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VANISHING RESULTS FOR THE COHOMOLOGY OF COMPLEX TORIC HYPERPLANE COMPLEMENTS

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Abstract: Suppose $\mathcal{R}$ is the complement of an essential arrangement of toric hyperlanes in the complex torus $(\mathbb{C}^*)^n$ and $\pi = \pi_1(\mathcal{R})$. We show that $H^*(\mathcal{R}; A)$ vanishes except in the top degree n when A is one of the following systems of local coefficients: (a) a system of nonresonant coefficients in a complex line bundle, (b) the von Neumann algebra $\mathcal{N}\pi$, or (c) the group ring $\mathbb{Z}\pi$. In case (a) the dimension of H^n is $|e(\mathcal{R})|$ where $e(\mathcal{R})$ denotes the Euler characteristic, and in case (b) the n^{th} ℓ^2 Betti number is also $|e(\mathcal{R})|$.

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1. Introduction

A *complex toric arrangement* is a family of complex subtori of a complex torus $(\mathbb{C}^*)^n$. The study of such objects is a relatively recent topic. Different versions of these arrangements, also known as toral arrangements, have been introduced and studied in works of Lehrer [**16**], [**17**], Dimca-Lehrer [**11**], Douglass [**12**], Looijenga [**18**], and Macmeikan [**20**], [**21**].

The foundation of the topic can be traced to the paper [**10**] by De Concini and Procesi. There the main objects are defined and the cohomology of the complement of a toric arrangement is studied. An explicit goal of [**10**] is to generalize the theory of hyperplane arrangements. (For an extensive account of the work of De Concini and Procesi see [**9**].)

The next step is the work of Moci, in particular his papers [**22**], [**23**], and [**24**], developing the theory with a special focus on combinatorics. In [**25**] Moci and the second author study the homotopy type of the

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complement of a special class of toric arrangements which they call *thick*. In [**3**] d'Antonio and Delucchi generalize results in [**25**] to a wider class of toric arrangements which they call *complexified* because of structural affinity with the case of hyperplane arrangements. They also prove that complements of complexified toric arrangements are minimal (see [**4**]).

In this paper we generalize to toric arrangements a well known result for affine arrangements: vanishing conditions for the cohomology of the complement $M(\mathcal{A})$ of an arrangement $\mathcal{A}$ with coefficients in a complex local system A. Necessary conditions for $H^k(M(\mathcal{A}); A) = 0$ if $k \neq n$, i.e., for the cohomology to be concentrated in top dimension, have been determined by a number of authors, including Kohno [**15**], Esnault, Schechtman and Viehweg [**13**], Davis, Januszkiewicz, and Leary [**5**], Schechtman, Terao and Varchenko [**28**], and Cohen and Orlik [**2**]. In particular, in [**28**] (see also [**2**]) it is proved that the cohomology of the complement $M(\mathcal{A})$ of an arrangement with coefficients in a complex local system is concentrated in top dimension provided certain *nonresonance* conditions for monodromies are fulfilled for a certain subset of edges (i.e., intersections of hyperplanes) that are called *denses*.

In order to generalize the above results we use techniques developed by the first author in a joint work with Januszkiewicz, Leary, and Okun, [**5**], [**6**], [**7**], [**8**]. One considers an open cover of the complement M by "small" open sets each homeomorphic to the complement of a central arrangement. In the cases of nonresonant rank one local coefficients or ℓ^2 coefficients, the E_1 page of the resulting Mayer-Vietoris spectral sequence is nonzero only along the bottom row, where it can be identified with the simplicial cochains with constant coefficients on a pair $(N(\mathcal{U}), N(\mathcal{U}_{\text{sing}}))$, which is homotopy equivalent to $(\mathbb{C}^n, \Sigma)$ where Σ is the union of all hyperplanes in the arrangement. (The simplicial complex $N(\mathcal{U})$ is the nerve of an open cover of $\mathbb{C}^n$ and $N(\mathcal{U}_{\text{sing}})$ is a subcomplex.)

It follows that the E_2 page can be nonzero only in position $(l, 0)$. One also can prove that for an affine hyperplane arrangement of rank l only the l^{th} ℓ^2-Betti number of the complement M can be nonzero and that it is equal to the rank of the reduced $(l-1)$-homology of Σ (cf. [**5**]). Similarly, with coefficients in the group ring, $\mathbb{Z}\pi$, for $\pi = \pi_1(M)$, $H^*(M; \mathbb{Z}\pi)$ is nonzero only in degree l (cf. [**6**]). We generalize all three of these vanishing results to the toric case in Theorems 5.1, 5.2 and 5.3.

In recent work [**27**], Papadima and Suciu generalize the result in [**2**] to arbitrary minimal CW-complex, i.e., a complex having as many k-cells as the k-th Betti number. It would be very interesting to decide if the complement of toric arrangement also could be minimal. In this case Theorem 5.1 would be a consequence of minimality.

Our paper begins with a review of some background about toric and affine arrangements. Then, in Section 3, we give a brief account of open covers by “small” convex sets. In Section 4 we recall basic definitions on systems of local coefficients. Finally in Section 5 we prove that the cohomology of the complement of a toric arrangement with coefficient in a local system A vanishes except in the top degree when A is a nonresonant local system, the von Neumann algebra $\mathcal{N}\pi$ or the group ring $\mathbb{Z}\pi$.

2. Affine and toric hyperplane arrangements

Affine hyperplanes arrangements. A *hyperplane arrangement* $\mathcal{A}$ is a finite collection of affine hyperplanes in $\mathbb{C}^n$. A *subspace* of $\mathcal{A}$ is a nonempty intersection of hyperplanes in $\mathcal{A}$. Denote by $L(\mathcal{A})$ the poset of subspaces, partially ordered by inclusion, and let $\overline{L}(\mathcal{A}) := L(\mathcal{A}) \cup \{\mathbb{C}^n\}$. An arrangement is *central* if $L(\mathcal{A})$ has a minimum element. Given $G \in L(\mathcal{A})$, its *rank*, $\mathrm{rk}(G)$, is the codimension of G in $\mathbb{C}^n$. The minimal elements of $L(\mathcal{A})$ form a family of parallel subspaces and they all have the same rank. The *rank* of an arrangement $\mathcal{A}$ is the rank of a minimal element in $L(\mathcal{A})$. $\mathcal{A}$ is *essential* if $\mathrm{rk}(\mathcal{A}) = n$.

The *singular set* $\Sigma(\mathcal{A})$ of $\mathcal{A}$ is the union of hyperplanes in $\mathcal{A}$. The complement of $\Sigma(\mathcal{A})$ in $\mathbb{C}^n$ is denoted $M(\mathcal{A})$.

Toric arrangements. Let $T = (\mathbb{C}^*)^n$ be a complex torus and let $\Lambda = \mathrm{Hom}(T, \mathbb{C}^*)$ denote the group of characters of T. Then $\Lambda \cong \mathbb{Z}^n$. A character is *primitive* if it is a primitive vector in Λ. Given a primitive character χ and an element $a \in \mathbb{C}^*$ put

$$H_{\chi,a} = \{t \in T \mid \chi(t) = a\}.$$

The subtorus $H_{\chi,a}$ is a *toric hyperplane*. A finite subset $X \subset \Lambda \times \mathbb{C}^*$ defines a *toric arrangement*,

$$\mathcal{T}_X := \{H_{\chi,a}\}_{(\chi,a)\in X}.$$

The projection of X onto the first factor is denoted $p(X)$ and is called the *character set* of $\mathcal{T}_X$. (Thus, $p(X) := \{\chi \mid (\chi, a) \in X\}$.) The *singular set*, Σ_X, is the union of toric hyperplanes in the arrangement. Its complement, $T - \Sigma_X$, is denoted $\mathcal{R}_X$. The *intersection poset* L_X is the set of nonempty intersections of toric hyperplanes and $\overline{L}_X = L_X \cup \{T\}$. $\overline{L}_X$ is partially ordered by inclusion. The *rank* of the arrangement is the dimension of the linear subspace of $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ spanned by $p(X)$. The arrangement is *essential* if its rank is n.

Suppose $G \in L_X$. Choose a point $x \in G$. The *tangential arrangement along* G is the arrangement $\mathcal{A}_G$ of linear hyperplanes which are tangent to the complex toric hyperplanes containing G (i.e., all hyperplanes of the form $T_x(H_{\chi,a})$ where $T_x(G) \subset T_x(H_{\chi,a})$). It is a central hyperplane arrangement of rank equal to $n - \dim G$.

Given a toric arrangement $\mathcal{T}_X$ of rank l, let K_X denote the identity component of the intersection of all kernels in $p(X)$, i.e., K_X is the identity component of

$$\bigcap_{\chi \in p(X)} \operatorname{Ker} \chi = \{t \in T \mid \chi(t) = 1,\ \forall\, \chi \in p(X)\}.$$

Put $\overline{T}_X := T/K_X$. Thus, K_X and $\overline{T}_X$ are tori of dimensions $n-l$ and l, respectively. ($K_X \cong (\mathbb{C}^*)^{n-l}$ and $\overline{T}_X \cong (\mathbb{C}^*)^l$.) Let $\overline{\Sigma}_X$ denote the image of Σ_X in $\overline{T}_X$. Since $T \to T/K_X$ is a trivial K_X-bundle, we have a homeomorphism of pairs,

$$(T, \Sigma_X) \cong K_X \times (\overline{T}_X, \overline{\Sigma}_X). \tag{1}$$

In other words, the arrangement in T is just the product of the arrangement in $\overline{T}_X$ with the torus K_X. We call $\overline{\mathcal{T}}_X$ the *essentialization* of $\mathcal{T}_X$. So, it is not restrictive to consider essential toric arrangements.

Lemma 2.1 (cf. [**5**, Proposition 2.1])**.** *Suppose $\mathcal{T}_X$ is an essential toric arrangement on T and $\Sigma = \Sigma_X$. Then $H_*(T, \Sigma)$ is free abelian and concentrated in degree n.*

Proof: We follow the "deletion-restriction" argument in [**5**, Proposition 2.1]) using induction on $\mathrm{Card}(\mathcal{T}_X)$. Choose a toric hyperplane $H \in \mathcal{T}_X$. Let $\mathcal{T}' = \mathcal{T}_X - \{H\}$ and let $\mathcal{T}''$ be the restriction of $\mathcal{T}_X$ to H, i.e., $\mathcal{T}'' = \{H \cap H' \mid H' \in \mathcal{T}_X\}$. Let Σ' and Σ'' denote the singular sets of $\mathcal{T}'$ and $\mathcal{T}''$, respectively. Consider the exact sequence of the triple (T, Σ, Σ'),

$$\to H_*(T, \Sigma') \to H_*(T, \Sigma) \to H_{*-1}(\Sigma, \Sigma') \to \tag{2}$$

There is an excision, $H_{*-1}(\Sigma, \Sigma') \cong H_{*-1}(H, \Sigma'')$. The rank of $\mathcal{T}'$ is either n or $n-1$, while the rank of $\mathcal{T}''$ is always $n-1$. The argument breaks into two cases depending on the rank of $\mathcal{T}'$.

Case 1: the rank of $\mathcal{T}'$ is n. By induction, $H_*(T, \Sigma')$ and $H_*(H, H \cap \Sigma)$ are free abelian and concentrated in degrees n and $n-1$, respectively. So, (2) becomes

$$0 \to H_n(T, \Sigma') \to H_n(T, \Sigma) \to H_{n-1}(H, H \cap \Sigma') \to 0$$

and all other terms are 0. Therefore, $H_*(T,\Sigma)$ is concentrated in degree n and $H_n(T,\Sigma)$ is free abelian.

Case 2: the rank of $\mathcal{T}'$ is $n-1$. Then the projection $T \to \overline{T}$ takes H isomorphically onto $\overline{T}$ and the arrangement $\mathcal{T}''$ on H maps isomorphically to the arrangement $\overline{\mathcal{T}}_X$ on $\overline{T}$. So, $(H, H\cap\Sigma') \cong (\overline{T},\overline{\Sigma})$. By (1), $(T,\Sigma') \cong K_X \times (H, H\cap\Sigma') \cong \mathbb{C}^* \times (H, H\cap\Sigma')$. By the Künneth Formula, $H_*(T,\Sigma') \cong H_*(\mathbb{C}^*) \otimes H_*(H, H\cap\Sigma')$. So,

$$H_{n-1}(T,\Sigma') \cong H_0(\mathbb{C}^*) \otimes H_{n-1}(H, H\cap\Sigma') \quad \text{and}$$
$$H_n(T,\Sigma') \cong H_1(\mathbb{C}^*) \otimes H_{n-1}(H, H\cap\Sigma');$$

moreover, the first isomorphism is induced by the inclusion $(H, H\cap\Sigma') \to (T,\Sigma')$. So, (2) becomes

$$0 \to H_1(\mathbb{C}^*) \otimes H_{n-1}(H, H\cap\Sigma') \to H_n(T,\Sigma) \to H_{n-1}(H, H\cap\Sigma') \\ \to H_0(\mathbb{C}^*) \otimes H_{n-1}(H, H\cap\Sigma')$$

where the last map is an isomorphism. It follows that $H_{n-1}(T,\Sigma) = 0$ and that $H_n(T,\Sigma) \cong H_1(\mathbb{C}^*) \otimes H_{n-1}(H, H\cap\Sigma')$, which, by inductive hypothesis, is free abelian. This proves the lemma. □

Complexified toric arrangements. In [**3**] d'Antonio-Delucchi consider the case of "complexified toric arrangements." This means that for each $(\chi, a) \in X$, the complex number a has modulus 1 (where $X \subset \Lambda\times\mathbb{C}^*$ is a set defining a toric arrangement $\mathcal{T}_X$). Let $T^{\text{cpt}} = (S^1)^n \subset \mathbb{C}^n$ be the compact torus. Then for each $H \in \mathcal{T}_X$, $H \cap T^{\text{cpt}}$ is a compact subtorus of T^{cpt}. The set of subtori, $\mathcal{T}_X^{\text{cpt}} := \{H \cap \mathcal{S} \mid H \in \mathcal{T}\}$, is called the associated *compact arrangement.*

Let $\Sigma^{\text{cpt}} := \Sigma_X \cap T^{\text{cpt}}$. We note that (T, Σ_X) deformation retracts onto $(T^{\text{cpt}}, \Sigma^{\text{cpt}})$. Here are a few obervations.

(i) The universal cover of T^{cpt} is $\mathbb{R}^n$ (actually the subspace $i\mathbb{R}^n \subset \mathbb{C}^n$). Let $\pi\colon \mathbb{R}^n \to T^{\text{cpt}}$ be the covering projection. Then for each $H^{\text{cpt}} \in \mathcal{T}_X^{\text{cpt}}$, each component of $\pi^{-1}(H^{\text{cpt}})$ is an affine hyperplane and the collection of these hyperplanes is a periodic affine hyperplane arrangement in $\mathbb{R}^n$.

(ii) If $\mathcal{T}_X$ is essential, then Σ^{cpt} cuts T^{cpt} into a disjoint union of convex polytopes, called *chambers* (see [**25**]) . The inverse images of these polytopes under π give a tiling of $\mathbb{R}^n$.

(iii) When $\mathcal{T}_X$ is essential, it follows from (ii) that for $n \geq 2$, Σ^{cpt} is connected and that for $n \geq 3$, $\pi_1(\Sigma^{\mathrm{cpt}}) = \pi_1(T^{\mathrm{cpt}})$.

(iv) It is easy to prove Lemma 2.1 in the case of a compact arrangement. We have an excision $H_*(T^{\mathrm{cpt}}, \Sigma^{\mathrm{cpt}}) \cong H_*(\coprod(P_i, \partial P_i))$ where each chamber P_i is an n-dimensional convex polytope. Hence, $H_*(T^{\mathrm{cpt}}, \Sigma^{\mathrm{cpt}})$ is concentrated in degree n and is free abelian. Moreover, the rank of $H_*(T^{\mathrm{cpt}}, \Sigma^{\mathrm{cpt}})$ is the number of chambers.

(v) Let $\widetilde{\Sigma}^{\mathrm{cpt}}$ denote the inverse image of Σ^{cpt} in $\mathbb{R}^n$ and let $\widetilde{\Sigma}_X$ be the induced cover of Σ_X. Suppose $\mathcal{T}_X$ is essential. Then $\widetilde{\Sigma}^{\mathrm{cpt}}$ cuts $\mathbb{R}^n$ into compact chambers. It follows that $\widetilde{\Sigma}^{\mathrm{cpt}}$ (and hence, $\widetilde{\Sigma}$) is homotopy equivalent to a wedge of $(n-1)$-spheres.

3. Certain covers and their nerves

Equip the torus $T = (\mathbb{C}^*)^n$ with an invariant metric. This lifts to a Euclidean metric on $\mathbb{C}^n$ induced from an inner product. Hence, geodesics in T lift to straight lines in $\mathbb{C}^n$ and each component of the inverse image of a subtorus of T is an affine subspace of $\mathbb{C}^n$. A *convex subset* of T means a geodesically convex subset. Thus, each component of the inverse image of a convex subset of T is a convex subset of $\mathbb{C}^n$.

The intersection of an open convex subset of T with the toric hyperplanes in $\mathcal{T}_X$ is equivalent to an affine arrangement. An open convex subset $U \subset T$ is *small* (with respect to $\mathcal{T}_X$) if this affine arrangement is central. In other words, U is *small* if the following two conditions hold (cf. [**5**], [**6**]):

(i) $\{G \in \overline{L}(\mathcal{T}_X) \mid G \cap U \neq \emptyset\}$ has a unique minimum element, $\mathrm{Min}(U)$.

(ii) A toric hyperplane $H \in \mathcal{T}_X$ has nonempty intersection with U if and only if $\mathrm{Min}(U) \subset H$.

If (i) and (ii) hold, then the arrangement in U is equivalent to the tangential arrangement along $\mathrm{Min}(U)$, which we denote by $\mathcal{A}_{\mathrm{Min}(U)}$. The intersection of two small convex open sets is also a small convex set; hence, the same is true for any finite intersection of such sets.

Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of T by small convex sets, put

$$\mathcal{U}_{\mathrm{sing}} := \{U \in \mathcal{U} \mid U \cap \Sigma_X \neq \emptyset\}.$$

Given a nonempty subset $\sigma \subset I$, put $U_\sigma := \bigcap_{i \in \sigma} U_i$. The *nerve* $N(\mathcal{U})$ of $\mathcal{U}$ is the simplicial complex defined as follows. Its vertex set is I and a finite, nonempty subset $\sigma \subset I$ spans a simplex of $N(\mathcal{U})$ if and only if U_σ is nonempty. We have the following lemma.

Lemma 3.1. *Suppose $\mathcal{T}_X$ is essential. $N(\mathcal{U})$ is homotopy equivalent to T and $N(\mathcal{U}_{\text{sing}})$ is a subcomplex homotopy equivalent to Σ_X. Moreover, $H_*(N(\mathcal{U}), N(\mathcal{U}_{\text{sing}}))$ is concentrated in degree n and $H_n(N(\mathcal{U}), N(\mathcal{U}_{\text{sing}}))$ is free abelian.*

Proof: $\mathcal{U}_{\text{sing}}$ is an open cover of a neighborhood of Σ_X which deformation retracts onto Σ_X. For each simplex σ of $N(\mathcal{U})$, U_σ is contractible (in fact, it is a small convex open set). By a well-known result (see [**14**, Corollary 4G.3 and Example 4G(4)]) $N(\mathcal{U})$ is homotopy equivalent to T and $N(\mathcal{U}_{\text{sing}})$ is homotopy equivalent to Σ_X. The last sentence of the lemma follows from Lemma 2.1. □

Definition 3.2. $\beta(\mathcal{T}_X)$ is the rank of $H_n(N(\mathcal{U}), N(\mathcal{U}_{\text{sing}}))$.

Equivalently, $\beta(\mathcal{T}_X)$ is the rank of $H_n(T, \Sigma_X)$. It is not difficult to see that, for essential arrangements, $(-1)^n\beta(\mathcal{T}_X) = e(T, \Sigma_X) = -e(\Sigma_X) = e(\mathcal{R}_X)$, where $e(\,)$ denotes Euler characteristic.

4. Local coefficients

Generic and nonresonant coefficients. Consider an affine arrangement $\mathcal{A}$. The fundamental group π of its complement, $M(\mathcal{A})$, is generated by loops a_H for $H \in \mathcal{A}$, where the loop a_H goes once around the hyperplane H in the "positive" direction. Let α_H denote the image of a_H in $H_1(M(\mathcal{A}))$. Then $H_1(M(\mathcal{A}))$ is free abelian with basis $\{\alpha_H\}_{H\in\mathcal{A}}$. So, a homomorphism $H_1(M(\mathcal{A})) \to \mathbb{C}^*$ is determined by an $\mathcal{A}$-tuple $\Lambda \in (\mathbb{C}^*)^{\mathcal{A}}$, where $\Lambda = (\lambda_H)_{H\in\mathcal{A}}$ corresponds to the homomorphism sending α_H to λ_H. Let $\psi_\Lambda \colon \pi \to \mathbb{C}^*$ be the composition of this homomorphism with the abelianization map $\pi \to H_1(M(\mathcal{A}))$. The resulting rank one local coefficient system on $M(\mathcal{A})$ is denoted A_Λ.

Returning to the case where $\mathcal{T}_X$ is a toric arrangement, for each simplex σ in $N(\widehat{\mathcal{U}})$, let $\mathcal{A}_\sigma := \mathcal{A}_{\text{Min}(U_\sigma)}$ be the corresponding central arrangement (so that $\widehat{U}_\sigma \cong M(\mathcal{A}_\sigma)$). Given $\Lambda_\sigma \in (\mathbb{C}^*)^{\mathcal{A}_\sigma}$, put

$$\lambda_\sigma := \prod_{H\in\mathcal{A}_\sigma} \lambda_H.$$

Let $A_{\Lambda_T} \in \text{Hom}(H_1(\mathcal{R}_X), \mathbb{C}^*)$ be a local coefficient system on $\mathcal{R}_X$. The localization of A_{Λ_T} on the open set $\widehat{U}_\sigma$ has the form A_{Λ_σ}, where Λ_σ is a $\mathcal{A}_\sigma$-tuple in $\mathbb{C}^*$. We call Λ_T *generic* if $\lambda_\sigma \neq 1$ for all $\sigma \in N(\mathcal{U}_{\text{sing}})$. We call Λ_T *nonresonant* if Λ_σ is nonresonant in the sense of [**2**] for all $\sigma \in N(\mathcal{U}_{\text{sing}})$ i.e., if the Betti numbers of $M(\mathcal{A}_\sigma)$ with coefficients in A_{Λ_σ} are minimal.

ℓ^2-cohomology and coeffiicients in a group von Neumann algebra. For a discrete group π, $\ell^2\pi$ denotes the Hilbert space of complex-valued, square integrable functions on π. There are unitary π-actions on $\ell^2\pi$ by either left or right multiplication; hence, $\mathbb{C}\pi$ acts either from the left or right as an algebra of operators. The *associated von Neumann algebra* $\mathcal{N}\pi$ is the commutant of $\mathbb{C}\pi$ (acting from, say, the right on $\ell^2\pi$).

Given a finite CW complex Y with fundamental group π, the space of ℓ^2-cochains on the universal cover $\widetilde{Y}$ is equal to $C^*(Y;\ell^2\pi)$, the cochains with local coefficients in $\ell^2\pi$. The image of the coboundary map need not be closed; hence, $H^*(Y;\ell^2\pi)$ need not be a Hilbert space. To remedy this, one defines the *reduced* ℓ^2-cohomology $H^*_{\text{red}}(Y;\ell^2\pi)$ to be the quotient of the space of cocycles by the closure of the space of coboundaries. The von Neumann algebra admits a trace. Using this, one can attach a "dimension," $\dim_{\mathcal{N}\pi} V$, to any closed, π-stable subspace V of a finite direct sum of copies of $\ell^2\pi$ (it is the trace of orthogonal projection onto V). The nonnegative real number $\dim_{\mathcal{N}\pi}(H^p_{\text{red}}(Y;\ell^2\pi))$ is the p^{th} *ℓ^2-Betti number* of Y.

A technical advance of Lück [**19**, Chapter 6] is the use local coefficients in $\mathcal{N}\pi$ in place of the previous version of ℓ^2-cohomology. He shows there is a well-defined dimension function on $\mathcal{N}\pi$-modules, $A \to \dim_{\mathcal{N}\pi} A$, which gives the same answer for ℓ^2-Betti numbers, i.e., for each p one has that $\dim_{\mathcal{N}\pi} H^p(Y;\mathcal{N}\pi) = \dim_{\mathcal{N}\pi} H^p_{\text{red}}(Y;\ell^2\pi)$.

Group ring coeffiicients. Let Y be a connected CW complex, $\pi = \pi_1(Y)$ and $r\colon \widetilde{Y} \to Y$ the universal cover. There is a well-defined action of π on $\widetilde{Y}$ and hence, on the cellular chain complex of $\widetilde{Y}$. Given the left π-module $\mathbb{Z}\pi$, define the cochain complex with *group ring coefficients*

$$C^*(Y;\mathbb{Z}\pi) := \operatorname{Hom}_\pi(C_*(\widetilde{Y}), \mathbb{Z}\pi).$$

Taking cohomology gives $H^*(Y;\mathbb{Z}\pi)$.

5. The Mayer-Vietoris spectral sequence

Statements of the main theorems. Suppose $\mathcal{T}_X$ is an essential toric arrangement in T and $\pi = \pi_1(\mathcal{R}_X)$.

Theorem 5.1. *Let Λ_T be a generic X-tuple with entries in k^*. Then $H^*(\mathcal{R}_X;A_{\Lambda_T})$ is concentrated in degree n and*

$$\dim_k H^n(\mathcal{R}_X;A_{\Lambda_T}) = \beta(\mathcal{T}_X).$$

Theorem 5.2 (cf. [**7**]). *The ℓ^2-Betti numbers of $\mathcal{R}_X$ are 0 except in degree n and $\ell^2 b_n(\mathcal{R}_X) = \beta(\mathcal{T}_X)$.*

Theorem 5.3 (cf. [**6**], [**8**]). *$H^*(\mathcal{R}_X; \mathbb{Z}\pi)$ vanishes except in degree n and $H^n(\mathcal{R}_X; \mathbb{Z}\pi)$ is free abelian.*

Remark 5.4. Suppose W is a Euclidean reflection group acting on $\mathbb{R}^n$ and that $\mathbb{Z}^n \subset W$ is the subgroup of translations. The quotient $W' := W/\mathbb{Z}^n$ is a finite Coxeter group. The reflection group W acts on the complexification $\mathbb{C}^n$ and W' acts on the torus $T = \mathbb{C}^n/\mathbb{Z}^n$. The image of the affine reflection arrangement in $\mathbb{C}^n$ gives a toric arrangement $\mathcal{T}_X$ in T. The fundamental group of $\mathcal{R}_X$ is the Artin group A associated to W and $\mathcal{R}_X$ is the Salvetti complex associated to A. The quotient of the compact torus by W' can be identified with the fundamental simplex Δ of W on $\mathbb{R}^n$. (If W is irreducible, then Δ is a simplex.) It follows that $\beta(\mathcal{T}_x)$ is the order of W' (i.e., the index of $\mathbb{Z}^n$ in W). So, in this case Theorem 5.2 is a special case of the main result of [**7**] and Theorem 5.3 is a special case of a result of [**8**, Theorem 4.1].

Lemma 5.5. *Suppose $\mathcal{A}$ is a finite, central arrangement of affine hyperplanes. Let $\pi' = \pi_1(M(\mathcal{A}))$. Then*

(i) (Cf. [**28**], [**2**], [**5**].) *For any generic system of local coefficients A, $H^*(M(\mathcal{A}); A)$ vanishes in all degrees.*

(ii) (Cf. [**5**].) *$H^*(M(\mathcal{A}); \mathcal{N}\pi')$ vanishes in all degrees. Hence, all ℓ^2-Betti numbers are 0.*

(iii) (Cf. [**6**].) *If the rank of $\mathcal{A}$ is l, then $H^*(M(\mathcal{A}); \mathbb{Z}\pi')$ vanishes except in the top degree, l.*

Proofs using the Mayer-Vietoris spectral sequence. The proofs of these three theorems closely follow the argument in [**7**], [**5**] and particularly, in [**6**]. For $\pi = \pi_1(\mathcal{R}_X)$, let A denote one of the left π-modules in Section 4.

Let $\mathcal{U} = \{U_i\}$ be an open cover of T by small convex sets. We may suppose that $\mathcal{U}$ is finite and that it is closed under taking intersections. For each $G \in \overline{L}_X$, put

$$\mathcal{U}_G := \{U \in \mathcal{U} \mid \mathrm{Min}(U) \leq G\},$$

$$\mathcal{U}_G^{\mathrm{sing}} := \{U \in \mathcal{U} \mid \mathrm{Min}(U) < G\} = \{U \in \mathcal{U}_G \mid U \cap \Sigma_X \cap G \neq \emptyset\}.$$

The open cover $\mathcal{U}$ restricts to an open cover $\widehat{\mathcal{U}} = \{U - \Sigma_X\}_{U \in \mathcal{U}}$ of $\mathcal{R}_X$. Any element $\widehat{U} = U - \Sigma_X$ of the cover is homotopy equivalent to the complement of a central arrangement $M(\mathcal{A}_{\mathrm{Min}(U)})$.

Suppose $N(\mathcal{U})$ is the nerve of $\mathcal{U}$ and $N(\mathcal{U}_G)$ is the subcomplex defined by $\mathcal{U}_G$. Since $N(\mathcal{U}_G)$ and $N(\mathcal{U}_G^{\text{sing}})$ are nerves of covers of G and $\Sigma_X \cap G$, respectively, by contractible open subsets, we have that for each $G \in \overline{L}(\mathcal{A})$,

$$H^*(N(\mathcal{U}_G), N(\mathcal{U}_G^{\text{sing}})) = H^*(G, \Sigma(\mathcal{T}_X \cap G)). \tag{3}$$

For each k-simplex $\sigma = \{i_0, \dots, i_k\}$ in $N(\mathcal{U})$, let

$$U_\sigma := U_{i_0} \cap \cdots \cap U_{i_k}$$

denote the corresponding intersection.

Let $r\colon \widetilde{\mathcal{R}}_X \to \mathcal{R}_X$ be the universal cover. The induced open cover $\{r^{-1}(\widehat{U})\}$ of $\widetilde{\mathcal{R}}_X$ has the same nerve $N(\widehat{\mathcal{U}})$ $(= N(\mathcal{U}))$. We have the Mayer-Vietoris double complex,

$$C_{i,j} := \bigoplus_{\sigma \in N^{(i)}} C_j(r^{-1}(\widehat{U}_\sigma)),$$

where $N^{(i)}$ denotes the set of i-simplices in $N(\mathcal{U})$ (cf. [**1**, Chapter VII].) We get a corresponding double cochain complex,

$$E_0^{i,j} := \operatorname{Hom}_\pi(C_{i,j}, A), \tag{4}$$

where $\pi = \pi_1(\mathcal{R}_X)$. The filtration on the double complex gives a spectral sequence converging to the associated graded module for cohomology:

$$\operatorname{Gr} H^m(\mathcal{R}_X; A) = E_\infty := \bigoplus_{i+j=m} E_\infty^{i,j}.$$

By first using the horizontal differential, there is a spectral sequence with E_1 page

$$E_1^{i,j} = C^i(N(\mathcal{U}); \mathcal{H}^j(A))$$

where $\mathcal{H}^j(A)$ is the coefficient system on $N(\mathcal{U})$ defined by

$$\sigma \mapsto H^j(\widehat{\mathcal{U}}_\sigma; A),$$

where $\widehat{\mathcal{U}}_\sigma \cong M(\mathcal{A}_{\mathrm{Min}(U_\sigma)})$. For $A = A_{\Lambda_T}$ or $A = \mathcal{N}\pi$ these coefficients are 0 for $G \neq T$. For $A = \mathbb{Z}\pi$, they are 0 for $j \neq \dim(G)$. Hence, in all cases, for any coface σ' of σ, if $G' := \mathrm{Min}(U_{\sigma'}) < G$, the coefficient homomorphism $H^j(M(\mathcal{A}_G); A) \to H^j(M(\mathcal{A}_{G'}); A)$ is the zero map.

Moreover, the E_1 page of the spectral sequence decomposes as a direct sum (cf. [**8**, Lemma 2.2]). In fact, for a fixed j, by using Lemma 5.5, we see that the $E_1^{i,j}$ term decomposes as

$$E_1^{i,j} = \bigoplus_{G \in \overline{L}_X^{n-j}} C^i(N(\mathcal{U}_G), N(\mathcal{U}_G^{\mathrm{sing}}); H^j(M(\mathcal{A}_G); A)),$$

where we have constant coefficients in each summand. Hence, at E_2 we have

$$\begin{aligned} E_2^{i,j} &= \bigoplus_{G \in \overline{L}_X^{n-j}} H^i(N(\mathcal{U}_G), N(\mathcal{U}_G^{\mathrm{sing}}); H^j(M(\mathcal{A}_G); A)) \\ &= \bigoplus_{G \in \overline{L}_X^{n-j}} H^i(G, \Sigma_X \cap G; H^j(M(\mathcal{A}_G); A)), \end{aligned} \tag{5}$$

where the second equation follows from (3).

When $A = A_{\Lambda_T}$ or $A = \mathcal{N}\pi$, all summands vanish for $G \neq T$ and $j \neq 0$. So, we are left with $E_2^{n,0} = H^n(T, \Sigma_X; A)$, which is isomorphic to the tensor product free abelian group of rank $\beta(\mathcal{T}_X)$ with A. It follows that $H^*(\mathcal{R}_X; A)$ is concentrated in degree n and that $\dim_{\mathbb{C}} H^n(\mathcal{R}_X; A_{\Lambda_T}) = \beta(\mathcal{T}_X) = \dim_{\mathcal{N}\pi} H^n(\mathcal{R}_X; \mathcal{N}\pi)$. This proves Theorems 5.1 and 5.2.

Consider formula (5) for $A = \mathbb{Z}\pi$. By Lemma 2.1, $H^i(G, \Sigma_X \cap G)$ is concentrated in degree $\dim G = n - j$. Hence, $E_2^{i,j}$ is nonzero (and free abelian) only for $i + j = n$. It follows that the spectral sequence degenerates at E_2, i.e., $E_2 = E_\infty$. This proves Theorem 5.3.

Remark 5.6. Let us remark that the statement of Theorem 5.1 holds even if the local system Λ_T is nonresonant or if it verifies the Schechtman, Terao and Varchenko nonresonance conditions in all small open convex sets , i.e. Λ_σ verifies the nonresonance conditions in [**28**] for all $\sigma \in N(\mathcal{U}_{\mathrm{sing}})$. Indeed under these conditions Lemma 5.5 holds.

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