SQUARES IN ARITHMETIC PROGRESSION OVER NUMBER FIELDS

XAVIER XARLES

Abstract. We show that there exists an upper bound for the number of squares in arithmetic progression over a number field that depends only on the degree of the field. We show that this bound is 5 for quadratic fields, and also that the result generalizes to k-powers for integers $k > 1$.

In this note we are dealing with the following natural problem: Given a number field K, is there a maximum for the number of distinct elements $a_0, \dots, a_n$ in K such that $a_i^2 - a_{i-1}^2 = a_{i+1}^2 - a_i^2$ for $i = 1, \dots, n-1$? We will prove that there is a bound for this maximum, and that this bound only depends on the degree of the field K over $\mathbb{Q}$. In fact we will show that the same result is also valid for k-powers, now the bound depending also on k.

The problem has a long history for $K = \mathbb{Q}$. In a letter written to Frénicle in 1640, Fermat proposed the problem of proving that there are no four squares in arithmetic progression. Euler gave the first published proof of this result in 1780. In a different direction, in 1970 Szemerédi proved that there exists at most $o(N)$ squares in an arithmetic progression of length N, and this result was improved by Bombieri, Granville and Pintz in 1992 in [2] to $O(N^{2/3}(\log N)^A)$ for a suitable constant A studying the arithmetic progressions that contain 5 squares, and by Bombieri and Zannier in 2002 in [3] to $O(N^{3/5}(\log N)^A)$ for a suitable constant A studying the ones that contain 4 squares.

The question for higher powers has also a long history. It is known that there does not exist a non-trivial three term arithmetic progression of k-th powers for $k \geq 3$. Observe that, when k is odd, we do have non-constant three term arithmetic progressions of k-th powers, the ones of the form $-a^k$, 0 and a^k for $a \in \mathbb{Q}$. In these cases, for non-trivial three term arithmetic progression we mean non-constant and with $a_1 \neq 0$. The cases $k = 3$ and $k = 4$ are mentioned in Carmichael's 1908 book on diophantine equations. The cases $k = 5, \dots 31$ were done by Denes in 1952 [6]. The cases where $k \geq 17$ is a prime number congruent to 1 modulo 4 were done by Ribet [16], and the rest of the cases by Darmon and Merel in 1997 [5].

The problem is related to some concrete curves having only trivial rational points (trivial in some sense). The rational points of these curves determine arithmetic progressions having squares at the first n terms, and the trivial points correspond to the constant arithmetic progression.

For example, four consecutive squares in an arithmetic progression determine a rational point in an elliptic curve, that one can show has only 8 solutions, all coming from the constant arithmetic progression.

2000 *Mathematics Subject Classification.* 11G30, 14H25, 11B25,11D45.
Key words and phrases. arithmetic progression, uniform bounds, gonality.
The author was partially supported by the grant MTM2009-10359.

The main result of this note is the following theorem.

Theorem 1. *For any $d \geq 1$, there exists a constant $S(d)$ depending only on d such that, if $K/\mathbb{Q}$ has degree $[K:\mathbb{Q}] = d$ and $a_i := a + i\,r$ is an arithmetic progression with a and $r \in K$, and a_i are squares in K for $i = 0, 1, 2, \cdots, S(d)-1$, then $r = 0$ (i.e. a_i is constant).*

Furthemore, if $d = 2$, then $S(2) = 6$.

In some other papers it is studied over which quadratic fields we have four squares in arithmetic progression (by E. González Jiménez and J. Steuding [9]), and five squares in arithmetic progression (by E. González Jiménez and X. Xarles [10]).

This note is organized as follows. In the first section we translate the problem to determining all the rational points of some algebraic curves C_n, and we prove some preliminary results. In the second section we give a lower bound for the gonality of these curves, which we use in section 3 in order to obtain the existence of the constant $S(d)$. In section 4 we investigate the value $S(2)$, proving some results concerning the rational points of C_4 and C_5 over quadratic fields. Finally, in the last section we show how to prove the result on k-powers, and we comment on some generalizations of the problem.

1. Translation to algebraic curves

We say that elements $a_0, \ldots, a_n$ in a field K are in arithmetic progression if there exist a and r, elements of K, $r \neq 0$, such that $a_i = a + i\,r$ for $i = 0, \ldots, n$. This is equivalent, of course, of having $a_i - a_{i-1} = r$ fixed for any $i = 1, \ldots, n$.

First of all, observe that, in order to study squares in arithmetic progressions, we can and will identify the arithmetic progressions $\{a_i\}$ and $\{a'_i\}$ such that there exists $c \in K^*$ with $a'_i = c^2 a_i$ for any i. In case that $K = \mathbb{Q}$, we can suppose then that a and r are coprime integers.

Now, suppose that there exist three squares x_0^2, x_1^2 and x_2^2 in arithmetic progression over K. This means that

$$x_1^2 - x_0^2 = x_2^2 - x_1^2$$

and so (x_0, x_1, x_2) is a solution of the equation

$$f(X_0, X_1, X_2) := X_0^2 - 2X_1^2 + X_2^2 = 0.$$

We are not interested in the trivial solution $(0,0,0)$, and solutions that are equal up to multiplication by an element in K^* (in fact, in $(K^*)^2$) we consider them equal. So we will work with solutions of the projective curve $f(X_0, X_1, X_2)$ in the projective plane $\mathbb{P}^2$.

Similarly, in order to consider $n+1$ squares in arithmetic progression, with $n \geq 2$, we will take the curve C_n in $\mathbb{P}^n$ determined by the $n-1$ equations

$$f(X_i, X_{i+1}, X_{i+2}) = 0 \quad \text{for } i = 0, \ldots, n-2.$$

The assignment of the arithmetic progression corresponding to any point induces the following map $\varphi_n \colon C_n \to \mathbb{P}^1$ given by $\varphi_n(X_0, \ldots, X_n) = [X_0^2 : X_1^2 - X_0^2]$.

The curve C_n is a non-singular projective curve over any field K of characteristic bigger that n, as we will prove in the following lemma (see also page 4 in [2]).

Lemma 2. *Let $n \geq 1$ and let K be any field, $p = char(K)$, with $p > n$ or $p = 0$. Then the curve C_n is a non-singular projective curve of genus $g_n := (n-3)2^{n-2}+1$.*

Moreover, let $\varphi_n\colon C_n \to \mathbb{P}^1$ *be given by* $\varphi_n(X_0,\dots,X_n) = [X_0^2 : X_1^2 - X_0^2]$. *Then* φ_n *has degree* 2^n, *and it is ramified at the points above* $[i : -1]$ *for* $i = 0,\dots,n$.

Proof. We use first the jacobian criterion in order to show nonsingularity. The Jacobian matrix of the system of equations defining C_n is

$$A := (\partial f(X_i, X_{i+1}, X_{i+2})/\partial X_j)_{0\le i\le n-2\,,\,0\le j\le n}.$$

For any $j_1 < j_2$, denote by A_{j_1,j_2} the matrix obtained from A by deleting the columns j_1 and j_2; it is a square matrix of size $(n-1)\times(n-1)$. It is easily shown that it has determinant

$$|A_{j_1,j_2}| = \pm 2^{n-1}\left(\prod_{i\ne j_1,j_2} X_i\right)(j_2 - j_1).$$

We want to show that, for any $[x_0 : \cdots : x_n] \in C(K)$, there exists $\{j_1, j_2\}$ such that $|A_{j_1,j_2}|(x_0 : \cdots : x_n) \ne 0$.

The first crucial observation is that any point $[x_0 : \cdots : x_n] \in C(K)$ can have at most one $i = 0,\dots,n$ such that $x_i = 0$. If the characteristic of the field is zero this is clear. If the characteristic is p, suppose that $x_i = x_j = 0$ with $i < j$. Since $x_i^2 = a + ir$ for certain a and $r \in K$, we will have that $(i-j)r = 0$, so $r = 0$, which implies $a = 0$, which is not possible, or $i - j = 0$ in K, so $i + kp = j$ for certain $k \in \mathbb{Z}$, which again is not possible if $p > n$.

So, if $p > n$ or $p = 0$, for any point $[x_0 : \cdots : x_n] \in C(K)$, if all x_i are different from 0 then $|A_{j_1,j_2}| \ne 0\ \forall j_1 \ne j_2$, and if $x_i = 0$, then $|A_{i,j_1}| \ne 0\ \forall j_1 \ne i$, hence the rank of A is $n-1$.

The genus of the curve C_n can be computed by induction on n applying the Hurwitz formulae to the natural forgetful cover $C_n \to C_{n-1}$ of degree 2, which is ramified on the 2^{n-1} points with $x_n = 0$. Or it can be computed by using the map $\varphi_n\colon C_n \to \mathbb{P}^1$ which has degree 2^n. The ramification points are the points $[x_0 : \cdots : x_n]$ such that there exists some i with $x_i = 0$. Such points have image by φ_n equal to $[i : -1]$, and ramification index equal to 2. □

We will call the 2^n points $[\pm1 : \cdots : \pm1]$ the trivial points. They correspond to the points P such that $\varphi_n(P) = [0 : 1] = \infty$, so giving the constant arithmetic progression. The first aim of this note is to prove that for n sufficiently large with respect to d, they are the only K-rational points for any extension $K/\mathbb{Q}$ of degree d.

Firstly, one can easily prove the existence of such a bound but depending on the field K. Observe that, if $n > 3$, then the genus is bigger than 1, so, by Faltings' Theorem (previously known as the Mordell Conjecture), for any number field K and $n > 3$, $C_n(K)$ is finite. One can prove even more.

Lemma 3. *Let* $K/\mathbb{Q}$ *be a finite extension. Then there exists a constant* n_K *such that* $C_{n_K}(K)$ *has only the trivial points.*

Proof. Consider first the finite set of points in $\varphi_4(C_4(K)) \subset \mathbb{P}^1(K)$. We want to show that $\varphi_n(C_n(K)) \subset \varphi_4(C_4(K))$ is equal to $\{\infty\}$ for n sufficiently large. This is equivalent to showing that for any $P := [a : r] \in \varphi_4(C(K))$ not equal to ∞, there exists some n_P such that P is not in $\varphi_n(C_n(K))$. But this is obvious from the following sublemma. □

Lemma 4. *Let $K/\mathbb{Q}$ be a finite extension, and let $\{a_i\}$ a non-constant arithmetic progression. Then there exists some n such that a_n is not a square in K.*

Proof. We can reduce to the case that all a_n are in the ring of integers of K. Consider a prime ideal $\mathfrak{p}$ in the ring of integers of K with residue field a prime field $\mathbb{F}_p$, $p > 2$, and such that the sequence $\{\widetilde{a_i}\}$, reduction modulo $\mathfrak{p}$ of $\{a_i\}$, is not constant. Then the sequence $\{\widetilde{a_i}\}$ can have only $(p+1)/2$ consecutive squares, and hence also $\{a_i\}$. So there exists $n \leq (p+3)/2$ such that a_n is not a square in K. $\square$

So we have proved that there exists a bound depending only on the field K. In order to show that one can find a bound depending only on the degree of the field, we will apply a criterion of Frey that is a consequence of Faltings' Theorem. To do this we need to give a lower bound for the gonality of the curves C_n.

2. The gonality of C_n over $\mathbb{Q}$.

Recall that the gonality $\gamma(C_K)$ of a curve C over a field K is the minimum m such that there exists a morphism $\phi : C \to \mathbb{P}^1$ of degree m defined over K. For example, hyperelliptic curves have gonality 2. See also the paper by Poonen [15] and references therein for more on the gonality. The aim of this section is to give a lower bound for the gonality γ_n of C_n over $\mathbb{Q}$.

From the forgetful map $C_n \to C_{n-1}$, and using that C_2 has genus 0, and hence gonality 1, we get the upper bound $2^{n-2} \geq \gamma_n$. In order to get a lower bound, we will use the following result which is well known for the experts (see, for example, Proposition 3 in [8]).

Proposition 5. *Let C be a curve over a number field, and let $\wp$ be a prime of good reduction of the curve, with residue field $\mathbb{F}_q$. Denote by C' the reduction of the curve C modulo $\wp$. Then the gonality γ of C satisfies that*

$$\gamma \geq \frac{\sharp C'(\mathbb{F}_{q^n})}{q^n + 1}$$

for any $n \geq 1$.

Proof. Suppose there is a map $f : C \to \mathbb{P}^1$ defined over K of degree γ. First, we want to show that there is a morphism $f' : C' \to \mathbb{P}^1$ of degree $\gamma' \leq \gamma$. This result is well-known, and can be deduced from results by Abhyankar (see for example [13]), and also from Deuring [7] or from Lemma 5.1 in [14], but we write a short proof for completeness.

I learned this proof from Q. Liu. By applying Lemma 4.14 in [12], with $X = C$, $Y = \mathbb{P}^1$, and $\mathcal{X}$ a smooth model of X, we get a model $\mathcal{Y}$ together with a rational map $\mathcal{X} \to \mathcal{Y}$ which is quasi-finite on the (good) reduction C of X. Considering the image D of C' in the reduction of $\mathcal{Y}$, then D is rational curve (because the reduction of $\mathcal{Y}$ has arithmetical genus 0). The degree of $C' \to D$ is less than the total degree γ. Now the normalization D' of D is smooth rational and geometrically irreducible because C' is geometrically irreducible. Since the curve C is a conic over a finite field, we get a rational point on D' (because of Waring's result), hence $D' = \mathbb{P}^1$.

Consider the morphism $f' : C \to \mathbb{P}^1$ defined over $\mathbb{F}_q$ of degree $\gamma' \leq \gamma$. At most γ' elements in $C(\mathbb{F}_{q^n})$ can go to the same point in $\mathbb{P}^1(\mathbb{F}_{q^n})$, which has $q^n + 1$ points. So $(q^n + 1)\gamma \geq (q^n + 1)\gamma' \geq \sharp C(\mathbb{F}_{q^n})$. $\square$

Corollary 6. *For any $n \geq 3$, let γ_n be the gonality of C_n. Then $2^{n-2} \geq \gamma_n \geq 2^{n-1}/n$.*

Proof. Let p be a prime such that $2n > p > n$. Then C_n has good reduction over p, and its reduction is given by the same curve C_n. Consider the 2^n trivial points $\varphi_n^{-1}(\infty) \subset C_n(\mathbb{F}_p)$. By applying the proposition we get that

$$\gamma \geq \frac{\sharp C_n(\mathbb{F}_p)}{p+1} \geq \frac{2^n}{p+1} \geq \frac{2^n}{2n}$$

by using that $p \leq 2n-1$. $\square$

Remark 7. *The result in the Corollary can be improved for some concrete values of n by considering some prime such that $p > n$ and some m (usually $m = 1$ or 2), and then considering all the points in $C_n(\mathbb{F}_{p^m})$. On the other hand, one can also use some results concerning the gonality over $\mathbb{C}$, since $\gamma(C_{\mathbb{Q}}) \geq \gamma(C_{\mathbb{C}})$. Recently we learned that $\gamma_n = \gamma(C_{n\mathbb{C}}) = 2^{n-2}$, a result that can be proved by using the theorem contained in Exercise 4.12 in* [11].

3. Proof of the main theorem, first part

First of all, observe that the results in the first section allow us to reduce the problem of the existence of the constant $S(d)$ (such that, for any degree d extension $K/\mathbb{Q}$, the only arithmetic progressions with $S(d)$ consecutive squares are the constant ones), to the existence of a constant $S(d)$ such that $C_{S(d)}(K)$ contains only the trivial points.

We will use the following criterion of Frey [8], proved also by Abramovich in his thesis.

Theorem 8 (Frey)**.** *Let C be a curve over a number field K, with gonality $\gamma > 1$ over K. Fix an algebraic closure $\overline{K}$ of K and consider the points of degree d of C,*

$$C^d(K) := \bigcup_{[L:K]\leq d} C(L) \ \subset C(\overline{K})$$

where the union is over all the finite extensions of K inside $\overline{K}$ of degree $\leq d$. Suppose that $2d < \gamma$. Then $C^d(K)$ is finite.

Hence, by Corollary 6, we get that, if $2d < 2^{n-1}/n \leq \gamma_n$, then there exists a finite number of points in $C_n^d(\mathbb{Q})$. So, there exists only a finite number of extensions $K_i/\mathbb{Q}$ of degree d such that $C(K_i)$ contains some non-trivial point, and for any other $K/\mathbb{Q}$ of degree d, $C(K)$ contains only the trivial points.

For any such extension, we apply now Lemma 3, so there exists some constant n_{K_i} such that $C_{n_{K_i}}$ contains only the trivial points. Hence, considering $S(d) := \max_i\{n_{K_i}\}$ we get the result.

Remark 9. *We do not know a method to get an upper bound on $S(d)$ in terms of d as defined above, and the proof given above of its existence does not help us because of the use of the non-effective Faltings' theorem.*

However, one could guess, using also Remark 7, that the constant $S(d)$ will not be much bigger than the one satisfying $d < 2^{S(d)-3}$, so the correct value of $S(d)$ should be $O(\log(d))$.

4. Proof of main theorem, second part: the case $d = 2$

In this section we want to calculate explicitly the constant $S(2)$. Observe that we do have 5 squares in arithmetic progression over quadratic fields, for example

$$7^2 = 49, 13^2 = 169, 17^2 = 289, 409, 23^2 = 529$$

over $\mathbb{Q}(\sqrt{409})$, so $S(2) > 5$. In fact, we have an infinite number of examples of such progressions, even though we do have only a finite number over any fixed field.

Lemma 10. *There is an infinite number of different quadratic extensions $K/\mathbb{Q}$ such that $C_4(K)$ contains non-trivial points.*

Proof. Consider the curve parametrizing the arithmetic progressions a_i such that a_i is an square for $i = 0, 1, 2, 4$. This curve is a genus 1 curve given by equations

$$X_0^2 - 2X_1^2 + X_2^2 = 0 \text{ and } 3X_0^2 - 4X_1^2 + X_4^2 = 0.$$

Since it has a (trivial) point, it is isomorphic to an elliptic curve, that can be given by the Weierstrass equation

$$y^2 = x(x + 2)(x + 6)$$

One shows by standard methods that this curve has an infinite number of rational points, and in fact $E(\mathbb{Q}) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, with $(2, 8)$ being a generator of the torsion free part.

Now, for any point $P \in E(\mathbb{Q})$, we consider the associated progression $a_0 := x_0^2$, $a_1 := x_1^2$, and so on. By considering the field $K := \mathbb{Q}(\sqrt{a_3})$, we get that in K the arithmetic progression $\{a_i\}$ has 5 consecutive squares. Equivalently, consider the degree 2 map $\psi : C_4 \to E$, and the points $Q \in C_4(K)$ such that $\psi(Q) = P$. Since by Faltings theorem there are only a finite number of points in $C_4(K)$, and we have infinitely many points in $E(\mathbb{Q})$, we get infinitely many such fields K. □

In a forthcoming paper [10], we study over which quadratic fields one has 5 squares in arithmetic progression, and we show, for example, that $\mathbb{Q}(\sqrt{409})$ is the smallest (in terms of the discriminant) of such fields.

Observe that the gonality γ of C_5 over $\mathbb{Q}$ is bounded below by $2^4/5$ by using corollary 6, but in fact one can show the gonality is equal to 8. So by Frey's result, Theorem 8, we get that there are only a finite number of quadratic fields K such that $C_5(K)$ contain non-trivial points. We will show that in fact there are none such fields.

In order to show this, we will study in more detail the case of 5 squares. We will prove that, if $K/\mathbb{Q}$ is a quadratic extension and $P \in C_4(K)$, then $\phi(P) \in \mathbb{P}^1(\mathbb{Q})$, hence the progression is defined over $\mathbb{Q}$ (as they are all the ones obtained from the lemma 10). This result will imply that $C_5(K)$ contains only the trivial points.

To show the result on C_4 we need first to study how are the points of $C_3(\mathbb{Q})$ and $C_3(K)$ for $K/\mathbb{Q}$ of degree 2. Observe that C_3 is isomorphic to an elliptic curve once fixed a rational point, and we will take always $[1 : 1 : 1 : 1]$. We get then a group operation $\oplus$ on C_3. The following lemma describes some easy cases.

Lemma 11. *For any field K, with characteristic distinct from 2 and 3, consider $P := [x_0 : x_1 : x_2 : x_3] \in C_3(K)$, and let $Q = [\pm 1 : \pm 1 : \pm 1 : \pm 1]$ be any trivial point. Then $\ominus P = [x_3 : x_2 : x_1 : x_0]$ and*

$$P \oplus Q = \begin{cases} \textit{Case 1:} & [\pm x_0 : \pm x_1 : \pm x_2 : \pm x_3], \\ \textit{Case 2:} & [\pm x_3 : \pm x_2 : \pm x_1 : \pm x_0], \end{cases}$$

where the Case 1 is when the number of -1 in Q is even, and the signs of the x_i's are the same as the signs of the corresponding coordinate of Q; and the Case 2 is when the number of -1 in Q is odd, with the same rule for the signs. Moreover, in the Case 1 the point Q has order 2, and in Case 2 it has order 4.

Proof. This is elementary, and we will merely outline the proof.

We will use the following observation: since C_3 has no CM (a fact that can be shown just by having a non integral j-invariant, which is $2^4 13^3/3^2$), the only automorphisms of order 2 of C_3 (as a genus 1 curve) which fix the 0 are the identity automorphism and negation, and the other automorphisms are the translations by a point, which do not have fixed points, and negation followed by translations by a point, which do have 2 fixed points. We get then that $[x_0 : x_1 : x_2 : x_3] \mapsto [x_3 : x_2 : x_1 : x_0]$ is the negation, since it fixes O and is not the identity. Once we know this, we get that $\ominus Q = Q$ if and only if Q is a trivial point with an even number of -1, hence in this case Q has order 2. The other trivial points must have order 4. Now, observe that the automorphisms determined in the remaining Cases 1 and 2 do not have fixed points, so they are translations by a point P. Just using that $O + Q = Q$ we get the correct choice of Q in any case. □

Now we need to recall the classical result of Fermat, concerning squares in arithmetic progressions over $\mathbb{Q}$, together with some other cases.

Lemma 12. *Let F be one of the following genus 1 curves, given as intersection of two quadrics in $\mathbb{P}^3$:*

$$C_3 : X_0^2 - 2X_1^2 + X_2^2 = 0 \quad \textit{and} \quad X_1^2 - 2X_2^2 + X_3^2 = 0,$$
$$F_1 : 2X_0^2 - 3X_1^2 + X_2^2 = 0 \quad \textit{and} \quad X_1^2 - 3X_2^2 + 2X_3^2 = 0,$$
$$F_2 : X_0^2 - 3X_1^2 + 2X_2^2 = 0 \quad \textit{and} \quad 2X_1^2 - 3X_2^2 + X_3^2 = 0,$$

Then $F(\mathbb{Q}) = \{\pm 1 : \pm 1 : \pm 1 : \pm 1]\}$.

As a consequence, if $\{a_i\}$ is an arithmetic progression over $\mathbb{Q}$ such that a_0, a_1, a_2 and a_3 are squares, or a_0, a_1, a_3 and a_4 are squares, or a_0, a_2, a_3 and a_5 are squares, then $\{a_i\}$ is constant.

Proof. We consider all the cases at the same time. Since F is a genus one curve with a rational point, F is isomorphic to its jacobian, which is given respectively by the Weierstrass equations $y^2 = x(x-1)(x+3)$, $y^2 = x(x-1)(x+8)$ and $y^2 = x(x-4)(x+5)$. These elliptic curves have only 8 rational points, as proved by standard descent methods. The assertion about the arithmetic progressions is easily deduced. □

Corollary 13. *Let $K/\mathbb{Q}$ be a degree 2 extension, and let σ be the generator of the Galois group. Let $P = [x_0 : x_1 : x_2 : x_3] \in C_3(K)$ be a non-trivial point. Then*

$$P^\sigma := [\sigma(x_0) : \sigma(x_1) : \sigma(x_2) : \sigma(x_3)] = \begin{cases} \textit{Case 1:} & [\pm x_3 : \pm x_2 : \pm x_1 : \pm x_0] \\ \textit{Case 2:} & [\pm x_0 : \pm x_1 : \pm x_2 : \pm x_3]. \end{cases}$$

Furthermore, in the case 2, $\phi_3(P) \in \mathbb{P}^1(\mathbb{Q})$.

Proof. Consider the point $Q := P \oplus P^\sigma$. Since $Q^\sigma = Q$, $Q \in C_3(\mathbb{Q})$, so $Q = [\pm 1 : \pm 1 : \pm 1 : \pm 1]$ for some choice of signs. Hence $P \ominus Q = \ominus P^\sigma$, and by the lemma above we have that either $P \ominus Q = [\pm x_0 : \pm x_1 : \pm x_2 : \pm x_3]$ in the Case 1, hence $P^\sigma = [\pm x_3 : \pm x_2 : \pm x_1 : \pm x_0]$, or $P \ominus Q = [\pm x_3 : \pm x_2 : \pm x_1 : \pm x_0]$; hence $P^\sigma = [\pm x_0 : \pm x_1 : \pm x_2 : \pm x_3]$. In this last Case we have that $\sigma(x_i^2) = x_i^2$ (maybe after rescaling the coordinates), and, therefore, the corresponding arithmetic progression is defined over $\mathbb{Q}$. $\square$

Example 14. *It is easy to give examples of the Case 2, for example the one given by $x_0 = 1$, $x_1 = 5$, $x_2 = 7$ and $x_3 = \sqrt{61}$, in $K = \mathbb{Q}(\sqrt{61})$. There are also examples of the Case 1: take $K = \mathbb{Q}(\sqrt{13})$, and consider $x_0 = 1$, $x_1 = 10 + 3\sqrt{13}$, $x_2 = 15 + 4\sqrt{13}$ and $x_3 = 18 + 5\sqrt{13}$. Then $P^\sigma = [-x_3 : x_2 : -x_1 : x_0]$.*

Now we apply this results to study the curve C_4. There are two different forgetful maps from C_4 to C_3, forgetting the first term and forgetting the last term. We will use this assertion in order to show the following result.

Proposition 15. *Let $K/\mathbb{Q}$ be a degree 2 extension, and let $P \in C_4(K)$ be a non-trivial point. Then $\phi_4(P) \in \mathbb{P}^1(\mathbb{Q})$.*

Proof. Let $P = [x_0 : x_1 : x_2 : x_3 : x_4] \in C_4(K)$, where we can suppose $x_i \neq 0$ for $i = 0, \dots, 4$, and consider $P_0 := [x_0 : x_1 : x_2 : x_3]$ and $P_1 = [x_1 : x_2 : x_3 : x_4] \in C_3(K)$. Suppose that P_i for some $i = 0, 1$ is in Case 2 of the previous corollary. Then, maybe after rescaling the coordinates, we have that x_1^2, x_2^2 and x_3^2 are in $\mathbb{Q}$, hence the arithmetic progression is defined over $\mathbb{Q}$, which is equivalent to $\phi_4(P)$ being in $\mathbb{P}^1(\mathbb{Q})$.

So we can suppose both P_i are in Case 1. Then one has that

$$[x_0^2 : x_1^2 : x_2^2] = [\sigma(x_3^2) : \sigma(x_2^2) : \sigma(x_1^2)] = [x_2^2 : x_3^2 : x_4^2].$$

But this implies that $x_i^2 = 1$ for all $i = 0, \dots, 4$, and hence P is already defined over $\mathbb{Q}$. $\square$

The content of the last proposition is that all the rational points of C_4 defined over quadratic extensions are obtained by taking square roots of some arithmetic progressions over $\mathbb{Q}$, and essentially with the method explained in the proof of Lema 10.

Now, we are going to use this result to show the non existence of 6 squares in arithmetic progressions over quadratic fields.

Theorem 16. *Let $K/\mathbb{Q}$ be a degree 2 extension, and suppose $P \in C_5(K)$. Then $P = [\pm 1 : \pm 1 : \pm 1 : \pm 1 : \pm 1 : \pm 1]$.*

Proof. Let D be a squarefree integer, and consider $K := \mathbb{Q}(\sqrt{D})$. Suppose we have a point $P = [x_0 : x_1 : x_2 : x_3 : x_4 : x_5] \in C_5(K)$. By the previous proposition, we have that $x_i^2 \in \mathbb{Q}$ for all $i = 0, \dots, 5$. So we can and will suppose that $x_i^2 = a_i := a + i\,r$ for some a and $r \in \mathbb{Z}$ coprime integers. If all the x_i are in $\mathbb{Q}$, then we are done by Fermat's result. So suppose we have some $x_i \notin \mathbb{Q}$, and hence $x_i^2 = Dy_i^2$ for some $y_i \in \mathbb{Q}$. Observe that, after rescaling, we can and will suppose that there are at most two values i in $\{0, \dots, 5\}$ with $x_i \notin \mathbb{Q}$.

First, we show that there should be exactly two values $0 \leq i_1 < i_2 \leq 5$ with $x_{i_j} \notin \mathbb{Q}$ for $j = 1, 2$. Since, if we have only one value $x_j \notin \mathbb{Q}$, and $j = 0$, 1, 4 or 5, then we will have 4 squares in arithmetic progression over $\mathbb{Q}$, so $x_i^2 = 1$ for all i by Fermat (or lemma 12), giving $x_j^2 = 1$, contrary to the hypothesis. If $j = 2$, then we will have a_0, a_1, a_3 and a_4 are squares over $\mathbb{Q}$, so again $x_i^2 = 1$ by lemma 12. The same argument shows the case $j = 3$.

Now, observe that no prime $p > 3$ can divide two of the x_i^2; since if $p > 5$ divides $x_i^2 - x_j^2 = (i - j)r$, then it divides $(i - j) < 6$, which is not possible, or it divides

r, and hence it divides $a = x_i^2 - i\ r$, which again is not posible since a and r are coprime. And, if 5 divides two elements of the progression, they must be a_0 and a_5, and then $a_1, \dots, a_4$ will be 4 squares in arithmetic progression over $\mathbb{Q}$.

A similar argument rules out the negative values of D. Since, if $D < 0$ and divides two of the x_i's, they must be the first ones or the last ones, so the rest will form an arithmetic progression of squares over $\mathbb{Q}$.

For the case that 3 divides two elements of the progression, we have three cases: Either a_0, a_1, a_3 and a_4 are squares over $\mathbb{Q}$, a case that we already considered, or a_0, a_2, a_3 and a_5 are squares over $\mathbb{Q}$, which is the third case considered in lemma 12, or, finally, a_1, a_2, a_4 and a_5 are squares, which is equivalent to the first case.

So we are reduced to the case $D = 2$, which needs a distinct but elementary argument. First, suppose that $a_0 = 2y_0^2$, $a_2 = 2y_2^2$ and $a_3 = y_3^2$ for some $y_i \in \mathbb{Q}$. Then we have that $y_0^2 + y_3^2 = 3y_2^2$, which has no solutions over $\mathbb{Q}$. This implies that we cannot have that 2 divides a_i and a_{i+2} for any $i = 0, \cdots, 3$. Second, suppose we have $a_0 = 2y_0^2$, $a_1 = y_1^2$ and $a_3 = y_3^2$. Then we have $4y_0^2 + y_3^2 = 3y_1^2$, which again has no solutions over $\mathbb{Q}$. With these facts we solve the remaining cases, proving that 2 cannot divide two terms of the $a_i = x_i^2$. $\square$

5. Some generalizations and conjectures

In order to compute explicitly the constant $S(d)$ for $d > 2$ one cannot use the same argument we did in the last section. In fact, we do not even know if $S(3) > 4$, since we don't know a way to produce 5-term arithmetic progressions of squares over cubic fields. Observe that we do have many examples of 4-term arithmetic progressions over cubic fields, obtained for example by putting the curve C_3 in Weierstrass form $y^2 = x(x-1)(x+3)$ and substituting any rational value of y. One possible idea could be to use the natural maps from C_n to sufficiently many elliptic curves with rank 0, and then showing that it is a formal immersion at p for some prime $p > 2$, as it is done (in a different context) in [8]. But this result will not be sufficient to deduce the result, because these curves contain always non-trivial points over finite fields with cardinality a power of p^2, since all the elements of $\mathbb{F}_p$ are squares over such fields. We do not know any other general argument to show these type of results, which essentially are the computation of all the rational points in the symmetric product of some curve C.

As we mention in the introduction, one can ask also for higher powers the same question we did for squares. And, in fact, the same type of arguments work for solving the existence of a uniform bound. So we get the following result.

Theorem 17. *Given $d \geq 1$ and $k \geq 2$, there exists a constant $S(d,k)$ depending only on d and k such that, if $K/\mathbb{Q}$ satisfies that $[K : \mathbb{Q}] = d$ and $a_i := a + i\ r$ is an arithmetic progression with a and $r \in K$, and a_i are k-powers in K for $i = 1, 2, \cdots, S(d,k)$, then $r = 0$ (i.e. a_i is constant).*

Proof. First, consider the case $d = 1$. In this case, as we mention in the introduction, it is known after the work of Denes [6], Ribet [16], and Darmon and Merel [5], that the only three term arithmetic progressions of k-th powers for $k \geq 3$ over $\mathbb{Q}$ are the constant ones if k is even, and the constant plus the ones of the form $-a^k$, 0 and a^k for $a \in \mathbb{Q}$ if k is odd. Using these it is clear that there are no non-constant four term arithmetic progression of k-th powers for k odd, and hence $S(1,k) = 3$ if k even and $S(1,k) = 4$ if k odd.

Now, fix $k > 1$ and $d > 1$. The assertion of the theorem is equivalent to showing that for n large enough in terms of d, the only points of degree d of the curves $C_{n,k}$ in $\mathbb{P}^n_{\mathbb{Q}}$ determined by the $n-1$ equations

$$f_k(X_i, X_{i+1}, X_{i+2}) = 0 \quad \text{for } i = 0, \ldots, n-2,$$

where $f_k(X,Y,Z) := X^k - 2Y^k + Z^k$, are the trivial points $[\pm 1 : \pm 1 : \cdots : \pm 1]$ if k is even, or $[1 : 1 : \cdots : 1]$ is k is odd. The same arguments used to show Lemma 2 show that $C_{n,k}$ has good reduction at any prime $p > n$ and not dividing k. Using the same argument as in section 3, we only need to show that the gonality of the curves $C_{n,k}$ tends to infinity when n goes to infinity, so Theorem 8 applies again.

If k is even, the exact same argument as in Corollary 6 concerning the lower bound of the gonality applies if $n > k$ (in order to avoid the primes p dividing k), so we get the result.

If k is odd, the curve $C_{n,k}$ contains only one trivial point in the reduction, unless the field $\mathbb{F}_p$ contains some k-roots of unity. In order to have these, we consider primes p of good reduction such that $p \equiv 1 \pmod{k}$. In this case we have that $C_{n,k}(\mathbb{F}_p)$ contains k^n points. Now, we apply a well-known consequence of Dirichlet's theorem on primes in arithmetic progressions asserting that there exists a constant $c(k)$ depending on k such that, for any $n > c(k)$, there exists a prime p satisfying that $n < p < 2n$ and $p \equiv 1 \pmod{k}$.

Now the argument works as follows: choose $n > c(k)$, $n > k$ and a prime $n < p < 2n$ with $p \equiv 1 \pmod{k}$. Then the gonality $\gamma_{n,k}$ of $C_{n,k}$ satisfies, by proposition 5, that

$$\gamma_{n,k} \geq \frac{\sharp C_{n,k}(\mathbb{F}_p)}{p+1} \geq \frac{k^n}{2n}.$$

Hence, for n large enough with respect to d we get $\gamma_{n,k} > 2d$, so $C_{n,k}$ contains only a finite number of points of degree d over $\mathbb{Q}$, and the proof is finished by using an analog of Lemma 3. □

A. Granville observed in a personal communication that one can use the main theorem to prove that there are always $o_d(N)$ squares in any arithmetic progression over any number field of degree d. In fact, it is just an application of Szeméredi's theorem on arithmetic progressions, as Szeméredi himself did for the case $d = 1$ using Fermat's result. More generally, we have the following result.

Corollary 18. *Let $d \geq 1$ and $k \geq 2$ be integers. Then there are $o_{d,k}(N)$ k-powers in any arithmetic progression $a + im$ with $m \neq 0$ and $1 \leq i \leq N$ over any field of degree d over $\mathbb{Q}$.*

Proof. Given a and $m \neq 0$ in a field K of degree d over $\mathbb{Q}$, let I be the set of i for which $a + im$ is a k-power. Then I does not contain any $S(d,k)$-term arithmetic progressions by Theorem 17. Hence $|I| = o_{d,k}(N)$ by Szeméredi's theorem, which states for any $1 > \delta > 0$ and for any M, if N is sufficiently large in terms of δ and M, any subset of $\{1, 2, .., N\}$ of size $> \delta N$ has an M-term arithmetic progression. □

This last result can probably be improved to some bound of the form $O_{d,k}(N^{1-c_{d,k}})$, for some constant $0 < c_{d,k} \leq \frac{k-1}{k}$, by using similar arguments as the ones used in [2] and [3]. One could even guess that $c_{d,k}$ can always be taken equal to $\frac{k-1}{k}$, that will generalize Rudin's conjecture, which asserts this in the case $k = 2$ and $d = 1$.

Finally, let us mention that one can ask a 2-dimensional (or even s-dimensional) analogous question. By this we mean the following natural question (which was asked of me, in a different form, by Ignacio Larrosa Cañestro and was the origin of this paper): is there a constant S such that the only degree 2 polynomials $f(x)$ with rational coefficients satisfying that $f(i)$ is a square for $i = 0, \ldots, S$ are the squares of a (degree 1) polynomials? This question can be translated to the computation of all the rational points of some algebraic surfaces. This question is investigated for example in [1] and in [4], and it seems that the natural guess is $S = 8$. But a positive answer of the existence of such S is related to the so called Lang's (and Bombieri) conjecture, which asserts that for a general type surface the rational points are all contained in a finite number of strict subvarieties, in our case curves and points. In fact, for $S = 8$, one gets that the corresponding surface is of general type. But the only known cases of Lang's conjecture, due essentially to Faltings and corresponding to subvarieties of abelian varieties, does not apply in these cases, being all these surfaces complete intersections in $\mathbb{P}^n$, hence with trivial albanese variety.

Acknowledgements

This paper had its origin in a question asked by Ignacio Larrosa Cañestro, which I could not answer satisfactorily (see the last section). I thank him, and also Adolfo Quirós, Enric Nart, Joaquim Roé, Javier Cilleruelo and Henri Darmon for some conversations and comments on the subject. I especially thank Andrew Granville and Qing Liu for some fruitful comments and for giving me some useful references and Enrique González Jiménez for some comments and corrections. Finally I thank an anonymous referee for some useful suggestions and corrections.

References

[1] D. Allison, *On square values of quadratics.* Math. Proc. Camb. Philos. Soc. **99** (1986), Issue 03, pp 381-383.

[2] E. Bombieri, A. Granville, J. Pintz, *Squares in arithmetic progressions.* Duke Math. J. **66** (1992), no. 3, 369–385.

[3] E. Bombieri, U. Zannier, *A note on squares in arithmetic progressions. II.* Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **13** (2002), no. 2, 69–75.

[4] A. Bremner, *On square values of quadratics.* Acta Arith. 108 (2003), no. 2, 95–111.

[5] H. Darmon, L. Merel, *Winding quotients and some variants of Fermat's last theorem.* J. Reine Angew. Math. **490** (1997), 81–100.

[6] P. Dénes, *Über die Diophantische Gleichung* $x^l + y^l = cz^l$. Acta Math. **88**, (1952). 241–251.

[7] M. Deuring, *Reduktion algebraischer Funktionkörper nach Primdivisoren des Konstantenkörper.* Math. Z. **47** (1942), 643-654.

[8] G. Frey, *Curves with infinitely many points of fixed degree.* Israel J. Math. **85** (1994), no. 1-3, 79–83.

[9] E. González Jiménez, J. Steuding, *Arithmetic progressions of four squares over quadratic fields*, Publicationes Mathematicae Debrecen 77, 1-2, 125-138 (2010).

[10] E. González Jiménez, X. Xarles, *Five squares in arithmetic progression over quadratic fields*, preprint 2010.

[11] R. Lazarsfeld, Lectures on Linear Series, in Complex Algebraic Geometry (Park City, UT, 1993), IAS/Park City Math. Ser., vol.**3**, Amer. Math. Soc., Providence, RI, 1997, pp. 161-219..

[12] D. Lorenzini and Q. Liu, *Models of curves and finite covers*, Compositio Math., **118** (1999), 61-102

[13] M. Nagata, *A Theorem on valuation rings and its applications*, Nagoya Math. J. **29** (1967), 85-91.

[14] K. V. Nguyen and M. H. Saito, D-*gonality of modular curves and bounding torsions*, preprint 1996.

[15] B. Poonen, Gonality of modular curves in characteristic p, Math. Res. Lett. 14 (2007) no. 4, 691-701.
[16] K. Ribet, *On the equation* $a^p + 2^\alpha b^p + c^p = 0$. Acta Arith. **79** (1997), no. 1, 7–16.

Departament de Matemàtiques
Universitat Autònoma de Barcelona
08193 Bellaterra, Barcelona, SPAIN
email: xarles@mat.uab.cat