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GROUP ALGEBRAS WITH CENTRALLY METABELIAN UNIT GROUPS

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Abstract

Given a field K of characteristic $p > 2$ and a finite group G, necessary and sufficient conditions for the unit group $U(KG)$ of the group algebra KG to be centrally metabelian are obtained. It is observed that $U(KG)$ is centrally metabelian if and only if KG is Lie centrally metabelian.

1. Introduction

Let G be a finite group and let K be a field of characteristic $p > 0$, $p \neq 2$. Necessary and sufficient conditions for the unit group $U(KG)$ to be metabelian were recently obtained by Shalev [**5**]. In $\operatorname{Char} K = p \geq 5$, it turns out that $U(KG)$ is metabelian if and only if G is abelian and in $\operatorname{Char} K = 3$, $U(KG)$ is metabelian if and only if either G is abelian or G' is central cyclic of order 3. The characterization of metabelian group algebras by Rosenberger and Levin [**2**] shows that for a finite group G and K a field with $\operatorname{Char} K \neq 2$, $U(KG)$ is metabelian if and only if the group algebra KG is Lie metabelian. Also, in this connection, we have an important result due to Sharma and Srivastava [**6**, Theorem 4.1], which is, $\delta^2(U(R))-1 \subseteq \delta^2(L(R))R$ for arbitrary rings R. This shows [**6**, Corollary 4.2] that the unit group of a Lie metabelian ring is a metabelian group.

The aim, in this paper, is to find necessary and sufficient conditions for the unit group $U(KG)$ to be centrally metabelian. Recall that a group G is centrally metabelian if the second derived term $\delta^2(G)$ is contained in the centre $\zeta(G)$, that is, $(\delta^2(G), G) = 1$. Recently Sharma and Srivastava [**6**] and Sahai and Srivastava [**4**] have obtained necessary and sufficient conditions for the group algebra KG to be Lie centrally metabelian. Our investigations show that $U(KG)$ is centrally metabelian as a group if and only if KG is Lie centrally metabelian, at least when $\operatorname{Char} K = p \neq 2$ and G is a finite group. This is not true in general as Tasic' [**7**] has given example of a Lie centrally metabelian algebra of characteristic 2 whose unit group is not centrally metabelian.

Our notations are standard. We use $(x, y) = x^{-1}y^{-1}xy$ for group commutators and $[x, y] = xy - yx$ for Lie commutators.

We now start with our work.

2. Sufficient conditions

Theorem 2.1. *Let K be a field,* $\operatorname{Char} K = p \neq 2$ *and let G be a group, finite or infinite. If KG is Lie centrally metabelian, then $U(KG)$ is centrally metabelian.*

Proof: Suppose that KG is Lie centrally metabelian. By [**4**, Theorem B] either G is abelian or $\operatorname{Char} K = 3$ and $G' = C_3$. If G is abelian, then clearly $U(KG)$ is abelian. Assume that $\operatorname{Char} K = 3$ and $G' = \langle t \rangle$, $t^3 = 1$. Since G' is normal in G, we see that $(t-1)^2 = t^2+t+1$ is central in KG. Also if G' is central, then by [**2**], KG is Lie metabelian and by [**6**, Corollary 4.2], $U(KG)$ is metabelian. This is also given in Shalev [**5**, Theorem B] for finite groups.

So we are left with the case when $G' = \langle t \rangle$, $t^3 = 1$, $\operatorname{Char} K = 3$ and t is not central in G. Now $\Delta(G')KG = (t-1)KG$. In this case, $\gamma_3(G) = G'$, $\delta^{(1)}(KG) = \Delta(G')KG$ and $\delta^{(2)}(KG) = \Delta(G')^2KG = (t-1)^2KG$. We know by [**6**, Theorem 4.1], $\delta^2(U(KG))-1 \subseteq \delta^2(L(KG))KG \subseteq \delta^{(2)}(KG)$. So $\delta^2(U(KG)) \subseteq 1 + (t-1)^2KG$. Let $u \in \delta^2(U(KG))$ and $g \in G$. Then $u - 1 \in (t-1)^2KG$ and we have $(u, g) - 1 = u^{-1}g^{-1}[u-1, g] \in KG[(t-1)^2KG, KG]$. Thus $(u, g) - 1 \in (t-1)^2\Delta(G')KG = 0$, since $(t-1)^2$ is central in KG and $\Delta(G')^3 = 0$. This shows that $(u, g) = 1$ for every $u \in \delta^2(U(KG))$ and for every $g \in G$ and hence $\delta^2(U(KG))$ is contained in the centre of KG, as desired. ■

3. Necessary conditions

We have seen in the previous section that for arbitrary groups G, KG Lie centrally metabelian implies either G is abelian or $\operatorname{Char} K = 3$ and $G' = C_3$ and this, in turn, implies that the unit group $U(KG)$ is centrally metabelian. For finite groups, now we assume that $U(KG)$ is centrally metabelian and establish the converse.

We first make the following observation:

Lemma 3.1. *$GL_2(Z_3)$ is not centrally metabelian.*

Proof: Let $A=\begin{pmatrix}-1 & 1\\ 0 & -1\end{pmatrix}$, $B=\begin{pmatrix}1 & -1\\ -1 & -1\end{pmatrix}$ in $SL_2(Z_3)=GL_2(Z_3)'$. Then $A^{-1}B^{-1}AB=\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$ belongs to $GL_2(Z_3)''$, however, $\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$ is not in the centre of $GL_2(Z_3)$. ■

Lemma 3.2. *Let G be a finite group and let* $\operatorname{Char} K = p \neq 2$ *such that the unit group $U(KG)$ is centrally metabelian. Then $G/O_p(G)$ is abelian.*

Proof: We have the exact sequence of groups

$$1 \to 1 + J(KG) \to U(KG) \to U(KG/J(KG)) \to 1.$$

Now $KG/J(KG) \cong \prod_{i=1}^{m} M_{n_i}(D_i)$ and so $U(KG/J(KG)) \cong \prod_{i=1}^{m} GL_{n_i}(D_i)$. But $U(KG)$ is centrally metabelian implies $U(KG/J(KG))$ is centrally metabelian. Thus $GL_{n_i}(D_i)$ is centrally metabelian for all i and therefore all D_i's are fields and in view of Lemma 3.1, $n_i = 1$ for all i. This is because $GL_n(D)$ is solvable, $n \neq 1$, $\operatorname{Char} D \neq 2$, implies $n = 2$, $D = Z_3$ and thus $GL_n(D) = GL_2(Z_3)$ but by Lemma 3.1, $GL_2(Z_3)$ is not centrally metabelian. Thus $U(KG/J(KG))$ is a direct product of multiplicative groups of fields and hence abelian. But then $U(KG)/\{1 + J(KG)\}$ is abelian and $U(KG)' \subseteq 1 + J(KG)$. We get $G' \subseteq G \cap \{1 + J(KG)\} = O_p(G)$ and therefore, $G/O_p(G)$ is abelian, as desired. ■

Corollary 3.3. *Let* $\operatorname{Char} K = p \neq 2$ *and let G be a finite group such that $O_p(G) = 1$ and $U(KG)$ is centrally metabelian. Then G must be abelian.*

Corollary 3.4. *Let* $\operatorname{Char} K = p \neq 2$ *and let G be a finite group such that $U(KG)$ is solvable. Then $G = P \rtimes H$, a split extension of a p-group P by a p'-group H.*

Proof: Since $U(KG)$ is solvable, either $G/O_p(G)$ is abelian or $p = 3$ and $G/O_3(G)$ is a 2-group, see [**3**]. In either case, Sylow p-subgroup of G is normal in G. Let it be P. Now $|P|$ and $|G : P|$ are relatively prime, hence by Schur-Zassenhaus Theorem $G = P \rtimes H$, with desired properties. ■

Lemma 3.5. *Let G be a finite p-group, $p \geq 5$ and let K be a field with* $\operatorname{Char} K = p$ *such that $U(KG)$ is centrally metabelian. Then G is abelian.*

Proof: If not, let G be a counter example of least order. Then $G = \langle x,y\rangle$, $z=(x,y)\neq 1$, $G'=\langle z\rangle$, $z^p=1$ and z central.

Let $u_1=(1+x,y)$ and $u_2=(1+y,x)$, then using centrality of z, we get

$$\begin{aligned}(u_1,u_2)-1 &= u_1^{-1}u_2^{-1}[u_1-1,u_2-1]\\ &= u_1^{-1}u_2^{-1}[(1+x)^{-1}y^{-1}[1+x,y],(1+y)^{-1}x^{-1}[1+y,x]]\\ &= u_1^{-1}u_2^{-1}[(1+x)^{-1}y^{-1}yx,(1+y)^{-1}x^{-1}xy]((x,y)-1)((y,x)-1)\\ &= -u_1^{-1}u_2^{-1}[(1+x)^{-1}(1+x-1),(1+y)^{-1}(1+y-1)](z-1)^2z^{-1}\\ &= -u_1^{-1}u_2^{-1}[(1+x)^{-1},(1+y)^{-1}](z-1)^2z^{-1}\\ &= -u_1^{-1}u_2^{-1}(1+x)^{-1}(1+y)^{-1}[1+x,1+y]\\ &\qquad (1+y)^{-1}(1+x)^{-1}(z-1)^2z^{-1}\\ &= -u_1^{-1}u_2^{-1}(1+x)^{-1}(1+y)^{-1}yx(1+y)^{-1}(1+x)^{-1}(z-1)^3z^{-1}\\ &= -u_1^{-1}u_2^{-1}\gamma(z-1)^3z^{-1},\end{aligned}$$

where $\gamma=(1+x)^{-1}(1+y)^{-1}yx(1+y)^{-1}(1+x)^{-1}$.

Since (u_1,u_2) is central in KG, so

$$\begin{aligned}0 &= [(u_1,u_2)-1,x]\\ &= -[u_1^{-1}u_2^{-1}\gamma,x](z-1)^3z^{-1}\\ &= -\{u_1^{-1}[u_2^{-1},x]+[u_1^{-1},x]u_2^{-1}\}\gamma(z-1)^3z^{-1}-u_1^{-1}u_2^{-1}[\gamma,x](z-1)^3z^{-1}.\end{aligned}$$

It is not difficult to see that both $[u_2^{-1},x]$ and $[u_1^{-1},x]$ belong to $KG(z-1)^2$. Now multiplying by $(z-1)^{p-5}$ and using $(z-1)^p=0$, given $p\geq 5$, we get $[\gamma,x](z-1)^{p-2}=0$. With routine calculations,

$$\begin{aligned}[\gamma,x] &= (1+x)^{-1}[(1+y)^{-1}yx(1+y)^{-1},x](1+x)^{-1}\\ &= (1+x)^{-1}(1+y)^{-1}[y,x]x(1+y)^{-1}+yx[(1+y)^{-1},x](1+x)^{-1}\\ &\quad +(1+x)^{-1}[(1+y)^{-1},x]yx(1+y)^{-1}(1+x)^{-1}.\end{aligned}$$

Using $[(1+y)^{-1},x]=-(1+y)^{-1}[1+y,x](1+y)^{-1}=(1+y)^{-1}yx(1+y)^{-1}(z-1)$, we get

$$\begin{aligned}[\gamma,x] &= (1+x)^{-1}(1+y)^{-1}\{-yx^2(1+y)^{-1}+yx(1+y)^{-1}yx(1+y)^{-1}\}\\ &\qquad (1+x)^{-1}(z-1)\\ &\quad +(1+x)^{-1}(1+y)^{-1}yx(1+y)^{-1}yx(1+y)^{-1}(1+x)^{-1}(z-1)\\ &= (1+x)^{-1}(1+y)^{-1}yx\{-x+2(1+y)^{-1}yx\}\\ &\qquad (1+y)^{-1}(1+x)^{-1}(z-1)\\ &= (1+x)^{-1}(1+y)^{-1}yx(1+y)^{-1}\{-(1+y)x+2yx\}\\ &\qquad (1+y)^{-1}(1+x)^{-1}(z-1)\\ &= -(1+x)^{-1}(1+y)^{-1}yx(1+y)^{-1}(y-1)x(1+y)^{-1}(1+x)^{-1}(z-1).\end{aligned}$$

Now $[\gamma, x](z-1)^{p-2} = 0$ implies $(y-1)(z-1)^{p-1} = 0$. So $y \in \langle z\rangle$ and y is central. But then $z = (x, y) = 1$, a contradiction. ■

We now apply this lemma to settle the case when $\operatorname{Char} K = p \geq 5$ and G is an arbitrary finite group.

Theorem 3.6. *Let* $\operatorname{Char} K = p \geq 5$ *and let* G *be any finite group such that* $U(KG)$ *is centrally metabelian. Then* G *is abelian.*

Proof: By Corollary 3.4, $G = P \rtimes H$, a split extension of a p-group P by a p'-group H. By Corollary 3.3, H is abelian and by Lemma 3.5, P is abelian. Suppose, if possible, G is non-abelian. Then $(P, h) \neq 1$ for some $1 \neq h \in H$. Since h induces a p'-automorphism on P, by [**1**, Theorem 5.3.6], $(P, h, h) = (P, h)$. Let $L = \langle (P, h), h\rangle$. Then $L' = (P, h, h) = (P, h) \neq 1$. The Jacobson radical $J = J(KL) = \Delta((P, h))KL$. Since $1 + J \subseteq U(KL)$, $(1 + J, h) \subseteq U(KL)'$ and $((1 + J, h), (P, h)) \subseteq U(KG)''$ which is central in $U(KG)$.

Let $x, y, z \in (P, h)$. Put $a = 1 - x$, then $a \in J$. Let $u_1 = (1 - ha, h)$. Then

$$\begin{aligned}
u_1 &= (1-ha)^{-1}(1-ha)^h \\
&= \{1 + ha + (ha)^2 + (ha)^3 + \cdots\}(1 - ha^h) \\
&= 1 + ha - ha^h + (ha)^2 - (ha)ha^h \\
&\quad + (ha)^3 - (ha)^2ha^h + (ha)^4 - (ha)^3ha^h + \cdots \\
&\equiv 1 + h(a - a^h) + h^2a^h(a - a^h) + h^3a^{h^2}a^h(a - a^h) \pmod{J^4}.
\end{aligned}$$

Now, since P is abelian, working modulo J^4, we have

$$\begin{aligned}
u_2 &= (u_1, y) \\
&= 1 + u_1^{-1}(u_1^y - u_1) \\
&\equiv 1 + u_1^{-1}\{(h^y - h)(a - a^h) + (h^{2y} - h^2)a^h(a - a^h) \\
&\quad + (h^{3y} - h^3)a^{h^2}a^h(a - a^h)\} \\
&\equiv 1 + u_1^{-1}h\{(h, y) - 1 + h((h^2, y) - 1)a^h + h^2((h^3, y) - 1)a^{h^2}a^h\}(a - a^h) \\
&\equiv 1 + u_1^{-1}h\{(h, y) - 1 + h((h^2, y) - 1)a^h\}(a - a^h), \pmod{J^4},
\end{aligned}$$

since $(h^3, y) - 1 \in J$. Now u_2 is central. So we have working modulo J^4

$$\begin{aligned}
0 &= [u_2 - 1, z] \\
&\equiv [u_1^{-1}h\{(h, y) - 1 + h((h^2, y) - 1)a^h\}(a - a^h), z] \\
&\equiv [u_1^{-1}, z]h\{(h, y) - 1 + h((h^2, y) - 1)a^h\}(a - a^h) \\
&\quad + u_1^{-1}[h((h, y) - 1) + h^2((h^2, y) - 1)a^h, z](a - a^h) \\
&\equiv u_1^{-1}\{[h, z]((h, y) - 1) + [h^2, z]((h^2, y) - 1)a^h\}(x^h - x),
\end{aligned}$$

since $[u_1^{-1}, z] = -u_1^{-1}[u_1, z]u_1^{-1} \in J^2$. Thus we get $((x,h)-1)((y,h)-1)((z,h)-1) \in J^4$ for all $x, y, z \in (P,h) = (P,h,h)$. So $J^3 \subseteq J^4$ and $J^3 = 0$, since J is nilpotent. Thus $(\Delta((P,h)))^3 = 0$ and $((P,h)-1)^3 = 0$. Now $\mathrm{Char}\, K = p \geq 5$ and (P,h) is a p-group implies $(P,h) = 1$, a contradiction to our assumption that $(P,h) \neq 1$. Thus G must be abelian. ■

Remark 3.7. The entire proof of Theorem 3.6 goes through upto $(\Delta((P,h)))^3 = 0$ in Char K=3 also if we assume that P is abelian.

We, now, turn to $\mathrm{Char}\, K = 3$.

Lemma 3.8. *Let* $\mathrm{Char}\, K = p = 3$ *and let* G *be a finite group of odd order such that* $U(KG)$ *is centrally metabelian. Then* $G = P \rtimes H$, P *a* p*-group,* H *an abelian* p'*-group. Further* $G' = P'$.

Proof: By Corollary 3.3 and 3.4, $G = P \rtimes H$, P a p-group, H an abelian p'-group. Assume, further, that P is abelian. Then $G' = (P, H)$. If $G' \neq 1$, choose $x \in P$, $h \in H$ such that $(x, h, h) \neq 1$ which is possible because $(P, h, h) = (P, h) \neq 1$ for some $h \in H$. By Remark 3.7, $(\Delta((P, h)))^3 = 0$. Now (P, h) is a p-group, $p = 3$, so (P, h) is cyclic of order 3. Then $(P, h) = \langle (x, h) \rangle$. It is easy to see that $(x, h)^h = (x, h)(x, h, h) \in (P, h)$, hence $(x, h)^h = (x, h)$ or $(x, h)^{-1}$. If $(x, h)^h = (x, h)^{-1}$, then $(x, h)^{h^2} = (x, h)$ implying $(x, h)^h = (x, h)$, because order of h is odd. Thus $(x, h, h) = 1$, a contradiction. Hence $G' = (P, H) = 1$ and G is abelian. So $G' = P'$.

Now let P be non-abelian. By applying the above case to the group G/P', we get $(P, H) \leq P'$ and so $G' = P'$ in this case also. ■

Next result is for finite 3-groups.

Proposition 3.9. *Suppose that* $\mathrm{Char}\, K = 3$ *and* P *is a finite* 3*-group such that* $U(KP)$ *is centrally metabelian. Then either* P *is abelian or* $P' = C_3$.

Proof: If not, let G be a minimal counter example. Then $|G'| = 9$ and we have the following three cases.

Case (i): G' is central cyclic of order 9.

Let $G' = \langle z, \rangle$, $z = (x, y)$, $x, y \in G$, $z^9 = 1$. Exactly as in the proof of Lemma 3.5, $G = \langle x, y \rangle$, $z = (x, y) \neq 1$, and we conclude that $(y-1)(z-1)^8 = 0$. Thus $y \in \langle z \rangle \subseteq \zeta(G)$ and so $(x, y) = 1$, a contradiction. Hence this case will not arise.

Case (ii): G' is central and $G' = C_3 \times C_3$.

Clearly $\Delta(G')^5 = 0$. Since G' is not cyclic, there exist elements $x, y_1, y_2 \in G$ such that $z_1 = (x, y_1) \neq 1$ and $z_2 = (x, y_2) \notin \langle z_1 \rangle$, see

[**4**, proof of Theorem B]. Let $u_1 = (1+x, y_1)$, $u_2 = (1+y_2, x)$. Then exactly as in the proof of Lemma 3.5, we get

$$(u_1, u_2) - 1 = -u_1^{-1}u_2^{-1}\gamma(z_1 - 1)(z_2 - 1)^2 z_2^{-1},$$

where $\gamma = (1+x)^{-1}(1+y_2)^{-1}y_2x(1+y_2)^{-1}(1+x)^{-1}$. Hence

$$\begin{aligned} 0 &= [(u_1, u_2) - 1, y_1] \\ &= -[u_1^{-1}u_2^{-1}\gamma, y_1](z_1 - 1)(z_2 - 1)^2 z_2^{-1} \\ &= -\{[u_1^{-1}u_2^{-1}, y_1]\gamma + u_1^{-1}u_2^{-1}[\gamma, y_1]\}(z_1 - 1)(z_2 - 1)^2 z_2^{-1}. \end{aligned}$$

We get $[\gamma, y_1](z_1 - 1)(z_2 - 1)^2 = 0$, first term above being 0 because $[u_1^{-1}u_2^{-1}, y_1] \in \Delta(G')^2KG$ and $\Delta(G')^5 = 0$. Now $-[\gamma^{-1}, y_1] = \gamma^{-1}[\gamma, y_1]\gamma^{-1}$ and z_1, z_2 are central. So $[\gamma^{-1}, y_1](z_1 - 1)(z_2 - 1)^2 = 0$. Now

$$\begin{aligned} \gamma^{-1} &= (1+x)(1+y_2)(y_2x)^{-1}(1+y_2)(1+x) \\ &= (1+x)(1+y_2)z_2y_2^{-1}x^{-1}(1+y_2)(1+x) \\ &= (1+x)(1+y_2^{-1})(z_2 + y_2^x z_2)(1+x^{-1}) \\ &= (1+x)(1+y_2^{-1})(z_2 + y_2)(1+x^{-1}). \end{aligned}$$

Since $G' = C_3 \times C_3$, let $(y_1, y_2) = z_1^i z_2^j$ for some $0 \le i, j \le 2$. Now using $(z_1^i - 1)(z_1 - 1) = i(z_1 - 1)^2$, $z_2(z_2 - 1)^2 = (z_2 - 1)^2$, and expanding $[\gamma^{-1}, y_1]$ in the usual way, we get

$$\begin{aligned} \{y_1x(1+y_2^{-1})(1+y_2)(1+x^{-1}) &+ i(1+x)y_1y_2^{-1}(1+y_2)(1+x^{-1}) \\ &- i(1+x)(1+y_2^{-1})y_2y_1(1+x^{-1}) \\ &- (1+x)(1+y_2^{-1})(1+y_2)x^{-1}y_1\}(z_1 - 1)^2(z_2 - 1)^2 = 0. \end{aligned}$$

Since $[\alpha, \beta] \in \Delta(G')KG$ for all $\alpha, \beta \in KG$ and $\Delta(G')^5 = 0$, on combining first term with last term and second term with third term, we get, using $[\alpha, \beta](z_1 - 1)^2(z_2 - 1)^2 = 0$, that

$$\begin{aligned} 0 &= \{(y_1x - x^{-1}y_1)(1+y_2^{-1})(1+y_2) \\ &\quad + i(1+x)(y_1y_2^{-1} - y_2y_1)(1+x^{-1})\}(z_1 - 1)^2(z_2 - 1)^2 \\ &= \{y_1x^{-1}(x^2 - 1)(1+y_2)^2y_2^{-1} \\ &\quad + i(1+x)^2(1 - y_2^2)y_2^{-1}y_1x^{-1}\}(z_1 - 1)^2(z_2 - 1)^2 \\ &= y_1x^{-1}(1+x)\{(x-1)(y_2+1) \\ &\quad + i(1+x)(1-y_2)\}(1+y_2)y_2^{-1}(z_1 - 1)^2(z_2 - 1)^2. \end{aligned}$$

We have $\{(x-1)(y_2+1)+i(x+1)(1-y_2)\}(z_1-1)^2(z_2-1)^2=0$. It is not difficult to see that this is not possible for any $i=0,1,2$.

Case (iii): G' is not central in G.

G is nilpotent, $|G'|=9$, $\gamma_3(G)\neq 1$ implies $\gamma_3(G)=C_3$ and $\gamma_4(G)=1$.

Choose $w\in G'$, $x\in G$ such that $z=(x,w)\neq 1$. Then $z\in\zeta(G)$, $z^3=1$ and $(1{+}x,w,w)$ is central in KG. Also $(x,G)\nsubseteq\gamma_3(G)$. For otherwise (x,G) will be in $\zeta(G)$ and then $(x,g^{-1},h)(g,h^{-1},x)(h,x^{-1},g)=1$, implies $(g,h^{-1},x)=1$ for all $g,h\in G$. So $(G',x)=1$ and $z=(x,w)=1$. Choose $y\in G$ such that $(x,y)\notin\gamma_3(G)$. Let $u=(1+x,w)$, then

$$\begin{aligned}(1+x,w,w)&=1+u^{-1}w^{-1}[u-1,w]\\&=1+u^{-1}w^{-1}[(1+x)^{-1}w^{-1}[1+x,w],w]\\&=1+u^{-1}w^{-1}[(1+x)^{-1}w^{-1}wx(z-1),w]\\&=1+u^{-1}w^{-1}(1+x)^{-1}[x,w](1+x)^{-1}(z-1)\\&=1+u^{-1}w^{-1}(1+x)^{-1}wx(1+x)^{-1}(z-1)^2\\&=1+u^{-1}(1+xz)^{-1}(1+x^{-1})^{-1}(z-1)^2.\end{aligned}$$

Now $[(u,w),y]=0$ implies $[(1{+}xz)^{-1}(1{+}x^{-1})^{-1},y](z{-}1)^2=0$, because $[u^{-1},y]\in(z-1)KG$ and $(z-1)^3=0$. Solving this further, we have

$$\begin{aligned}0&=[(1+x^{-1})(1+xz),y](z-1)^2\\&=[x^{-1}+xz,y](z-1)^2\\&=[x^{-1}+x,y](z-1)^2.\end{aligned}$$

Hence

$$\begin{aligned}0&=\{-x^{-1}[x,y]x^{-1}+[x,y]\}(z-1)^2\\&=\{-x^{-1}yx((x,y)-1)x^{-1}+yx((x,y)-1)\}(z-1)^2\\&=\{-x^{-1}y((x,y)^{x^{-1}}-1)+yx((x,y)-1)\}(z-1)^2\\&=\{-x^{-1}y((x,y)(x,y,x^{-1})-1)+yx((x,y)-1)\}(z-1)^2\\&=\{-x^{-1}y+yx\}((x,y)-1)(z-1)^2,\end{aligned}$$

since $(x,y,x^{-1})\in\gamma_3(G)=\langle z\rangle$. This gives $yx\{x^{-1}y^{-1}x^{-1}y-1\}((x,y)-1)(z-1)^2=0$ and so $\{(y,x)^x x^{-2}-1\}((x,y)-1)(z-1)^2=0$. Since $(x,y)\notin\gamma_3(G)$, it follows that $(y,x)^x x^{-2}$ is in G' and so $x^{-2}\in G'$. But then $x\in G'$ as order of x is odd. This is a contradiction as $(x,y)\notin\gamma_3(G)$.

Thus we have a contradiction in all the three cases, so either P is abelian or $P'=C_3$. ■

Corollary 3.10. *Let* $\mathrm{Char}\, K = 3$ *and let* G *be a finite group of odd order such that* $U(KG)$ *is centrally metabelian. Then either* G *is abelian or* G' *is cyclic of order* 3.

Proof: It can be deduced easily from Lemma 3.8 and Proposition 3.9. ∎

Now we shall study the case when G is a group of even order.

Lemma 3.11. *Let* $G = P \rtimes \langle h \rangle$*,* P *a finite* 3*-group,* $o(h)$ *is even and coprime to* 3 *and let* $\mathrm{Char}\, K = 3$*, such that* $U(KG)$ *is centrally metabelian. Then either* $G' = 1$ *or* $G' = C_3$*.*

Proof: If $(P,h) \subseteq P'$, then $G' = P'$ and by Proposition 3.9, $G' = 1$ or C_3. So we are through. Assume that $(P,h) \nsubseteq P'$. Then $z = (x,h) \notin P'$ for some $x \in P$.

First suppose that P is abelian. Consider the group $L = \langle (P,h), h \rangle$. By Remark 3.7, $(\Delta((P,h)))^3 = 0$ and hence (P,h) is cyclic of order 3. So $G' = (P,h) = C_3$, since $P' = 1$.

Now let P be non-abelian. Then $P' = C_3 = \langle t \rangle$, say. Applying the above case to G/P', we have $G'/P' = (P,h)P'/P' \cong C_3$. Then $G'/P' = \langle zP' \rangle$, since $z \notin P'$. Thus $z^3 \in P'$. This gives that $|G'| = 9$ and hence G' is abelian.

Again take $L = \langle (P,h), h \rangle$, then $L' = (P,h,h) = (P,h) = C_3$, since $U(KL)$ is centrally metabelian, (P,h) is abelian and we can apply Remark 3.7. So $(P,h) = \langle z \rangle$, $z^3 = 1$. Also $(z,h) \in (P,h)$, $(z,h) \neq 1$ and so $(z,h) = z$ and $z^h = z^{-1}$. Clearly $G' = (P,h)P' = \langle z \rangle \times \langle t \rangle$, $z^3 = t^3 = 1$ and $\Delta(G')^5 = 0$. Since P is nilpotent, $t \in \zeta(P)$. Further, $(t,h) \in (P,h) \cap P' = 1$ and t is central in G.

Case (i): $(z,P) \neq 1$.

There exists $y \in P$ with $1 \neq (z,y) \in P' = \langle t \rangle$. So we may take $(z,y) = t$. Let $a = 1 - z \in \Delta(P)$. Then $1 - ha$ is a unit. Let $u_1 = (1+z, y)$ and $u_2 = (1 - ha, h)$. Then

$$\begin{aligned} u_2 &= (1-ha)^{-1}(1-ha)^h \\ &= \{1 + ha + (ha)^2\}(1 - ha^h) \\ &= 1 + h(z-1)z + h^2(z-1)^2, \end{aligned}$$

using $(z-1)^3=0$, $z^h=z^{-1}$ and $(ha)^3=0$. Now

$$\begin{aligned}(u_1,u_2)-1 &= u_1^{-1}u_2^{-1}[u_1-1,u_2-1]\\ &= u_1^{-1}u_2^{-1}[(1+z)^{-1}y^{-1}[1+z,y],h(z-1)z+h^2(z-1)^2]\\ &= u_1^{-1}u_2^{-1}[(1+z)^{-1}y^{-1}yz((z,y)-1),h(z-1)z+h^2(z-1)^2]\\ &= u_1^{-1}u_2^{-1}(1+z)^{-1}[z,h(z-1)z+h^2(z-1)^2](1+z)^{-1}(t-1)\\ &= u_1^{-1}u_2^{-1}h(z-1)^2(t-1),\end{aligned}$$

because $z^{h^2}=z$, and $(1+z)^{-1}=-(1-z+z^2)$.

Now since $[u_1^{-1}u_2^{-1},y]\in\Delta(G')^2KG$ and $\Delta(G')^5=0$, we get

$$\begin{aligned}0 &= [(u_1,u_2)-1,y]\\ &= [u_1^{-1}u_2^{-1}h(z-1)^2,y](t-1)\\ &= u_1^{-1}u_2^{-1}\{yh((h,y)-1)(z-1)^2+h[z+z^2,y]\}(t-1)\\ &= -u_1^{-1}u_2^{-1}hyz(z-1)(t-1)^2\end{aligned}$$

because $(h,y)\in\langle z\rangle$. It follows that $(z-1)(t-1)^2=0$ implying that $z\in\langle t\rangle$, a contradiction.

Case (ii): $(z,P)=1$.

Let $t=(a,b)$ for some $a,b\in P$. If $(y,h)\neq 1$ implies $(y,P)=1$ for all $y\in P$, then take $\pi=xa$. Otherwise we may take $\pi=a=x$. Let $g=hb$. Then $(\pi,g)=zt$ and $(\pi,b)=t$ always. Set $\alpha=1-\pi\in\Delta(P)$, then $1-g\alpha$ is a unit. We have

$$\begin{aligned}u_1 &= (1-g\alpha,g)\\ &= 1+g(\pi^g-\pi)+g^2(1-\pi^g)(\pi^g-\pi)+\cdots\\ &= (1+g+g^2(1-\pi^g)+\cdots)\pi(zt-1)\end{aligned}$$

and

$$\begin{aligned}(u_1,z) &= 1+u_1^{-1}(u_1^z-u_1)\\ &= 1+u_1^{-1}\{(g^z-g)+(g^{2z}-g^2)(1-\pi^g)+\cdots\}(zt-1)\pi.\end{aligned}$$

Now $(g,z)=z^2$ and $(g^2,z)=1$. Therefore,

$$\begin{aligned}(u_1,z) &= 1+u_1^{-1}\{g(z^2-1)+g^3(z^2-1)(1-\pi^{g^2})(1-\pi^g)+\cdots\}(zt-1)\pi\\ &= 1+u_1^{-1}g\{1+g^2(1-\pi^{g^2})(1-\pi^g)+\cdots\}(z^2-1)(zt-1)\pi.\end{aligned}$$

Since (u_1, z) is central, we have

$$\begin{aligned}
0 &= [(u_1,z),z] \\
&= [u_1^{-1}g\{1+g^2(1-\pi^{g^2})(1-\pi^g)+\cdots\}(z^2-1)(zt-1)\pi, z] \\
&= ([u_1^{-1},z]g\{1+g^2(1-\pi^{g^2})(1-\pi^g)+\cdots\} \\
&\quad + u_1^{-1}[g+g^3(1-\pi^{g^2})(1-\pi^g)+\cdots, z])(z^2-1)(zt-1)\pi \\
&\equiv u_1^{-1}zg(1+g^2(1-\pi^{g^2})(1-\pi^g)+\cdots)(z^2-1)^2(zt-1)\pi \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{mod } \Delta(G')^4KG),
\end{aligned}$$

because $[u_1^{-1}, z] = -u_1^{-1}[u_1, z]u_1^{-1}$ and $[u_1, z] = zu_1((u_1, z) - 1) \in \Delta(G')^2KG$. Since $\Delta(G')^5 = 0$, on multiplying by $(t-1)$, we get

$$(1+g^2(1-\pi^{g^2})(1-\pi^g)+\cdots)(z-1)^2(zt-1)(t-1)=0.$$

Once again on multiplying by $(1-\pi)^{o(\pi)-1}$ from the right this gives $(1-\pi)^{o(\pi)-1}(z-1)^2(t-1)^2 = 0$, since $1-\pi^g = 1-\pi((\pi,g)-1)-\pi$. Thus $\pi \in G' \subseteq \zeta(P)$ and hence $1 = (\pi, b) = t$, a contradiction.

So $(P, h) \le P'$ and $G' = P'$ with $|G'| = 1$ or 3. ■

Proposition 3.12. *Let K be a field with* $\operatorname{Char} K = 3$ *and let $G = P \rtimes H$, P a 3-group, H a $3'$-group of even order, such that $U(KG)$ is centrally metabelian. Then either $G' = 1$ or $G' = C_3$.*

Proof: By Corollary 3.3, H is abelian and so $G' = (P,H)P'$. Let $h \in H$. Consider the group $L = \langle P, h\rangle$. Then by Corollary 3.10 and Lemma 3.11, either $L' = 1$ or $L' = (P,h)P' = C_3$. If P is non-abelian, this gives $(P,h) \le P'$. Since this is true for any $h \in H$, we get $G' = P' = C_3$.

Let P be abelian. Then $L' = (P,h)$ is cyclic of order 3. Thus $G' = (P,H)$ is an elementary abelian 3-group. Also (P,h) is normal in G, for $(\pi,h)^{\pi'h'} = (\pi, h^{\pi'})^{h'} = (\pi, (\pi', h^{-1})h)^{h'} = (\pi,h)^{h'} = (\pi^{h'}, h)$ is in (P,h). Let $h_1, h_2 \in H$ such that $(P,h_1) \ne 1$ and $(P,h_2) \ne 1$. Suppose that $(P,h_1) = \langle z_1\rangle$, $(P,h_2) = \langle z_2\rangle$ and $\langle z_1\rangle \cap \langle z_2\rangle = 1$. Then since (P,h_1) is normal, $(z_1,h_2) \in (P,h_1)$. Also $(z_1,h_2) \in (P,h_2)$ and so $(z_1,h_2) = 1$. Similarly $(z_2,h_1) = 1$. Set $\pi = z_1z_2$, $g = h_1h_2$ and $\alpha = 1-\pi$. Then

$$\begin{aligned}
u = (1-g\alpha, g) &= 1 + g(\pi^g - \pi) + g^2(1-\pi^g)(\pi^g-\pi) + \cdots \\
&= 1 + g\pi(\pi-1) + g^2(\pi-1)^2, \\
(u, z_1) &= 1 + u^{-1}z_1^{-1}[u, z_1] \\
&= 1 + u^{-1}g(z_1^2-1)\pi(\pi-1)
\end{aligned}$$

and

$$\begin{aligned}0 &= [(u, z_1), z_1]\\ &= [u^{-1}, z_1]g(z_1^2 - 1)\pi(\pi - 1) + u^{-1}[g, z_1](z_1^2 - 1)\pi(\pi - 1)\\ &= u^{-1}\{-[u, z_1]u^{-1}g + [g, z_1]\}(z_1^2 - 1)\pi(\pi - 1)\\ &= u^{-1}\{-z_1 g(z_1^2 - 1)\pi(\pi - 1)u^{-1}g + z_1 g(z_1^2 - 1)\}(z_1^2 - 1)\pi(\pi - 1).\end{aligned}$$

Let $Q = \langle z_1\rangle \times \langle z_2\rangle$. Then $\Delta(Q)^5 = 0$. Since $u^{-1} \in 1 + \Delta(Q)KG$, it follows from the above equation that

$$\{-(z_1^2 - 1)\pi(\pi - 1)g + z_1^2 - 1\}(z_1^2 - 1)(\pi - 1) = 0.$$

On multiplying by $(z_2 - 1)$, we get

$$\begin{aligned}0 &= (z_1^2 - 1)^2(z_1 z_2 - 1)(z_2 - 1)\\ &= (z_1 - 1)^2(z_2 - 1)^2.\end{aligned}$$

This gives $z_1 \in \langle z_2\rangle$, a contradiction as $\langle z_1\rangle \cap \langle z_2\rangle = 1$. Therefore G' must be cyclic. so $G' = C_3$. ■

We are now in a position to state our main results of this section.

Theorem 3.13. *Let G be a finite group and let K be a field with* $\operatorname{Char} K = p \neq 2$. *Then $U(KG)$ is centrally metabelian if and only if either G is abelian or* $\operatorname{Char} K = 3$ *and $G' = C_3$.*

Proof: First let $U(KG)$ be centrally metabelian. If $\operatorname{Char} K = 0$ then G is abelian, see [**3**]. So let $\operatorname{Char} K = p > 0$. If $p \geq 5$, then Theorem 3.6 gives that G is abelian.

Now let $p = 3$. By Corollary 3.4, we have $G = P \rtimes H$, where P is a 3-group and H is a $3'$-group. Also since $U(KH) \leq U(KG)$ is centrally metabelian, by Corollary 3.3, H is abelian. Finally by Corollary 3.10 and Proposition 3.12, we get that either G is abelian or $G' = C_3$.

Thus if $U(KG)$ is centrally metabelian, then either G is abelian or $\operatorname{Char} K = 3$ and $G' = C_3$. But then KG is Lie centrally metabelian (see [**4**]). The converse now follows from Theorem 2.1. ■

Corollary 3.14. *Let K and G be as in Theorem 3.13. Then $U(KG)$ is centrally metabelian if and only if KG is Lie centrally metabelian.*

Proof: KG is Lie centrally metabelian if and only if either G is abelian or $\operatorname{Char} K = 3$ and $G' = C_3$. Rest follows from Theorem 3.13. ■

Corollary 3.15. *Let K be a field with* $\operatorname{Char} K = 3$ *and let G be a finite group of odd order. Then the following are equivalent:*

(i) *$U(KG)$ is centrally metabelian;*
(ii) *G is either abelian or nilpotent with $G' = C_3$;*
(iii) *$U(KG)$ is metabelian;*
(iv) *KG is Lie metabelian.*

Proof: Let $U(KG)$ be centrally metabelian. By Theorem 3.13, if G is non-abelian, then $G' = C_3$. Let $G' = \langle t \rangle$. If $t^g \neq t$ for some $g \in G$, then $t^g = t^{-1}$ and so $t^{g^2} = (t^{-1})^g = t$. Now G has odd order so g, also, is of odd order and $g \in \langle g^2 \rangle$. This gives $t^g = t$. Thus $t^g = t$ for all $g \in G$ and so $G' = \langle t \rangle$ is central in G. Now by using [**2**] and Theorem 2.1, we get that statements (i), (ii) and (iv) given above are equivalent. Further by [**6**, Corollary 4.2], KG Lie metabelian implies $U(KG)$ is metabelian. Now [**5**, Theorem B] gives that either G is abelian or G is nilpotent with $G' = C_3$.

It is easy to see that Corollary 3.15 is parallel to what we have for Lie centrally metabelian group algebras KG, $\operatorname{Char} K = 3$ and G torsion having no element of order 2 (see [**4**, Theorem A]). ■

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