AN EXTENSION OF SUB-FRACTIONAL BROWNIAN MOTION

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Abstract: In this paper, firstly, we introduce and study a self-similar Gaussian process with parameters $H \in (0,1)$ and $K \in (0,1]$ that is an extension of the well known sub-fractional Brownian motion introduced by Bojdecki et *al.* [**4**]. Secondly, by using a decomposition in law of this process, we prove the existence and the joint continuity of its local time.

1. Introduction

The sub-fractional Brownian motion $S^H := \{S_t^H; t \geq 0\}$ (sfBm for short) with parameter $H \in (0,2)$ was introduced by Bojdecki et *al.* [**4**]. It is a continuous centered Gaussian process, starting from zero, with covariance function

$$\mathbb{E}(S_t^H S_s^H) = t^H + s^H - \frac{1}{2}[(t+s)^H + |t-s|^H].$$

The case $H = 1$ corresponds to the standard Brownian motion (Bm for short).

Bojdecki et *al.* [**4**] have proved that the increments of sfBm satisfy

$$C_H'|t-s|^H \leq \mathbb{E}[S_t^H - S_s^H]^2 \leq C_H|t-s|^H.$$

On the other hand, Ruiz de Chávez and Tudor [**8**] have obtained for $H \in (0,1)$ the following decomposition in law of sfBm

$$(1) \qquad\qquad S_t^H \stackrel{d}{=} B_t^H + C_1(H)X_t^H,$$

where $C_1(H) = \sqrt{\frac{H}{2\Gamma(1-H)}}$ and $X_t^H = \int_0^{+\infty}(1 - e^{-\theta t})\theta^{-\frac{H+1}{2}} \, dW_\theta$, and the Bm W and the fractional Brownian motion (fBm for short) B^H are independent.

The case $H \in (1, 2)$ is given by Bardina and Bascompte [1]. They proved that
$$B_t^H \stackrel{d}{=} S_t^H + C_2(H)X_t^H,$$
where $C_2(H) = \sqrt{\frac{H(H-1)}{2\Gamma(2-H)}}$, and the Bm W and the sfBm S^H are independent.

They also proved that the process X^H is Gaussian, centered, and that its covariance function is
$$\mathbb{E}(X_t^H X_s^H) = \begin{cases} \frac{\Gamma(1-H)}{H}[t^H + s^H - (t+s)^H], & \forall\, H \in (0, 1), \\ \frac{\Gamma(2-H)}{H(H-1)}[(t+s)^H - t^H - s^H], & \forall\, H \in (1, 2). \end{cases}$$

Moreover, Mendy [7] proved that there exists a constant $C_K > 0$ such that
$$(2) \qquad\qquad \mathbb{E}[X_t^K - X_s^K]^2 \le C_K|t - s|^2.$$

The self similarity and stationarity of the increments are two main properties for which fBm enjoyed success as modeling tool in telecommunications and finance. The sfBm is an extension of Bm which preserves many properties of fBm, but not the stationarity of the increments. This property makes sfBm a possible candidate for models which involve long dependence, self similarity and non stationarity of increments. It is, thus, very natural to explore the existence of processes which keep some of the properties of sfBm, specially a decomposition in law that includes sfBm, but also enlarge our modelling tool kit. The same motivation is given by Houdré and Villa [6] in case of the bifractional Brownian motion (bfBm for short), which generalizes the fBm.

Definition 1.1. We denote by $S^{H,K} := \{S_t^{H,K}; t \ge 0\}$ a centered Gaussian process, starting from zero, with covariance function
$$S(t, s) := \mathbb{E}(S_t^{H,K} S_s^{H,K}) = (t^H + s^H)^K - \frac{1}{2}[(t+s)^{HK} + |t - s|^{HK}],$$
where $H \in (0, 1)$ and $K \in (0, 1]$.

The case $K = 1$ corresponds to sfBm with parameter $H \in (0, 1)$.

Existence of $S^{H,K}$ can be shown in the following two ways: 1) Consider the process
$$Y_t := \frac{B_t^{H,K} + B_{-t}^{H,K}}{2^{2-K}}, \quad t \ge 0,$$
where $\{B_t^{H,K}; t \in \mathbb{R}\}$ is the bfBm on the whole real line with parameters $H \in (0, 1)$ and $K \in (0, 1]$, introduced by Houdré and Villa [6]. It is easy to see that Y_t and $S^{H,K}$ have the same covariance function. Therefore

$S^{H,K}$ exists. 2) It can be shown by the following theorem which gives also a decomposition in law of the process. It will be useful in the sequel.

Theorem 1.2. *1) For any $H \in (0,1)$ and $K \in (0,1]$, the function $S(.,.)$ is symmetric and positive definite.*

2) We have the following decomposition in law

$$(3) \qquad S_t^{HK} \stackrel{d}{=} S_t^{H,K} + C_3(K) X_{t^H}^K,$$

where $C_3(K) = \sqrt{\frac{K}{\Gamma(1-K)}}$, $H \in (0,1)$, and $K \in (0,1)$, and $S^{H,K}$ and the Bm W are independent. S^{HK} is sfBm with parameter $HK \in (0,1)$.

Proof: 1) The case $K=1$ corresponds to sfBm with parameter $H \in (0,1)$.

First recall the following (easily verified) identity

$$\lambda^K = \frac{K}{\Gamma(1-K)} \int_0^{+\infty} (1 - e^{-\lambda x}) x^{-1-K} \, dx,$$

valid for $\lambda \geq 0$ and $K \in (0,1)$. Then, for any $c_1, c_2, \ldots, c_n \in \mathbb{R}$,

$$\sum_{i=1}^n \sum_{j=1}^n c_i c_j S(t_i, t_j)$$

$$= \frac{K}{\Gamma(1-K)} \int_0^{+\infty} \sum_{i=1}^n \sum_{j=1}^n c_i c_j \left(-e^{-x(t_i^H + t_j^H)} + \frac{1}{2} e^{-x(t_i+t_j)^H} + \frac{1}{2} e^{-x|t_i-t_j|^H} \right) x^{-1-K} dx$$

$$= \frac{K}{2\Gamma(1-K)} \int_0^{+\infty} \sum_{i=1}^n \sum_{j=1}^n c_i c_j e^{-x(t_i^H + t_j^H)} \left(e^{x(t_i^H + t_j^H - (t_i+t_j)^H)} - 1 \right) x^{-1-K} dx$$

$$+ \frac{K}{2\Gamma(1-K)} \int_0^{+\infty} \sum_{i=1}^n \sum_{j=1}^n c_i c_j e^{-x(t_i^H + t_j^H)} \left(e^{x(t_i^H + t_j^H - |t_i-t_j|^H)} - 1 \right) x^{-1-K} dx.$$

Since the functions $t^H + s^H - (t+s)^H$ and $t^H + s^H - |t-s|^H$ are positive definite, so are

$$\left(e^{x(t^H + s^H - (t+s)^H)} - 1 \right) \quad \text{and} \quad \left(e^{x(t^H + s^H - |t-s|^H)} - 1 \right),$$

for all $x \geq 0$. Therefore the function $S(.,.)$ is positive definite.

2) Using the fact that $S^{H,K}$ is a Gaussian process, it suffices to see that

$$\mathbb{E}[S_t^{HK} S_s^{HK}] = \left(t^{HK} + s^{HK} - (t^H + s^H)^K \right) + \mathbb{E}[S_t^{H,K} S_s^{H,K}]. \qquad \square$$

In the sequel C and C_p denote constants which will be different even when they vary from one line to the next.

2. Local time of $S^{H,K}$

We begin this section by the definition of local time. For a complete survey on local time, we refer to Geman and Horowitz [5] and the references therein.

Let $X := \{X_t; t \geq 0\}$ be a real-valued separable random process with Borel sample functions. For any Borel set $B \subset \mathbb{R}^+$, the occupation measure of X on B is defined as

$$\mu_B(A) = \lambda\{s \in B; X_s \in A\}, \quad \forall A \in \mathcal{B}(\mathbb{R}),$$

where λ is the one-dimensional Lebesgue measure on $\mathbb{R}^+$. If μ_B is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$, we say that X has a local time on B and define its local time, $L(B,.)$, to be the Radon-Nikodym derivative of μ_B. Here, x is the so-called space variable and B is the time variable. By standard monotone class arguments, one can deduce that the local time has a measurable modification that satisfies the occupation density formula: for every Borel set $B \subset \mathbb{R}^+$ and every measurable function $f \colon \mathbb{R} \to \mathbb{R}^+$,

$$\int_B f(X_t)\, dt = \int_{\mathbb{R}} f(x) L(B, x)\, dx.$$

Sometimes, we write $L(t, x)$ instead of $L([0, t], x)$.

Here is the outline of the analytic method used by Berman [2] for the calculation of the moments of local time.

For fixed sample function at fixed t, the Fourier transform on x of $L(t, x)$ is the function

$$F(u) = \int_{\mathbb{R}} e^{iux} L(t, x)\, dx.$$

Using the density of occupation formula, we get

$$F(u) = \int_0^t e^{iuX_s}\, ds.$$

Therefore, we may represent the local time as the inverse Fourier transform of this function, i.e.,

$$(4) \qquad L(t, x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left(\int_0^t e^{iu(X_s - x)}\, ds \right)\, du.$$

We end this section by the definition of the concept of local nondeterminism, (LND for short). Let J be an open interval on the t axis. Assume that $\{X_t; t \geq 0\}$ is a zero mean Gaussian process without singularities

in any interval of the length δ, for some $\delta > 0$, and without fixed zeros, i.e., there exists $\delta > 0$, such that

$$
\begin{cases}
\mathbb{E}[X_t - X_s]^2 > 0, & \text{whenever } 0 < |t - s| < \delta, \\
\mathbb{E}(X_t)^2 > 0, & \text{for } t \in J.
\end{cases}
$$

To introduce the concept of LND, Berman [3] defined the relative conditioning error,

$$
V_p = \frac{\mathrm{Var}\{X_{t_p} - X_{t_{p-1}}/X_{t_1}, \ldots, X_{t_{p-1}}\}}{\mathrm{Var}\{X_{t_p} - X_{t_{p-1}}\}},
$$

where for $p \geq 2$, $t_1 < \cdots < t_p$ are arbitrary ordered points in J.

We say that the process X is LND on J if for every $p \geq 2$,

$$
\lim_{c \to 0^+} \inf_{0 < t_p - t_1 \leq c} V_p > 0.
$$

This condition means that a small increment of the process is not almost relatively predictable on the basis of a finite number of observations from the immediate past. Berman [3] has proved, for Gaussian processes, that the LND is characterized as follows.

Proposition 2.1. *A Gaussian process X is LND if and only if for every integer $p \geq 2$, there exist two positive constants δ and C_p such that*

$$
\mathrm{Var}\left(\sum_{i=1}^{p} u_j (X_{t_j} - X_{t_{j-1}}) \right) \geq C_p \sum_{i=1}^{m} u_j^2 \, \mathrm{Var}(X_{t_j} - X_{t_{j-1}}),
$$

for all orderer points $t_1 < \cdots < t_p$ that are arbitrary points in J with $t_0 = 0$, $t_p - t_1 \leq \delta$ and $(u_1, \ldots, u_j) \in \mathbb{R}$.

Remark 2.2. Mendy [7] proved by using (1) that the sfBm is LND on $[0, 1]$ for any $H \in (0, 1)$.

The purpose of this section is to present sufficient conditions for the existence of the local time of $S^{H,K}$. Furthermore, using the LND approach, we show that the local time of $S^{H,K}$ has a jointly continuous version.

Theorem 2.3. *Assume $H \in (0, 1)$ and $K \in (0, 1)$. On each (time-)interval $[a, b] \subset [0, \infty)$, $S^{H,K}$ admits a local time which satisfies*

$$
\int_{\mathbb{R}} L^2([a, b], x) \, dx < \infty \quad a.s.
$$

For the proof of Theorem 2.3 we need the following lemma. This result on the regularity of the increments of $S^{H,K}$ will be the key for the existence and the regularity of local times.

Lemma 2.4. *Assume $H \in (0,1)$ and $K \in (0,1)$. There exists $\delta > 0$ and, for any integer $p \geq 2$, there exists a constant $0 < C_p < +\infty$, such that*

$$\text{(5)} \qquad \mathbb{E}[S_t^{H,K} - S_s^{H,K}]^p \geq C_p |t - s|^{pHK},$$

for all $s, t \geq 0$ such that $|t - s| < \delta$.

Proof: By virtue of (3) and the elementary inequality $(a+b)^2 \leq 2a^2 + 2b^2$, we have

$$\mathbb{E}[S_t^{HK} - S_s^{HK}]^2 \leq 2\mathbb{E}[S_t^{H,K} - S_s^{H,K}]^2 + 2C_3^2(K)\mathbb{E}[X_{t^H}^K - X_{s^H}^K]^2.$$

Then, (2) implies that

$$\mathbb{E}[S_t^{H,K} - S_s^{H,K}]^2 \geq C_{H,K}|t - s|^{2HK} - C_3^2(K)C_K|t^H - s^H|^2.$$

For any $H \in (0,1)$, we have $|t^H - s^H| \leq |t - s|^H$, then

$$\mathbb{E}[S_t^{H,K} - S_s^{H,K}]^2 \geq |t - s|^{2HK} \left[C_{H,K} - C_3^2(K)C_K|t - s|^{2H(1-K)} \right].$$

Since $0 < K < 1$, we can choose $\delta > 0$ small enough such that for all $t, s \geq 0$ with $|t - s| < \delta$, we have

$$\left[C_{H,K} - C_3^2(K)C_K|t - s|^{2H(1-K)} \right] > 0.$$

Indeed, it suffices to choose

$$\delta < \left(\frac{C_{H,K}}{C_3^2(K)C_K} \wedge 1 \right)^{1/2H(1-K)}.$$

Finally,

$$\mathbb{E}[S_t^{H,K} - S_s^{H,K}]^2 \geq C|t - s|^{2HK},$$

with $|t - s| < \delta$ and

$$C = \left[C_{H,K} - C_3^2(K)C_K\delta^{2H(1-K)} \right].$$

Since $S^{H,K}$ is a centered Gaussian process, then the proof of Lemma 2.4. $\qquad \square$

Proof of Theorem 2.3: It is well known by Berman [2] that, for a jointly measurable zero-mean Gaussian process $X := \{X(t); t \in [0,1]\}$ with bounded variance, the variance condition

$$\int_0^1 \int_0^1 \left(\mathbb{E}[X(t) - X(t)]^2 \right)^{-1/2} ds \, dt < \infty$$

is sufficient for the local time $L(t,u)$ of X to exist on $[0,1]$ a.s. and to be square integrable as a function of u. For any $[a,b] \subset [0,\infty)$, and for $I = [a',b'] \subset [a,b]$ such that $|b' - a'| < \delta$, according to (5), we have,

$$\int_I \int_I \left(\mathbb{E}[S^{H,K}(t) - S^{H,K}(s)]^2 \right)^{-1/2} ds\, dt < C \int_I \int_I |t - s|^{-HK} ds\, dt.$$

The last integral is finite because $0 < HK < 1$. Then $S^{H,K}$ possesses, on any interval $I \subset [a,b]$ with length $|I| < \delta$, a local time which is square integrable as function of u. Finally, since $[a,b]$ is a finite interval, we can obtain the local time on $[a,b]$ by a patch-up procedure, i.e. we partition $[a,b]$ into $\cup_{i=1}^n [a_{i-1}, a_i]$, such that $|a_i - a_{i-1}| < \delta$, and define $L([a,b],x) = \sum_{i=1}^n L([a_{i-1}, a_i], x)$, where $a_0 = a$ and $a_n = b$. $\qquad\square$

Proposition 2.5. *Assume $H \in (0,1)$ and $K \in (0,1)$. Then $S^{H,K}$ is LND on $[0,1]$.*

Proof: By virtue of (3), we have

$$[S_t^{HK} - S_s^{HK}] = [S_t^{H,K} - S_s^{H,K}] + C_3(K)[X_{t^H}^K - X_{s^H}^K].$$

Therefore, the elementary inequality $(a+b)^2 \leq 2a^2 + 2b^2$ implies that

$$\mathrm{Var}\left(\sum_{j=1}^p u_j [S_{t_j}^{H,K} - S_{t_{j-1}}^{H,K}] \right) \geq \frac{1}{2} \mathrm{Var}\left(\sum_{j=1}^p u_j [S_{t_j}^{HK} - S_{t_{j-1}}^{HK}] \right)$$
$$- C_3^2(K) \mathrm{Var}\left(\sum_{j=1}^p u_j [X_{t_j^H}^K - X_{t_{j-1}^H}^K] \right).$$

According to Remark 2.2, the sfBm S^{HK} is LND on $[0,1]$, then there exist two constants $\delta > 0$ and $0 < C_p < +\infty$ such that for any $t_0 = 0 < t_1 < t_2 < \cdots < t_p < 1$ with $t_p - t_1 < \delta$, we have

$$\mathrm{Var}\left(\sum_{j=1}^p u_j [S_{t_j}^{H,K} - S_{t_{j-1}}^{H,K}] \right) \geq C_p \sum_{j=1}^p u_j^2 \mathrm{Var}\left(S_{t_j}^{HK} - S_{t_{j-1}}^{HK} \right)$$
$$- p C_3^2(K) \sum_{j=1}^p u_j^2 \mathrm{Var}\left(X_{t_j^H}^K - X_{t_{j-1}^H}^K \right).$$

Moreover (2) and the fact that $H \in (0,1)$ imply that

$$\text{Var}\left(\sum_{j=1}^{p} u_j [S_{t_j}^{H,K} - S_{t_{j-1}}^{H,K}]\right)$$

$$\geq C_p \sum_{j=1}^{p} u_j^2 |t_j - t_{j-1}|^{2HK} - pC_3^2(K)C_K \sum_{j=1}^{p} u_j^2 |t_j - t_{j-1}|^{2H}$$

$$\geq \left[C_p - pC_3^2(K)C_K \delta^{2H(1-K)}\right] \sum_{j=1}^{p} u_j^2 |t_j - t_{j-1}|^{2HK}.$$

In addition, the elementary inequality $(a+b)^2 \geq \frac{1}{2}a^2 - b^2$ implies that

$$\mathbb{E}[S_t^{H,K} - S_s^{H,K}]^2 \leq 2\mathbb{E}[S_t^{HK} - S_s^{HK}]^2 + 2C_3^2(K)\mathbb{E}[X_{t^H}^K - X_{s^H}^K]^2$$

$$\leq C_{H,K}|t-s|^{2HK} + 2C_3^2(K)C_K|t^H - s^H|^2$$

$$\leq [C_{H,K} + 2C_3^2(K)C_K \delta^{2H(1-K)}]|t-s|^{2HK}$$

$$\leq C(H,K,\delta)|t-s|^{2HK}.$$

Therefore, it suffices now to choose

$$\tilde{\delta} < \left(\frac{C_p}{pC_3^2(K)C_K}\right)^{\frac{1}{2H(1-K)}} \wedge \delta$$

and to consider

$$C = \frac{1}{C(H,K,\delta)}[C_p - pC_3^2(K)C_K \tilde{\delta}^{2H(1-K)}].$$

This with Proposition 2.1 complete the proof of Proposition 2.5.   □

Now, we are in position to give the main result of this section.

Theorem 2.6. *Assume $H \in (0,1)$ and $K \in (0,1)$ and let $\delta > 0$ the constant appearing in Lemma 2.4. For any integer $p \geq 2$ there exists a constant $C_p > 0$ such that, for any $t \geq 0$, any $h \in (0,\delta)$, all $x,y \in \mathbb{R}$, and any $0 < \xi < \frac{1-HK}{2HK}$,*

(6) $$\mathbb{E}[L(t+h,x) - L(t,x)]^p \leq C_p \frac{h^{p(1-HK)}}{\Gamma(1+p(1-HK))},$$

(7) $$\mathbb{E}[L(t+h,y) - L(t,y) - L(t+h,x) + L(t,x)]^p$$

$$\leq C_p |y-x|^{p\xi} \frac{h^{p(1-HK(1+\xi))}}{\Gamma(1+p(1-HK(1+\xi)))}.$$

Proof: We will prove only (7), the proof of (6) is similar. It follows from (4) that for any $x, y \in \mathbb{R}$, $t, t + h \geq 0$ and for any integer $p \geq 2$,

$$\mathbb{E}[L(t + h, y) - L(t, y) - L(t + h, x) + L(t, x)]^p$$

$$= \frac{1}{(2\pi)^p} \int_{[t,t+h]^p} \int_{\mathbb{R}^p} \prod_{j=1}^{p} [e^{-iyu_j} - e^{-ixu_j}]$$

$$\times \mathbb{E}\left(e^{i \sum_{j=1}^{p} u_j S_{s_j}^{H,K}} \right) \prod_{j=1}^{p} du_j \prod_{j=1}^{p} ds_j.$$

Using the elementary inequality $|1 - e^{i\theta}| \leq 2^{1-\xi} |\theta|^\xi$ for all $0 < \xi < 1$ and any $\theta \in \mathbb{R}$, we obtain

(8) $\mathbb{E}[L(t+h, y) - L(t, y) - L(t+h, x) + L(t, x)]^p \leq (2^\xi \pi)^{-p} p! |x - y|^{p\xi}$

$$\times \int_{t<t_1<\cdots<t_p<t+h} \int_{\mathbb{R}^p} \prod_{j=1}^{p} |u_j|^\xi \mathbb{E}\left[\exp\left(i \sum_{j=1}^{p} u_j S_{t_j}^{H,K} \right) \right] \prod_{j=1}^{p} du_j \prod_{j=1}^{p} t_j,$$

where in order to apply the LND property of $S_t^{H,K}$, we replaced the integration over the domain $[t, t + h]$ by over the subset $t < t_1 < \cdots < t_p < t + h$. We deal now with the inner multiple integral over the u's. Change the variables of integration by mean of the transformation

$$u_j = v_j - v_{j+1}, j = 1, \ldots, p - 1; u_p = v_p.$$

Then, the linear combination in the exponent in (8) is transformed according to

$$\sum_{j=1}^{p} u_j S_{t_j}^{H,K} = \sum_{j=1}^{p} v_j (S_{t_j}^{H,K} - S_{t_{j-1}}^{H,K}),$$

where $t_0 = 0$. Since $S^{H,K}$ is a Gaussian process, the characteristic function in (8) has the form

(9) $$\exp\left(-\frac{1}{2} \text{Var}\left[\sum_{j=1}^{p} v_j (S_{t_j}^{H,K} - S_{t_{j-1}}^{H,K}) \right] \right).$$

Since $|x - y|^\xi \leq |x|^\xi + |y|^\xi$ for all $0 < \xi < 1$, it follows that

(10) $$\prod_{j=1}^{p} |u_j|^\xi \leq \prod_{j=1}^{p-1} (|v_j|^\xi + |v_{j+1}|^\xi) |v_p|^\xi.$$

Moreover, the last product is at most equal to a finite sum of 2^{p-1} terms of the form $\prod_{j=1}^{p} |x_j|^{\xi \varepsilon_j}$, where $\varepsilon_j = 0, 1$ or 2 and $\sum_{j=1}^{p} \varepsilon_j = p$.

Let us write for simplicity $\sigma_j^2 = \mathbb{E}\left(S_{t_j}^{H,K} - S_{t_{j-1}}^{H,K}\right)^2$. Combining the result of Proposition 2.5, (9) and (10), we get that the integral in (8) is dominated by the sum over all possible choices of $(\varepsilon_1, \ldots, \varepsilon_m) \in \{0,1,2\}^m$ of the following terms

$$\int_{t<t_1<\cdots<t_p<t+h} \int_{\mathbb{R}^p} \prod_{j=1}^{p} |v_j|^{\xi\varepsilon_j} \exp\left(-\frac{C_p}{2}\sum_{j=1}^{p} v_j^2 \sigma_j^2\right) \prod_{j=1}^{p} dt_j\, dv_j,$$

where C_p is the constant given in Proposition 2.5. The change of variable $x_j = \sigma_j v_j$ converts the last integral to

$$\int_{t<t_1<\cdots<t_p<t+h} \prod_{j=1}^{p} \sigma^{-1-\xi\varepsilon_j}\, dt_1 \ldots dt_p$$

$$\times \int_{\mathbb{R}^p} \prod_{j=1}^{p} |x_j|^{\xi\varepsilon_j} \exp\left(-\frac{C_p}{2}\sum_{j=1}^{p} x_j^2\right) \prod_{j=1}^{p} dx_j.$$

Let us denote

$$J(p,\xi) = \int_{\mathbb{R}^p} \prod_{j=1}^{p} |x_j|^{\xi\varepsilon_j} \exp\left(-\frac{C_p}{2}\sum_{j=1}^{p} x_j^2\right) \prod_{j=1}^{p} dx_j.$$

Consequently

(11) $\mathbb{E}[L(t+h,y) - L(t,y) - L(t+h,x) + L(t,x)]^p$

$$\leq J(p,\xi)C_p|y-x|^{p\xi}\int_{t<t_1<\cdots<t_p<t+h} \prod_{j=1}^{m} \sigma_j^{-1-\xi\varepsilon_j}\, dt_1 \ldots dt_p.$$

According to (5), for h sufficiently small, namely $0 < h < \inf(\delta,1)$, we have

$$\mathbb{E}[S_{t_i}^{H,K} - S_{t_j}^{H,K}]^2 \geq C|t_i - t_j|^{2HK}, \quad \forall\, t_i, t_j \in [t, t+h].$$

It follows that the integral on the right hand side of (11) is bounded, up to a constant, by

(12) $\int_{t<t_1<\cdots<t_p<t+h} \prod_{j=1}^{p} (t_j - t_{j-1})^{-HK(1+\xi\varepsilon_j)}\, dt_1 \ldots dt_p.$

Since, $(t_j - t_{j-1}) < 1$, for all $j \in \{2, \ldots, p\}$, we have

$$(t_j - t_{j-1})^{-HK(1+\xi\varepsilon_j)} \leq (t_j - t_{j-1})^{-HK(1+2\xi)}, \quad \forall\, \epsilon_j \in \{0,1,2\}.$$

Since by hypothesis $0 < \xi < \frac{1}{2HK} - \frac{1}{2}$, the integral in (12) is finite. Moreover, by an elementary calculation, for all $p \geq 1$, $h > 0$ and $b_j < 1$,

$$\int_{t<s_1<\cdots<s_p<t+h} \prod_{j=1}^{p} (s_j-s_{j-1})^{-b_j} ds_1 \ldots ds_p = h^{p-\sum_{j=1}^{p} b_j} \frac{\prod_{j=1}^{p} \Gamma(1-b_j)}{\Gamma(1+h-\sum_{j=1}^{p} b_j)}$$

where $s_0 = t$. It follows that (12) is dominated by

$$C_p \frac{h^{p(1-HK(1+\xi))}}{\Gamma(1 + p(1 - HK(1 + \xi)))}$$

where $\sum_{j=1}^{p} \varepsilon_j = p$. Consequently

$$\mathbb{E}[L(t+h,y) - L(t,y) - L(t+h,x) + L(t,x)]^p$$
$$\leq C_p |y - x|^{p\xi} \frac{h^{p(1-HK(1+\xi))}}{\Gamma(1+p(1-HK(1+\xi)))}. \quad \square$$

Remark 2.7. Using the fact that $L(0,x) = 0$ a.s. for any $x \in \mathbb{R}$ and (7) by changing $t + h$ by t and t by 0, we get

$$\mathbb{E}[L(t,x) - L(t,y)]^p \leq C_p \frac{|x - y|^{p\xi}}{\Gamma(1 + p(1 - HK(1 + \xi)))}.$$

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