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The scheme of connected components of the Néron model of an algebraic torus

By *Xavier Xarles* at Barcelona

Given a local field K with ring of integers $\mathcal{O}$ and perfect residual field k, denote by $\eta = \operatorname{Spec} K$, $S = \operatorname{Spec} \mathcal{O}$, $s = \operatorname{Spec} k$, and $j: \eta \to S$, $i: s \to S$ the canonical open and closed immersions. The Néron model of a torus $T_{|K}$ is a smooth group scheme $\mathcal{T}$ over $\mathcal{O}$ representing the sheaf $j_* T$ for the smooth topology. $\mathcal{T}$ fits into an exact sequence of étale sheaves:

$$0 \to \mathcal{T}^\circ \to \mathcal{T} \to \mathcal{T}/\mathcal{T}^\circ \to 0$$

where $\mathcal{T}^\circ$ denotes the connected component of $\mathcal{T}$. The aim of this paper is to describe $\mathcal{T}/\mathcal{T}^\circ$, the sheaf of connected components of $\mathcal{T}$, in terms of the character group $X := \operatorname{Hom}_{\bar{K}}(T_{\bar{K}}, G_m)$. Since $\mathcal{T}/\mathcal{T}^\circ$ is a skyscraper sheaf, it is equivalent to describe $\phi := i^*(\mathcal{T}/\mathcal{T}^\circ)$, the scheme of connected components of the reduction of $\mathcal{T}$. It is well-known that ϕ is represented by an étale scheme of finite type, hence totally determined by the G_k-module $\phi := \phi(k^s)$, where G_k is the absolute Galois group of k and k^s is the separable clausure of k.

Bégueri has studied this question in the case where K has an algebraically closed residual field. In this case, ϕ has trivial Galois action and its structure as an abstract group is determined by $H^0(G_K, X)$ and $H^1(G_K, X)$ ([Be], Th. 7.2.1, 7.2.2), where $G_K = \operatorname{Gal}(\bar{K}/K)$.

In this paper we solve this problem in the general case by giving an explicit description of ϕ as an étale scheme in terms of the complex of G_k-modules $R\Gamma(I, X)$, where I is the inertia subgroup of G_K. Our proofs are independent of the result of Bégueri. The methods are also different. We use essentially homological methods.

The structure as G_k-module of the torsion and free parts of ϕ is determined in Section 2. We prove (Corollary (2.12)) that there is an exact sequence of G_k-modules:

$$0 \to \operatorname{Hom}_Z(H^1(I, X), Q/Z) \to \phi \to \operatorname{Hom}_Z(X^I, Z) \to 0.$$

This result does not determine ϕ as G_k-module. The computation of ϕ is achieved in Section 3 where we prove:

Theorem. *There is an isomorphism respecting G_k-action*:

$$R\operatorname{Hom}_Z(\phi, \mathbb{Z}) \cong \tau_{\leqq 1} R\Gamma(I, X).$$

In particular, we can compute ϕ as $R\operatorname{Hom}_Z(\tau_{\leqq 1} R\Gamma(I, X), \mathbb{Z})$ using any I-acyclic resolution of X. The functor $\tau_{\leqq 1}$ truncates the complex by degree 1, but maintaining the 0 and 1 cohomology.

This result can be considered as a duality assertion. Essentially, we are saying that the functors:

$$T \to \phi,$$
$$X \to \tau_{\leqq 1} R\Gamma(I, X)$$

commute with duality. Cartier duality on the left and linear duality in the derived category on the right.

In the last section we give an example of two tori having isomorphic G_k-action on the 0 and 1 cohomology, but whose schemes of connected components of their Néron model are not isomorphic.

It is a pleasure to thank Uwe Jannsen for some helpful conversation on this problem.

1. The case of good reduction

We say that the torus T has good reduction if the connected component of its Néron model is a torus, or equivalently if the inertia subgroup I of G_K acts trivially on X (see [Na-Xa], (1.1)). In this case the description of ϕ is easy.

(1.1) Theorem. *If T has good reduction, then $\phi \cong X^\vee$ as G_k-modules.*

Here Hom_Z will denote homomorphism as abelian groups, and $X^\vee := \operatorname{Hom}_Z(X, \mathbb{Z})$ has the natural structure of G_k-module.

Proof. The sheaf $\mathscr{T}/\mathscr{T}^\circ$ for the étale topology is a skyscraper sheaf. Hence,

$$\mathscr{T}/\mathscr{T}^\circ \cong i_* i^*(\mathscr{T}/\mathscr{T}^\circ) \cong i_*(\phi)$$

and we have an exact sequence:

$$0 \to \mathscr{T}^\circ \to \mathscr{T} \to i_*\phi \to 0.$$

I acts trivially on $X^\vee$, hence, $i^* j_*(X) \cong X$ as étale sheaves, and

$$i_*(X^\vee) \cong i_* \underline{\operatorname{Hom}}_{\text{ét}}(i^* j_* X, \mathbb{Z}) \cong \underline{\operatorname{Hom}}_{\text{ét}}(j_* X, i_* \mathbb{Z}).$$

Hence, it suffices to show that $i_* \phi \cong \underline{\mathrm{Hom}}_{\acute{e}t}(j_* X, i_* \mathbb{Z})$. To see this, we shall prove the existence of an exact sequence

$$(1.2) \qquad 0 \to \mathscr{T}^\circ \to j_* T \xrightarrow{v} \underline{\mathrm{Hom}}_{\acute{e}t}(j_* X, i_* \mathbb{Z}) \to 0 .$$

We define first the homomorphism v. By Cartier duality,

$$j_* T \cong j_* \underline{\mathrm{Hom}}_{\acute{e}t}(X, \mathbb{G}_m) \cong \underline{\mathrm{Hom}}_{\acute{e}t}(j_* X, j_* \mathbb{G}_m) .$$

On the other hand, the Néron model of $\mathbb{G}_m$ fits into an exact sequence:

$$(1.3) \qquad 0 \to \mathbb{G}_m \to j_* \mathbb{G}_m \xrightarrow{v} i_* \mathbb{Z} \to 0 .$$

The map v induces a homomorphism:

$$v : j_* T \cong \underline{\mathrm{Hom}}_{\acute{e}t}(j_* X, j_* \mathbb{G}_m) \to \underline{\mathrm{Hom}}_{\acute{e}t}(j_* X, i_* \mathbb{Z}) .$$

We prove now that (1.2) is exact in the fibers $\bar{s}$ and $\bar{\eta}$. Denote also by v the valuation of $\mathcal{O}$.

The generic fiber $\bar{\eta}$ of (1.2) is

$$0 \to (\bar{K})^{*n} \to (\bar{K})^{*n} \to 0 \to 0 .$$

Since $\mathcal{O}_{S,\bar{s}} = \mathcal{O}^{\mathrm{sh}}$, the strict Henselization of $\mathcal{O}$, the fiber at $\bar{s}$ of the sequence (1.2) is:

$$0 \to (\mathcal{O}^{\mathrm{sh}})^{*n} \to (K^{\mathrm{nr}})^{*n} \xrightarrow{v} \mathbb{Z}^n \to 0 .$$

where K^{nr} is the no-ramification closure of K and v is the valuation of K^{nr}. □

2. Free and torsion parts of ϕ

We state now a generalization of the theorems of Bégueri [Be], Th. 7.2.1, 7.2.2.

(2.1) Theorem. *There are natural isomorphims as G_k-modules*:

$$\mathrm{Hom}_{\mathbb{Z}}(\phi, \mathbb{Z}) \cong H^0(I, X) ,$$
$$\mathrm{Ext}^1_{\mathbb{Z}}(\phi, \mathbb{Z}) \cong H^1(I, X) .$$

We need the following lemmas:

(2.2) Lemma. *Let $\mathscr{X}$ be an S-group scheme of finite type which is smooth and connected. Then*

$$\underline{\mathrm{Hom}}_{\acute{e}t}(\mathscr{X}, i_* \mathbb{Z}) = 0 .$$

Proof. Since $i_* \underline{\mathrm{Hom}}_{\text{ét}}(i^*\mathscr{X}, \mathbb{Z}) \cong \underline{\mathrm{Hom}}_{\text{ét}}(\mathscr{X}, i_*\mathbb{Z})$, we only need to show that $\underline{\mathrm{Hom}}_{\text{ét}}(i^*\mathscr{X}, \mathbb{Z}) = 0$, or, equivalently, that $\underline{\mathrm{Hom}}_{\text{ét}}((i^*\mathscr{X})_{k'}, \mathbb{Z}) = 0$ for any finite separable extension k' of k. Since $(i^*\mathscr{X}_{k'})(k^s) = \mathscr{X}(\mathcal{O}^{\text{sh}})$ ([Mi1], II.2.9(d)), we have that

$$\mathrm{Hom}_{\text{ét}}((i^*\mathscr{X})_{k'}, \mathbb{Z}) \cong \mathrm{Hom}_{G_{k'}}(\mathscr{X}(\mathcal{O}^{\text{sh}}), \mathbb{Z}),$$

using the equivalence between the category of étale sheaves over k' and the category of $G_{k'}$-modules. Then the assertion holds because $\mathscr{X}(\mathcal{O}^{\text{sh}})$ is ℓ-divisible for any positive integer ℓ prime to the characteristic of k (see [Mi1], II.2.19 and [Bo-Lü-Ra], 7.3.2). □

The functor j_* is not exact in general, but in our case and for the étale topology we have

(2.3) Lemma. *$R^i j_* T = 0$ for the étale topology for all $i > 0$.*

Proof. We only need to check that the geometric fibers are zero. One has:

$$(R^i j_* T)_s = H^i(K^{\text{nr}}, T) = 0,$$
$$(R^i j_* T)_{\bar\eta} = H^i(\bar K, T) = 0,$$

by [Mi2], III 4.1, or [Be], 3.2.1. □

Thus j_* is exact for the étale topology when is applied to tori.

Let us prove now the first assertion of Theorem (2.1):

(2.4) Proposition. $\mathrm{Hom}_{\mathbb{Z}}(\phi, \mathbb{Z}) \cong X^I$ *as G_k-modules.*

Proof. Denote by T^I the torus over K having X^I as character group. The torus T^I has good reduction. Le ϕ^I be the corresponding scheme of connected components of the reduction of its Néron model. By (1.1), we have $\mathrm{Hom}_{\mathbb{Z}}(\phi, \mathbb{Z}) \cong X^I$ as G_k-modules. We show that

$$\mathrm{Hom}_{\mathbb{Z}}(\phi, \mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(\phi^I, \mathbb{Z})$$

to conclude the proof.

Consider the G_k-module $X' = X/X^I$. It is easy to show that X' is torsion-free and that $X'^I = 0$. Hence, it determines a torus T' over K. Since it has no characters fixed by I, the torus $T' \otimes_K K^{\text{nr}}$ does not contain $\mathbb{G}_m$. Hence, by a theorem of Raynaud [Bo-Lü-Ra], 10.2.1, its Néron model is of finite type, and in consequence the scheme of connected component ϕ' is finite.

Consider now the sequence:

$$0 \to T' \to T \to T^I \to 0.$$

By (2.3), the corresponding sequence of étale sheaves:

$$(2.5) \qquad 0 \to j_* T' \to j_* T \to j_* T^I \to 0,$$

is exact. Applying $\underline{\mathrm{Hom}}_{\text{ét}}(-, i_* \mathbb{Z})$ to (2.5), we obtain the sequence

$$0 \to \underline{\mathrm{Hom}}_{\text{ét}}(\mathcal{T}^I, i_* \mathbb{Z}) \to \underline{\mathrm{Hom}}_{\text{ét}}(\mathcal{T}, i_* \mathbb{Z}) \to \underline{\mathrm{Hom}}_{\text{ét}}(\mathcal{T}', i_* \mathbb{Z}) \to \underline{\mathrm{Ext}}^1_{\text{ét}}(\mathcal{T}^I, i_* \mathbb{Z}) \to \dots,$$

where $\mathcal{T}^I$ and $\mathcal{T}'$ are the Néron models of T^I and T'.

By (2.2), applying $\underline{\mathrm{Hom}}_{\text{ét}}(-, i_* \mathbb{Z})$ to the exact sequence

$$0 \to \mathcal{T}^\circ \to \mathcal{T} \to i_* \phi \to 0,$$

we obtain natural isomorphisms as étale sheafs

$$\underline{\mathrm{Hom}}_{\text{ét}}(\mathcal{T}, i_* \mathbb{Z}) \cong \underline{\mathrm{Hom}}_{\text{ét}}(i_* \phi, i_* \mathbb{Z}).$$

Since $\mathrm{Hom}_{\mathbb{Z}}(\phi, \mathbb{Z})$ are the G_k-modules associated to the sheaves $\underline{\mathrm{Hom}}_{\text{ét}}(\phi, \mathbb{Z})$, we obtain the exact sequence

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(\phi^I, \mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(\phi, \mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(\phi', \mathbb{Z}) = 0;$$

the group $\mathrm{Hom}_{\mathbb{Z}}(\phi', \mathbb{Z})$ being trivial because ϕ' is finite. □

We prove now the the second assertion of Theorem (2.1) for the torus $T = R_{L/K}(\mathbb{G}_m)$, the Weil restriction of $\mathbb{G}_m$ under a finite extension L/K of local fields. For the definition and properties of this torus, see [Bo-Lü-Ra], 7.6 or [Na-Xa], 2. The group of characters of T is the induced G_K-module $\mathrm{Ind}^1_{\mathrm{Gal}(L/K)}(\mathbb{Z})$. Hence, it is cohomologically trivial. Thus, we want to show that $\mathrm{Ext}^1_{\mathbb{Z}}(\phi, \mathbb{Z}) = 0$ in this case.

(2.6) Lemma. *If $T = R_{L/K}(\mathbb{G}_m)$, then ϕ is torsion-free.*

Proof. The Weil restriction commutes with the formation of Néron model, and the Weil restriction of a torus is connected ([Na-Xa], (2.4)). Then, since the Weil restriction is an exact functor, we have from (1.2) that $i_* \phi = R_{\mathcal{O}_L/\mathcal{O}_K}(i_* \mathbb{Z})$. Thus, when we take the special fiber, we obtain that $\phi \cong R_{\mathcal{O}_L \times_{\mathcal{O}_K} k/k}(\mathbb{Z})$.

Let $K' = K^{\mathrm{nr}} \cap L$ and denote by $T' = R_{L/K'}(\mathbb{G}_m)$. We have $T = R_{K'/K}(T')$ and $\mathcal{T}^\circ = R_{\mathcal{O}_{K'}/\mathcal{O}_K}(\mathcal{T}'^\circ)$ because T' is a good reduction torus, where $\mathcal{T}'$ is the Néron model of T'. If ϕ' denotes the corresponding scheme of connected components of the reduction of $\mathcal{T}'$ then $\phi \cong R_{l/k}(\phi')$, since K'/K is not ramified and $\mathcal{O}_{K'} \times_{\mathcal{O}_K} k = l$, where l is the residual field of K' and L.

Since L/K' is totally ramified, there exists an Einsenstein polynomial

$$q(x) = x^e + a_1 x^{n-1} + \cdots + a_n, \quad a_i \in \mathbf{m}, \quad \forall i = 1, \dots, n$$

such that

$$\mathcal{O}_L = \mathcal{O}_{K'}[x]/q(x),$$

where $e = e(L/K') = [L:K']$ and $\mathbf{m}$ is the maximal ideal of $\mathcal{O}_{K'}$. Hence,

$$\mathcal{O}_L \otimes_{\mathcal{O}_{K'}} l \cong l[x]/x^e.$$

Thus,

$$\phi' \cong R_{\frac{l[x]}{x^e}/l}\left(i_* \mathbb{Z} \otimes_{\mathcal{O}_{K'}} \frac{l[x]}{x^e}\right)$$

and therefore

$$\phi'(k^s) = \mathbb{Z}\left(\frac{k^s[x]}{x^e}\right) = \mathbb{Z}$$

as an abelian group. Finally, since

$$\phi \otimes_k k^s = R_{l/k}(\phi') \otimes_k k^s \cong (\phi')^{[l:k]},$$

ϕ is torsion-free. □

We can now deduce the second assertion of Theorem (2.1) in the case that $H^1(I, X(T)) = 0$.

(2.7) Proposition. *If* $H^1(I, X) = 0$ *then* ϕ *is torsion-free.*

Proof. Let d be the dimension of T and let K_T denote the splitting field. We consider the exact sequence:

$$0 \to T' \to R_{K_T/K}(\mathbb{G}_m^d) \to T \to 0$$

corresponding under Cartier duality to the sequence of G_K-modules:

$$(2.8) \qquad 0 \to X \to X_R \to X' \to 0,$$

where X_R denotes the G_K-module $\mathrm{Ind}^1_{\mathrm{Gal}(K_T/K)}(\mathbb{Z}^d)$ and X' is torsion-free.

We can proceed as in the proof of (2.4) to obtain an exact sequence

$$(2.9) \qquad 0 \to \mathrm{Hom}_{\mathbb{Z}}(\phi, \mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(\phi_R, \mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(\phi', \mathbb{Z}),$$

where ϕ_R and ϕ' are the G_k-modules corresponding to the schemes of connected components of the Néron models of $R_{K_T/K}(\mathbb{G}_m^d)$ and T'.

Consider the cohomological sequence deduced from (2.8):

$$(2.10) \qquad 0 \to H^0(I, X) \to H^0(I, X_R) \to H^0(I, X') \to H^1(I, X) = 0.$$

By (2.4) there is a natural commutative diagram linking (2.9) and (2.10), and the three terms of these sequences are isomorphic; hence the morphism

$$\mathrm{Hom}_Z(\phi_R, \mathbb{Z}) \to \mathrm{Hom}_Z(\Phi', \mathbb{Z})$$

is surjective. We can deduce that the morphism of étale sheaves

$$\underline{\mathrm{Ext}}^1_{\acute{e}t}(i^* j_* T, \mathbb{Z}) \to \underline{\mathrm{Ext}}^1_{\acute{e}t}(i^* j_* T_R, \mathbb{Z})$$

is injective and hence we have a commutative diagram

$$\begin{array}{ccccc} 0 \to & \underline{\mathrm{Ext}}^1_{\acute{e}t}(i^* j_* T, \mathbb{Z}) & \to & \underline{\mathrm{Ext}}^1_{\acute{e}t}(i^* j_* T_R, \mathbb{Z}) \\ & \uparrow & & \uparrow \\ & \underline{\mathrm{Ext}}^1_{\acute{e}t}(\phi, \mathbb{Z}) & \to & \underline{\mathrm{Ext}}^1_{\acute{e}t}(\phi_R, \mathbb{Z}) \\ & \uparrow & & \uparrow \\ & 0 & & 0 \end{array}$$

where $\underline{\mathrm{Ext}}^1_{\acute{e}t}(\phi_R, \mathbb{Z}) = 0$ by Lemma (2.6). Hence we obtain that $\mathrm{Ext}^1_Z(\phi, \mathbb{Z}) = 0$. □

To compute the torsion part of ϕ in the general case we need to work in the smooth topology. We need later the following lemmas.

(2.11) Lemma. *If T is a good reduction torus then $R^1 j_* T_{sm} = 0$ for the smooth topology.*

Proof. The assertion is clear from the fact that $R^1 j_* \mathcal{G}m_{sm} = 0$ ([Mi 2], III. C. 10). □

(2.12) Lemma. *Let $\mathcal{X}$ be an S-group scheme of finite type which is smooth and connected. Then*

$$\underline{\mathrm{Hom}}_{sm}(\mathcal{X}, i_* \mathbb{Z}) = 0,$$

$$\underline{\mathrm{Ext}}^1_{sm}(\mathcal{X}, i_* \mathbb{Z}) = 0.$$

Proof. The first assertion is due to the fact that there are no morphisms between a connected group scheme and an étale group scheme.

Since $\underline{\mathrm{Hom}}_{sm}(\mathcal{X}, i_* \mathbb{Z}) = 0$, we have $\underline{\mathrm{Ext}}^1_{\acute{e}t}(\mathcal{X}, i_* \mathbb{Z})(U) = \mathrm{Ext}^1(\mathcal{X}_{|U}, i_* \mathbb{Z}_{|U})$ for any S-smooth scheme U, $\mathrm{Ext}^1_{sm}(\mathcal{X}_{|U}, i_* \mathbb{Z}_{|U}) = \mathrm{Ext}^1_{U_s}(\mathcal{X}_{|U}, i_* \mathbb{Z}_{U_s})$ by [SGA 7], VIII. 5.9, and this last group is trivial by [SGA 7], VIII. 5.1. □

(2.13) Lemma. *Let T be a torus over K. There exists an exact sequence of*

$$0 \to T' \to R \to T \to 0$$

where T' is a good reduction torus and $H^1(I, X(R)) = 0$.

Proof. Let d be the dimension of T and let K_T denote the splitting field. We consider the exact sequence of G_K-modules:

$$0 \to X \to X_R \to N \to 0,$$

where X_R denotes the G_K-module $\mathrm{Ind}^1_{\mathrm{Gal}(K_T/K)}(\mathbb{Z}^d)$ and N is torsion-free. Denote by M the pull-back of the diagram

$$\begin{array}{ccc} M & \to & N^I \\ \downarrow & & \downarrow \\ X_R & \to & N. \end{array}$$

The the tori R and T' are the respective Cartier duals of M and N^I. □

We can prove now the second assertion of the Theorem (2.1) in the general case.

(2.14) Proposition. $\mathrm{Ext}^1_{\mathbb{Z}}(\phi, \mathbb{Z}) \cong H^1(I, X)$.

Proof. We consider the exact sequence defined in Lemma (2.13),

$$0 \to T' \to R \to T \to 0. \tag{2.15}$$

By (2.11), the corresponding sequence of smooth sheaves,

$$0 \to j_* T' \to j_* R \to j_* T \to 0, \tag{2.16}$$

is exact. Applying $\underline{\mathrm{Hom}}_{\mathrm{sm}}(-, i_* \mathbb{Z})$ to the sequence of smooth sheaves

$$0 \to \mathscr{T}^\circ \to j_* T \to i_* \phi \to \phi \to 0,$$

and using Lemma (2.12) we obtain natural isomorphisms as smooth sheaves

$$\underline{\mathrm{Hom}}_{\mathrm{sm}}(j_* T, i_* \mathbb{Z}) \cong i_* \underline{\mathrm{Hom}}_{\mathrm{sm}}(\phi, \mathbb{Z}),$$
$$\underline{\mathrm{Ext}}^1_{\mathrm{sm}}(j_* T, i_* \mathbb{Z}) \cong i_* \underline{\mathrm{Ext}}^1_{\mathrm{sm}}(\phi, \mathbb{Z}).$$

Hence, applying the functor $\underline{\mathrm{Hom}}_{\mathrm{sm}}(-, i_* \mathbb{Z})$ to the sequence (2.16) we obtain

$$0 \to \underline{\mathrm{Hom}}_{\mathrm{sm}}(\phi, \mathbb{Z}) \to \underline{\mathrm{Hom}}_{\mathrm{sm}}(\phi_R, \mathbb{Z}) \to \underline{\mathrm{Hom}}_{\mathrm{sm}}(\phi', \mathbb{Z}) \to \underline{\mathrm{Ext}}^1_{\mathrm{sm}}(\phi', \mathbb{Z}) \to 0$$

where ϕ_R and ϕ' denote the schemes of connected components of the Néron models of R and T', and $\underline{\mathrm{Ext}}^1_{\mathrm{sm}}(\phi_R, \mathbb{Z}) = 0$ by the Proposition (2.7). Using now the fact that the sheaves $\underline{\mathrm{Hom}}_{\mathrm{sm}}(\phi, \mathbb{Z})$ and $\underline{\mathrm{Ext}}^1_{\mathrm{sm}}(\phi, \mathbb{Z})$ are representable by the étale schemes $\mathrm{Hom}_{\mathbb{Z}}(\phi, \mathbb{Z})$ and $\mathrm{Ext}^1_{\mathbb{Z}}(\phi, \mathbb{Z})$, and the first assertion of the Theorem (2.1) (Proposition (2.4)), we have a commutative diagram

$$\begin{array}{ccccccccc} 0 \to & \mathrm{Hom}_{\mathbb{Z}}(\phi, \mathbb{Z}) & \to & \mathrm{Hom}_{\mathbb{Z}}(\phi_R, \mathbb{Z}) & \to & \mathrm{Hom}_{\mathbb{Z}}(\phi', \mathbb{Z}) & \to & \mathrm{Ext}^1_{\mathbb{Z}}(\phi, \mathbb{Z}) & \to 0 \\ & \uparrow\wr & & \uparrow\wr & & \uparrow\wr & & & \\ 0 \to & X^I & \to & X(R)^I & \to & X(T') & \to & H^1(I, X) & \to 0 \end{array}$$

and hence we can deduce the assertion. □

We end this section giving, in Corollary (2.18) below, a more compact version of Theorem (2.1).

(2.17) Lemma. *Let G be a group and let ϕ be a finitely generated abelian group with a G-action. Then, there is a natural exact sequence of G-modules*

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{Ext}^1_{\mathbb{Z}}(\phi, \mathbb{Z}), \mathbb{Z}) \to \phi \to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_{\mathbb{Z}}(\phi, \mathbb{Z}), \mathbb{Z}) \to 0 .$$

Proof. The G-homomorphism $\varphi : \phi \to H := \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_{\mathbb{Z}}(\phi, \mathbb{Z}), \mathbb{Z})$ is defined as follows:

$$\varphi(x)(f) = f(x), \quad x \in \phi \quad \text{and} \quad f \in \mathrm{Hom}_{\mathbb{Z}}(\phi, \mathbb{Z}) .$$

It is easy to prove that $\mathrm{Ker}(\varphi)$ is isomorphic to $\tau(\phi)$, the torsion part of ϕ, and that φ is an epimorphism. Now, applying $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Z})$ to the sequence

$$0 \to \tau(\phi) \to \phi \to H \to 0 ,$$

we obtain that $\mathrm{Ext}^1_{\mathbb{Z}}(\tau(\phi), \mathbb{Z}) \cong \mathrm{Ext}^1_{\mathbb{Z}}(\phi, \mathbb{Z})$. But $\tau(\phi)$ is finite, and hence

$$\tau(\phi) \cong \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{Ext}^1_{\mathbb{Z}}(\tau(\phi), \mathbb{Z}), \mathbb{Z}) . \quad \square$$

(2.18) Corollary. *There is an exact sequence of G_k-modules*

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(H^1(I, X), \mathbb{Q}/\mathbb{Z}) \to \phi \to \mathrm{Hom}_{\mathbb{Z}}(X^I, \mathbb{Z}) \to 0 .$$

(2.19) Corollary. (a) *ϕ is torsion-free if and only if $H^1(I, X) = 0$. In this case $\phi \cong \mathrm{Hom}_{\mathbb{Z}}(X^I, \mathbb{Z})$ as G_k-modules.*

(b) *ϕ is finite if and only if $X^I = 0$. In this case $\phi \cong \mathrm{Hom}_{\mathbb{Z}}(H^1(I, X), \mathbb{Q}/\mathbb{Z})$ as G_k-modules (Pontrjagin dual).*

3. The main result

We want to proof now the theorem announced in the introduction. Fix a torus T over K and denote by X its group of characters.

If Φ is a complex of G_k-modules, finitely generated as abelian groups, then the complex $R\,\mathrm{Hom}_{\mathbb{Z}}(\Phi, \mathbb{Z})$ has a natural action of G_k. This action is canonical since for any single G_k-module ϕ (finitely generated as an abelian group), we can identify the natural G_k-structure of $\mathrm{Hom}_{\mathbb{Z}}(\phi, \mathbb{Z})$ with $\underline{\mathrm{Hom}}_{G_k}(\phi, \mathbb{Z})$ (internal homomorphisms).

Take an I-acyclic resolution of X by G_K-modules which are torsion-free as $\mathbb{Z}$-modules:

$$0 \to X \to M \to M' \to M'' \to \dots .$$

If we denote by $X' = \operatorname{Ker}(M' \to M'')$, then $\tau_{\leqq 1} R\Gamma(I, X)$ is the complex of G_k-modules

$$M^I \to (X')^I .$$

We want to prove that $R\operatorname{Hom}_Z(\phi, \mathbb{Z}) \cong \tau_{\leqq 1} R\Gamma(I, X)$, or, equivalently, that

$$\phi \cong R\operatorname{Hom}_Z(\tau_{\leqq 1} R\Gamma(I, X), \mathbb{Z})$$

as G_k-modules. Since M^I and X'^I are $\mathbb{Z}$-free, $R\operatorname{Hom}_Z(\tau_{\leqq 1} R\Gamma(I, X), \mathbb{Z})$ is the complex of G_k-modules

$$\operatorname{Hom}_Z(X'^I, \mathbb{Z}) \to \operatorname{Hom}_Z(M^I, \mathbb{Z}) .$$

Hence, we need only to prove the following:

(3.1) Theorem. *There is an exact sequence of G_k-modules*:

$$0 \to \operatorname{Hom}_Z(X'^I, \mathbb{Z}) \to \operatorname{Hom}_Z(M^I, \mathbb{Z}) \to \phi \to 0 .$$

Proof. Let T_M and T' be the tori over K, respective Cartier dual of M and X'. Denote by ϕ_M and ϕ' the corresponding schemes of connected components of the reduction of the Néron models of T_M and T'.

By (2.19), $\phi_M \cong \operatorname{Hom}_Z(M^I, \mathbb{Z})$ is torsion free. On the other hand, by Cartier duality we obtain an exact sequence:

$$0 \to T' \to T_M \to T \to 0 .$$

From this we obtain the following exact sequence

$$(3.2) \qquad 0 \to j_* T' \to j_* T_M \to j_* T \to 0 .$$

Hence, from the homomorphisms

$$\mathcal{T}'^{\circ} \to \mathcal{T}_M^{\circ} \to \mathcal{T}^{\circ} ,$$

we obtain G_k-homomorphisms:

$$\phi' \to \phi_M \xrightarrow{\beta} \phi .$$

Since ϕ_M is free, the homomorphism $\phi' \to \phi_M$ induces a G_k-homomorphism:

$$\operatorname{Hom}_Z(\operatorname{Hom}_Z(\phi', \mathbb{Z}), \mathbb{Z}) \xrightarrow{\alpha} \operatorname{Hom}_Z(\operatorname{Hom}_Z(\phi_M, \mathbb{Z}), \mathbb{Z}) \cong \phi_M .$$

Using now Theorem (2.1) we obtain the complex:

$$0 \to \operatorname{Hom}_Z(X'^I, \mathbb{Z}) \xrightarrow{\alpha} \operatorname{Hom}_Z(M^I, \mathbb{Z}) \xrightarrow{\beta} \phi \to 0 .$$

We want to show that this is an exact sequence. The morphism β is obviously an epimorphism, since (3.2) is exact, and α is a monomorphism since $\operatorname{Hom}_Z(H^1(I, X), \mathbb{Z}) = 0$.

We only need to show that $\operatorname{Ker}\beta = \operatorname{Im}\alpha$. The inclusion $\operatorname{Im}\alpha \subset \operatorname{Ker}\beta$ is obvious. Let us check the opposite inclusion.

By (2.18) we have the following exact sequence of G_k-modules:

$$0 \to \operatorname{Ext}^1_Z(H^1(I,X),\mathbb{Z}) \to \phi \to \operatorname{Hom}_Z(X^I,\mathbb{Z}) \to 0\,.$$

Consider now the exact sequence

$$0 \to X^I \to M^I \to X'^I \to H^1(I,X) \to 0\,, \tag{3.3}$$

and denote by N the module $\operatorname{Im}(M^I \to X'^I) = \operatorname{Ker}(X'^I \to H^1(I,X))$. The sequence (3.3) splits into two exact sequences:

$$0 \to X^I \to M^I \to N \to 0\,,$$
$$0 \to N \to X'^I \to H^1(I,X) \to 0\,.$$

Applying now the functor $\operatorname{Hom}_Z(-,\mathbb{Z})$ to both sequences we obtain:

$$0 \to \operatorname{Hom}_Z(N,\mathbb{Z}) \to \operatorname{Hom}_Z(M^I,\mathbb{Z}) \to \operatorname{Hom}_Z(X^I,\mathbb{Z}) \to \operatorname{Ext}^1_Z(N,\mathbb{Z}) = 0\,,$$
$$0 \to \operatorname{Hom}_Z(X'^I,\mathbb{Z}) \to \operatorname{Hom}_Z(N,\mathbb{Z}) \to \operatorname{Ext}^1_Z(H^1(I,X),\mathbb{Z}) \to 0\,.$$

Since N is torsion-free, it determines a torus T_N. The scheme of connected components of the Néron model of T_N is isomorphic to $\operatorname{Hom}_Z(N,\mathbb{Z})$. By the naturality of the isomorphisms of Theorem (2.1), we obtain a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \to & \operatorname{Ext}^1_Z(H^1(I,X),\mathbb{Z}) & \to & \phi & \to & \operatorname{Hom}_Z(X^I,\mathbb{Z}) & \to & 0 \\
 & & \uparrow & & {\scriptstyle\beta}\uparrow & & \uparrow & & \\
0 & \to & \operatorname{Hom}_Z(N,\mathbb{Z}) & \to & \operatorname{Hom}_Z(M^I,\mathbb{Z}) & \to & \operatorname{Hom}_Z(X^I,\mathbb{Z}) & \to & 0 \\
 & & \uparrow & & {\scriptstyle\alpha}\uparrow & & & & \\
 & & \operatorname{Hom}_Z(X'^I,\mathbb{Z}) & \xrightarrow{\sim} & \operatorname{Hom}_Z(X'^I,\mathbb{Z}) & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0\,. & & & &
\end{array}$$

A diagram chase shows that the middle vertical sequence is exact. □

4. Some examples

In this section we give an example showing that Theorem (2.1) (or Corollary (2.18)) is not sufficient to determine the scheme ϕ of connected components of the Néron model of a torus. In fact, we find below two tori T_1 and T_2 of dimension three, with character groups X_1 and X_2, admitting G_k-isomorphisms:

$$H^i(I, X_1) \cong H^i(I, X_2), \quad \text{for } i = 0, 1,$$

but with non-isomorphic ϕ.

In dimension one it is easy to check that ϕ is isomorphic either to $\mathbb{Z}$ (with trivial or non trivial action) or to $\mathbb{Z}/2\mathbb{Z}$. For tori of dimension two the list of possible ϕ's is larger but easily computable from our results. In all cases, however, the exact sequence of Corollary (2.18) splits. Let us check this assertion; recall that $(\ )^\vee$ denotes linear dual and $(\ )^*$ denotes Pontrjagin dual.

(4.1) Proposition. *If T is a 2-dimensional torus, then $\phi \cong (X^I)^\vee \oplus H^1(I, X)^*$ as G_k-modules.*

Proof. If $X^I = 0$ or $H^I(I, X) = 0$ the assertion is given by Corollary (1.12). Let T be a torus such that $0 \subsetneq X^I \subsetneq X$. It is clear that $(X^\vee)^I \neq 0$ and that the action of I in $X^\vee$ is not trivial. Hence, the quotient $X' = X^\vee/(X^\vee)^I$ is a free-torsion G_K-module of rank one. Taking I-cohomology on the exact sequence

$$0 \to (X^\vee)^I \to X^\vee \to X' \to 0,$$

we obtain the exact sequence

$$0 \to X^I \to ((X^\vee)^I)^\vee \to H^1(I, X'^\vee) \to H^1(I, X) \to 0.$$

The G_K-module $X'^\vee$ has rank one and no characters fixed by I. Hence

$$H^1(I, X'^\vee) = \mathbb{Z}/2\mathbb{Z}$$

and the morphism $((X^\vee)^I)^\vee \to H^1(I, X'^\vee)$ is surjective or zero. If it is surjective, then $H^1(I, X) = 0$ and the position is clear. If it is zero, then the isomorphism $X^I \cong ((X^\vee)^I)^\vee$ composed by the natural morphism $X \to ((X^\vee)^I)^\vee$ gives a split of the exact sequence

$$0 \to X^I \to X \to X'' \to 0.$$

Thus $X \cong X^I \oplus X''$, and therefore $T \cong T^I \times T''$, where T^I and T'' are the tori with character groups X^I and X'' respectively. Since the formation of Néron model commutes with the product, the proposition is clear. □

In dimension three, however, this is no more true. Let us give an example.

Let K be a local field and L/K a finite Galois extension with $G := \mathrm{Gal}(L/K) \cong \mathbb{Z}/4\mathbb{Z}$ and inertia subgroup $I \cong \mathbb{Z}/2\mathbb{Z}$. We define the 3-dimensional tori T_1 and T_2 over K which decomposes over L determined by the Galois representations:

$$\varrho_i : G \to \mathrm{Aut}(X(T_i)) \cong GL_3(\mathbb{Z}), \quad i = 1, 2,$$

$$\varrho_1(\sigma) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \varrho_2(\sigma) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$

where σ is a generator of G. Denote by $X_i = X(T_i)$, $i = 1, 2$.

To compute X_i^I, $H^1(I, X_i)$ and $\phi_i = \phi(T_i)$ we consider the following exact sequence:

$$0 \to X_i \to M_i \to N_i \to 0,$$

where M_i is the cohomologically trivial G-module $\mathrm{Ind}^1_{\varrho_i(G)}(X_i)$ and $X_i \to M_i$ is the natural inclusion. Now, it is easy to show that $X_i^I \cong \mathbb{Z}$ with trivial action and $H^1(I, X_i) = \mathbb{Z}/2\mathbb{Z}$. Observe that $X_1 = X_1^I \oplus' X_1'$. Hence, as we have seen above,

$$\phi_1 = (X_1^I)^\vee \oplus H^1(I, X_1)^* \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$$

with trivial action of G/I.

Applying now Theorem (3.1), we have that

$$\phi_2 = \mathrm{cokernel}\left((N_2^I)^\vee \xrightarrow{\alpha} (M_2^I)^\vee\right).$$

Easy calculations show that $(M_2^I)^\vee = \mathbb{Z}^6$ with the following action of $\bar{\sigma}$:

$$(a, b, c, a', b', c')^{\bar{\sigma}} = (a', b', c', a, b, c),$$

and $(N_2^I)^\vee$ has rank 5. The morphism α is

$$\alpha(a, b, c, d, e) = (-a, -a + c + d + 2e, -b - d, a, b, c).$$

Hence $\phi_2 \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ with the non-splitting action $(a, \bar{b})^{\bar{\sigma}} = (a, \overline{a + b})$.

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Departament de Mathemàtiques, Universitat Autònoma de Barcelona,
08193 Bellaterra, Barcelona, Catalunya, Spain.

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