j|^{-2p(x)} u(x) \, dx \leq \prod_{l=1,3} \left( \frac{\sigma_{\rho(l)}(Q^k_j)}{\|w_{\rho(l)}^{-1} \chi_{Q^k_j}\|_{p_l^\prime(\cdot)}} \right)^{q(Q^k_j)}
$$

$$
\times \left( \prod_{l=1,3} \|w_{\rho(l)}^{-1} \chi_{Q^k_j}\|_{p_l^\prime(\cdot)}^{q(Q^k_j)-p(x)} \right) \left( \prod_{l=1,3} \|w_{\rho(l)}^{-1} \chi_{Q^k_j}\|_{p_l^\prime(\cdot)}^{p(x)} |Q^k_j|^{-2p(x)} u(x) \right) \, dx.
$$

By the $A_{\vec{p}(\cdot)}$ condition we have that there is a constant $c$ such that

$$
\|c|Q^k_j|^{-2} \prod_{l=1}^2 \|w_{l}^{-1} \chi_{Q^k_j}\|_{p_l^\prime(\cdot)} w \chi_{Q^k_j}\|_{p(\cdot)} \leq 1,
$$

which by Lemma 3.5 implies that

$$
(6.6) \quad \int Q^k_j \prod_{l=1}^2 \|w_{l}^{-1} \chi_{Q^k_j}\|_{p_l^\prime(\cdot)} |Q^k_j|^{-2p(x)} u(x) \, dx \lesssim 1.
$$

Hence, to prove (6.4) it will suffice to show that for $l = 1, 2,$

$$
(6.7) \quad \left( \frac{\sigma_l(Q^k_j)}{\|w_{l}^{-1} \chi_{Q^k_j}\|_{p_l^\prime(\cdot)}} \right)^{q(Q^k_j)} \lesssim \sigma_l(Q^k_j)^{q(Q^k_j)/(p_l)-(Q^k_j)}
$$

and

$$
(6.8) \quad \|w_{l}^{-1} \chi_{Q^k_j}\|_{p_l^\prime(\cdot)}^{q(Q^k_j)-p(x)} \lesssim 1.
$$

We first prove (6.7). Suppose that $\|w_{l}^{-1} \chi_{Q^k_j}\|_{p_l^\prime(\cdot)} > 1.$ Then by Lemma 3.5, since $(p_l')_{\pm}(Q^k_j) = (p_l)_{\pm}(Q^k_j)'$, we have that

$$
\left( \frac{\sigma_l(Q^k_j)}{\|w_{l}^{-1} \chi_{Q^k_j}\|_{p_l^\prime(\cdot)}} \right)^{q(Q^k_j)} \leq \left( \sigma_l(Q^k_j)^{1-(Q^k_j)/(p_l)-(Q^k_j)'} \right)^{q(Q^k_j)} = \sigma_l(Q^k_j)^{q(Q^k_j)/(p_l)-(Q^k_j)}.
$$
On the other hand, if \( \|w_i^{-1}Q_k^j\|_{p(')} \leq 1, \)
\[
\frac{\sigma_i(Q_k^j)}{\|w_i^{-1}Q_k^j\|_{p(')}} \leq \sigma_i(Q_k^j)^{\frac{1}{(p_1)_++(Q_k^j)^{q(Q_k^j)}}} = \sigma_i(Q_k^j)^{\frac{1}{(p_1)_++(Q_k^j)^{q(Q_k^j)}}} = \sigma_i(Q_k^j)^{\frac{1}{(p_1)^-(Q_k^j)^{q(Q_k^j)}}} \sigma_i(Q_k^j)^{\frac{1}{(p_1)_+-(Q_k^j)^{q(Q_k^j)}}}.
\]

Again by Lemma 3.5, and then by Lemma 3.3 and Lemma 4.5,
\[
\sigma_i(Q_k^j)^{\frac{1}{(p_1)^+(Q_k^j)^{q(Q_k^j)}} - \frac{1}{(p_1)^-(Q_k^j)^{q(Q_k^j)}}} \leq \|w_i^{-\frac{1}{2}}Q_k^j\|_{2p(')}^{\frac{2(p'i)^-}{(p_1)^+-(Q_k^j)^{q(Q_k^j)}}} - \frac{1}{(p_1)^-(Q_k^j)^{q(Q_k^j)}}
\]
\[
= \|w_i^{-\frac{1}{2}}Q_k^j\|_{2p(')}^{\frac{2(p'i)^-}{(p_1)^+-(Q_k^j)^{q(Q_k^j)}}} - \frac{1}{(p_1)^-(Q_k^j)^{q(Q_k^j)}}
\]
\[
= \|w_i^{-\frac{1}{2}}Q_k^j\|_{2p(')}^{\frac{2(p'i)^-}{(p_1)^+-(Q_k^j)^{q(Q_k^j)}}} - \frac{1}{(p_1)^-(Q_k^j)^{q(Q_k^j)}}
\]
\[
\leq \|w_i^{-\frac{1}{2}}Q_k^j\|_{2p(')}^{\frac{c(2p'i)^--(Q_k^j)^{q(Q_k^j)}-2(p'i)^+(Q_k^j))}{2p(')}
\]
\[
\leq 1.
\]

Hence,
\[
(6.9) \quad \left(\frac{\sigma_i(Q_k^j)}{\|w_i^{-1}Q_k^j\|_{p(')}}\right)^{q(Q_k^j)} \lesssim \sigma_i(Q_k^j)^{\frac{q(Q_k^j)}{(p_1)_+-(Q_k^j)^{q(Q_k^j)}}}.
\]

We now prove (6.8). If \( \|w_i^{-1}Q_k^j\|_{p(')} \geq 1, \) then this is immediate. If \( \|w_i^{-1}Q_k^j\|_{p(')} < 1, \) then by Lemma 3.3, and then by Propositions 4.7 and 4.10,
\[
\|w_i^{-1}Q_k^j\|_{p(')}^{q(Q_k^j)-p(x)} = \|w_i^{-\frac{1}{2}}Q_k^j\|_{2p(')}^{2q(Q_k^j)-2p(x)} \lesssim 1.
\]
This completes the estimate of \( I_1. \)

**The estimate for \( I_2. ** We first form the bilinear Calderón–Zygmund cubes associated with \( \mathcal{M}^d(h_1\sigma_1, h_4\sigma_2) \) and we use the same notation as we did in the estimate for \( I_1. \) To estimate this term \( I_2 \) we need to divide the cubes \( Q_j^k \) into three sets: small cubes close to the origin, large cubes close to the origin, and cubes (of all sizes) far from the origin. To make this precise, let \( \{P_i\}_{i=1}^{2^n} \) be the \( 2^n \) dyadic cubes adjacent to the origin, \( |P_i| \geq 1, \) that are so large that if \( Q \) is any dyadic cube equal to or adjacent to \( P_i \) in the same quadrant, and \( |P_i| = |Q|, \) then \( u(Q) \geq 1 \) and \( \sigma_i(Q) \geq 1, \) \( l = 1, 2. \) The existence of such cubes follows from Lemma 3.1
and Corollary 4.9. Let $P = \bigcup_i P_i$. We can then partition the cubes $\{Q_j^k\}$ into three disjoint sets:

\[
\mathcal{F} = \{(k, j) : Q_j^k \subset P_i \text{ for some } i\},
\]

\[
\mathcal{G} = \{(k, j) : P_i \subset Q_j^k \text{ for some } i\},
\]

\[
\mathcal{H} = \{(k, j) : Q_j^k \cap P_i = \emptyset \text{ for all } i\}.
\]

We now estimate $I_2$, arguing as we did at the beginning of the estimate for $I_1$:

\[
\int_{\mathbb{R}^n} \mathcal{M}^d(h_1 \sigma_1, h_4 \sigma_2)(x)^{p(x)} u(x) \, dx \lesssim \sum_{k,j} \int_{E_j^k} \prod_{l=1,4} \langle h_l \sigma_l \rangle_{Q_j^k}^{p(x)} u(x) \, dx
\]

\[
= \sum_{(k,j) \in \mathcal{F}} + \sum_{(k,j) \in \mathcal{G}} + \sum_{(k,j) \in \mathcal{H}}
\]

\[
= J_1 + J_2 + J_3.
\]

We will estimate each sum in turn.

Remark 6.2. Throughout the rest of this proof, we will allow the implicit constants to depend on $\sigma_l(P)$ or $u(P)$. The choice of the $P_i$ is the one place where the proof depends on the fact that we are working with the dyadic grid $D_0$. For the grids $D_t$ we will replace the origin by its translate $\pm t$, where the sign will depend on the scale at which we choose the $P_i$. See Remark 6.3 below for where the dyadic grid and the choice of the $P_i$ affects the proof.

The estimate for $J_1$. Since $h_4 \leq 1$ and $p_+ < \infty$, we have that

\[
J_1 = \sum_{(k,j) \in \mathcal{F}} \int_{E_j^k} \prod_{l=1,4} \langle h_l \sigma_l \rangle_{Q_j^k}^{p(x)} u(x) \, dx
\]

\[
\leq \sum_{(k,j) \in \mathcal{F}} \int_{E_j^k} \langle h_1 \sigma_1 \rangle_{Q_j^k}^{p(x)} \langle \sigma_2 \rangle_{Q_j^k}^{p(x)} u(x) \, dx
\]

\[
= \sum_{(k,j) \in \mathcal{F}} \int_{E_j^k} \left( \int_{Q_j^k} h_1 \sigma_1 \, dy \right)^{p(x)}
\]

\[
\times \sigma_2(Q_j^k)^{p(x)-q(Q_j^k)} \sigma_2(Q_j^k) q(Q_j^k) |Q_j^k|^{-2p(x)} u(x) \, dx;
\]
by inequalities (6.2) and (6.3),
\[
\leq \sum_{(k,j) \in \mathcal{F}} \int_{E_j^k} \left( \int_{Q_j^k} h_1 \sigma_1 \, dy \right)^{q(Q_j^k)}
\times \sigma_2(Q_j^k)^{p(x) - q(Q_j^k)} \sigma_2(Q_j^k)^{q(Q_j^k)} |Q_j^k|^{-2p(x)} u(x) \, dx
\]
\[
= \sum_{(k,j) \in \mathcal{F}} \int_{E_j^k} \langle h_1 \rangle^{q(Q_j^k)} \sigma_2(Q_j^k)^{p(x) - q(Q_j^k)}
\times \sigma_1(Q_j^k)^{q(Q_j^k)} \sigma_2(Q_j^k)^{q(Q_j^k)} |Q_j^k|^{-2p(x)} u(x) \, dx
\]
\[
\leq \sum_{(k,j) \in \mathcal{F}} (\sigma_2(Q_j^k) + 1)^{p_+ - q_-} \sum_{(k,j) \in \mathcal{F}} \langle h_1 \rangle_{\sigma_1, Q_j^k}^{(p_1)(-)} \sigma_2(Q_j^k)^{q(Q_j^k)}
\times \sigma_1(Q_j^k)^{q(Q_j^k)} \sigma_2(Q_j^k)^{q(Q_j^k)} |Q_j^k|^{-2p(x)} u(x) \, dx.
\]

Let \( q_- \) be defined as in (4.3). By (6.4) we can estimate the integral:
\[
\lesssim (\sigma_2(P) + 1)^{p_+ - q_-} \sum_{(k,j) \in \mathcal{F}} \langle h_1 \rangle_{\sigma_1, Q_j^k}^{(p_1)(-)} \sigma_2(Q_j^k)^{q(Q_j^k)}
\times \sigma_1(Q_j^k)^{q(Q_j^k)} \sigma_2(Q_j^k)^{q(Q_j^k)} |Q_j^k|^{-2p(x)} u(x) \, dx.
\]

Therefore, by Young’s inequality and by Lemma 3.2,
\[
\leq (\sigma_2(P) + 1)^{p_+ - q_-} \sum_{(k,j) \in \mathcal{F}} \left[ \langle h_1 \rangle_{\sigma_1, Q_j^k}^{(p_1)(-)} \sigma_1(Q_j^k) + \sigma_2(Q_j^k) \right]
\]
\[
\lesssim (\sigma_2(P) + 1)^{p_+ - q_-} \sum_{(k,j) \in \mathcal{F}} \left[ \langle h_1 \rangle_{\sigma_1, Q_j^k}^{(p_1)(-)} \sigma_1(E_j^k) + \sigma_2(E_j^k) \right]
\]
\[
\leq \sum_{(k,j) \in \mathcal{F}} \int_{E_j^k} \mathcal{M}_{\sigma_1}^d \left( f_1 \right)^{p_1} \sigma_1(x) \, dx + \sum_{(k,j) \in \mathcal{F}} \sigma_2(E_j^k)
\]
\[
\lesssim \int_{\mathbb{R}^n} f_1(x)^{p_1} \sigma_1(x) \, dx + \sigma_2(P)
\]
\[
\lesssim 1.
\]
The estimate for $J_2$. Given $(k, j) \in \mathcal{G}$, since $P_i \subset Q_j^k$ we have that $1 \leq \sigma_2(P_i) \leq \sigma_2(Q_j^k)$. Therefore, by Lemma 4.3 applied twice to $w_2^{-\frac{k}{2}} \in A_{2p_2'}(\cdot)$, we get

$$
\frac{1}{|Q_j^k|} \leq \frac{|P_i|}{|Q_j^k|} \lesssim \left( \frac{\sigma_2(P_i)}{\sigma_2(Q_j^k)} \right)^{\frac{1}{2(p_2')}} \lesssim \sigma_2(Q_j^k)^{-\frac{1}{2(p_2')}}
$$

$$
\lesssim \|w_2^{-\frac{k}{2}} \chi_{Q_j^k}\|_{2p_2'}^{-1} \lesssim \|w_2^{-\frac{k}{2}} \chi_{Q_j^k}\|_{2p_2'}^{-\frac{1}{2}}.
$$

Hence, by Lemma 3.6,

$$
\frac{1}{|Q_j^k|^2} \int_{Q_j^k} h_4(y) \sigma_2(y) \, dy \lesssim \|w_2^{-1} \chi_{Q_j^k}\|_{2p_2'}^{-1} \int_{Q_j^k} h_4(y) \sigma_2(y) \frac{1}{p_2'(y)} \sigma_2(y) \frac{1}{p_2'(y)} \, dy
$$

$$
\lesssim \|w_2^{-1} \chi_{Q_j^k}\|_{2p_2'}^{-1} \|h_4\|_{L^p_{\sigma_2}(\cdot)} \|\chi_{Q_j^k}\|_{L^p_{\sigma_2}(\cdot)}
$$

$$
\leq \|w_2^{-1} \chi_{Q_j^k}\|_{2p_2'}^{-1} \|f_2\|_{L^p_{\sigma_2}(\cdot)} \|w_2^{-\frac{1}{2}} \chi_{Q_j^k}\|_{2p_2'}
$$

$$
\leq c_0.
$$

We can now estimate $J_2$. By inequality (6.2) and Lemmas 3.13 and 4.6, there exists $t > 1$ such that

$$
J_2 = \sum_{(k, j) \in \mathcal{G}} \int_{E_j^k} c_0^{p(x)} \left( \int_{Q_j^k} h_1 \sigma_1 \, dy \right)^{p(x)} \left( \frac{c_0^{-1}}{|Q_j^k|^2} \int_{Q_j^k} h_4 \sigma_2 \, dy \right)^{p(x)} u(x) \, dx
$$

$$
\lesssim \sum_{(k, j) \in \mathcal{G}} c_0^{p+} \int_{E_j^k} \left( \int_{Q_j^k} h_1 \sigma_1 \, dy \right)^{p(x)} \left( \frac{c_0^{-1}}{|Q_j^k|^2} \int_{Q_j^k} h_4 \sigma_2 \, dy \right)^{p(x)} u(x) \, dx
$$

$$
+ \sum_{(k, j) \in \mathcal{G}} \int_{E_j^k} \frac{u(x)}{(e + |x|)^{ntp}} \, dx
$$

$$
\lesssim \sum_{(k, j) \in \mathcal{G}} \prod_{l=1,4} \langle h_l \rangle_{\sigma(p(l), Q_j^k)}^{'p_{\infty}} \sigma_1(Q_j^k)^{p_{\infty}} \sigma_2(Q_j^k)^{p_{\infty}} |Q_j^k|^{-2p_{\infty}} u(E_j^k) + 1.
$$

We estimate each term in the product separately. First, we claim

$$
(6.10) \quad \sigma_1(Q_j^k)^{p_{\infty}} \sigma_2(Q_j^k)^{p_{\infty}} |Q_j^k|^{-2p_{\infty}} u(E_j^k) \lesssim \sigma_1(Q_j^k)^{\frac{p_{\infty}}{(p_1)_{\infty}}} \sigma_2(Q_j^k)^{\frac{p_{\infty}}{(p_2)_{\infty}}}.
$$
Since $\sigma_l(Q^k_j), u(Q^k_j) \geq 1$, by Lemma 4.3 (applied several times) and the definition of $A_{p(\cdot)}$, we have

\[
\left[ \sigma_1(Q^k_j)\sigma_2(Q^k_j) \right]^{p_\infty} \lesssim \left( \|w_1^{-\frac{1}{2}} \chi_{Q^k_j} \|_{2p'_1(\cdot)} \right)^{2(p'_1(\cdot))\infty} \left( \|w_2^{-\frac{1}{2}} \chi_{Q^k_j} \|_{2p'_2(\cdot)} \right)^{2(p'_2(\cdot))\infty} \\
= \left( \|w_1^{-1} \chi_{Q^k_j} \|_{p'_1(\cdot)} \right)^{p_\infty} \left( \left( \prod_{l=1}^{2} \|w_l^{-1} \chi_{Q^k_j} \|_{p'_l(\cdot)} \right)^{p_\infty} \right)^{p_\infty} \\
\lesssim \left( \|w_1^{-1} \chi_{Q^k_j} \|_{p'_1(\cdot)} \right)^{p_\infty} \left( \left( \prod_{l=1}^{2} \|w_l^{-1} \chi_{Q^k_j} \|_{p'_l(\cdot)} \right)^{p_\infty} \right)^{p_\infty} \left( \frac{|Q^k_j|^{2p_\infty}}{u(Q^k_j)} \right) \\
\lesssim \left( \sigma_1(Q^k_j) \right)^{(p'_1(\cdot))\infty} \sigma_2(Q^k_j) \left( \frac{(p'_2(\cdot))\infty}{p_\infty} \right) \left( \frac{|Q^k_j|^{2p_\infty}}{u(E^k_j)} \right)
\]

This proves (6.10).

Second, by Lemma 3.6 and again by Lemma 4.3 we have

\[
\frac{1}{\sigma_1(Q^k_j)} \int_{Q^k_j} h_1(y)\sigma_1(y) \, dy \lesssim \sigma_1(Q^k_j)^{-1} \|h_1\|_{L^{p'_1(\cdot)} \sigma_1(Q^k_j)} \|\chi_{Q^k_j}\|_{L^{p'_1(\cdot)} \sigma_1(Q^k_j)} \\
\leq \sigma_1(Q^k_j)^{-1} \|f\|_{L^{p'_1(\cdot)} \sigma_1(Q^k_j)} \|w_1^{-1} \chi_{Q^k_j}\|_{p'_1(\cdot)} \\
\lesssim \sigma_1(Q^k_j) \left( \frac{1}{p'_1(\cdot)} \right)^{\infty} \lesssim \sigma_1(P) \left( \frac{1}{p'_1(\cdot)} \right)^{\infty} \lesssim 1.
\]

(6.11)

We can now continue the estimate of $J_2$. Since

\[
1 = \frac{p_\infty}{(p_1)_{\infty}} + \frac{p_\infty}{(p_2)_{\infty}},
\]

by (6.10) and Young’s inequality,

\[
J_2 \lesssim \sum_{(k,j) \in \mathcal{G}} \prod_{l=1,4} \langle h_1 \rangle^{p_\infty}_{\sigma_{p(\cdot)},Q^k_j} \sigma_{p(l)}(Q^k_j)^{\frac{p_\infty}{(p_l)_{\infty}}} + 1 \\
\lesssim \sum_{(k,j) \in \mathcal{G}} \langle h_1 \rangle^{(p_1)_{\infty}}_{\sigma_1,Q^k_j} \sigma_1(Q^k_j) + \sum_{(k,j) \in \mathcal{G}} \langle h_4 \rangle^{(p_2)_{\infty}}_{\sigma_2,Q^k_j} \sigma_2(Q^k_j) + 1 \\
\lesssim \sum_{(k,j) \in \mathcal{G}} \langle h_1 \rangle^{(p_1)_{\infty}}_{\sigma_1,Q^k_j} \sigma_1(Q^k_j) + \sum_{(k,j) \in \mathcal{G}} \langle h_4 \rangle^{(p_2)_{\infty}}_{\sigma_2,Q^k_j} \sigma_2(Q^k_j) + 1;
\]

(6.12)
by Lemmas 3.13 and 4.6 there exists $t > 1$ such that,

$$\lesssim \sum_{(k,j)\in\mathcal{G}} \int_{E_j^k} \langle c_0^{-1} h_1 \rangle_{p_1(x)}^{p_1(x)} \sigma_1(x)\,dx$$

$$+ \sum_{(k,j)\in\mathcal{G}} \int_{E_j^k} \frac{\sigma_1(x)}{(e + |x|)^{ln(p_1)-}}\,dx$$

$$+ \sum_{(k,j)\in\mathcal{G}} \int_{E_j^k} M_{\sigma_2}^d h_4(x)^{(p_2)\infty} \sigma_2(x)\,dx + 1$$

$$\lesssim \sum_{(k,j)\in\mathcal{G}} \int_{E_j^k} \langle c_0^{-1} h_1 \rangle_{p_1(x)}^{p_1(x)} \gamma^{(p_1)-} \sigma_1(x)\,dx$$

$$+ \int_{\mathbb{R}^n} M_{\sigma_2}^d h_4(x)^{(p_2)\infty} \sigma_2(x)\,dx + 1;$$

by Lemma 3.2 applied twice,

$$\lesssim \int_{\mathbb{R}^n} M_{\sigma_1}^d \left(\frac{\gamma^{(p_1)-}}{(p_1)-}\right)(x)^{(p_1)-} \sigma_1(x)\,dx$$

$$+ \int_{\mathbb{R}^n} h_4(x)^{(p_2)\infty} \sigma_2(x)\,dx + 1$$

$$\lesssim \int_{\mathbb{R}^n} h_1(x)^{p_1(x)} \sigma_1(x)\,dx + \int_{\mathbb{R}^n} h_4(x)^{(p_2)\infty} \sigma_2(x)\,dx + 1$$

$$\lesssim \int_{\mathbb{R}^n} h_4(x)^{(p_2)\infty} \sigma_2(x)\,dx + 1.$$

Finally, we again apply Lemmas 3.13 and 4.6 to get

$$\lesssim \int_{\mathbb{R}^n} h_4(x)^{p_2(x)} \sigma_2(x)\,dx + \int_{\mathbb{R}^n} \frac{\sigma_2(x)}{(e + |x|)^{ln(p_2)-}}\,dx + 1$$

$$\lesssim 1,$$

which completes the estimate for $J_2$.

**The estimate for $J_3$.** If $Q_j^k$ is such that $(k, j) \in \mathcal{H}$, then $Q_j^k$ does not contain the origin. Since it is a dyadic cube, we have that $\text{dist}(Q_j^k, 0) \geq$
Therefore, there exists a constant $R > 1$ depending only on $n$ such that

\begin{equation}
\sup_{x \in Q_j^k} |x| \leq R \inf_{x \in Q_j^k} |x|.
\end{equation}

Remark 6.3. The estimate $\text{dist}(Q_j^k, 0) \geq \ell(Q_j^k)$ holds because we are working with the grid $D_0$. For an arbitrary grid $D_t$, since the origin will be contained in one of the cubes $P_i$, we will have that for some $c > 0$, $\text{dist}(Q_j^k, 0) \geq c\ell(Q_j^k)$, and so (6.13) will hold with a possibly larger constant $R$.

By the continuity of $p(\cdot)$, there exists $x_+$ in the closure of $Q_j^k$ such that $p_+(Q_j^k) = p(x_+)$. Hence, since $p(\cdot) \in LH$, for all $x \in Q_j^k$, by (6.13),

\begin{equation}
0 \leq p_+(Q_j^k) - p(x) \leq |p(x_+) - p(x)| + |p(x) - p_\infty| \leq \frac{C_\infty}{\log(e + |x_+|)} + \frac{C_\infty}{\log(e + |x|)} \lesssim \frac{1}{\log(e + |x|)}.
\end{equation}

In the same way, for $l = 1, 2$ we have that $p_l(\cdot)$ satisfies

\begin{equation}
|p_l(Q_j^k) - p_l(x)| \lesssim \frac{1}{\log(e + |x|)}.
\end{equation}

To estimate $J_3$ we need to divide $\mathcal{H}$ into two subsets depending on the size of the cubes $Q_j^k$ with respect to $\sigma_2$:

\[ \mathcal{H}_1 = \{(k, j) \in \mathcal{H} : \sigma_2(Q_j^k) \leq 1\}, \quad \mathcal{H}_2 = \{(k, j) \in \mathcal{H} : \sigma_2(Q_j^k) > 1\}. \]

We first estimate the sum over $\mathcal{H}_1$. By (6.14) and Lemmas 3.13 and 4.6,

\[
\sum_{(k, j) \in \mathcal{H}_1} \int_{E_j^k} \prod_{l=1,4} \langle h_l \sigma_{\rho(l)} \rangle_{Q_j^k}^{p(x)} u(x) \, dx \\
\lesssim \sum_{(k, j) \in \mathcal{H}_1} \int_{E_j^k} \prod_{l=1,4} \langle h_l \sigma_{\rho(l)} \rangle_{Q_j^k}^{p_+(Q_j^k)} u(x) \, dx + \sum_{(k, j) \in \mathcal{H}_1} \int_{E_j^k} \frac{u(x)}{(e + |x|)^{1 + p -}} \, dx \\
\leq \sum_{(k, j) \in \mathcal{H}_1} \int_{E_j^k} \prod_{l=1,4} \langle h_l \sigma_{\rho(l)} \rangle_{Q_j^k}^{p_+(Q_j^k)} u(x) \, dx + 1.
\]
By Lemma 3.11, (6.2), and (6.4), and since $h_1 \geq 1$, $h_4 \leq 1$, and $\sigma_2(Q^k_j) \leq 1$,

$$
\sum_{(k,j) \in \mathcal{H}_1} \int_{E^k_j} \left( \int_{Q^k_j} h_1 \sigma_1 \, dy \right)^{p_+(Q^k_j)} \left( \frac{1}{\sigma_2(Q^k_j)} \int_{Q^k_j} h_4 \sigma_2 \, dy \right)^{p_+(Q^k_j)}
\times |Q^k_j|^{-2p_+(Q^k_j)} \sigma_2(Q^k_j)^{p_+(Q^k_j)} u(x) \, dx + 1
\leq \sum_{(k,j) \in \mathcal{H}_1} \langle h_1 \rangle_{\sigma_1, Q^k_j}^q(Q^k_j) \langle h_4 \rangle_{\sigma_2, Q^k_j}^q(Q^k_j)
\times \int_{E^k_j} |Q^k_j|^{-2p(x)} \sigma_1(Q^k_j)^{q(Q^k_j)} \sigma_2(Q^k_j)^{q(Q^k_j)} u(x) \, dx + 1
\leq \sum_{(k,j) \in \mathcal{H}_1} \prod_{l=1,4} \langle h_l \rangle_{\sigma_{p(l), Q^k_j}}^{q(Q^k_j)} \sigma_1(Q^k_j)^{q(Q^k_j)} \sigma_2(Q^k_j)^{q(Q^k_j)} + 1
\leq \sum_{(k,j) \in \mathcal{H}_1} \langle h_1 \rangle_{\sigma_1, Q^k_j}^{\frac{1}{p_1} - (Q^k_j)} \langle h_4 \rangle_{\sigma_2, Q^k_j}^{\frac{1}{p_2} - (Q^k_j)} \sigma_1(Q^k_j)^{\frac{q(Q^k_j)}{p_1} - (Q^k_j)} \sigma_2(Q^k_j)^{\frac{q(Q^k_j)}{p_2} - (Q^k_j)} + 1
\leq \sum_{(k,j) \in \mathcal{H}_1} \langle h_1 \rangle_{\sigma_1, Q^k_j}^{\frac{1}{p_1} - (Q^k_j)} \langle h_4 \rangle_{\sigma_2, Q^k_j}^{\frac{1}{p_2} - (Q^k_j)} \sigma_1(Q^k_j)^{\frac{q(Q^k_j)}{p_1} - (Q^k_j)} \sigma_2(Q^k_j)^{\frac{q(Q^k_j)}{p_2} - (Q^k_j)} + 1
\leq \sum_{(k,j) \in \mathcal{H}_1} \langle h_1 \rangle_{\sigma_1, Q^k_j}^{\frac{1}{p_1} - (Q^k_j)} \sigma_1(Q^k_j)^{\frac{q(Q^k_j)}{p_1} - (Q^k_j)} + \sum_{(k,j) \in \mathcal{H}_1} \langle h_4 \rangle_{\sigma_2, Q^k_j}^{\frac{1}{p_2} - (Q^k_j)} \sigma_2(Q^k_j)^{\frac{q(Q^k_j)}{p_2} - (Q^k_j)} + 1
= K_1 + K_2 + 1.
$$

by Hölder’s inequality and Young’s inequality,

$$
K_2 \lesssim \sum_{(k,j) \in \mathcal{H}_1} \int_{E^k_j} \langle h_4 \rangle_{\sigma_2, Q^k_j}^{\frac{1}{p_2} - (Q^k_j)} \sigma_2(x) \, dx
\lesssim \sum_{(k,j) \in \mathcal{H}_1} \int_{E^k_j} \langle h_4 \rangle_{\sigma_2, Q^k_j}^{\frac{1}{p_2} - (Q^k_j)} \sigma_2(x) \, dx + \sum_{(k,j) \in \mathcal{H}_1} \int_{E^k_j} \frac{\sigma_2(x)}{(e + |x|)^{nt(p_2)} - 2} \, dx
\leq \int_{\mathbb{R}^n} M^d_{\sigma_2} h_4(x)^{\frac{1}{p_2} - (Q^k_j)} \sigma_2(x) \, dx + \int_{\mathbb{R}^n} \frac{\sigma_2(x)}{(e + |x|)^{nt(p_2)} - 2} \, dx
$$

The proof that $K_1$ is bounded is exactly the same as the final estimate for $I_1$, beginning at (6.5). Therefore, to complete the estimate for the sum over $\mathcal{H}_1$, we need to bound $K_2$. By Lemmas 3.13 and 4.6 (applied twice) and by Lemma 3.2,
\[
\lesssim \int_{\mathbb{R}^n} h_4(x)^{(p_2)\infty} \sigma_2(x) \, dx + 1
\]
\[
\lesssim \int_{\mathbb{R}^n} h_4(x)^{p_2(x)} \sigma_2(x) \, dx + \int_{\mathbb{R}^n} \frac{\sigma_2(x)}{(e + |x|)^{nt(p_2)-}} \, dx + 1
\]
\[
\lesssim 1.
\]

To estimate the sum over \( \mathcal{H}_2 \), first note that by Lemma 3.6 we have
\[
(6.16) \quad \int_{Q_j^k} h_1 \sigma_1 \, dy \lesssim \|h_1\|_{L_{\sigma_1}^{p_1}(\cdot)} \|\chi_{Q_j^k}\|_{L_{\sigma_1}^{p'_1}(\cdot)}
\]
\[
\lesssim \|f_1\|_{L_{\sigma_1}^{p_1}(\cdot)} \|w_1^{-1} \chi_{Q_j^k}\|_{p'_1(\cdot)} \leq c_0 \|w_1^{-1} \chi_{Q_j^k}\|_{p'_1(\cdot)};
\]
similarly, we have
\[
(6.17) \quad \int_{Q_j^k} h_4 \sigma_2 \, dy \leq c_0 \|w_2^{-1} \chi_{Q_j^k}\|_{p'_2(\cdot)}.
\]

We now divide the cubes in \( \mathcal{H}_2 \) into two subsets depending on the size of \( \sigma_1(Q_j^k) \):
\[
\mathcal{H}_{2a} = \{(k, j) \in \mathcal{H}_2 : \sigma_1(Q_j^k) \geq 1\}, \quad \mathcal{H}_{2b} = \{(k, j) \in \mathcal{H}_2 : \sigma_1(Q_j^k) < 1\}.
\]
We first estimate the sum over \( \mathcal{H}_{2a} \). Given (6.16) and (6.17), by Lemma 3.13,
\[
\sum_{(k, j) \in \mathcal{H}_{2a}} \int_{E_j^k} \prod_{l=1,4} \langle h_l \sigma_{\rho(l)} \rangle_{Q_j^k}^{p(x)} u(x) \, dx
\]
\[
\leq c_0^{2p_+} \sum_{(k, j) \in \mathcal{H}_{2a}} \int_{E_j^k} \prod_{l=1,4} \left( c_0^{-1} \|w_{l(l)}^{-1} \chi_{Q_j^k}\|_{p_{\rho(l)}(\cdot)} \int_{Q_j^k} h_l \sigma_{\rho(l)} \, dy \right)^{p(x)}
\]
\[
\times \prod_{l=1}^2 \left( \frac{\|w_l^{-1} \chi_{Q_j^k}\|_{p'_l(\cdot)}}{|Q_j^k|} \right)^{p(x)} u(x) \, dx
\]
\[
\lesssim \sum_{(k, j) \in \mathcal{H}_{2a}} \int_{E_j^k} \prod_{l=1,4} \left( c_0^{-1} \|w_{l(l)}^{-1} \chi_{Q_j^k}\|_{p_{\rho(l)}(\cdot)} \int_{Q_j^k} h_l \sigma_{\rho(l)} \, dy \right)^{p_{\infty}}
\]
\[
\times \prod_{l=1}^2 \left( \frac{\|w_l^{-1} \chi_{Q_j^k}\|_{p'_l(\cdot)}}{|Q_j^k|} \right)^{p(x)} u(x) \, dx
\]
\[
+ \sum_{(k, j) \in \mathcal{H}_{2a}} \int_{E_j^k} \prod_{l=1}^2 \left( \frac{\|w_l^{-1} \chi_{Q_j^k}\|_{p'_l(\cdot)}}{|Q_j^k|} \right)^{p(x)} u(x) \frac{1}{(e + |x|)^{tnp_{-}}} \, dx
\]
\[
= L_1 + L_2.
\]
We first estimate $L_2$. Since $\sigma_2(E_j^k) \gtrsim \sigma_2(Q_j^k) \geq 1$, by (6.13), (6.6), and Lemma 4.6,

$$L_2 \leq \sum_{(k,j) \in \mathcal{H}_2} \sup_{x \in Q_j^k} (e + |x|)^{-ntp-} \int_{Q_j^k} \prod_{l=1}^2 \| w_l^{-1} \chi_{Q_j^k} \|_{p_l}^{p(x)} |Q_j^k|^{-2p(x)} u(x) dx$$

$$\lesssim \sum_{(k,j) \in \mathcal{H}_2} \inf_{x \in Q_j^k} (e + |x|)^{-ntp-} \sigma_2(E_j^k)$$

$$\lesssim \int_{\mathbb{R}^n} (e + |x|)^{ntp-} dx$$

$$\lesssim 1.$$

In order to estimate $L_1$ we first note that for $l = 1, 2$, since $\sigma_l(Q_j^k) \geq 1$, by Lemma 4.3,

$$(6.18) \left( \frac{\sigma_l(Q_j^k)}{\| w_l^{-1} \chi_{Q_j^k} \|_{p_l}} \right)^{p_\infty} \lesssim \left( \frac{\sigma_l(Q_j^k)}{\sigma_l(Q_j^k)^{\frac{1}{p_1} \infty}} \right)^{p_\infty} = \sigma_l(Q_j^k)^{\frac{p_\infty}{p_1} \infty}.$$

Given this estimate, by (6.6) and Young’s inequality we have that

$$L_1 \lesssim \sum_{(k,j) \in \mathcal{H}_2} \int_{E_j^k} \prod_{l=1}^2 \langle h_l \rangle_{\sigma_l(p_l), Q_j^k}^{p_\infty} \sigma_1(Q_j^k)^{\frac{p_\infty}{p_1} \infty} \sigma_2(Q_j^k)^{p_\infty}$$

$$\times \prod_{l=1}^2 \left( \frac{\| w_l^{-1} \chi_{Q_j^k} \|_{p_l}}{|Q_j^k|} \right)^{p(x)} u(x) dx$$

$$\leq \sum_{(k,j) \in \mathcal{H}_2} \prod_{l=1}^4 \langle h_l \rangle_{\sigma_l(p_l), Q_j^k}^{p_\infty} \sigma_1(Q_j^k)^{\frac{p_\infty}{p_1} \infty} \sigma_2(Q_j^k)^{p_\infty}$$

$$\times \int_{Q_j^k} \prod_{l=1}^2 \| w_l^{-1} \chi_{Q_j^k} \|_{p_l}^{p(x)} |Q_j^k|^{-2p(x)} u(x) dx$$

$$\lesssim \sum_{(k,j) \in \mathcal{H}_2} \langle h_1 \rangle_{\sigma_1, Q_j^k}^{(p_1)} \sigma_1(Q_j^k) + \sum_{(k,j) \in \mathcal{H}_2} \langle h_4 \rangle_{\sigma_2, Q_j^k}^{(p_2)} \sigma_2(Q_j^k).$$

The estimate of the last term is identical to the estimate for $J_2$ above, beginning at inequality (6.12); here we use the fact that $\sigma_1(Q_j^k) \geq 1$ to get (6.11).

The estimate over $\mathcal{H}_{2b}$ is similar, but we must replace the exponent $p_\infty$ with $r(Q_j^k)$, which is defined by

$$\frac{1}{r(Q_j^k)} = \frac{1}{(p_1) - (Q_j^k)} + \frac{1}{(p_2) \infty}.$$
Then by (6.15), for \( x \in Q^k_j \),
\[
\left| \frac{1}{p(x)} - \frac{1}{r(Q^k_j)} \right| \leq \left| \frac{1}{p_1(x)} - \frac{1}{(p_1)_-(Q^k_j)} \right| + \left| \frac{1}{p_2(x)} - \frac{1}{(p_2)_\infty} \right| \lesssim \log(e + |x|).
\]
We can then argue as we did for the sum over \( \mathcal{H}_{2a} \) above to get
\[
\sum_{(k,j) \in \mathcal{H}_{2b}} \int_{E_j^k} \prod_{l=1,4} \langle h_l \sigma_{\rho(l)} \rangle_{Q^k_j}^{p(x)} u(x) \, dx
\]
\[
\lesssim \sum_{(k,j) \in \mathcal{H}_{2b}} \int_{E_j^k} \prod_{l=1,4} \left( c_0^{-1} \| w_l^{-1} \chi_{Q^k_j} \|_{p_l}' \right)^{-1} \int_{Q^k_j} h_l \sigma_{\rho(l)} \, dy \langle Q^k_j \rangle_{r(Q^k_j)}
\times \prod_{l=1}^2 \left( \frac{\| w_l^{-1} \chi_{Q^k_j} \|_{p_l}'(\cdot)}{|Q^k_j|} \right)^{p(x)} u(x) \, dx
\]
\[
+ \sum_{(k,j) \in \mathcal{H}_{2b}} \int_{E_j^k} \prod_{l=1}^2 \left( \frac{\| w_l^{-1} \chi_{Q^k_j} \|_{p_l}'(\cdot)}{|Q^k_j|} \right)^{p(x)} u(x) \, dx
\]
\[
\times (e + |x|)^{t p_\infty} \, dx = M_1 + M_2.
\]

The estimate for \( M_2 \) is identical to the estimate for \( L_2 \). To estimate \( M_1 \), we again use (6.18) for \( \sigma_2 \), replacing \( p_\infty \) with \( r(Q^k_j) \). Because \( \sigma_1(Q^k_j) < 1 \) we need to replace (6.18) with a different estimate. Since \( (p_1)'(Q^k_j) = (p_1)'(Q^k_j) \), by the estimate (6.9), replacing \( q(Q^k_j) \) with \( r(Q^k_j) \), we get
\[
\left( \frac{\sigma_1(Q^k_j)}{\| w_1^{-1} \chi_{Q^k_j} \|_{p_1}'(\cdot)} \right)^{r(Q^k_j)} \lesssim \sigma_1(Q^k_j)^{r(Q^k_j)/(p_1)_-(Q^k_j)}.
\]

We can now modify the estimate for \( L_1 \) to estimate \( M_1 \):
\[
M_1 \lesssim \sum_{(k,j) \in \mathcal{H}_{2b}} \int_{E_j^k} \prod_{l=1,4} \langle h_l \rangle_{\sigma_{\rho(l)},Q^k_j}^{r(Q^k_j)} \sigma_1(Q^k_j)^{r(Q^k_j)/(p_1)_-(Q^k_j)} \sigma_2(Q^k_j)^{r(Q^k_j)/(p_2)_\infty}
\times \prod_{l=1}^2 \left( \frac{\| w_l^{-1} \chi_{Q^k_j} \|_{p_l}'(\cdot)}{|Q^k_j|} \right)^{p(x)} u(x) \, dx
\]
\[
\leq \sum_{(k,j) \in \mathcal{H}_{2b}} \prod_{l=1,4} \langle h_l \rangle_{\sigma_{\rho(l)},Q^k_j}^{r(Q^k_j)} \sigma_1(Q^k_j)^{r(Q^k_j)/(p_1)_-(Q^k_j)} \sigma_2(Q^k_j)^{r(Q^k_j)/(p_2)_\infty}
\times \int_{Q^k_j} \prod_{l=1}^2 \| w_l^{-1} \chi_{Q^k_j} \|_{p_l}'(\cdot) |Q^k_j|^{-2p(x)} u(x) \, dx.
\]
\[
\lesssim \sum_{(k,j) \in \mathcal{H}_{2b}} \langle h_1 \rangle_{\sigma_1, Q_j^k}^{(p_1)(Q_j^k)} \sigma_1(Q_j^k) + \sum_{(k,j) \in \mathcal{H}_{2b}} \langle h_4 \rangle_{\sigma_2, Q_j^k}^{(p_2)\infty} \sigma_2(Q_j^k).
\]

The estimate for the second term in the last line is the same as the final estimate for \(J_2\); we use the same argument above to estimate \(L_1\). The estimate for the first term is the same as the estimate for \(K_1\) above, noting that since \(h_1 \geq 1\) and by Hölder’s inequality,

\[
\langle h_1 \rangle_{\sigma_1, Q_j^k}^{(p_1)-Q_j^k} \leq \langle h_1 \rangle_{\sigma_1, Q_j^k}^{(p_1)-Q_j^k} \sigma_1^{-1}(Q_j^k) \leq \langle h_1 \rangle_{\sigma_1, Q_j^k}^{(p_1)-Q_j^k}.
\]

This completes the estimate of \(M_1\) and so of \(I_2\).

**Remark 6.4.** As noted above, the argument for \(I_3\) is the same as that for \(I_2\), replacing \(h_1 \sigma_1\) with \(h_3 \sigma_2\) and \(h_4 \sigma_2\) with \(h_2 \sigma_1\).

**The estimate for \(I_4\).** The estimate for \(I_4\) parallels that for \(I_2\). In particular, we will decompose \(I_4\) into essentially the same parts as we did above. For some parts the estimate is very similar to the corresponding part \(I_2\), and so we give the key inequalities but will omit some of the details. For other parts we will need to modify the argument and we will present these in more detail.

Begin by forming the bilinear Calderón–Zygmund cubes associated with \(\mathcal{M}^{d}(h_2 \sigma_1, h_4 \sigma_2)\). We then decompose the collection of these cubes into the sets \(\mathcal{F}, \mathcal{G}, \text{ and } \mathcal{H}\), defined as above. Denote the sums over these sets by \(N_1\), \(N_2\), and \(N_3\).

**The estimate for \(N_1\).** The estimate for \(N_1\) is very similar to that for \(J_1\) above. We replace the arguments used for the \(h_1\) term and estimate the \(h_2\) term and the \(h_4\) term in the same way, using the fact that \(h_2, h_4 \leq 1\):

\[
N_1 = \sum_{(k,j) \in \mathcal{F}} \int_{E_j^k} \prod_{l=2}^{4} \langle h_l \sigma_{p(l)} \rangle_{Q_j^k}^{p(x)} u(x) \, dx
\]

\[
\leq \sum_{(k,j) \in \mathcal{F}} \int_{E_j^k} \prod_{l=1}^{2} \langle \sigma_{l} \rangle_{Q_j^k}^{p(x)} u(x) \, dx
\]

\[
= \sum_{(k,j) \in \mathcal{F}} \int_{E_j^k} \prod_{l=1}^{2} \sigma_{l} (Q_j^k)^{p(x)-q(Q_j^k)}
\]

\[
\times \sigma_1 (Q_j^k)^{q(Q_j^k)} \sigma_2 (Q_j^k)^{q(Q_j^k)} |Q_j^k|^{-2p(x)} u(x) \, dx
\]

\[
\leq \sum_{(k,j) \in \mathcal{F}} \prod_{l=1}^{2} (1 + \sigma_{l} (Q_j^k))^{p_+ (Q_j^k)-q(Q_j^k)}
\]

\[
\times \int_{E_j^k} \sigma_1 (Q_j^k)^{p(Q_j^k)} \sigma_2 (Q_j^k)^{q(Q_j^k)} |Q_j^k|^{-2p(x)} u(x) \, dx
\]
\[ \sum_{l=1}^{2} (1 + \sigma_l(P))^{p_-} \sum_{(k,j) \in \mathcal{F}} \sigma_1(Q_j^k)^{q(Q_j^k)_{(p_1) - (Q_j^k)}} \sigma_2(Q_j^k)^{q(Q_j^k)_{(p_2) - (Q_j^k)}} \]

\[ \lesssim \sum_{(k,j) \in \mathcal{F}} \sigma_1(Q_j^k) + \sum_{(k,j) \in \mathcal{F}} \sigma_2(Q_j^k) \]

\[ \lesssim \sum_{(k,j) \in \mathcal{F}} \sigma_1(E_j^k) + \sum_{(k,j) \in \mathcal{F}} \sigma_2(E_j^k) \]

\[ \leq \sigma_1(P) + \sigma_2(P) \]

\[ \lesssim 1. \]

**The estimate for \( N_2 \).** To estimate \( N_2 \) we modify the argument for \( J_2 \).

By the definition of \( \mathcal{A}_{\rho(-)} \) and by Lemma 3.6 we have

\[
\frac{1}{|Q_j^k|^2} \int_{Q_j^k} h_2 \sigma_1 \, dy \int_{Q_j^k} h_4 \sigma_2 \, dy
\lesssim \left\| w \chi_{Q_j^k} \right\|_{p(-)}^{-1} \prod_{l=1}^{2} \left\| w_l^{-1} \chi_{Q_j^k} \right\|_{p'_1(-)}^{-1} \left\| h_2 \right\|_{L_{\sigma_1}^p(Q_j^k)}
\times \left\| h_4 \right\|_{L_{\sigma_2}^p(Q_j^k)} \left\| \chi_{Q_j^k} \right\|_{L_{\rho_1}^p(Q_j^k)} \left\| \chi_{Q_j^k} \right\|_{L_{\rho_2}^p(Q_j^k)}
= \left\| w \chi_{Q_j^k} \right\|_{p(-)}^{-1} \prod_{l=1}^{2} \left\| w_l^{-1} \chi_{Q_j^k} \right\|_{p'_1(-)}^{-1} \left\| h_2 \right\|_{L_{\sigma_1}^p(Q_j^k)}
\times \left\| h_4 \right\|_{L_{\sigma_2}^p(Q_j^k)} \left\| w_1^{-1} \chi_{Q_j^k} \right\|_{p'_1(-)} \left\| w_2^{-1} \chi_{Q_j^k} \right\|_{p'_2(-)};
\]

since \( u(Q) \geq u(P_i) \geq 1 \), by Lemma 3.5,

\[ \lesssim \left\| w \chi_{Q_j^k} \right\|_{p(-)}^{-1} \]

\[ \lesssim c_0. \]

Therefore, by Lemma 3.13,

\[ N_2 = \sum_{(k,j) \in \mathcal{F}} \int_{E_j^k} \prod_{l=2,4} \left\langle h_l \sigma_{\rho(l)} \right\rangle_{Q_j^k} u(x) \, dx \]

\[ \lesssim \sum_{(k,j) \in \mathcal{F}} \int_{E_j^k} \left( \frac{c_0^{-1}}{|Q_j^k|^2} \int_{Q_j^k} h_2 \sigma_1 \, dy \int_{Q_j^k} h_4 \sigma_2 \, dy \right)^{p(x)} u(x) \, dx \]

\[ \lesssim \sum_{(k,j) \in \mathcal{F}} \int_{E_j^k} \prod_{l=2,4} \left\langle h_l \sigma_{\rho(l)} \right\rangle_{Q_j^k} u(x) \, dx + \sum_{(k,j) \in \mathcal{F}} \int_{E_j^k} \frac{u(x)}{(e + |x|)^{tnp}} \, dx. \]
By Lemma 4.6, the second term on the last line is bounded by a constant 1. We estimate the first term using (6.10):

\[
\sum_{(k,j) \in G} \int_{E_j^k} \prod_{l=2,4} \langle h_l \sigma_{\rho(l)} \rangle_{Q_j^k}^{p_\infty} u(x) \, dx
= \sum_{(k,j) \in G} \int_{E_j^k} \prod_{l=2,4} \langle h_l \rangle_{Q_j^k}^{p_\infty} \sigma_1(Q_j^k)^{p_\infty} \sigma_2(Q_j^k)^{p_\infty} |Q_j^k|^{-2p_\infty} u(x) \, dx
\approx \sum_{(k,j) \in G} \langle h_2 \rangle_{\sigma_1, Q_j^k}^{p_\infty} \sigma_1(Q_j^k)^{p_\infty} \sigma_2(Q_j^k)^{p_\infty} \langle h_4 \rangle_{\sigma_2, Q_j^k}^{p_\infty} \langle h_4 \rangle_{\sigma_2, Q_j^k}^{p_\infty} \langle h_4 \rangle_{\sigma_2, Q_j^k}^{p_\infty} ;
\]

by Young’s inequality and Lemmas 3.2, 3.13, and 4.6,

\[
\approx \sum_{(k,j) \in G} \langle h_2 \rangle_{\sigma_1, Q_j^k}^{p_\infty} \sigma_1(Q_j^k) + \sum_{(k,j) \in G} \langle h_4 \rangle_{\sigma_2, Q_j^k}^{p_\infty} \sigma_2(Q_j^k)
\approx \sum_{(k,j) \in G} \langle h_2 \rangle_{\sigma_1, Q_j^k}^{p_\infty} \sigma_1(E_j^k) + \sum_{(k,j) \in G} \langle h_4 \rangle_{\sigma_2, Q_j^k}^{p_\infty} \sigma_2(E_j^k)
\leq \int_{\mathbb{R}^n} M_{\sigma_1}^{d} h_2(x)^{(p_1)} \sigma_1(x) \, dx + \int_{\mathbb{R}^n} M_{\sigma_2}^{d} h_4(x)^{(p_2)} \sigma_2(x) \, dx
\approx \int_{\mathbb{R}^n} h_2(x)^{(p_1)} \sigma_1(x) \, dx + \int_{\mathbb{R}^n} h_4(x)^{(p_2)} \sigma_2(x) \, dx
\approx \int_{\mathbb{R}^n} h_2(x)^{p_1} \sigma_1(x) \, dx + \int_{\mathbb{R}^n} h_4(x)^{p_2} \sigma_2(x) \, dx
+ \int_{\mathbb{R}^n} \frac{\sigma_1(x)}{(e + |x|)^{tn(p_1)}} \, dx + \int_{\mathbb{R}^n} \frac{\sigma_2(x)}{(e + |x|)^{tn(p_2)}} \, dx
\approx 1.
\]

The estimate for \( N_3 \). The estimate for \( N_3 \) is broadly similar to the estimate for \( J_3 \) above, but it differs considerably in the details. We first begin by dividing the cubes in \( \mathcal{H} \) into the sets \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) as before. However, we now have to subdivide both of these sets and not just \( \mathcal{H}_2 \). Define

\[
\mathcal{H}_{1a} = \{(k, j) \in \mathcal{H}_1 : \sigma_1(Q_j^k) \leq 1, \sigma_2(Q_j^k) \leq 1\}
\]

and

\[
\mathcal{H}_{1b} = \{(k, j) \in \mathcal{H}_1 : \sigma_1(Q_j^k) > 1, \sigma_2(Q_j^k) \leq 1\}.
\]
The estimate for the sum over $\mathcal{H}_1$ is similar to the estimate over $\mathcal{H}_1$ above for $J_3$, but we use the fact that both $h_2, h_4 \leq 1$. By Lemmas 3.13 and 4.6,

$$
\sum_{(k,j) \in \mathcal{H}_{1a}} \int_{E_j^k} \prod_{l=2,4} \langle h_1 \sigma_{\rho(l)} \rangle_{Q_j^k}^{p(x)} u(x) \, dx
$$

$$
\lesssim \sum_{(k,j) \in \mathcal{H}_{1a}} \int_{E_j^k} \prod_{l=2,4} \langle h_1 \sigma_{\rho(l)} \rangle_{Q_j^k}^{p+(Q_j^k)} u(x) \, dx
$$

$$
+ \sum_{(k,j) \in \mathcal{H}_{1a}} \int_{E_j^k} \frac{u(x)}{(e + |x|)^{m(p)}} \, dx
$$

$$
\leq \sum_{(k,j) \in \mathcal{H}_{1a}} \prod_{l=2,4} \langle h_1 \rangle_{\sigma_{\rho(l)}, Q_j^k}^{p+(Q_j^k)} \langle Q_j^k \rangle^{-2p+(Q_j^k)} \prod_{l=2,4} \sigma_{\rho(l)}(Q_j^k)^{p+(Q_j^k)} u(x) \, dx + 1.
$$

Since $h_2, h_4 \leq 1$ and $\sigma_l(Q_j^k) \leq 1$, $l = 1, 2$, by (6.4), replacing $|Q_j^k|^{-2p(x)}$ with $|Q_j^k|^{-2p+(Q_j^k)}$ (which we can do by Lemma 3.11),

$$
\leq \sum_{(k,j) \in \mathcal{H}_{1a}} \prod_{l=2,4} \langle h_1 \rangle_{\sigma_{\rho(l)}, Q_j^k}^{q(Q_j^k)} \langle Q_j^k \rangle^{-2p+(Q_j^k)} \prod_{l=2,4} \sigma_{\rho(l)}(Q_j^k)^{q(Q_j^k)} u(x) \, dx + 1
$$

$$
\lesssim \sum_{(k,j) \in \mathcal{H}_{1a}} \prod_{l=2,4} \langle h_1 \rangle_{\sigma_{\rho(l)}, Q_j^k}^{q(Q_j^k)} \sigma_1(Q_j^k)^{q(Q_j^k)} \sigma_2(Q_j^k)^{q(Q_j^k)} + 1;
$$

by Young’s inequality,

$$
\lesssim \sum_{(k,j) \in \mathcal{H}_{1a}} \langle h_2 \rangle_{\sigma_1, Q_j^k}^{(p_1)_{-}(Q_j^k)} \sigma_1(Q_j^k) + \sum_{(k,j) \in \mathcal{H}_{1a}} \langle h_4 \rangle_{\sigma_2, Q_j^k}^{(p_2)_{-}(Q_j^k)} \sigma_2(Q_j^k) + 1.
$$

Both of the final terms are estimated as $K_2$ above.
To estimate the sum over $\mathcal{H}_{1b}$, we first define the exponent $s(Q_j^k)$ by

$$\frac{1}{s(Q_j^k)} = \frac{1}{(p_1)_\infty} + \frac{1}{(p_2) + (Q_j^k)}.$$ 

Then, arguing as we did for (6.14), we get that for $x \in Q_j^k$,

$$\left| \frac{1}{p(x)} - \frac{1}{s(Q_j^k)} \right| \leq \left| \frac{1}{p_1(x)} - \frac{1}{(p_1)_\infty} \right| + \left| \frac{1}{p_2(x)} - \frac{1}{(p_2) + (Q_j^k)} \right| \lesssim \frac{1}{\log(e + |x|)}.$$

Given this, by (6.16) (for $h_2$ instead of $h_1$), (6.17), and Lemma 3.13,

$$\sum_{(k, j) \in \mathcal{H}_{1b}} \int_{E_j^k} \prod_{l=2,4} \left( h_l \sigma_{\rho(l)} Q_j^k \right)^{p(x)} u(x) \, dx \leq \sum_{(k, j) \in \mathcal{H}_{1b}} \int_{E_j^k} \prod_{l=2,4} \left( c_0^{-1} \| w_{\rho(l)}^{-1} \chi_{Q_j^k} \|_{p'_{\rho(l)(\cdot)}}^{-1} \int_{Q_j^k} h_l \sigma_{\rho(l)} \, dy \right)^{s(Q_j^k)} \times \prod_{l=1}^2 \left( \frac{\| w_{l}^{-1} \chi_{Q_j^k} \|_{p'_{l}(\cdot)}}{|Q_j^k|} \right)^{p(x)} u(x) \, dx
$$

$$+ \sum_{(k, j) \in \mathcal{H}_{1b}} \int_{E_j^k} \prod_{l=1}^2 \left( \frac{\| w_{l}^{-1} \chi_{Q_j^k} \|_{p'_{l}(\cdot)}}{|Q_j^k|} \right)^{p(x)} u(x) \, dx \frac{1}{(e + |x|)^{tn_{p_-}}} \, dx = R_1 + R_2.$$

The estimate for $R_2$ is identical to the estimate for $L_2$. To estimate $R_1$, we again use (6.18) for $\sigma_1$, replacing $p_{\infty}$ with $s(Q_j^k)$. Because $\sigma_2(Q_j^k) < 1$ we use a different estimate. Since $(p_2') - (Q_j^k) = (p_2) + (Q_j^k)'$, by Lemma 3.5,

$$\left( \frac{\sigma_2(Q_j^k)}{\| w_{2}^{-1} \chi_{Q_j^k} \|_{p_2(\cdot)}} \right)^{s(Q_j^k)} \leq \left( \sigma_2(Q_j^k)^{1 - \frac{1}{(p_2) + (Q_j^k)'}} \right)^{s(Q_j^k)} = \sigma_2(Q_j^k)^{\frac{s(Q_j^k)}{(p_2) + (Q_j^k)'}}.$$

We can now argue as in the estimate of $L_1$ to get

$$R_1 \lesssim \sum_{(k, j) \in \mathcal{H}_{1b}} \int_{E_j^k} \prod_{l=2,4} \left( h_l \right)^{s(Q_j^k)} \sigma_{\rho(l), Q_j^k} \sigma_1(Q_j^k)^{\frac{s(Q_j^k)}{(p_1)_\infty}} \sigma_2(Q_j^k)^{\frac{s(Q_j^k)}{(p_2) + (Q_j^k)'}} \times \prod_{J=1}^2 \left( \frac{\| w_{J}^{-1} \chi_{Q_j^k} \|_{p_J'(\cdot)}}{|Q_j^k|} \right)^{p(x)} u(x) \, dx$$
\[
\leq \sum_{(k,j) \in \mathcal{H}_1} \prod_{l=2,4} (h_l)^{s(Q_j^k)}_{\sigma_{(l)}}, Q_j^k \sigma_1(Q_j^k)^{s(Q_j^k)}_{(p_1)\infty} \sigma_2(Q_j^k)^{s(Q_j^k)}_{(p_2)\infty} Q_j^k)
\times \int_{Q_j^k} \prod_{j=1}^2 \|w_j^{-1} x^{Q_j^k} \|_{p_{j}^{(x)}} |Q_j^k|^{-2p(x)} u(x) \, dx
\lesssim \sum_{(k,j) \in \mathcal{H}_1} (h_4)^{p_2} + (Q_j^k) \sigma_2(E_j^k) \sigma_1(Q_j^k) + \sum_{(k,j) \in \mathcal{H}_1} (h_4)^{p_2} + (Q_j^k) \sigma_2(E_j^k) \sigma_1(Q_j^k),
\]

The estimate for the first term in the last line is the same as the estimate for the $h_4$ term in $J_2$. Arguing as we did for (6.14) and (6.15), we get
\[
|(p_2) + (Q_j^k) - p_\infty| \lesssim \frac{1}{\log(e + |x|)}.
\]

Then, since $\langle h_4 \rangle_{\sigma_1, Q_j^k} \leq 1$, the estimate for the second term follows by (6.15), and by Lemmas 3.13, 3.2, and 4.6:
\[
\sum_{(k,j) \in \mathcal{H}_1} (h_4)^{p_2} + (Q_j^k) \sigma_2(E_j^k) \sigma_1(Q_j^k) \lesssim \sum_{(k,j) \in \mathcal{H}_1} \int_{E_j^k} (h_4)^{p_2} \sigma_2(x) \, dx + \int_{\mathbb{R}^n} \frac{\sigma_2(x)}{(e + |x|) \ln(p_2) - } \, dx + 1
\lesssim \sum_{(k,j) \in \mathcal{H}_1} \int_{E_j^k} (h_4)^{p_2} \sigma_2(x) \, dx + 1.
\]

Again, we estimate this last sum as in the final estimate for $J_2$.

To estimate the sum over $\mathcal{H}_2$, we argue as we did before for $J_3$, dividing it into sums over $\mathcal{H}_{2a}$ and $\mathcal{H}_{2b}$. The estimate over $\mathcal{H}_{2a}$ is identical to the estimate over this set as before, replacing $h_1$ by $h_2$. This yields terms just like $L_1$ and $L_2$ above. The estimate for the $L_2$ term is the same as is the estimate for the $L_1$ term, except that in the final line the $h_2$ term is estimated like the $h_4$ term since both $h_2, h_4 \leq 1$.

To estimate the sum over $\mathcal{H}_{2b}$, we can argue as before, getting terms like $M_1$ and $M_2$, replacing $h_1$ by $h_2$. The estimate of the $M_2$ term is again the same. To estimate the $M_1$ term we argue as before except that we replace the exponent $r(Q_j^k)$ by $p_\infty$. But then the final line of the estimate becomes
\[
\sum_{(k,j) \in \mathcal{H}_{2b}} (h_1)^{p_1} \sigma_1(Q_j^k) + \sum_{(k,j) \in \mathcal{H}_{2b}} (h_4)^{p_2} \sigma_2(Q_j^k),
\]
and both of these sums are estimated like the final estimate for $J_2$. This completes the estimate for $N_3$ and so of $I_4$. This completes the proof of Theorem 2.4.
7. Proof of Theorem 2.8

Theorem 2.8 follows almost directly from Theorem 2.4. To prove it, we will need two estimates for the Fefferman–Stein sharp maximal operator and an extrapolation theorem in the scale of weighted variable Lebesgue spaces. We first recall the definition of the sharp maximal operator. Given $f \in L^1_{\text{loc}}$, let

$$M^\# f(x) = \sup_Q \int_Q |f(y) - \langle f \rangle_Q| \, dy \chi_Q(x),$$

where the supremum is taken over all cubes $Q$. For $\delta > 0$, define $M^\#_\delta f(x) = M^\#(|f|^{\delta})(x)^{\frac{1}{\delta}}$. The first estimate relates the norm of $f$ and $M^\#$. For a proof, see Journé [20] or [8].

Proposition 7.1. Given $w \in A_\infty$, $0 < p < \infty$, and $0 < \delta < 1$,

$$\|f\|_{L^p_w} \lesssim \|M^\#_\delta f\|_{L^p_w}.$$  

The implicit constant depends on $p$, $n$, $\delta$, and $w$.

The second estimate is a pointwise inequality proved in [22].

Proposition 7.2. Given $0 < \delta < 1/2$ and a bilinear Calderón–Zygmund singular integral $T$, for all $f_1, f_2 \in L^\infty_c$,

$$M^\#_\delta (T(f_1, f_2))(x) \lesssim M(f_1, f_2)(x).$$

The implicit constant depends only on $T$, $\delta$, and $n$.

To apply these results we need to extend Proposition 7.1 to the scale of variable Lebesgue spaces. The following result was proved in [13, Theorem 2.25]. The hypotheses are somewhat technical, but they are the right generalization to prove $A_\infty$ extrapolation in this setting ([8]). The result is stated in the abstract language of extrapolation pairs; for more on this approach to Rubio de Francia extrapolation, see [9].

Proposition 7.3. Suppose for some $0 < p < \infty$ and every $w_0 \in A_\infty$,

$$\|f\|_{L^{p}_{w_0}} \lesssim \|g\|_{L^{p}_{w_0}}$$

for every pair of functions $(f, g)$ in a family $\mathcal{F}$ such that $\|f\|_{L^p(w)} < \infty$. Given $p(\cdot) \in \mathcal{P}_0$, suppose there exists $s \leq p_-$ such that $w^s \in A_{p(\cdot)/s}$ and the maximal operator is bounded on $L^{p(\cdot)/s}(w^{-s})$. Then for $(f, g) \in \mathcal{F}$ such that $\|f\|_{L^{p(\cdot)}(w)} < \infty$,

$$\|f\|_{L^{p(\cdot)}(w)} \lesssim \|g\|_{L^{p(\cdot)}(w)}.$$
Proof of Theorem 2.8: Fix $\vec{p}(\cdot)$ as in the hypotheses and $\vec{w} \in A_{\vec{p}(\cdot)}$. Since $$(p_j)_- > 1, p_- > 1/2.$$ Let $s = 1/2$. Then by Proposition 4.7 $w^s \in A_{\vec{p}(\cdot)/s}$, so $w^{-s} \in A_{(\vec{p}(\cdot)/s)'}$. Since $p(\cdot) \in LH$, so is $(p(\cdot)/s)'$. Thus, by the weighted bounds for the maximal operator on variable Lebesgue spaces (see [7]), $M$ is bounded on $L^{(p(\cdot)/s)'}(w^{-s})$. Therefore, the main hypothesis of Proposition 7.3 holds.

Fix $0 < \delta < 1/2$ and define the family of extrapolation pairs

$$\mathcal{F} = \{ (\min(|T(f_1, f_2)|, N)\chi_{B(0, N)}, M^\delta_\#(T(f_1, f_2)) ) : f_1, f_2 \in L^\infty_c, N > 1 \}.$$ 

Since 

$$\min(|T(f_1, f_2)|, N)\chi_{B(0, N)} \in L^\infty_c \subset L^p_{w_0}$$

for any $p > 0$ and $w_0 \in A_\infty$, it follows from Proposition 7.1 that (7.1) holds for every pair in $\mathcal{F}$. Similarly, we have

$$\min(|T(f_1, f_2)|, N)\chi_{B(0, N)} \in L^{p(\cdot)}(w),$$

and so by Propositions 7.3 and 7.2,

$$\| \min(|T(f_1, f_2)|, N)\chi_{B(0, N)} \|_{L^{p(\cdot)}(w)} \lesssim \| M^\delta_\#(T(f_1, f_2)) \|_{L^{p(\cdot)}(w)} \lesssim \| M(f_1, f_2) \|_{L^{p(\cdot)}(w)}.$$ 

If we take the limit as $N \to \infty$, then by Fatou’s Lemma (Lemma 3.4) and Theorem 2.4,

$$\| T(f_1, f_2) \|_{L^{p(\cdot)}(w)} \lesssim \| f_1 \|_{L^{p(\cdot)}(w_1)} \| f_2 \|_{L^{p(\cdot)}(w_2)}.$$ 

The desired conclusion now follows by a standard approximation argument since $L^\infty_c$ is dense in $L^{p(\cdot)}(w_j), j = 1, 2$ [13, Lemma 3.1].

References


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