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ON THE JACOBSON RADICAL AND UNIT GROUPS OF GROUP ALGEBRAS

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Abstract

In this paper, we study the situation as to when the unit group $U(KG)$ of a group algebra KG equals $K^*G(1+J(KG))$, where K is a field of characteristic $p>0$ and G is a finite group.

1. Introduction

Let R be any associative ring with identity $1 \neq 0$. Then R may be treated as a Lie ring under the Lie multiplication $[x,y] = xy - yx$, x, $y \in R$. The Lie ring thus obtained is denoted by $L(R)$ and is called the associated Lie ring of R. The lower central chain $\{\gamma_n(L(R)) \mid n = 1,2,\dots\}$ and the derived chain $\{\delta^n(L(R)) \mid n = 0,1,2,\dots\}$ of $L(R)$ are defined inductively as follows:

$$\gamma_1(L(R)) = \delta^0(L(R)) = L(R),$$
$$\gamma_{n+1}(L(R)) = [\gamma_n(L(R)), L(R)],$$
$$\delta^n(L(R)) = [\delta^{n-1}(L(R)), \delta^{n-1}(L(R))].$$

The Lie ring $L(R)$ is solvable of length n if $\delta^n(L(R)) = (0)$ but $\delta^{n-1}(L(R)) \neq (0)$. Let $J(R)$ denote the Jacobson radical of R. Then $1 + J(R)$ is a normal subgroup of the unit group $U(R)$ and we have the exact sequence of groups

$$1 \to 1 + J(R) \to U(R) \to U(R/J(R)) \to 1.$$

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Thus $U(R)/(1 + J(R)) \cong U(R/J(R))$. If further 2 and 3 are invertible in R and the associated Lie ring $L(R)$ is solvable, then $\gamma_2(L(R))R = \delta^1(L(R))R$ is a nil ideal of R by Sharma and Srivastava [**7**, Theorem 2.4]. Since nil ideals are always contained in the Jacobson radical, we have, in this situation, $\gamma_2(L(R))R \subseteq J(R)$ and thus $R/J(R)$ is commutative. Thus the commutator subgroup $U(R)'$ of $U(R)$ is contained in $1+J(R)$. If $J(R)$ is nilpotent as an ideal, then $1+J(R)$ is nilpotent as a group and so $U(R)$ is solvable. In particular, in the above situation, if $(J(R))^2 = 0$, then $U(R)$ is metabelian.

We wish to study, in this paper, some connections in the above direction when $R = KG$ is the group algebra of the group G over the field K, where $\operatorname{Char} K = p > 0$ and G is finite. Throughout the paper, Z_p denotes the field with p elements.

2. Preliminaries

Let KG be the group algebra of the group G over the field K. We denote by $\Delta(G)$, the augmentation ideal of KG. Clearly $1 + J(KG)$ defines a normal subgroup of the unit group $U(KG)$. Also there are the trivial units of the form kg, $0 \neq k \in K$, $g \in G$, in $U(KG)$. Our aim, in this paper, is to investigate situations where $U(KG) = K^*G(1+J(KG))$, $K^* = K\backslash\{0\}$. Obviously $U(KG)$ can not be smaller than this as the right hand side is always contained in $U(KG)$.

Almost in all the known cases the Jacobson radical $J(KG)$ of a group algebra KG is a nil ideal; (see Passman [**5**, Chap. 8]), and at least, for sure, this is the case for the class of solvable, linear and locally finite groups. Suppose $\operatorname{Char} K = p$, $p > 0$ and $J(KG)$ is nil. Then for any $\alpha \in J(KG)$, $\alpha^{p^n} = 0$ for some $n \geq 0$ and thus $(1+\alpha)^{p^n} = 1 + \alpha^{p^n} = 1$. This shows that $1 + J(KG)$ is a normal p-subgroup of $U(KG)$ if $J(KG)$ is a nil ideal.

We make the following observations.

Lemma 2.1. *Let K be a field with $\operatorname{Char} K = p > 0$ and let G be a group. Then $G \cap \{1 + J(KG)\}$ is a normal p-subgroup of G. Further if G is locally finite, then $O_p(G) = G \cap \{1 + J(KG)\}$.*

Proof: Clearly $G \cap \{1 + J(KG)\}$ is a normal subgroup of G. Let $1 \neq x \in G \cap \{1+J(KG)\}$. Then $x - 1 \in J(KG)$ and $\Delta(\langle x\rangle) = (x-1)K\langle x\rangle \subseteq J(K\langle x\rangle)$. Thus $J(K\langle x\rangle) \neq 0$ and so $\langle x\rangle$ is finite. Also $J(K\langle x\rangle) \supseteq \Delta(\langle x\rangle)$ is nilpotent, since $K\langle x\rangle$ is Artinian. Hence $\langle x\rangle$ is a finite p-group and $G \cap \{1 + J(KG)\}$ is a normal p-subgroup.

If G is locally finite, then $O_p(G)$ is a locally finite normal p-subgroup and so $\Delta(O_p(G)) = J(KO_p(G)) \subseteq J(KG)$. Thus $O_p(G) \subseteq G \cap \{1 + J(KG)\}$ and by the first part, we get $G \cap \{1 + J(KG)\} = O_p(G)$, as desired. ∎

This result easily yields

Corollary 2.2. *If G is locally finite and* $\mathrm{Char}\, K = p > 0$*, then $\Delta(N)KG \subseteq J(KG)$ for every normal p-subgroup N of G and equality holds if N is a normal Sylow p-subgroup of G.*

It may be noted that $\Delta(G) = J(KG)$ for any locally finite p-group G if $\mathrm{Char}\, K = p > 0$ (Passman [**5**, Chap. 8]).

3. Main results

Now we start our study of the problem: When is $U(KG) = K^*G(1 + J(KG))$?

Proposition 3.1. *Let K be a field with* $\mathrm{Char}\, K = p > 0$ *and let G be a locally finite group having a normal Sylow p-subgroup P. Then $U(KG) = K^*G(1 + J(KG))$ if and only if one of the following holds:*

(i) *$G = P$;*

(ii) *$K = Z_2$ and $G/P \cong C_3$;*

(iii) *$K = Z_3$ and $G/P \cong C_2$.*

Proof: First suppose that $U(KG) = K^*G(1 + J(KG))$. By Corollary 2.2, $J(KG) = \Delta(P)KG$ and $KG/J(KG) \cong KG/P$. Further $U(KG/J(KG)) \cong U(KG)/(1+J(KG)) = K^*G(1+J(KG))/(1+J(KG))$. So $U(KG/J(KG)) \cong K^*G/(G \cap \{1+J(KG)\})$. Also $U(KG/J(KG)) \cong U(KG/P)$. Since by Lemma 2.1, $G \cap \{1 + J(KG)\} = O_p(G) = P$, we see that $U(KG/P) = K^* \cdot G/P$ using the natural epimorphism $U(KG) \to U(KG/P)$. Thus the group algebra KG/P has only trivial units. So by Passman [**5**, Lemma 13.1.1], either G/P is trivial, that is, $G = P$ or $K = Z_2$ and $G/P \cong C_3$ since G/P is a p'-group or $K = Z_3$ and $G/P \cong C_2$.

Conversely if $G = P$, then $J(KG) = \Delta(G)$ and we are through as $U(KG) = K^*(1 + J(KG))$. In the other two cases, the units of KG/P are trivial, $J(KG) = \Delta(P)KG$ and $G \cap \{1 + J(KG)\} = P$. Hence clearly $U(KG) = K^*G(1 + J(KG))$.

In fact, $1 \neq G/P = G/(G \cap \{1+J(KG)\}) \cong G(1+J(KG))/(1+J(KG))$ and this is a subgroup of $U(KG)/(1+J(KG)) \cong U(KG/J(KG)) = U(KG/\Delta(P)KG) \cong U(KG/P)$. But $U(Z_2C_3) = C_3$ and $U(Z_3C_2) = \pm C_2$, hence the result. ■

Now we turn to finite groups. If $\operatorname{Char} K = p > 0$ and G has no p-elements, then $J(KG) = 0$, so our problem $U(KG) = K^*G(1 + J(KG))$ reduces to $U(KG) = K^*G$. This is the case of trivial units. So we assume that G is finite, it has p-elements and hence $J(KG) \neq 0$. Also if G is a finite p-group or G has a normal Sylow p-subgroup, then Proposition 3.1 above gives the answer.

Theorem 3.2. *If* $\operatorname{Char} K = p > 0$ *and G is a finite solvable group having no normal Sylow p-subgroup, then* $U(KG) = K^*G(1 + J(KG))$ *if and only if* $K = Z_2$ *and* $G/O_2(G) \cong S_3$.

Proof: Suppose $U(KG) = K^*G(1+J(KG))$. Then $U(KG)$ is solvable. Further $G/O_p(G)$ is not abelian, otherwise Sylow p-subgroup will be normal. By Passman's Theorem (see Karpilovsky [**4**, Theorem 3.8.9] or Bateman [**2**, Theorem 5]), $K = Z_2$ or Z_3. But $K = Z_3$ case gives that $G/O_3(G)$ is a 2-group, so Sylow 3-subgroup is normal. Hence we are left with only one case when $K = Z_2$ and $G/O_2(G) = A\langle x \rangle$, where A is an elementary abelian 3-group and x is an element of order 2 such that $x^{-1}ax = a^{-1}$ for all $a \in A$. We wish to show that $A = C_3$. Now

$$U(KG/J(KG)) \cong \frac{U(KG)}{1 + J(KG)} = \frac{K^*G(1 + J(KG))}{1 + J(KG)}$$

$$\cong \frac{K^*G}{K^*G \cap (1 + J(KG))} = \frac{G}{G \cap (1 + J(KG))}$$

$$= \frac{G}{O_2(G)} = A\langle x \rangle.$$

Here $K = Z_2$, so if $KG/J(KG) \cong \prod_{i=1}^{r} M_{n_i}(D_i)$, by Bateman [**2**, Theorem 5], $U(KG/J(KG)) \cong \prod_{i=0}^{s} K_i^* \times \prod_{j=1}^{t} GL_2(Z_2)$, where K_i are finite fields of characteristic 2 and second term is a direct product of t-copies of $GL_2(Z_2) \cong S_3$. Also $|U(KG/J(KG))| = |G/O_2(G)| = |A|\,|\langle x \rangle| = 3^m \cdot 2$ where $A = C_3 \times C_3 \times \cdots \times C_3$ (m-copies). Thus clearly $t = 1$. Also $|K_i| = 2^{n_i}$ for some n_i, so $|K_i^*| = 2^{n_i} - 1$ for $i = 0, 1, 2, \ldots, s$. Thus $n_i = 2$ for every i. We show that $s = 0$ and $U(KG/J(KG)) \cong G/O_2(G) \cong GL_2(Z_2) \cong S_3$.

Suppose $|A| = 3^m$ and $m > 1$. Then there exist a, $b \in A$ such that $\langle a\rangle \times \langle b\rangle \subseteq A$, $a^3 = b^3 = 1$, $x^{-1}ax = a^{-1}$, $x^{-1}bx = b^{-1}$. We have $A\langle x\rangle = G/O_2(G) \cong \prod_{i=0}^{s} K_i^* \times GL_2(Z_2)$ and denote by ϕ the isomorphism. Then $\phi(a) = (\prod_{i=0}^{s} k_i, g_1), \phi(b) = (\prod_{i=0}^{s} k_i', g_2)$, a, b non-central implies $g_1 \neq 1$, $g_2 \neq 1$. Also $a^3 = b^3 = 1$ gives $g_1^3 = g_2^3 = 1$. In $GL_2(Z_2) \cong S_3$, either $g_1 = g_2$ or $g_2 = g_1^{-1} = g_1^2$. If $g_1 = g_2$, then $\phi(a^2b)$ is central and so a^2b is central. But $x^{-1}a^2bx = (a^2b)^{-1}$, so $(a^2b)^{-1} = a^2b$ and we get $a = b$. If $g_2 = g_1^{-1}$, then $\phi(ab)$ is central, so ab is central and $x^{-1}abx = (ab)^{-1} = ab$. Thus $a = b^{-1}$. In both cases we get a contradiction, since $\langle a\rangle \cap \langle b\rangle = 1$. Thus $A = \langle a\rangle = C_3$ and $G/O_2(G) = GL_2(Z_2) \cong S_3$, as desired.

Conversely, let $K = Z_2$ and $G/O_2(G) \cong S_3$. By [**6**, 6.2, p. 215]

$$\left|\frac{U(Z_2G)}{1 + \Delta(O_2(G))Z_2G}\right| = |U(Z_2G/O_2(G))| = |U(Z_2S_3)| = 12.$$

Also
$$\frac{U(Z_2G)}{1 + J(Z_2G)} \cong \frac{U(Z_2G)/\{1 + \Delta(O_2(G))Z_2G\}}{\{1 + J(Z_2G)\}/\{1 + \Delta(O_2(G))Z_2G\}}$$
and so
$$\left|\frac{U(Z_2G)}{1 + J(Z_2G)}\right| = \frac{12}{|\{1 + J(Z_2G)\}/\{1 + \Delta(O_2(G))Z_2G\}|}.$$

Since the Sylow 2-subgroups are not normal, $G/O_2(G)$ contains 2-elements and $J(Z_2G) \supset \Delta(O_2(G))Z_2G$. Further

$$\frac{U(Z_2G)}{1 + J(Z_2G)} \cong U\left(\frac{Z_2G}{J(Z_2G)}\right) = GL_2(Z_2) \times \prod_{i=0}^{s} K_i^*, \quad K_i = 2^{n_i}, \quad K^* = K\backslash\{0\}$$

since $U(Z_2G)$ is solvable and $U(Z_2G/J(Z_2G))$ is non-abelian, otherwise $G' \subseteq G \cap \{1 + J(Z_2G)\} = O_2(G)$ implies that a Sylow 2-subgroup is normal. All this forces $|(1 + J(Z_2G))/\{1 + \Delta(O_2(G))Z_2G\}| = 2$ and $\frac{U(Z_2G)}{1+J(Z_2G)} \cong GL_2(Z_2) \cong S_3 \cong G/O_2(G) = \frac{G}{G\cap(1+J(Z_2G))}$. Thus $U(Z_2G) = G(1 + J(Z_2G))$, as desired. ■

In general if G is a finite group and K is a field with $\operatorname{Char} K = p$ such that $U(KG) = K^*G(1 + J(KG))$, then $U(KG)^n \subseteq \zeta(U(KG))$, the center of $U(KG)$, for some fixed n. This can be seen as follows. Since $J(KG)$ is nilpotent, we have $J(KG)^{p^l} = 0$ for some fixed l. Now let $u \in U(KG)$, then $u = kg(1 + \alpha)$ for some $k \in K^*$, $g \in G$, $\alpha \in J(KG)$.

It is easy to see that for all m, we have

$$u^m = k^m g^m (1+\alpha^{g^{m-1}})(1+\alpha^{g^{m-2}})\dots(1+\alpha^g)(1+\alpha).$$

Thus if $n_0 = |G|$, then $u^{n_0} = k^{n_0}(1+\beta)$, for some $\beta \in J(KG)$. Furthermore $u^{n_0 p^l} = k^{n_0 p^l}$ and thus if $n = n_0 p^l$, then u^n is central. Thus $U(KG)^n \subseteq \zeta(U(KG))$ and we can use Coelho [**3**, Lemma 1.1].

Let $A = \{g \in G \mid g \text{ is a } p'\text{-element}\}$. If A consists of central elements alone, then A is a normal subgroup of G and $G = AP$ for any Sylow p-subgroup P of G. Clearly then $P \lhd G$ and Proposition 3.1 handles the situation $U(KG) = K^*G(1 + J(KG))$. We wish to tackle, now, the case when G has a non-central p'-element. By Coelho [**3**, Lemma 1.1] and the above discussion we must have that K is a finite field.

Lemma 3.3. *Let G be a finite group and let $\operatorname{Char} K = p > 0$ such that $U(KG) = K^*G(1 + J(KG))$. Then $U(K\bar{G}) = K^*\bar{G}(1 + J(K\bar{G}))$, where $\bar{G} = G/O_p(G)$.*

Proof: Since

$$\Delta(O_p(G))KG \subseteq J(KG),$$
$$U(KG/J(KG)) \cong U(K\bar{G}/J(K\bar{G})).$$

Therefore,

$$\frac{U(KG)}{1+J(KG)} = \frac{K^*G(1+J(KG))}{1+J(KG)} \cong \frac{K^*G}{G\cap(1+J(KG))}$$
$$= \frac{K^*G}{O_p(G)} \cong \frac{U(K\bar{G})}{1+J(K\bar{G})}.$$

This clearly shows that $U(K\bar{G}) = K^*\bar{G}(1 + J(K\bar{G}))$. ■

When p'-elements are not central, A need not form a subgroup. Even when A forms a subgroup, Sylow p-subgroup need not be normal. However, we have the following.

Theorem 3.4. *Let G be a finite group such that A forms a non-central subgroup and $\operatorname{Char} K = P > 0$. If $U(KG) = K^*G(1 + J(KG))$ then G is solvable and K is finite.*

Proof: Since $U(KG) = K^*G(1 + J(KG))$ and G is finite, K is a finite field. Hence in the decomposition $KG/J(KG) \cong \prod_{i=1}^{r} M_{n_i}(D_i)$, each $D_i = K_i$ is a field, being finite division rings. Thus $U(KG/J(KG)) \cong \prod_{i=1}^{r} GL_{n_i}(K_i)$, K_i finite, $\mathrm{Char}\, K_i = p$. If $\bar{G} = G/O_p(G)$ is solvable, then clearly G is solvable. In view of Lemma 3.3, we can assume that $O_p(G) = 1$.

Now

$$U\left(\frac{KG}{J(KG)}\right) \cong \frac{U(KG)}{1 + J(KG)} \cong \frac{K^*G}{G \cap \{1 + J(KG)\}} = K^*G.$$

Let A_i denote the set of p'-elements of $GL_{n_i}(K_i)$ for all $i = 1, 2, \ldots, r$. Clearly, A_i is a subgroup of $GL_{n_i}(K_i)$ for all $i = 1, 2, \ldots, r$. Also A_i is non-central in $GL_{n_i}(K_i)$, if $n_i > 1$. Therefore, $n_i = 1$ or 2 and $K_i \cong K$ if $n_i = 2$, where $K = Z_2$ or Z_3 (see Artin [**1**, p. 165]). Since both $GL_2(Z_2)$ and $GL_2(Z_3)$ are solvable, $U(KG/J(KG))$ is solvable and so $G \leq U(KG)$ is solvable, as desired. ■

We now discuss finite p-solvable groups:

Let K be a field with $\mathrm{Char}\, K = p > 0$ and G a finite group such that $U(KG)$ is p-solvable. Then $U(Z_pG)$ is p-solvable and hence $U(Z_pG/J(Z_pG))$ is p-solvable. But $U(Z_pG/J(Z_pG)) = \prod_{i=1}^{r} GL_{n_i}(D_i)$, so each D_i is a field, being a finite division ring. Thus for each i, $GL_{n_i}(D_i) = GL_{n_i}(GF(q_i))$, $q_i = p^{n_i}$ and p-solvabiblity forces each $n_i = 1$ or $n_i = 2$, $q_i = p$, $p = 2$ or 3. But $GL_2(Z_2)$ and $GL_2(Z_3)$ are solvable. Thus $U(Z_pG/J(Z_pG))$ is solvable and therefore, $U(Z_pG)$ is solvable. This gives that G is solvable. Thus $U(KG)$ is p-solvable implies G is solvable. In particular, we have

Theorem 3.5. *If* $\mathrm{Char}\, K = p > 0$ *and* G *is a* p*-solvable group such that* $U(KG) = K^*G(1 + J(KG))$*, then* G *is solvable.*

Proof: Clearly $U(KG)$ is p-solvable. Rest follows from the above discussion. ■

4. Conclusion

We have covered most of the cases for finite groups except for finite groups which are not p-solvable, in which the p'-elements are non-central and do not form a subgroup. This problem is still open. Some preliminary results have been obtained in this direction by the author and will be taken up separately in a subsequent paper.

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