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On symmetric square values of quadratic polynomials

by

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1. Introduction. In this note we are dealing with the following problem. Given a degree two polynomial $f(x) = ax^2 + bx + c \in \mathbb{Z}[x]$ which is not a square of a degree one polynomial, how many consecutive integer values $f(i)$ can be squares in $\mathbb{Z}$? This problem has been considered by D. Allison in [1] and [2], who found infinitely many examples with eight consecutive values, and by A. Bremner in [3], who found more examples with seven consecutive values.

The examples found by Allison are all by polynomials which are symmetric with an axis of symmetry midway between two integers. This means that, after some easy translation, all the examples are of the form $f(x) = a(x^2 + x) + c$ and the values are $f(i)$ for $i = -3, -2, -1, 0, 1, 2, 3$ and 4. This result was obtained by translating the problem to computing rational points on some elliptic curve which has rank one.

On the other hand, Bremner [3] shows that there does not exist any example which is symmetric about an integral value and with seven values, by showing that these examples would be described by rational points on some rank zero elliptic curve, which has 12 points, all corresponding to the polynomial $f(x)$ being the square of a polynomial.

In the same paper, Bremner asks if there are examples as the ones found by Allison, but with ten consecutive squares. The problem translates to finding all the rational points of a genus 5 curve, a fact already noticed by Allison and by Bremner. He conjectures that there is no such example.

In this note we prove this conjecture, and so, together with the results of Bremner and Allison, we get the following theorem.

Theorem 1. *Let N be a positive integer and $\mathcal{B}_N$ the set consisting of non-square quadratic polynomials $f(x) = ax^2 + bx + c \in \mathbb{Z}[x]$ that takes*

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square values for N consecutive integer values $x = r, r+1, \dots, r+N-1$, and $f(r) = f(r+N-1)$ for some $r \in \mathbb{Z}$. Then

$$\#\mathcal{B}_N = \begin{cases} \infty & \text{if } N \leq 6 \text{ or } N = 8, \\ 0 & \text{if } N = 7 \text{ or } N \geq 9. \end{cases}$$

To show this result we will use similar techniques to the one we use in [13] to study arithmetic progressions of squares over quadratic fields. In fact, the problem we study here is in some sense a generalization to higher dimensions of the old result by Fermat about arithmetic progressions of squares, and the problems in [13] and in [20] are generalizations to higher degrees (in the sense of the number field involved). We can say almost nothing about whether there exists a maximum number of consecutive square values taken by a non-square quadratic polynomial, or even if this number is 8, as the known examples suggest. However, it is possible that an argument similar to the one given by Vojta in [18] could be used to show the existence of this maximum under the Bombieri–Lang conjecture.

Remark 2. Let $f(x) = ax^2 + bx + c$ be a quadratic polynomial. Then for any x and y we have $f(x+y) - 2f(x+y+1) + f(x+y+2) = 2a$. Hence, $f(x+i)$ for $i \in \mathbb{Z}$ form a sequence whose second differences are constant and equal to $2a$. In particular, if $f(x) \in \mathbb{Z}[x]$ and there exists $r \in \mathbb{Q}$ such that f takes square values for $x = r, r+1, \dots, r+N-1$ then $f(r+i)$ is a sequence of squares of length N whose second differences are constant and equal to $2a$. Conversely, if $a_1^2, a_2^2, \dots, a_n^2 \in \mathbb{Z}$ is a sequence of squares whose second differences are constant and equal to $2L$, then $f(x) = L(x^2 - x) + (a_2^2 - a_1^2)x + a_1^2$ takes square values for $x = 0, 1, \dots, n-1$. Note that squares of arithmetic progressions correspond, by the above characterization, to squares of linear polynomials.

Büchi [14] asked if there exists a positive integer n such that any sequence of integer squares of length at least n with constant second differences 2 is a sequence of squares of consecutive integers. A positive answer to the above question is known as Büchi's conjecture or n-squares conjecture. He also proved that this conjecture gives a negative answer to Hilbert's tenth problem (for systems of second degree equations). Vojta [18] proved that the n-squares conjecture is a consequence of the Bombieri–Lang conjecture. For the case of second difference greater than 2, several authors [7, 15, 4] have treated this problem. Taking into account these results, Browkin and Brzeziński [4] generalized the n-squares conjecture to: Any sequence of integer squares of length greater than eight whose second differences are constant and greater than 2, is equal to a sequence of squares of an arithmetic progression. In the present paper, we prove the above conjecture in the symmetric case.

2. Translation to geometry. Fix a polynomial $f(x) = ax^2 + bx + c \in \mathbb{Z}[x]$ which takes square values for $x = r, r+1, \ldots, r+N-1$, and $f(r) = f(r+N-1)$. Suppose that N is even. After translation by $-r - N/2$, we can suppose that $r = -N/2$. Then $f(x)$ has the form $f(x) = a(x^2 + x) + c$, and we are asking to have $f(i) = x_i^2$ for $i = 0, \ldots, N/2 - 1$ and $x_i \in \mathbb{Z}$.

Now, suppose that $N = 10$. The conditions we get from $f(i) = x_i^2$ for $i = 0, \ldots, 4$ are given by the following equations:

$$C : \begin{cases} 2x_0^2 - 3x_1^2 + x_2^2 = 0, \\ 5x_0^2 - 6x_1^2 + x_3^2 = 0, \\ 9x_0^2 - 10x_1^2 + x_4^2 = 0, \end{cases}$$

which determine a genus 5 curve C in $\mathbb{P}^4$. Any point $P := [x_0 : x_1 : x_2 : x_3 : x_4]$ of this curve defined over $\mathbb{Q}$ will give us a polynomial $f(x)$ as above, by setting $c = x_0^2$ and $a = (x_1^2 - x_0^2)/2$. Observe that multiplying a pair (a, c) by a square number will produce the same polynomial but multiplied by a square number, a case that we consider equivalent.

Now, the solutions given by $P = [\pm 1 : \pm 1 : \pm 1 : \pm 1 : \pm 1]$ correspond to $a = 0$, so the polynomial is in fact constant. There are also the solutions given by $P = [\pm 1 : \pm 3 : \pm 5 : \pm 7 : \pm 9]$, which correspond to the case $a = 4$ and $c = 1$, so $f(x) = (2x + 1)^2$. Our aim will be to show that these are the only rational points.

First of all, observe that the curve C has degree 2 maps Φ_n to five distinct genus one curves F_n, for $n = 0, 1, 2, 3, 4$. They can be described easily as intersections of two quadrics in $\mathbb{P}^3$, by taking the first two of the three quadrics describing C, which gives three of these curves, and transforming the equations in order to get more quadratic forms involving only three variables, which gives the other two. The curve F_n is the one given by equations not involving the variable x_n.

All these genus one curves have rational points over $\mathbb{Q}$, therefore they are isomorphic to elliptic curves over $\mathbb{Q}$. Denoting by E_n the Weierstrass model of the curve F_n and using the labeling of Cremona's tables [11], one can check that $E_0 =$ `1680G2`, $E_1 =$ `20160BG2`, $E_2 =$ `960H2`, $E_3 =$ `840H2` and $E_4 =$ `360E2`.

So, if one of such elliptic curves has a finite number of rational points, then the problem of computing $C(\mathbb{Q})$ becomes easy. Now, it is a straightforward computation to check that the torsion subgroup of $E_n(\mathbb{Q})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. On the other hand, the set $\Phi_n([\pm 1 : \pm 1 : \pm 1 : \pm 1 : \pm 1])$ has cardinality eight. Therefore the rank of $E_n(\mathbb{Q})$ is greater than one, that is, we cannot use this argument to determine $C(\mathbb{Q})$. In fact, we can easily compute by descent (or, better, using some algebraic computational system like `Magma` [8] or `Sage` [17], or still better, using Cremona's tables) that

$\operatorname{rank}_{\mathbb{Z}} E_n(\mathbb{Q}) = 1$ for $n \neq 1$ and $\operatorname{rank}_{\mathbb{Z}} E_1(\mathbb{Q}) = 2$. This result also precludes the use of Chabauty's method or Dem'yanenko–Manin's method to compute $C(\mathbb{Q})$.

3. Two descent and covering collections. In order to actually compute the rational points on the curve C, we will apply the covering collections technique, as developed by Coombes and Grant [10], Wetherell [19] and others, and specifically a modification of what is now called the elliptic Chabauty method developed by Flynn, Wetherell and Bruin [12, 5, 6].

The method has two steps. Suppose we have a curve C over a number field K and an unramified map $\chi : C' \to C$ of degree greater than one and may be defined over a finite extension L of K. We consider all the distinct unramified coverings $\chi^{(s)} : C'^{(s)} \to C$ formed by twists of the given one, and we get

$$C(K) = \bigcup_s \chi^{(s)}(\{P \in C'^{(s)}(L) : \chi^{(s)}(P) \in C(K)\}),$$

the union being disjoint. Only a finite number of twists have rational points, and the finite (larger) set of twists having points locally everywhere can be explicitly described. The first step is to compute this set of twists, and the second to compute the points $P \in C'^{(s)}(L)$ such that $\chi^{(s)}(P) \in C(K)$. The second step depends on having nice quotients of the curves $C'^{(s)}$, for example genus one quotients, where it is possible to do the computations. In this section we will concentrate on the first step.

The coverings we are going to consider are Galois coverings with Galois group isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. Such coverings are in principle easy to construct. One only needs to have an isogeny map from an abelian variety A to the jacobian $\operatorname{Jac}(C)$ of the curve C with kernel isomorphic, as group scheme, to the group $(\mathbb{Z}/2\mathbb{Z})^2$. Since in our case the jacobian $\operatorname{Jac}(C)$ is isogenous to a product of elliptic curves E_i, the coverings can be constructed by choosing two such elliptic curves and one degree two isogeny in each of them.

Moreover, the elliptic curves E_i have all the 2-torsion points defined over $\mathbb{Q}$, hence the coverings we are searching for will be defined over $\mathbb{Q}$. On the other hand, the genus one quotients of such coverings that we will use in the next section are, in general, not defined over $\mathbb{Q}$, but in a quadratic or biquadratic extension. The way we will construct the coverings, by using a factorization of quartic polynomials, will also give us directly the genus one quotient and the field where it is defined.

In order now to construct the coverings of the curve C, we first rewrite the equations of the curve in the following form:

$$C : \begin{cases} y^2 = q(t) = 36t^4 - 72t^3 + 72t^2 - 60t + 25, \\ z^2 = p(t) = 36t^4 + 96t^3 - 236t^2 + 80t + 25, \end{cases}$$

by setting $t = \frac{x_3+x_1}{x_3-x_0}$. This model has two natural maps to the genus one curves $F_4 : y^2 = q(t)$ and $F_2 : z^2 = p(t)$, whose jacobians are E_4 and E_2 respectively.

First, we concentrate on the unramified degree two coverings of the genus one curve given by a quartic model. If F is such a genus one curve defined over a field K, given by the equation $y^2 = r_1(x)r_2(x)$, where $r_1(x)$ and $r_2(x)$ are degree two polynomials defined over an extension L of K, we consider the degree two unramified covering $\chi : F' \to F$ with affine part in $\mathbb{A}^3$ given by the zeros of the polynomials $y_1^2 = r_1(x)$ and $y_2^2 = r_2(x)$, the map being given by $\chi(x, y_1, y_2) = (x, y_1y_2)$. For any $\delta \in L^*$, we consider the curve $F'^{(\delta)}$ given by the equations $\delta y_1^2 = p_1(x)$ and $\delta y_2^2 = p_2(x)$, and the map to F defined by $\chi^{(\delta)}(x, y_1, y_2) = (x, y_1y_2/\delta^2)$. Then $F'^{(\delta)}$ are all the quadratic twists of F', and there exists a finite set $\Delta_L(\chi) \subset L^*$ such that

$$F(K) \subseteq \bigcup_{\delta\in\Delta_L(\chi)} \chi^{(\delta)}(\{(x, y_1, y_2) \in F'^{(\delta)}(L) : x \in K \text{ or } x = \infty\}).$$

First, we consider the case of F_2, given as $y^2 = p(t)$, where

$$p(t) = 36t^4 + 96t^3 - 236t^2 + 80t + 25 = (6t^2 - 4t - 1)(6t^2 + 20t - 25).$$

Lemma 3. *Consider the degree two covering defined over $\mathbb{Q}$ given by*

$$F_2' : \begin{cases} z_1^2 = p_1(t) = 6t^2 - 4t - 1, \\ z_2^2 = p_2(t) = 6t^2 + 20t - 25, \end{cases}$$

together with the natural map $\psi_2 : F_2' \to F_2$ given by $\psi_2(t, z_1, z_2) = (t, z_1z_2)$. Then $\Delta_{\mathbb{Q}}(\psi_2) = \{\pm 1, \pm 6\}$, hence

$$F_2(\mathbb{Q}) \subseteq \bigcup_{\delta\in\{\pm1,\pm6\}} \psi_2^{(\delta)}(\{(t, z_1, z_2) \in F_2'^{(\delta)}(\mathbb{Q})\}).$$

Proof. It is easy to show and well-known that ψ_2 is an unramified degree two covering of F_2. Since $F_2'(\mathbb{Q}) \neq \emptyset$, because it contains the point $P' := (1, 1, 1)$, we can identify F_2' with an elliptic curve E_2', by sending P' to O, and identify F_2 with the elliptic curve E_2 by sending $P := \chi(P') = (1, 1)$ to O. We then get an unramified degree two covering $\phi_2 : E_2' \to E_2$, which must be a degree two isogeny. With appropriate choices of the identifications, we can get this isogeny in the standard form (see, for example, [16, III.4.5] or [9, §8.2]). After some computations we get the map $\phi_2 : E_2' \to E_2$ defined by

$$\phi_2(x, y) = \left(\frac{y^2}{4x^2}, \frac{y(x^2 - 2500)}{8x^2}\right),$$

where $E_2 : y^2 = x(x + 4)(x + 54)$ and $E_2' : y^2 = x(x^2 - 116x + 2500)$.

Now, the quadratic twists $F_2'^{(\delta)}$ which locally have rational points correspond to the elements of the Selmer group $\mathrm{Sel}(\phi_2)$. After identifying $\mathrm{Sel}(\phi_2)$ with a subgroup of $\mathbb{Q}^*/(\mathbb{Q}^*)^2$ in the standard way, the identification sends $\delta \in \mathbb{Q}^*$ to its class modulo squares. A standard 2-descent calculation gives that $\mathrm{Sel}(\phi_2) = \{\pm 1, \pm 6\}$. But now, by using the fact that $E_2(\mathbb{Q})$ contains the points $(0,0)$, $(-54,0)$ and $(36,-360)$, one can see that all the elements of the Selmer group $\mathrm{Sel}(\phi_2)$ correspond to elements of $E(\mathbb{Q})$. These are exactly the δ's such that $F_2'^{(\delta)}(\mathbb{Q}) \neq \emptyset$. ∎

Now we consider the curve F_4, given as $y^2 = q(t)$, where $q(t) = 36t^4 - 72t^3 + 72t^2 - 60t + 25$. Observe that the polynomial $q(t)$ is irreducible over $\mathbb{Q}$, but it factorizes over some quadratic extensions as product of two degree two polynomials. Over $\mathbb{Q}(\sqrt{6})$ we have

$$q(t) = (5 + 2\sqrt{6})(6t^2 - 2(3+\sqrt{6})t + 5) \cdot (5 - 2\sqrt{6})(6t^2 - 2(3-\sqrt{6})t + 5).$$

Lemma 4. *Consider the degree two covering defined over $\mathbb{Q}(\sqrt{6})$ given by*

$$F_4' : \begin{cases} y_1^2 = q_1(t) = (5 + 2\sqrt{6})(6t^2 - 2(3+\sqrt{6})t + 5), \\ y_2^2 = q_2(t) = (5 - 2\sqrt{6})(6t^2 - 2(3-\sqrt{6})t + 5), \end{cases}$$

together with the natural map $\psi_4 : F_4' \to F_4$ given by $\psi_4(t, y_1, y_2) = (t, y_1 y_2)$. Then $\Delta_{\mathbb{Q}(\sqrt{6})}(\psi_4) = \{1, 2, 5, 10\}$, hence

$$F_4(\mathbb{Q}) \subseteq \bigcup_{\delta \in \{1,2,5,10\}} \psi_4^{(\delta)}(\{(t, y_1, y_2) \in F_4'^{(\delta)}(\mathbb{Q}(\sqrt{6})) : t \in \mathbb{Q} \text{ or } t = \infty\}).$$

Proof. As in the proof of the lemma above, observe that $F_4'(\mathbb{Q}(\sqrt{6}))$ contains the point $(1,1,1)$ such that $\psi_4(1,1,1) = (1,1) \in F_4(\mathbb{Q})$. Then the degree two covering $\psi_4 : F_4' \to F_4$ defined over $\mathbb{Q}(\sqrt{6})$ corresponds, by taking some isomorphisms to the respective jacobians, to the 2-isogeny $\phi_4 : E_4' \to E_4$ defined by

$$\phi_4(x, y) = \left(\frac{y^2}{4x^2}, \frac{y(x^2 - 9)}{8x^2} \right),$$

where $E_4 : y^2 = x(x-12)(x-15)$ and $E_4' : y^2 = x(x^2 + 54x + 9)$, which is the dual isogeny of the 2-isogeny corresponding to the 2-torsion point $P = (0,0) \in E_4(\mathbb{Q})$. Now, a descent computation shows that $\mathrm{Sel}(\phi_4) = \{1, 3, 2, 6, 5, 15, 10, 30\}$. But observe now that two δ and δ' in $\mathbb{Q}$ that are equivalent modulo squares over $\mathbb{Q}(\sqrt{6})^*$ give isomorphic coverings $\psi_4^{(\delta)}$. Hence we only need to consider the set $\Delta_{\mathbb{Q}(\sqrt{6})}(\psi_4)$ which is $\mathrm{Sel}(\phi_4)$ modulo $(\mathbb{Q}(\sqrt{6})^*)^2$, which gives the result. ∎

We now take the unramified covering $\chi : C' \to C$ defined by the equations:

$$C' : \begin{cases} y_1^2 = q_1(t) = (5+2\sqrt{6})(6t^2 - 2(3+\sqrt{6})t + 5), \\ y_2^2 = q_2(t) = (5-2\sqrt{6})(6t^2 - 2(3-\sqrt{6})t + 5), \\ z_1^2 = p_1(t) = 6t^2 - 4t - 1, \\ z_2^2 = p_2(t) = 6t^2 + 20t - 25, \end{cases}$$

which is a curve of genus 17.

The lemmas above yield the twists that have to be considered.

COROLLARY 5. *The set of relevant twists is equal to*

$$\Delta := \{(\delta_2, \delta_4) \in \mathbb{Q}(\sqrt{6})^* : \delta_i \in \Delta_{\mathbb{Q}(\sqrt{6})}(\phi_i),\ i = 2, 4\},$$

where $\Delta_{\mathbb{Q}(\sqrt{6})}(\phi_2) = \{\pm 1\}$ *and* $\Delta_{\mathbb{Q}(\sqrt{6})}(\phi_4) = \{1, 2, 5, 10\}$, *which correspond to a set of representatives in* $\mathbb{Q}(\sqrt{6})$ *of the images of Selmer groups of* ϕ_i $(i = 2, 4)$ *in* $\mathbb{Q}(\sqrt{6})^*/(\mathbb{Q}(\sqrt{6})^*)^2$ *via the natural maps. Hence,*

$$C(\mathbb{Q}) \subseteq \bigcup_{\delta \in \Delta} \chi^{(\delta)}(\{(t, y_1, y_2, z_1, z_2) \in C'^{(\delta)}(\mathbb{Q}(\sqrt{6})) : t \in \mathbb{Q} \text{ or } t = \infty\}),$$

where $C'^{(\delta_2,\delta_4)}$ *is the curve defined by*

$$C'^{(\delta_2,\delta_4)} : \{\delta_4 y_1^2 = q_1(t),\ \delta_4 y_2^2 = q_2(t),\ \delta_2 z_1^2 = p_1(t),\ \delta_2 z_2^2 = p_2(t)\},$$

where $\chi^{(\delta_2,\delta_4)}(t, y_1, y_2, z_1, z_2) = (x, y_1 y_2/\delta_4, z_1 z_2/\delta_2)$.

Proof. By the lemmas above, we only need to observe that $\Delta_{\mathbb{Q}}(\phi_2) = \{\pm 1, \pm 6\}$ becomes, after taking the image in $\mathbb{Q}(\sqrt{6})^*/(\mathbb{Q}(\sqrt{6})^*)^2$, the set $\Delta_{\mathbb{Q}(\sqrt{6})}(\phi_2) = \{\pm 1\}$. ∎

One can reduce the set of twists even further by using the natural automorphisms of C given by interchanging the sign of one of the coordinates (in the first model of C).

COROLLARY 6. *Let* τ_i *be the automorphisms of* C *given by* $\tau_i(x_i) = -x_i$, *and* $\tau_i(x_j) = x_j$ *if* $j \neq i$, *for* $i = 0, 1, 2, 3, 4$, *and let* Υ *be the subgroup they generate. Let* $\Delta' = \{(1, 1), (-1, 1)\}$. *Then, for any* $P \in C(\mathbb{Q})$, *there exist* $\tau \in \Upsilon$ *and* $\delta \in \Delta'$ *such that*

$$\tau(P) \in \chi^{(\delta)}(\{(t, y_1, y_2, z_1, z_2) \in C'^{(\delta)}(\mathbb{Q}(\sqrt{6})) : t \in \mathbb{Q} \text{ or } t = \infty\}),$$

where $\chi^{(\pm 1,1)}(t, y_1, y_2, z_1, z_2) = (x, \pm y_1 y_2, z_1 z_2)$.

Proof. It is enough to show that for any $P \in C(\mathbb{Q})$ and any $\delta_4 \in \Delta_{\mathbb{Q}(\sqrt{6})}(\phi_4)$, there exist $\delta_2 \in \Delta_{\mathbb{Q}(\sqrt{6})}(\phi_2)$ and $\tau \in \Upsilon$ such that $\tau(P) \in \chi^{(\delta_2,\delta_4)}(C'^{(\delta_2,\delta_4)}(\mathbb{Q}(\sqrt{6})))$. Therefore, the problem reduces to showing that for any $\delta_4 \in \Delta_{\mathbb{Q}(\sqrt{6})}(\phi_4)$, there exists $\tau \in \Upsilon$ such that the image of $\varrho_2(\tau(P))$

in $\mathrm{Sel}(\phi_2)$ is equal to δ_2 modulo $(\mathbb{Q}(\sqrt{6})^*)^2$, where $\varrho_2 : C \to F_2$ is the map given in Section 2.

Observe that the involutions τ_i for $i = 0, 1, 3, 4$ determine involutions in F_4, which in turn determine involutions τ_i' in E_4 on fixing an isomorphism between F_4 and E_4. These involutions must be of the form $\tau_i'(Q) = -Q + Q_i$ for some $Q_i \in E(\mathbb{Q})$, since they have fixed points. Hence, the involutions τ_i' are determined once we know the image of a single point Q. Thus, if we know the result for just one point $P \in C(\mathbb{Q})$, we will obtain the result for all points in $C(\mathbb{Q})$.

Take $P := [1 : 1 : 1 : 1 : 1]$. Then one shows easily that the image of $\varrho_2(\tau_i(P))$ in the Selmer group $\mathrm{Sel}(\phi_2)$ for $i = 0, 1, 3$, together with $\varrho_2(P)$, covers all the group, which proves the result. ∎

Remark 7. An easy computation shows that the involutions τ_i take the following form in the model of C given by $y^2 = q(t)$ and $z^2 = p(t)$:

$$\tau_0(t, y, z) = \left(\frac{6t-5}{6(t-1)}, \frac{y}{6(t-1)^2}, \frac{z}{6(t-1)^2}\right),$$

$$\tau_1(t, y, z) = \left(\frac{5(t-1)}{6t-5}, \frac{5y}{(6t-5)^2}, \frac{5z}{(6t-5)^2}\right),$$

$$\tau_3(t, y, z) = \left(\frac{5}{6t}, \frac{5y}{6t^2}, \frac{5z}{6t^2}\right),$$

$\tau_2(t, y, z) = (t, y, -z)$ and $\tau_4(t, y, z) = (t, -y, z)$. These can also be used to show the last corollary.

Observe that the known points in $C(\mathbb{Q})$, corresponding to the points $[1 : \pm1 : \pm1 : \pm1 : \pm1]$ and $[1 : \pm3 : \pm5 : \pm7 : \pm9]$, give rise to the points in $C'^{(1,1)}(\mathbb{Q}(\sqrt{6}))$ with $t = 1$ and in $C'^{(-1,1)}(\mathbb{Q}(\sqrt{6}))$ with $t = 1/2$, respectively.

Now, to compute the points (t, y_1, y_2, z_1, z_2) in $C'^{(\pm1,1)}(\mathbb{Q}(\sqrt{6}))$ such that $t \in \mathbb{Q}$, we consider the natural genus one quotients of $C'^{(\pm1,1)}$ defined by

$$H_{i,j}^{\pm} : \pm w^2 = q_i(t)p_j(t)$$

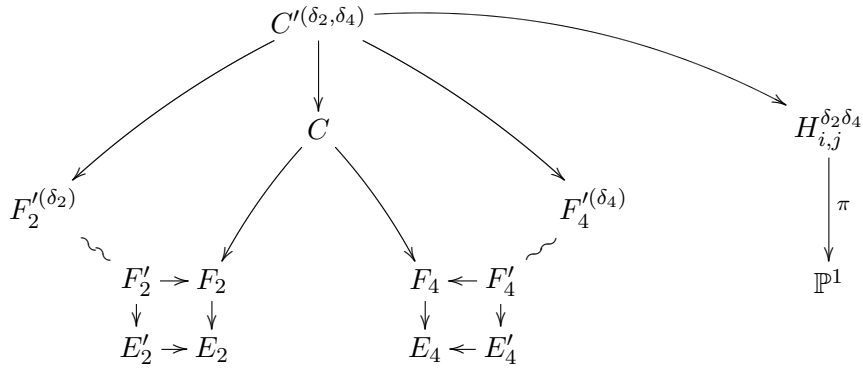
for $i, j = 1, 2$. We have four of them for any sign, corresponding in fact to the factors of a natural genus four quotient of any of the curves $C'^{(\pm1,1)}$, which is defined over $\mathbb{Q}$.

Hence, we only need to compute the points

$$\{(t, w) \in H_{(i,j)}^{\pm} : t \in \mathbb{Q} \text{ or } t = \infty\}$$

for some (i, j). But this can be done by using the elliptic Chabauty method.

The following diagram illustrates all the curves and morphisms involved in our problem:



4. An elliptic Chabauty argument. Our aim in this section is to compute, for any choice of sign, all the $\mathbb{Q}(\sqrt{6})$-rational points on some of the curves $H_{i,j}^{\pm} : \pm w^2 = q_i(t)p_j(t)$ such that $t \in \mathbb{Q}$. We will be able to do this once we establish that the jacobians of the curves have rank 0 or 1 over $\mathbb{Q}(\sqrt{6})$, a condition coming from the Chabauty technique.

Since we only need to show this result for just one (i,j), we will do it for $(1,1)$ for both signs. This choice is not totally arbitrary, since one can show that all the cases with $j = 2$ have rank 2, hence they do not satisfy the necessary conditions.

We will denote by $H^{\pm} = H_{1,1}^{\pm}$ the genus one curves defined over $\mathbb{Q}(\sqrt{6})$ by the equations

$$\pm w^2 = q_1(t)p_1(t) = (5 + 2\sqrt{6})(6t^2 - 2(3 + \sqrt{6})t + 5)(6t^2 - 4t - 1).$$

LEMMA 8. *Consider the points* $P_{\pm}^{+} := (1, \pm 1) \in H^{+}(\mathbb{Q}(\sqrt{6}))$ *and* $P_{\pm}^{-} := (1/2, \pm(-2\sqrt{6} - 3)/2) \in H^{-}(\mathbb{Q}(\sqrt{6}))$. *Then the curves* $H^{\pm}$ *are isomorphic over* $\mathbb{Q}(\sqrt{6})$ *to their corresponding jacobians* $J^{\pm} := \mathrm{Jac}(H^{\pm})$, *which are given by the equations*

$$J^{+} : y^2 + (-2\sqrt{6} - 10)xy + (38\sqrt{6} + 22)y = x^3 + (-24\sqrt{6} - 34)x^2 + (448\sqrt{6} + 1253)x$$

and

$$J^{-} : y^2 + (42\sqrt{6} + 422)xy + (-291822\sqrt{6} - 113902)y = x^3 + (-8076\sqrt{6} - 33466)x^2 + (67635708\sqrt{6} + 141575953)x,$$

by isomorphisms $\mu^{\pm} : H^{\pm} \to J^{\pm}$ *sending the points* $P_{+}^{\pm}$ *to the zero point in* $J^{\pm}$. *Moreover,* $\mu^{\pm}(P_{-}^{\pm}) = -(0,0)$.

Finally, a point $(t,w) \in H^{\pm}(\mathbb{Q}(\sqrt{6}))$ *has* $t \in \mathbb{Q}$ *if and only if* $\pi^{\pm}(x,y) \in \mathbb{P}^1(\mathbb{Q})$, *where* $(x,y) = \mu^{\pm}(t,w) \in J^{\pm}(\mathbb{Q}(\sqrt{6}))$ *and*

$$\pi^{+}(x,y) = \frac{y}{2x+y}, \quad \pi^{-}(x,y) = \frac{y}{300x+2y}.$$

Proof. The first part is a standard computation (see for example [9, §7.2.3]). The inverses of the maps $\mu^{\pm}$ are defined by

$$\nu^{+}(x,y) := \left(\frac{y}{2x+y}, \frac{2x^3+(-24\sqrt{6}-34)x^2+(2\sqrt{6}+10)xy-y^2}{2x+y}\right)$$

and

$$\nu^{-}(x,y) := \left(\frac{y}{300x+150y}, \frac{-2(2\sqrt{6}+3)x^3+(91160\sqrt{6}+197310)x^2+(970\sqrt{6}+1770)xy+(2\sqrt{6}+3)y^2}{150^3(150x+y)}\right).$$

The map $\pi^{\pm}$ is the composition of $\nu^{\pm}$ and the map $H^{\pm} \to \mathbb{P}^1$ sending (t,w) to t. ∎

Remark 9. Note that $j(J^{\pm}) = 140608/245 \in \mathbb{Q}$. Moreover, $J^{\pm}$ is isomorphic to the $\pm(\sqrt{6}-3)$-twist of the elliptic curve defined over $\mathbb{Q}$ given by the Weierstrass model $y^2 = x^3 + 312x - 3008$ (which is `80640CU2` in Cremona's tables).

Lemma 10. *The elliptic curves $J^{\pm}$ both have rank 1 over $\mathbb{Q}(\sqrt{6})$. We have*

$$J^{\pm}(\mathbb{Q}(\sqrt{6})) \supset S^{\pm} = \langle T^{\pm}, (0,0)\rangle,$$

where

$$T^{+} := (2\sqrt{6}-7, -16\sqrt{6}-34), \quad T^{-} := (-462\sqrt{6}-2767, 301500\sqrt{6}+699000)$$

are 2-torsion points and $(0,0)$ is of infinite order. The subgroups $S^{\pm}$ are of finite index not divisible by any prime < 14.

Proof. This is shown by a standard 2-descent argument, using either `Magma`, `Sage` or `PARI` (see for example [9, §8.3]). The non-divisibility property of the index can be shown easily by proving that in both cases the point $(0,0)$ of infinite order is not a p-multiple of another point in $J^{\pm}(\mathbb{Q}(\sqrt{6}))$ for any prime $p < 14$. ∎

Now, we are in a position to apply the Chabauty technique. We need to choose a prime p of good reduction for $J^{\pm}$, and also inert in $\mathbb{Q}(\sqrt{6})$ (the technique can also be used for split primes, but in a slightly different form, see for example [5]). Denote by $J_p^{\pm}$ the reduction modulo p of $J^{\pm}$, which is an elliptic curve over $\mathbb{F}_{p^2} := \mathbb{F}_p(\sqrt{6})$, and by $\mathrm{red}_p : J^{\pm}(\mathbb{Q}(\sqrt{6})) \to J_p^{\pm}(\mathbb{F}_{p^2})$ the reduction map. Then the elliptic Chabauty method will allow us to bound, for each point R in $J_p^{\pm}(\mathbb{F}_{p^2})$, the number of points Q in $J^{\pm}(\mathbb{Q}(\sqrt{6}))$ such that $\mathrm{red}_p(Q) = R$ and $\pi^{\pm}(Q) \in \mathbb{P}^1(\mathbb{Q})$. Denote this set of points by

$$\Omega_{\pm,p}(R) := \{Q \in J^{\pm}(\mathbb{Q}(\sqrt{6})) : \pi^{\pm}(Q) \in \mathbb{P}^1(\mathbb{Q}) \text{ and } \mathrm{red}_p(Q) = R\}.$$

Clearly, we have

$$\{Q \in J^{\pm}(\mathbb{Q}(\sqrt{6})) : \pi^{\pm}(Q) \in \mathbb{Q}\} = \bigsqcup_{R \in J_p^{\pm}(\mathbb{F}_{p^2})} \Omega_{\pm,p}(R),$$

for any choice of an inert good reduction prime p. So, if we compute these sets $\Omega_{\pm,p}(R)$ for some p and all R, we will have computed the sets we are interested in.

We will choose the primes $p = 11$ and $p = 13$, depending on the sign of the case considered.

PROPOSITION 11. *Set $p_+ = 11$ and $p_- = 13$. Then $\Omega_{\pm,p_\pm}(\widetilde{R}) \neq \emptyset$ if and only if $\widetilde{R} = O$ or $\widetilde{R} = -(0,0)$.*

Proof. First of all, observe that $\Omega_{\pm,p_\pm}(O) \neq \emptyset$ and $\Omega_{\pm,p_\pm}(-(0,0)) \neq \emptyset$ since they contain the points O and $-(0,0)$, respectively.

In order to show that the remaining subsets are empty, we will argue modulo $p_\pm^m$ for various powers of $p_\pm$. Denote by $\mathcal{O}$ the ring of integers of $\mathbb{Q}(\sqrt{6})$, by $\mathcal{J}$ the Néron model of J over $\mathcal{O}$ and by $\pi^{\pm}_{p_\pm^n} : \mathcal{J}^{\pm}_{\mathcal{O}/p_\pm^n\mathcal{O}} \to \mathbb{P}^1$ the reduction modulo $p_\pm^n$ of the map $\pi^{\pm}$, which is a well-defined map of schemes over $\mathcal{O}/p_\pm^n\mathcal{O}$. Observe that for good reduction primes p as we have, $\mathcal{J}^{\pm}_{\mathcal{O}/p^n\mathcal{O}}$ is an abelian scheme.

First we work modulo $p_\pm$. We find that $(0,0)$ has order 8 in $J^+_{11}(\mathbb{F}_{11^2})$ and order 12 in $J^-_{13}(\mathbb{F}_{13^2})$. Since the point $(0,0)$ is not divisible by 2 and 3 in $J^{\pm}(\mathbb{Q}(\sqrt{6}))$ as shown in Lemma 10, in both cases we get $\mathrm{red}_{p_\pm}(S^{\pm}) = \mathrm{red}_{p_\pm}(J^{\pm}(\mathbb{Q}(\sqrt{6})))$, so we can work with the subgroup $S^{\pm}$.

One easily computes that the only points R in $\mathrm{red}_{p_\pm}(S^{\pm})$ such that $\pi^{\pm}_{p_\pm}(R) \in \mathcal{O}/p_\pm\mathcal{O} = \mathbb{F}_{p_\pm^2}$ are

$$O, -(0,0), T^+ + 2(0,0), T^+ - 3(0,0) \in \mathrm{red}_{p_+}(S^+)$$

and

$$O, -(0,0), 4(0,0), -5(0,0) \in \mathrm{red}_{p_-}(S^-).$$

Since $\Omega_{\pm,p_\pm}(R)$ is obviously empty if $\pi^{\pm}_{p_\pm}(R)$ is not in $\mathbb{F}_{p_\pm^2}$, we only need to show that $\Omega_{+,p_+}(R) = \emptyset$ if $R = T^+ + 2(0,0)$ or $T^+ - 3(0,0)$, and that $\Omega_{-,p_-}(R) = \emptyset$ if $R = 4(0,0)$ or $-5(0,0)$.

We start with the $+$ case. In this case one computes all the points in $\mathcal{J}^+_{\mathcal{O}/11^2\mathcal{O}}$ which are equal to $R = T^+ + 2(0,0)$ or to $T^+ - 3(0,0)$ modulo 11 (there are 22 of them), and then we compute their images under $\pi^+_{11^2}$. But for any of these 22 points, the image is not in $\mathbb{Z}/11^2\mathbb{Z} \hookrightarrow \mathcal{O}/11^2\mathcal{O}$, hence $\Omega_{+,p_+}(R) = \emptyset$ for both points.

The $-$ case is done similarly, but one needs to work modulo 13^3, since all the lifts to 13^2 have image in $\mathbb{Z}/13^2\mathbb{Z}$ with respect to the map $\pi^+_{13^2}$. Modulo 13^3, the total number of points considered is $2 \cdot 13^2$. ∎

PROPOSITION 12. *The sets $\Omega_{\pm,p_\pm}(O)$ and $\Omega_{\pm,p_\pm}(-(0,0))$ contain, for any sign, only one point.*

Proof. We use the Chabauty argument. First of all, recall that the order of $(0,0)$ modulo $p_\pm$ is $m_+ := 8$ in the $+$ case and $m_- := 12$ in the $-$ case. Hence, any point in $\Omega_{\pm,p_\pm}(R)$ must be of the form $R+n(m_\pm(0,0))$ for some $n \in \mathbb{Z}$. Since in both cases, $R = O$ and $R = -(0,0)$, the point R is in $\Omega_{\pm,p_\pm}(R)$, we want to show that the only solution is $n = 0$ in all cases. We will work modulo $p_\pm^2$.

We define the *z-coordinate* of the point (x,y) (with respect to the given equation of $J^\pm$) to be $z := -x/y$ (as a point in $\mathbb{P}^1$).

Denote by $z_{\pm,p}$ the z-coordinate of the point $m_\pm(0,0)$ modulo $p_\pm^2$. We get $z_{+,11} = 11-55\sqrt{6} \in \mathcal{O}/11^2\mathcal{O}$ and $z_{-,13} = 26-39\sqrt{6} \in \mathcal{O}/13^2\mathcal{O}$. Because we are working modulo $p_\pm^2$, we have

$$z\text{-coord}(n(m_\pm(0,0))) = nz_{\pm,p_\pm} \pmod{p_\pm^2\mathcal{O}}$$

(a fact that can be proved using the formal logarithm and exponential of the elliptic curves $J^\pm$).

Now, we can express the function $\pi_\pm(P)$ at any point P as a power series in the z-coordinate of P. We get, for the $+$ case,

$$\pi_+(z) = 1 + 2z + 4z^2 + 8z^3 + O(z^4),$$

and for the $-$ case,

$$\pi_-(z) = 1/2 + 75z + 11250z^2 + 1687500z^3 + O(z^4).$$

First we treat the point O. We find that $\pi_\pm(n(m_\pm(0,0)))$ can be expressed as a power series $\Theta(n)$ in n with coefficients in $\mathbb{Q}(\sqrt{6})$. We express this power series as $\Theta(n) = \Theta_0(n)+\sqrt{6}\,\Theta_1(n)$, with $\Theta_i(n)$ now being a power series with coefficients in $\mathbb{Q}$. Then $\pi_\pm(n(m_\pm(0,0))) \in \mathbb{Q}$ for some $n \in \mathbb{Z}$ if and only if $\Theta_1(n) = 0$ for that n. Observe also that since $\pi_\pm(O) \in \mathbb{Q}$, we get $\Theta_1(0) = 0$, so $\Theta_1(n) = j_1 n + j_2 n^2 + j_3 n^3 + \cdots$. To conclude, we use the Strassmann Theorem: if the $p_\pm$-adic valuation of j_1 is strictly smaller than the $p_\pm$-adic valuation of j_i for any $i > 1$, then this power series has only one zero in the p-adic ring $\mathbb{Z}_{p_\pm}$, and this zero is $n = 0$. In fact, one can easily show that this power series is such that the $p_\pm$-adic valuation of j_i is always greater than or equal to i, so if we show that $j_1 \not\equiv 0 \pmod{p_\pm^2}$ we are done.

Since the z-coordinate of $m_\pm(0,0)$ is congruent to 0 modulo $p_\pm$, to compute $\pi_\pm(z\text{-coord}(n(m_\pm(0,0))))$ modulo $p_\pm^2$ we only need the power series up to degree 1. We get

$$\pi_+(z\text{-coord}(n(m_+(0,0)))) = (22 - 44\sqrt{6})n + 1 \pmod{11^2}$$

and

$$\pi_-(z\text{-coord}(n(m_-(0,0)))) = (-78 - 52\sqrt{6})n - 84 \pmod{13^2}.$$

Hence, for the + case, $\Theta_1(n) = -44n$ modulo 11^2, thus the valuation of j_1 is 1, and we are done. Similarly, for the − case, $\Theta_1(n) = -52n$ modulo 13^2, and again we are done.

In order to consider the other point $-(0,0)$ one can either compute directly $\pi_\pm(z\text{-coord}(-(0,0)+n(m_\pm(0,0))))$ as a power series in n and apply the same type of argument, or observe that there is an involution in $J^\pm$ (as a genus one curve) that interchanges the points O and $-(0,0)$ and preserves the function $\pi_\pm$; it corresponds to the hyperelliptic involution on $H^\pm$ sending (t,w) to $(t,-w)$. ∎

Hence, by using the results just proved, we finally obtain the following.

COROLLARY 13. *The only points* $(t_\pm, w) \in H^\pm(\mathbb{Q}(\sqrt{6}))$ *with* $t_\pm \in \mathbb{Q}$ *are the points with* $t_+ = 1$ *and with* $t_- = 1/2$.

5. Proof of Theorem 1. By using the results proved in the last two sections, we finally obtain all the rational points on the curve C.

THEOREM 14. $C(\mathbb{Q}) = \{[\pm1 : \pm1 : \pm1 : \pm1 : \pm1], [\pm1 : \pm3 : \pm5 : \pm7 : \pm9]\}$.

Proof. We review briefly the steps we followed. By using Corollary 6, we deduce that, for any $P \in C(\mathbb{Q})$, there exists a sign $\pm$ and an involution $\tau \in \Upsilon$ of C such that

$$\tau(P) \in \chi^{((\pm1,1))}(\{(t, y_1, y_2, z_1, z_2) \in C'^{((\pm1,1))}(\mathbb{Q}(\sqrt{6})) : t \in \mathbb{Q}\}).$$

Moreover, the points $[\pm1 : \pm1 : \pm1 : \pm1 : \pm1]$ and $[\pm1 : \pm3 : \pm5 : \pm7 : \pm9]$ come, respectively, from the points in $C'^{((1,1))}$ with $t = 1$ and the points in $C'^{((-1,1))}$ with $t = 1/2$.

Next, we consider the genus one quotients $H^\pm$ of the curves $C'^{((\pm1,1))}$ with quotient maps defined by $(t, y_1, y_2, z_1, z_2) \mapsto (t, y_1 z_1)$. We find that the points $(t, y_1, y_2, z_1, z_2) \in C'^{((\pm1,1))}(\mathbb{Q}(\sqrt{6}))$ go to points $(t, w) \in H^\pm(\mathbb{Q}(\sqrt{6}))$. Finally, by Corollary 13, the only points in H^+ with $t \in \mathbb{Q}$ are the ones with $t = 1$, and the only points in H^- with $t \in \mathbb{Q}$ are the ones with $t = 1/2$. This proves the result. ∎

Proof of Theorem 1. If N is odd, then translating by $-r-(N-1)/2$ we can suppose $r = (N-1)/2$. Thus, $b = 0$ and $f(x) = ax^2+c$ is symmetric with respect to $x = 0$. In the even case, we can apply translation by $-r - N/2$, and suppose that $r = -N/2$. Thus, $b = a$ and $f(x) = a(x^2 + x) + c$ is symmetric with respect to $x = -1/2$. Now, the existence of $f(x)$ is equivalent to the existence of $x_k \in \mathbb{Z}$ such that $f(k) = x_k^2$, $k = 0, 1, \ldots, s_N$, where $s_N = (N-1)/2$ or $s_N = N/2 - 1$ depending on whether N is odd or even, respectively. Note that for $N \le 4$ it is trivial to prove that there are infinitely many non-square quadratic polynomials satisfying the hypothesis. If $N = 5$

(resp. $N = 6$), then they satisfy $3x_0^2+x_2^2 = 4x_1^2$ (resp. $2x_0^2+x_2^2 = 3x_1^2$), which is a conic in $\mathbb{P}^2$ with infinitely many rational points. If $N = 7$ (resp. $N = 8$), then they satisfy $3x_0^2+x_2^2 = 4x_1^2$ and $8x_0^2+x_3^2 = 9x_1^2$ (resp. $2x_0^2+x_2^2 = 3x_1^2$ and $5x_0^2+x_3^2 = 6x_1^2$), which is isomorphic to the elliptic curve $y^2 = x(x-5)(x+27)$ (resp. $y^2 = x(x-12)(x-15)$) and it is denoted by `30A2` (resp. `360E2`) in Cremona's table with Mordell–Weil group isomorphic to $\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/6\mathbb{Z}$ (resp. $\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}$). Therefore if $N \geq 7$ is odd the proof of the theorem is finished. If $N = 8$, since the rank of the underlying elliptic curve is non-zero, there are infinitely many non-square quadratic polynomials satisfying the hypothesis. The remaining case is when $N \geq 10$ even. The characterization given in Section 2 shows that if $N = 10$ then any quadratic polynomial $f(x) \in \mathbb{Z}[x]$ satisfying the hypothesis of the theorem corresponds to a point $P \in C(\mathbb{Q})$. In Theorem 14 we have proved that the unique points of $C(\mathbb{Q})$ are $[\pm1 : \pm1 : \pm1 : \pm1 : \pm1]$, $[\pm1 : \pm3 : \pm5 : \pm7 : \pm9]$, which correspond to the constant polynomials and to $f(x) = (2x+1)^2$ respectively. ∎

Data. All the `Magma` sources are available from the first author's webpage.

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