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ON p-ADIC INTERMEDIATE JACOBIANS

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ABSTRACT. For an algebraic variety X of dimension d with totally degenerate reduction over a p-adic field (definition recalled below) and an integer i with $1 \leq i \leq d$, we define a rigid analytic torus $J^i(X)$ together with an Abel-Jacobi mapping to it from the Chow group $CH^i(X)_{hom}$ of codimension i algebraic cycles that are homologically equivalent to zero modulo rational equivalence. These tori are analogous to those defined by Griffiths using Hodge theory over $\mathbf{C}$. We compare and contrast the complex and p-adic theories. Finally, we examine a special case of a p-adic analogue of the Generalized Hodge Conjecture.

INTRODUCTION

Let E be an elliptic curve over the complex number field $\mathbf{C}$. The group $E(\mathbf{C})$ of $\mathbf{C}$-points of E may be written as

$$E(\mathbf{C}) = \mathbf{C}/\Lambda,$$

where Λ is a lattice of periods of the form $\mathbf{Z} \oplus \mathbf{Z}\tau$, with τ in the upper half plane. Setting $q = e^{2\pi i\tau}$, we can also write:

$$E(\mathbf{C}) = \mathbf{C}^*/q^{\mathbf{Z}}.$$

In 1959, Tate showed that one can make sense of this latter description in a general complete valued field K if q is of absolute value less than 1. Such elliptic curves are now called *Tate elliptic curves*, and if K is nonarchimedian, then they have split multiplicative reduction modulo the maximal ideal of the valuation ring of K. Let ℓ be a prime number, $E[\ell^n]$ be the group of points of order ℓ^n on E over an algebraic closure of K and T_ℓ denote the Tate module

$$\varprojlim_n E[\ell^n].$$

Then we have an exact sequence of G-modules, where G is the absolute Galois group of K:

$$0 \to \mathbf{Z}_\ell(1) \to T_\ell \to \mathbf{Z}_\ell \to 0.$$

The class of this extension in $Ext^1_G(\mathbf{Z}_\ell, \mathbf{Z}_\ell(1))$ is given by the class of q in the completion of K^* with respect to its subgroups $K^{*\,\ell^n}$. The Galois module structure of

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such extensions is reasonably simple but still very interesting. Mumford, Raynaud and many others have generalized this theory to other types of varieties such as abelian varieties with totally multiplicative reduction and certain (Mumford) curves of higher genus (see [Mu1], [Mu2] and [Ray1]). Manin and Drinfeld have given an analytic definition of the Abel-Jacobi mapping in the case of Mumford curves [MD].

Recall that the important ingredients in the theory of p-adic uniformization of an abelian variety A (see for example [Ray1]) are two isogenous lattices of periods Λ and Λ' and a nondegenerate pairing (where $\Lambda'^{\vee} = \mathrm{Hom}(\Lambda', \mathbf{Z})$):

$$\Lambda \times \Lambda'^{\vee} \to K^*.$$

Then the pairing gives a uniformization of A:

$$A \cong \mathrm{Hom}(\Lambda'^{\vee}, \mathbf{G}_m)/\Lambda$$

as rigid analytic varieties.

Now let K be a finite extension of $\mathbf{Q}_p$ and X be a smooth, projective, geometrically connected variety over K. Let $\overline{K}$ be an algebraic closure of K and $\overline{X} = X \times_K \overline{K}$. In this paper, we use the methods of our earlier paper [RX] to reinterpret these lattices and the pairing in a purely algebro-geometric way using comparison theorems between p-adic étale cohomology and log-crystalline cohomology (that is, Tsuji's proof of the semi-stable conjecture C_{st} [Tsu]; see also Faltings' approach in [Fa2]). The utility of these methods is that they generalize quite easily to higher cohomology groups (with the lattices for abelian varieties being the case of H^1), whereas the uniformization methods have met with somewhat less success here (but see [dS]).

In [RX], we formalized the notion of a variety X over K with *totally degenerate reduction* (we will recall the definition below) and we showed that for *all* prime numbers ℓ, the étale cohomology groups $H^*(\overline{X}, \mathbf{Q}_\ell)$ are, after a finite unramified extension, successive extensions of direct sums of Galois modules of the form $\mathbf{Q}_\ell(r)$ for various r. The main result of the present paper associates to each odd étale cohomology group $\prod_\ell H^{2i-1}(\overline{X}, \mathbf{Q}_\ell)$ of such a variety a p-adic analytic torus $J^i(X)$ of dimension equal to the Hodge number $h^{i,i-1}$ and an Abel-Jacobi mapping:

$$CH^i(X)_{hom} \to J^i(X)(K) \otimes_{\mathbf{Z}} \mathbf{Q},$$

where the group on the left is that of cycles of codimension i that are homologically equivalent to zero, modulo rational equivalence (see Remark 17(i) in §3 below for why we have to tensor with $\mathbf{Q}$ on the right). We have that $J^1(X)$ is the Picard variety, $Pic^0(X)$, and $J^d(X)$ is the Albanese variety, $Alb(X)$, where d is the dimension of X.

We call the $J^i(X)$ "p-adic intermediate Jacobians", and we expect them to provide a useful first step in the study of algebraic cycles on such varieties, in a similar way as the intermediate Jacobians of Griffiths [Gr] do for varieties $X/\mathbf{C}$. Our original motivation for studying these p-adic intermediate Jacobians was to provide, at least in some cases, an algebro-geometric interpretation for the intermediate Jacobians $\tilde{J}^i(X)$ of Griffiths and the Abel-Jacobi map

$$CH^i(X)_{hom} \to \tilde{J}^i(X).$$

These are defined by Griffiths in a complex analytic way using the Hodge decomposition, and they have no known purely algebro-geometric interpretation, in

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general. Weil had earlier introduced similar objects, but with a different complex structure [W1], [W2].

Recall that $\tilde{J}^i(X)$ is a (compact) complex torus, not necessarily an abelian variety, whose dimension is one half of $B_{2i-1} = \dim_{\mathbf{C}} H^{2i-1}(X, \mathbf{C})$. For X a generic quintic 3-fold, Griffiths (loc. cit.) used the Abel-Jacobi map to $\tilde{J}^2(X)$ to detect codimension two cycles that are homologically equivalent to zero, but not algebraically equivalent to zero.

The big advantage of our J^i is that it is defined over K and therefore has arithmetic meaning and applications. A disadvantage is that the dimension of J^i is in general strictly smaller than that of the analogous complex intermediate Jacobian $\tilde{J}^i$. To help compensate for this deficiency, we relate the "missing pieces" to higher odd algebraic K-theory, which we are not able to do at present in the complex case (see Remark 17 (ii) for more on this). In fact, the methods of this paper suggest going back to the complex case and trying to reinterpret $\tilde{J}^i$ in a similar way, using the idea that complex varieties should have (totally degenerate) "semi-stable" reduction in the same way as the varieties we study here do over complete discretely valued fields (see [Ma] for what this means in the case of curves). In work in preparation, the second author and Infante compare and contrast the p-adic and complex cases for abelian varieties. In [R2], the first author formulates a "generalized Hodge-Tate conjecture" for varieties of the type considered in this paper and in [R3], he proves a slightly weaker form of this conjecture for divisors. We consider a special case of the generalized Hodge-Tate conjecture in §4 below.

This paper is organized as follows: in §1, we recall the necessary preliminaries from [RX]. We then define the intermediate Jacobians in §2 and the Abel-Jacobi mapping in §3. Finally, in §4 we consider an example of the product of two and then three Tate elliptic curves, and we determine in some cases the dimension of the image in $J^2(X)$ of the Abel-Jacobi mapping restricted to cycles algebraically equivalent to zero. This dimension conjecturally depends on the rank of the space of multiplicative relations between the Tate parameters of the curves. This is a special case of the generalized Hodge-Tate conjecture mentioned above. Our analysis shows that $J^2(X)$ is never equal to this image. This agrees with the complex case, since there we have that this image is contained in $H^{1,2}/[H^{1,2} \cap H^3(X, \mathbf{Z})]$, and is all of this last group iff the three elliptic curves are isogenous and have complex multiplication (see [Li], p. 1197). This cannot happen for Tate elliptic curves, since CM curves have everywhere potentially good reduction. The authors would like to thank, respectively, the Universitat Autònoma de Barcelona and the University of Southern California for their hospitality. This paper was completed while the first author enjoyed the hospitality of Université de Paris-Sud, SFB Heidelberg and Université Louis Pasteur (Strasbourg). We also thank J.-L. Colliot-Thélène and A. Werner for helpful comments and information.

1. Notation and preliminaries

1.1. **Notation.** Let K be a finite extension of $\mathbf{Q}_p$, with valuation ring R and residue field F, and let X be a smooth, projective, geometrically connected variety over K. We denote by $\overline{K}$ an algebraic closure of K, $\overline{X} = X \times_K \overline{K}$ and $G = Gal(\overline{K}/K)$. If r is a nonnegative integer, then $\mathbf{Z}_\ell(r)$ denotes the Galois module $\mathbf{Z}_\ell$, Tate-twisted r times. If $r < 0$, then $\mathbf{Z}_\ell(r) = \mathrm{Hom}(\mathbf{Z}_\ell(-r), \mathbf{Z}_\ell)$. $\mathbf{Q}_\ell(r)$ is defined by tensoring these by $\mathbf{Q}_\ell$.

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By an *analytic torus* over K, we mean the quotient of an algebraic torus over K by a lattice as rigid analytic varieties (see for example [Ray1] for the exact meaning of this).

Let S be any domain with field of fractions $\mathrm{frac}(S)$ of characteristic zero. If M is an S-module, we denote by $M/tors$ the torsion free quotient of M. That is, M/M_{tors}, where M_{tors} denotes the submodule of torsion elements in M. We say that a map between S-modules is an isomorphism modulo torsion if it induces an isomorphism between the torsion free quotients. If M and M' are torsion free S-modules of finite rank, we say that a map $\phi\colon M \to M'$ is an isogeny if it is injective with cokernel of finite exponent (as an abelian group). In this case, there exists a unique map $\psi\colon M' \to M$, the dual isogeny, such that $\phi \circ \psi = [e]$ and $\psi \circ \phi = [e]$, where e is the exponent of the cokernel of ϕ and $[e]$ denotes the map multiplication by e. In general, if M and M' are S-modules with torsion free quotients of finite rank, we say that a morphism $\psi\colon M' \to M$ is an isogeny if the induced map on the torsion free quotients is an isogeny.

1.2. **Totally degenerate reduction.** Let X be a smooth projective geometrically connected variety over K. We assume that X has a regular proper model $\mathcal{X}$ over R which is strictly semi-stable, which means that the following conditions hold:

(*) Let Y be the special fibre of $\mathcal{X}$. Then Y is reduced; write

$$Y = \bigcup_{i=1}^{n} Y_i,$$

with each Y_i irreducible. For each nonempty subset $I = \{i_1, \dots, i_k\}$ of $\{1, \dots, n\}$, we set

$$Y_I = Y_{i_1} \cap ... \cap Y_{i_k},$$

scheme theoretically. Then Y_I is smooth and reduced over F of pure codimension $|I|$ in $\mathcal{X}$ if it is nonempty. See [DeJ], 2.16, and [Ku], §1.9, 1.10, for a clear summary of these conditions, as well as comparison with other notions of semi-stability.

Let $\overline{F}$ be an algebraic closure of F. We set $\overline{Y}_I = Y_I \times_F \overline{F}$. Note that these need not be connected.

Definition 1. We say that Y is *totally degenerate over* F if it is projective and the following conditions are satisfied for each Y_I:

a) For every $i = 0, ..., d$, the Chow groups $CH^i(\overline{Y}_I)$ are finitely generated abelian groups.

   The groups $CH^i_{\mathbf{Q}}(\overline{Y}_I)$ satisfy the Hodge index theorem: let

$$\xi : CH^i(\overline{Y}_I) \to CH^{i+1}(\overline{Y}_I)$$

   be the map given by intersecting with the class of a hyperplane section for a fixed embedding of Y in a projective space. Suppose we have $x \in CH^i_{\mathbf{Q}}(\overline{Y}_I)$ such that $\xi^{d-i}(x) = 0$. Then we have $(-1)^i \mathrm{tr}(x\xi^{d-2i}(x)) \geq 0$, with equality iff $x = 0$.

b) For every prime number ℓ different from p, the étale cohomology groups $H^{2i+1}(\overline{Y}_I, \mathbf{Z}_\ell)$ are torsion, and the cycle map induces an isomorphism

$$CH^i(\overline{Y}_I)/tors \otimes \mathbf{Z}_\ell \cong H^{2i}(\overline{Y}_I, \mathbf{Z}_\ell(i))/tors.$$

   Note that this is compatible with the action of the Galois group.

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c) If $p > 0$, let W be the ring of Witt vectors of $\overline{F}$. Denote by $H^*(\overline{Y}_I/W)$ the crystalline cohomology groups of $\overline{Y}_I$. Then the groups $H^{2i+1}_{\text{crys}}(\overline{Y}_I/W)$ are torsion, and $CH^i(\overline{Y}_I) \otimes W(-i)/tors \cong H^{2i}_{\text{crys}}(\overline{Y}_I/W)/tors$ via the cycle map. Here $W(-i)$ is W with the action of Frobenius multiplied by p^i.

d) Y is ordinary, in that $H^r(\overline{Y}, B\omega^s) = 0$ for all i, r and s. Here $B\omega$ is the subcomplex of exact forms in the logarithmic de Rham complex on $\overline{Y}$ (see e.g. [I3], Définition 1.4). By [H] and [I2], Proposition 1.10, this is implied by the $\overline{Y}_I$ being ordinary in the usual sense, in that $H^r(\overline{Y}_I, d\Omega^s) = 0$ for all I, r and s. For more on the condition of ordinary, see [I2], Appendice and [BK1], Proposition 7.3.

We will say that X has *totally degenerate reduction* if it has a regular proper model $\mathcal{X}$ over R which is strictly semi-stable and whose special fibre Y is totally degenerate over F. If Y is totally degenerate and the natural maps

$$CH^i(Y_I) \to CH^i(\overline{Y}_I)$$

are isomorphisms modulo torsion for all I, we shall say that Y is *split totally degenerate*. Since the $CH^i(\overline{Y}_I)$ are all finitely generated abelian groups and there is a finite number of them, there is a finite extension of the field of definition where all the cycles given basis for any of the $CH^i(\overline{Y}_I)$ modulo torsion are defined. So, after a finite extension, any totally degenerate variety becomes split totally degenerate.

Examples of varieties with totally degenerate reduction include abelian varieties A such that the special fibre of the Néron model of A over the ring of integers R of K is a torus, and products of Mumford curves or other p-adically uniformizable varieties, such as Drinfeld modular varieties [Mus] and some unitary Shimura varieties (see Example 1 in [RX] for further details and references). We expect that condition c) implies b).

1.3. **Chow complexes.** For more details on this section, please see [RX], §3. We write $Y^{(m)} := \bigsqcup Y_I$, where the disjoint union is taken over all subsets I of $\{1, ..., n\}$ with $\#I = m$ and $Y_I \neq \emptyset$.

For each pair of integers (i, j) we define

$$C^i_j(Y) = \bigoplus_{k \geq \max\{0,i\}} CH^{i+j-k}(Y^{(2k-i+1)}).$$

Note that there are only a finite number of summands here, because k runs from $\max\{i, 0\}$ to $i + j$. Note also that $C^i_j(Y)$ can be nonzero only if $i = -d, ..., d$ and $j = -i, ..., d - i$.

Then define differentials $d^i_j : C^i_j(Y) \to C^{i+1}_j(Y)$ by using the natural restriction and Gysin maps, and define $T^i_j(Y) = \text{Ker} d^i_j/\text{Im} d^{i-1}_j$, the homology in degree i of the complex $C^i_j(Y)$.

The monodromy operator $N : C^i_j(Y) \to C^{i+2}_{j-1}(Y)$ is defined as the identity map on the summands in common, and the zero map on different summands. N commutes with the differentials, and so induces an operator on the $T^i_j(Y)$, which we also denote by N. We have that N^i is the identity on $C^i_j(Y)$ for $i \geq 0$. The following result is a direct consequence of a result of Guillen and Navarro ([GN], Prop. 2.9 and Théorème 5.2 or [BGS], Lemma 1.5 and Theorem 2), using the

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fact that the Chow groups of the components of our Y satisfy the hard Lefschetz theorem and the Hodge index theorem. It is a crucial result for this paper.

Proposition 2. *Suppose that Y satisfies the assumptions of §1. Then the monodromy operator N^i induces an isogeny:*

$$N^i : T_{j+i}^{-i}(\overline{Y}) \to T_j^i(\overline{Y})$$

for all $i \geq 0$ and j.

In the next theorem we summarize the main results in our first paper on the étale cohomology of X (see [RX], §4 Corollary 1 and §6, Theorem 3).

Theorem 3. *Let X be a variety over K of dimension d with totally degenerate reduction and with special fiber Y. Then there is a monodromy filtration $M_\bullet$ on $H^*(\overline{X}, \mathbf{Z}_\ell)$ for all ℓ such that we have a canonical isogeny*

$$T_j^i(\overline{Y}) \otimes \mathbf{Z}_\ell(-j) \to Gr_{-i}^M H^{i+2j}(\overline{X}, \mathbf{Z}_\ell)$$

as G-modules, where G is the absolute Galois group of K. These isogenies are isomorphisms for almost all ℓ. We also have that $Gr_{-i+1}^M H^{i+2j}(\overline{X}, \mathbf{Z}_\ell)$ is torsion. Moreover, for $\ell \neq p$, this filtration coincides with the usual monodromy filtration on $H^i(\overline{X}, \mathbf{Q}_\ell)$, and the map N tensored with $\mathbf{Q}_\ell$ is the ℓ-adic monodromy map on the graded quotients. For $\ell = p$, this filtration is obtained from the monodromy filtration on the log-crystalline cohomology of $\overline{Y}$ by applying the functor D_{st} (see ibid. for more details).

Remark 4. The idea of looking at the Chow complexes was suggested by analogies with mixed Hodge theory, as described in the work of Deligne [De1], Rapoport-Zink [RZ1], Steenbrink [St], and in work of Bloch-Gillet-Soulé [BGS] and Consani [Con] on the monodromy filtration and Euler factors of L-functions.

2. p-adic intermediate Jacobians

The goal of this section is to define, for every $j = 1, \ldots, d := \dim(X)$, rigid analytic tori $J^j(X)$ associated to $\mathcal{X}$ with totally degenerate special fibre in the sense of §1.2. To do this, we will look for a subquotient of the étale cohomology group $H^{2j-1}(\overline{X}, \mathbf{Z}_\ell(j))$ that is an extension of $\mathbf{Z}_\ell$'s by $\mathbf{Z}_\ell(1)$'s, as the ℓ-Tate module of an analytic torus must be. The monodromy filtration gives a natural way of finding such subquotients. Using Theorem 3, we see that the graded quotients for this filtration have natural $\mathbf{Z}$-structures. The torsion free quotients of these $\mathbf{Z}$-structures will give us lattices Λ and Λ' (to be defined below), and we will define a nondegenerate period pairing

$$\Lambda \times \Lambda'^{\vee} \to K^*,$$

where $\Lambda'^{\vee} = \mathrm{Hom}(\Lambda', \mathbf{Z})$.

Then $J^j(X)$ is defined to be $\mathrm{Hom}(\Lambda^\vee, K^*)/\Lambda'$. To define the Abel-Jacobi mapping, we use the groups $H_g^1(K, -)$ of Bloch-Kato ([BK2], §3), and the fact that

$$J^j(X) \cong \prod_\ell H_g^1(K, M_{1,\ell}/M_{-3,\ell}),$$

where $M_{\bullet,\ell}$ is the monodromy filtration on $H^{2j-1}(\overline{X}, \mathbf{Z}_\ell(i))$ modulo torsion (see Theorem 3 for this filtration), and the product is taken over all prime numbers ℓ. Then the Abel-Jacobi mapping is obtained from the ℓ-adic Abel-Jacobi mapping

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([Bl], [R1]). It would be helpful to define the Abel-Jacobi mapping by analytic means, so that it could be defined directly over a field such as $\mathbf{C}_p$ (the completion of an algebraic closure of K). For example, Besser [Be] has interpreted the p-adic Abel-Jacobi mapping for varieties with good reduction over p-adic fields in terms of p-adic integration.

Note also that our $J^j(X)$ depend *a priori* on the choice of regular proper model $\mathcal{X}$ of X over R. We expect that they can be defined directly from the monodromy filtration on X, but we have not managed to do this. We can show that modulo isogeny, they only depend on X (see Remark 10(ii) in §2).

To shorten notation, we will denote $T^i_j(\overline{Y})$ by T^i_j. Since all the $T^i_j/tors$ are abelian groups of finite rank, the action of the Galois group, which by definition of the T^i_j is unramified, will factor through a finite group. Suppose first of all that the absolute Galois group of F acts trivially on $T^{-1}_j/tors$ and on $T^1_{j-1}/tors$ for any j; this will happen after a finite unramified extension of K. We define a pairing for every j:

$$\{-,-\}_l : T^{-1}_j/tors \times T^{1\vee}_{j-1}/tors \to K^{*(\ell)} := \varprojlim_n K^*/K^{*\ell^n}.$$

This is done by using properties of the monodromy filtration $M_i(H)$ on $H := H^{2j-1}(\overline{X}, \mathbf{Z}_\ell(j))$. The idea is to look at the quotient

$$\widetilde{M} := (M_1(H)/M_{-3}(H))/tors,$$

which is a subquotient of $H^{2j-1}(\overline{X}, \mathbf{Z}_\ell(j))$ which is, modulo torsion, an extension of $\mathbf{Z}_\ell$'s by $\mathbf{Z}_\ell(1)$'s. But due to the possible torsion, we have to do a slight modification of $\widetilde{M}$.

Consider the natural epimorphism from $\widetilde{M}$ to $W_1 := \mathrm{Gr}^M_1 H/tors$. Then define W_{-1} as the kernel of this map. Observe that W_{-1} is isogenous to $\mathrm{Gr}^M_{-1}H/tors$, and the natural isogeny $\psi\colon \mathrm{Gr}^M_{-1}H/tors \to W_{-1}$ has cokernel isomorphic to the cokernel of $M_1(H)/M_{-3}(H) \to \mathrm{Gr}^M_1 H = M_1(H)/M_{-1}(H)$ and with exponent e_ℓ. Considering the dual isogeny φ from W_{-1} to $\mathrm{Gr}^M_{-1}H/tors$ such that $\psi \circ \varphi = [e_\ell]$ (see §1.1), we can push out the extension

$$0 \to W_{-1} \to \widetilde{M} \to W_1 \to 0,$$

with respect to this isogeny. Recalling that we have canonical isogenies

$$\psi_1\colon \mathrm{Gr}^M_{-1}H/tors \to T^1_{j-1}/tors \otimes \mathbf{Z}_\ell(1) \text{ and } \psi_2\colon \mathrm{Gr}^M_1 H/tors \to T^{-1}_j/tors \otimes \mathbf{Z}_\ell,$$

we push out with respect to ψ_1 and pull back with respect to the dual isogeny associated to ψ_2 to get an extension:

$$0 \to T^1_{j-1}/tors \otimes \mathbf{Z}_\ell(1) \to E'_\ell \to T^{-1}_j/tors \otimes \mathbf{Z}_\ell \to 0.$$

Observe that for almost all ℓ the isogenies ψ_1 and ψ_2 are all isomorphisms.

Denote by $e := \prod_\ell e_\ell$; it is well defined because $e_\ell = 1$ for almost all ℓ since the graded quotients for the monodromy filtration are torsion-free for almost all ℓ (this follows from Lemma 1 of [RX]). We use these numbers to normalize these extensions: for any ℓ consider the extension E_ℓ obtained from E'_ℓ by pushing out with respect to the map

$$[e/e_\ell] : T^1_{j-1}/tors \otimes \mathbf{Z}_\ell(1) \to T^1_{j-1}/tors \otimes \mathbf{Z}_\ell(1).$$

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Lemma 5. *Consider the $\mathbf{Q}_\ell$-vector space $E_\ell \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell$ constructed above. Then, for $\ell \neq p$, we have that the monodromy map is given by the composition*

$$E_\ell \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell \to T_j^{-1} \otimes \mathbf{Q}_\ell \xrightarrow{\widetilde{N}} T_{j-1}^1 \otimes \mathbf{Q}_\ell \to E_\ell \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell$$

where the left and right hand side maps are the natural ones and the map $\widetilde{N}$ is the composition of the map N defined in §1.3 and the multiplication-by-e map. For $\ell = p$, applying the functor D_{st} (see the introduction to [Tsu]*) to this composition, we get the monodromy map on the corresponding filtered (Φ, N)-module.*

Proof. First consider the case $\ell \neq p$. Observe that for a $\mathbf{Q}_\ell$-module V with action of the absolute Galois group such that monodromy filtration satisfies that $M_{i-1} = 0 \subseteq M_i \subseteq M_{i+1} \subseteq M_{i+2} = V$, then the monodromy map N is the composition of

$$V \to V/M_i \xrightarrow{N'} M_i(1) \to V,$$

where $N' \colon V/M_i \to M_i$ is the induced map by N.

Now by ([RX],§4, Corollary 2) if X has totally degenerate reduction, and $M_\bullet$ is the monodromy filtration on the ℓ-adic cohomology $H^{i+2j}(\overline{X}, \mathbf{Q}_\ell)$, the induced map

$$\mathrm{Gr}^M_{-i} H^{i+2j}(\overline{X}, \mathbf{Q}_\ell) \xrightarrow{N'} \mathrm{Gr}^M_{-i-2} H^{i+2j}(\overline{X}, \mathbf{Q}_\ell)(1)$$

coincides with the map $N \colon T_j^i \otimes \mathbf{Q}_\ell(-j) \to T_{j-1}^{i+2} \otimes \mathbf{Q}_\ell(-j)$ that we defined in section §2, by composing with the isomorphism

$$T_j^i(\overline{Y}) \otimes \mathbf{Q}_\ell(-j) \cong \mathrm{Gr}^M_{-i} H^{i+2j}(\overline{X}, \mathbf{Q}_\ell)$$

deduced from the weight spectral sequence and the cycle maps on the components of the special fiber $\overline{Y}^{(k)}$. In particular, we have that, using the notation in the construction above, the composition

$$\widetilde{M} \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell \to W_1 \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell \cong T_j^{-1} \otimes \mathbf{Q}_\ell \xrightarrow{N} T_{j-1}^1 \otimes \mathbf{Q}_\ell \cong W_{-1} \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell \to \widetilde{M} \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell$$

is the monodromy map on $\widetilde{M}$.

To get the result observe that the $\mathbf{Q}_\ell$-vector space $E \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell$ was obtained from $\widetilde{M} \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell$ by pushing out with respect to the map multiplication by e on the $T_{j-1}^1 \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell$, so we have a commutative diagram

$$\begin{array}{ccccccc}
0 \to & T_{j-1}^1 \otimes \mathbf{Q}_\ell(1) & \to & E_\ell \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell & \to & T_j^{-1} \otimes \mathbf{Q}_\ell & \to 0 \\
 & \uparrow [e] & & \uparrow g & & \| & \\
0 \to & T_{j-1}^1 \otimes \mathbf{Q}_\ell(1) & \to & \widetilde{M} \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell & \to & T_j^{-1} \otimes \mathbf{Q}_\ell & \to 0,
\end{array}$$

and the monodromy operator N on E_ℓ is obtained from the monodromy operator on $\widetilde{M}$ by composing with the isomorphism g and its inverse:

$$E_\ell \xrightarrow{g^{-1}} \widetilde{M} \xrightarrow{N} \widetilde{M}(-1) \xrightarrow{g} E_\ell(-1).$$

Finally, to obtain the result in the case $\ell = p$, one uses a similar argument for the log-crystalline cohomology by applying [RX], Corollary 3 and Theorem 3.

Now let $\Lambda = T_j^{-1}/tors$ and $\Lambda' = T_{j-1}^1/tors$. We are going to define a pairing

$$\Lambda \times \Lambda'^{\vee} \to K^*$$

as follows: let $\alpha \in \Lambda, \beta \in \Lambda'^{\vee}$, and tensor the subgroups that these elements generate with $\mathbf{Z}_\ell$ and $\mathbf{Z}_\ell(1)$, respectively. Pull back the extension E_ℓ (which was

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introduced just before Lemma 5) by the former and push it out by the latter to get an extension of $\mathbf{Z}_\ell$ by $\mathbf{Z}_\ell(1)$, hence an element of

$$\mathrm{Ext}^1_G(\mathbf{Z}_\ell, \mathbf{Z}_\ell(1)) = K^{*(\ell)},$$

the ℓ-completion of the multiplicative group of K. We denote this element by $\{\alpha, \beta\}_\ell$ and the product of the pairings over all ℓ by $\{\alpha, \beta\}$. This pairing *a priori* takes values in $\widehat{K^*} = \varprojlim_n K^*/K^{*\,n}$.

Proposition 6. *The image of the pairing $\{-,-\}$ in $\widehat{K^*}$ is contained in the image of the natural map $K^* \to \widehat{K^*}$.*

In the proof, we shall use the following well-known lemma:

Lemma 7. *Let K be a finite extension of $\mathbf{Q}_p$ with normalized valuation v. Denote by $\widehat{v}$ the "completed valuation"*

$$\widehat{K^*} \to \widehat{\mathbf{Z}}.$$

Then the completed valuation of an element $\alpha \in \widehat{K^*}$ lies in the image of $\mathbf{Z}$ in $\widehat{\mathbf{Z}}$ iff α lies in the image of K^* in $\widehat{K^*}$.

Proof of lemma 7. Let R be the ring of integers in K. Then since K is finite over $\mathbf{Q}_p$, the groups R^*/R^{*n} are finite for every integer n, and R^* is complete with respect to the topology induced by its subgroups of finite index. Thus the natural map $R^* \to \widehat{R^*}$ is an isomorphism and the map $K^* \to \widehat{K^*}$ is injective. Hence the induced map

$$\widehat{K^*}/K^* \to \widehat{\mathbf{Z}}/\mathbf{Z}$$

is an isomorphism, which proves the lemma.

Remark 8. The use of this lemma is the one place where our construction will not work over a larger field such as $\mathbf{C}_p$. What seems to be needed is to define our extensions in some sort of category $\mathcal{C}$ of "analytic mixed motives" over $\mathbf{C}_p$, in which we would have $Ext^1_{\mathcal{C}}(\mathbf{Z}, \mathbf{Z}(1)) = \mathbf{C}_p^*$. As is well-known, extensions of $\mathbf{C}_p$ by $\mathbf{C}_p(1)$ in the category of continuous $Gal(\overline{K}/K)$-modules are trivial, so this category won't do.

Proof of Proposition 6. It is sufficient to show that for each ℓ, the diagram of pairings:

$$\begin{array}{ccccccc}
T_j^{-1} & \times & T_{j-1}^{1\vee} & \to & K^{*(\ell)} & \to & \mathbf{Z}_\ell \\
\| & & \| & & & & \uparrow \\
T_j^{-1} & \times & T_{j-1}^{1\vee} & \to & & & \mathbf{Z}
\end{array}$$

is commutative. The pairing in the top row is that just defined, and the pairing in the bottom row is constructed from the isogeny

$$\widetilde{N} : T_j^{-1} \to T_{j-1}^1$$

defined in Lemma 5. This shows that the top pairing takes values in $\mathbf{Z}$, hence factors through the image of K^* in $K^{*(\ell)}$ for every ℓ. Clearly, we only need to prove that this diagram is commutative after tensoring by $\mathbf{Q}$.

Now, the commutativity of the diagram is deduced from Lemma 5 by using the same argument as in the paper of Raynaud ([Ray2], section 4.6). Concretely, given an exact sequence $0 \to E' \to E \to E/E' \to 0$ as Galois $\mathbf{Q}_\ell$-modules such

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that $E'(-1)$ and E/E' are $\mathbf{Q}_\ell$-vector spaces of finite dimension with trivial Galois action, one can construct as before a pairing

$$\{-,-\} : E/E' \times E'(-1)^\vee \to K^{*(\ell)} \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell \xrightarrow{v_\ell} \mathbf{Q}_\ell.$$

Then the map $N' \colon E/E' \to E'(-1)$ constructed from this pairing coincides with the map induced from the monodromy on E.

Definition 9. If the Galois action on $T_j^{-1}(\overline{Y})$ and $T_{j-1}^1(\overline{Y})$ is trivial, we define the j-th intermediate Jacobian $J^j(X)$ as follows: consider the pairing that we just defined:

$$\{-,-\} : \Lambda \times \Lambda^{'\vee} \to K^*.$$

Now define:

$$J^j(X) := \mathrm{Hom}(\Lambda^{'\vee}, \mathbf{G}_m)/\Lambda,$$

where we are viewing Λ as a subgroup of $\mathrm{Hom}(\Lambda^{'\vee}, \mathbf{G}_m)$ via the pairing $\{-,-\}$. This quotient makes sense in a rigid-analytic setting due to Proposition 2, which tells us that Λ is a lattice in the algebraic torus $\mathrm{Hom}(\Lambda^{'\vee}, \mathbf{G}_m)$.

Now we consider the case where there may be a non-trivial action of the absolute Galois group G_F on $T_j^{-1}/tors$ and on $T_{j-1}^1/tors$, necessarily through a finite quotient. After a finite unramified Galois extension L/K, we have a pairing

$$T_j^{-1}/tors \times T_{j-1}^{1\vee}/tors \to L^*.$$

If we show that this pairing respects the action of the Galois group of L/K on both sides, we will have an identification of $T_j^{-1}/tors$ and a lattice in the torus with character group $T_{j-1}^1/tors$. We can then define the $J^j(X)$ as the *non-split* analytic torus defined as the quotient. But this pairing was defined by using an extension in $\mathrm{Ext}^1_G(T_j^{-1}/tors \otimes \mathbf{Z}_\ell, T_{j-1}^1/tors \otimes \mathbf{Z}_\ell(1))$ for any ℓ, so it is clear that the pairing

$$\{-,-\} : T_j^{-1}/tors \times T_{j-1}^{1\vee}/tors \to \prod_\ell L^{*(\ell)}$$

respects the Galois action of $\mathrm{Gal}(L/K)$.

Remark 10. (i) Observe that the dimension of $J^j(X)$ is equal to the rank of T_{j-1}^1, which is equal to $\dim_{\mathbf{Q}_p}(H^j(\overline{Y}, W\omega^{j-1}_{\overline{Y},log}) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p)$ (see [RX], Corollary 5). This last dimension is equal to $h^{j,j-1} := \dim_K(H^j(X, \Omega_X^{j-1}))$ by [I3], Proposition 2.6 (c).

(ii) Note that our intermediate Jacobian, which *a priori* depends on the chosen regular proper model of X over R, actually only depends modulo isogeny on the variety X. To show this, observe first that the monodromy filtration we constructed in the p-adic cohomology $H^n(\overline{X}, \mathbf{Q}_p)$ is uniquely defined, because there are no maps between $\mathbf{Q}_p(j)$ and $\mathbf{Q}_p(i)$ if $i \neq j$. On the other hand, the Tate module $V_p(J^j(X))$ is isomorphic to $M_1(H)/M_{-3}(H)$, where $H = H^{2j-1}(\overline{X}, \mathbf{Q}_p(j))$ and $M_\bullet$ is the monodromy filtration. But an analytic torus over K is determined modulo isogeny by its Tate module (the proof of this is a generalization of the argument in [Se], Ch. IV, §A.1.2). Observe that for $j = 1$ and $j = d$, the subquotient of $H^{2j-1}(\overline{X}, \mathbf{Z}_\ell)$ that we use to construct the intermediate Jacobian is, in fact, the whole torsion free quotient of this cohomology group. This fact, together with similar arguments just given to determine an analytic torus by its Tate module,

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then show that for $j = 1$ and $j = d$, $J^j(X)$ is uniquely determined by X and is equal to, respectively, the Picard and Albanese varieties of X.

3. The Abel-Jacobi mapping

Let K be a finite extension of $\mathbf{Q}_p$, and let X be a smooth projective variety over K. We assume that X has a regular proper model $\mathcal{X}$ over the ring of integers R of K, with special fibre satisfying the assumptions in §1.2. In the last section, we defined intermediate Jacobians $J^j(X)$, which we will denote by J^j when X is fixed. In this section, we define the Abel-Jacobi mapping:

$$CH^j(X)_{hom} \to J^j(X) \otimes_{\mathbf{Z}} \mathbf{Q},$$

where $CH^j(X)_{hom}$ is the subgroup of the Chow group $CH^j(X)$ consisting of cycles in the kernel of the cycle map:

$$CH^j(X) \to \prod_\ell H^{2j}(\overline{X}, \mathbf{Z}_\ell(j)),$$

where the product is taken over all prime numbers ℓ.

Let $T_\ell = T_\ell(J^j(X))$, the Tate module of $J^j(X)$, and let $T = \prod_\ell T_\ell$. Put $V_\ell = T_\ell \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell$, and let $V = \prod_\ell V_\ell$. Recall the groups $H^1_g(K,T)$, $H^1_g(K,V)$ of Bloch-Kato ([BK2], 3.7). We have

$$H^1_g(K, V_p) = \mathrm{Ker}[H^1(K, V_p) \to H^1(K, V_p \otimes_{\mathbf{Q}_p} B_{DR})],$$

where B_{DR} is the field of "p-adic periods" of Fontaine. For $\ell \neq p$, we have:

$$H^1_g(K, V_\ell) = H^1(K, V_\ell).$$

$H^1_g(K,T)$ is defined as the inverse image of $H^1_g(K,V)$ under the natural map

$$H^1(K,T) \to H^1(K,V).$$

The following lemma is proved as in the case of an abelian variety in [BK2], §3.

Lemma 11. *We have an isomorphism:*

$$J^j(K) \to H^1_g(K,T),$$

compatible with passage to a finite extension of K and with norms.

Remark 12. The content of the lemma is that the extensions of $\mathbf{Z}_p$ by $\mathbf{Z}_p(1)$ that give pairings with values in K^* (as opposed to $\widehat{K^*}$) are precisely the de Rham extensions.

Now, the following result is essentially due to Faltings [Fa2]: it is a consequence of his results that the étale cohomology of a smooth variety (not necessarily projective) is de Rham in the sense defined by Fontaine (see for example [I1]).

Theorem 13 (Faltings)**.** *Let X be a smooth projective variety. Let*

$$H := \prod_\ell H^{2j-1}(\overline{X}, \mathbf{Z}_\ell(j))$$

and let $cl' : CH^j(X)_{hom} \to H^1(K,H)$ be the canonical étale Abel-Jacobi mapping (see [Bl] *and* [R1]*). Then the image of cl' is contained in $H^1_g(K,H)$.*

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Corollary 14. *Let X be a smooth projective variety over K with a regular proper model $\mathcal{X}$ over R satisfying the assumptions of §1.2. Let M be the monodromy filtration on $H^{2j-1}(\overline{X}, \mathbf{Q}_\ell(j))$, and let $V = \prod_\ell V_\ell$. Then the ℓ-adic Abel-Jacobi map cl' factorizes through a map $cl' : CH^j(X)_{hom} \to H^1_g(K, M_1)$.*

Proof. Observe first that for every ℓ we have that V_ℓ/M_1 is a successive extension of $\mathbf{Q}_\ell$-vector spaces of the form $\mathbf{Q}_\ell(i)$, with $i < 0$. Then the result follows from the fact that $[V_\ell/M_1]^G = 0$ and $H^1_g(K, \mathbf{Q}_\ell(i)) = 0$ for $i < 0$ ([BK2], Example 3.5). □

Definition 15. We define the rigid-analytic Abel-Jacobi mapping as the composition

$$CH^j(X)_{hom} \to H^1_g(K, M_1) \to H^1_g(K, M_1/M_{-3}) \cong J^j(X)(K) \otimes \mathbf{Q}$$

using Lemma 11 and Corollary 14.

Example 16. Let $\mathcal{X}$ be a regular proper scheme of relative dimension d over R, whose special fibre satisfies the assumptions of §1. Then $J^1(X)$ is the Jacobian variety of X, $J^d(X)$ is the Albanese variety of X and the maps defined above are the classical Abel-Jacobi mappings, tensored with $\mathbf{Q}$.

Remark 17. (i) Recall that by definition, $H^1_g(K, T)$ contains the torsion subgroup of $H^1(K, T)$ for any finitely generated $\mathbf{Z}_\ell$-module T. Now for $i < 0$, $H^1(K, \mathbf{Z}_\ell(i))$ has nontrivial torsion, in general, and so it is not clear that we can define the Abel-Jacobi mapping to $J^j(X)$ integrally. Of course, we can define the map on the inverse image of $H^1(K, T \cap M_1)$ under the Abel-Jacobi mapping, but it is not clear how this subgroup changes as we extend the base field K.

(ii) As mentioned in Remark 10, the dimension of $J^j(X)$ is the Hodge number $h^{j,j-1}$. This differs in general from the dimension of the complex intermediate Jacobian of Griffiths, which is always equal to $\frac{B_{2j-1}}{2}$. Using higher odd K-theory, one can in some sense "recover" the "rest" of the classical intermediate Jacobian. For example, for $j = 2$, we have the exact sequence:

$$0 \to M_{-3} \to M_1 \to M_1/M_{-3} \to 0.$$

The term on the right is the Tate module of our $J^2(X)$. We then get a natural map:

$$\ker \alpha_2 \to H^1(K, M_{-3}),$$

where α_2 is the Abel-Jacobi mapping we have just defined. Now M_{-3} modulo torsion is a direct sum of $\mathbf{Z}_\ell(2)$'s, and we have

$$\varprojlim_n K_3(K, \mathbf{Z}/\ell^n) \cong H^1(K, \mathbf{Z}_\ell(2))$$

(see [Le] and [MS]).

Thus we can relate $\ker \alpha_2$ to $K_3(K)$. Using work of Hesselholt-Madsen (see [HM], Theorem A), we can relate the kernel of α_j successively to higher odd continuous algebraic K-groups $\varprojlim_n K_{2r-1}(K, \mathbf{Z}/\ell^n)$, for $r = 2, \cdots, j$. We hope to pursue this point in another paper.

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4. An example

In this section we describe in detail an example that motivated our theory.

4.1. **Product of two Tate elliptic curves.** To warm up, we analyze the case of the product X of two Tate elliptic curves E_1, E_2, that is, elliptic curves with split multiplicative reduction, or, equivalently, rigid analytic tori of dimension 1. We can write $E_i = \mathbf{G}_m/q_i^{\mathbf{Z}}$, for some $q_i \in K^*$ with $|q_i| < 1$. The results we shall describe here are due to Serre and Tate (see [Se], Ch. IV §A.1.4), but we interpret them in terms of our monodromy filtration, which will put them in a form suitable for generalization to higher cohomology groups.

Let $V = H^1(\overline{E}_1, \mathbf{Q}_p(1))$ and $W = H^1(\overline{E}_2, \mathbf{Q}_p)$. By the Künneth formula, we have that $H^2(\overline{X}, \mathbf{Q}_p(1))$ is the direct sum of $V \otimes W$ and two copies of the trivial representation $\mathbf{Q}_p$. The monodromy filtration on $M = V \otimes W$ is the tensor product of the monodromy filtrations on V and W. We thus have

$$M_0 = \sum_{i+j=0} V_i \otimes W_j.$$

We have exact sequences:

$$0 \to \mathbf{Q}_p(1) \to V \to \mathbf{Q}_p \to 0,$$
$$0 \to \mathbf{Q}_p \to W \to \mathbf{Q}_p(-1) \to 0,$$

which encode all of the information about the monodromy filtrations. Let q_i $(i = 1, 2)$ be the class of the extension V (resp. W) in

$$Ext^1_G(\mathbf{Q}_p, \mathbf{Q}_p(1)) = K^{*\,(p)} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p.$$

q_i is in fact the image of the Tate parameter of E_i under the natural map:

$$K^* \to K^{*\,(p)} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p,$$

which hopefully explains this abuse of notation.

Tensoring the first sequence with W and the second by V, we get exact sequences

$$0 \to W(1) \to V \otimes W \to W \to 0,$$
$$0 \to V \to V \otimes W \to V(-1) \to 0.$$

This gives us a map:

$$V \oplus W(1) \to V \otimes W,$$

and since $V = V_1, W = W_1$, we get a surjection

$$V \oplus W(1) \to M_0.$$

Note that the dimension of the space on the left is 4 and the dimension of that on the right is 3. Thus the kernel is of dimension 1, and is easily seen to be $V \cap W(1) = \mathbf{Q}_p(1)$, where these two spaces are viewed as subspaces of M_0 via the exact sequences above. We also have that $M_{-2} = V_{-1} \otimes W_{-1} = \mathbf{Q}_p(1)$. Summarizing, we have an exact sequence:

$$0 \to \mathbf{Q}_p(1) \to M_0 \to M_0/M_{-2} \to 0,$$

where the Galois group acts on the right hand space through a finite quotient. Passing to an open subgroup H of finite index that acts trivially on this space and taking the standard basis of $\mathbf{Q}_p^2$, we see that the map

$$M_0/M_{-2} \to H^1(H, \mathbf{Q}_p(1))$$

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sends the vector (a_1, a_2) to the class of $q_1^{a_1} q_2^{a_2}$ in $K^{*\,(p)} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$. Thus $M_0^H \neq 0$ iff there exist nonzero p-adic numbers a_1, a_2 such that $q_1^{a_1} q_2^{a_2} = 1$. We now have:

Theorem 18 (see [Se], Ch. IV, §A.1.2)**.** *E_1 and E_2 are isogenous iff there exist nonzero integers a_1, a_2 such that $q_1^{a_1} q_2^{a_2} = 1$. The map*

$$Hom(E_1, E_2) \otimes_{\mathbf{Z}} \mathbf{Q}_p \to Hom_G(V_p(E_1), V_p(E_2))$$

is an isomorphism.

Taking into account the other contributions to $H^2(\overline{X}, \mathbf{Q}_p(1))$, we get

Corollary 19. *For X the product of two Tate elliptic curves, the cycle map*

$$Pic(X) \otimes \mathbf{Q}_p \to H^2(\overline{X}, \mathbf{Q}_p(1))^G$$

is surjective.

This last statement is the "p-adic Tate conjecture" for the product of two Tate elliptic curves (see Remark 25 ii) below).

4.2. **Product of three Tate elliptic curves.** Now we generalize this to H^3 of the product X of three Tate elliptic curves, $E_i, i = 1, 2, 3$, with parameters q_i. We study $J^2(X)$. Now X is a 3-dimensional abelian variety, so we have:

$$\dim H^3(\overline{X}, \mathbf{Q}_\ell) \cong \dim \bigwedge^3 H^1(X, \mathbf{Q}_\ell) = \binom{6}{3} = 20.$$

Thus the complex intermediate Jacobian would be of dimension 10. Our $J^2(X)$ will be of dimension 9, and we explain below how to recover the part that has been "lost" .

We calculate the monodromy filtration on $H^3(\overline{X}, \mathbf{Q}_p(2))$. As above, the monodromy filtration on each $V_i = H^1(\overline{E_i}, \mathbf{Q}_p(1))$ is given by:

$$V_{i,-1} = \mathbf{Q}_p(1), V_{i,0} = \mathbf{Q}_p(1), V_{i,1} = V,$$

and $V_i/V_{i,-1} = \mathbf{Q}_p$. The class of the extension

$$0 \to V_{i,-1} \to V_i \to V_i/V_{i,-1} \to 0$$

in $H^1(K, \mathbf{Q}_p(1)) = K^{*\,(p)} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ is given by the class of the Tate parameter q_i of E_i in this group. The monodromy filtration on $M = V_1 \otimes V_2 \otimes V_3(-1)$ is the tensor product of the monodromy filtrations on the V_i. Thus

$$M_n = \sum_{r+s+t=n} V_{1,r} \otimes V_{2,s} \otimes V_{3,t}(-1);$$

note that this sum is not direct. We easily calculate the following table, in which the top row is the level of the filtration and the bottom row is the dimension:

-3	-2	-1	0	1	2	3
1	1	4	4	7	7	8.

If we include the other contributions to $H^3(\overline{X}, \mathbf{Q}_p(2))$ from the Künneth formula, we find the following table, which gives the dimension of the spaces in the various steps of the monodromy filtration on $H^3(\overline{X}, \mathbf{Q}_p(2))$:

-3	-2	-1	0	1	2	3
1	1	10	10	19	19	20.

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The p-Tate vector space $V_p(J^2(X))$ of our $J^2(\overline{X})$ is M_1/M_{-3}, which is of dimension 18, and thus the dimension of $J^2(X)$ is 9. The "lost" dimension comes from the fact that $Gr_{-2}M \cong \mathbf{Q}_p(2)$ as Galois modules, and hence this cannot contribute to the Tate module of a p-adic analytic torus. As mentioned above in Remark 17(ii), the lost part can be directly related to

$$K_3(K, \mathbf{Q}_p)^{ind} = [\varprojlim_n K_3(K, \mathbf{Z}/p^n)] \otimes \mathbf{Q}_p,$$

via extensions of $\mathbf{Q}_p$ by $\mathbf{Q}_p(2)$.

In some cases, we can also determine the image of the restriction of the Abel-Jacobi mapping to $CH^2(X)_{alg}$; this is an abelian variety, which is the universal abelian variety into which $CH^2(X)_{alg}$ maps via a regular homomorphism (see [Mur]); we denote it by $J^2_a(X)$. Its dimension should depend on the multiplicative relations between the q_i with integer exponents, as we now explain. To simplify the exposition, **we assume that the Galois group acts trivially on the** $T^i_j(\overline{Y})$, where $\overline{Y}$ is the geometric special fibre of a strictly semi-stable model $\mathcal{X}$ of X over the ring of integers of K. As we already said, this will always happen after a finite unramified extension of K because all of the $T^i_j(\overline{Y})$ are finitely generated abelian groups.

4.3. **Enriched monodromy operators.** With notation as in the previous sections, we have an extension for each prime number ℓ:

$$0 \to T^3_0 \otimes \mathbf{Q}_\ell(1) \to M_{-1} \to T^1_1 \otimes \mathbf{Q}_\ell \to 0.$$

As in the definition of the intermediate Jacobian, we can define a pairing:

$$(T^1_1 \otimes_{\mathbf{Z}} \mathbf{Q}) \times (T^{3\vee}_0 \otimes_{\mathbf{Z}} \mathbf{Q}) \to K^{*(\ell)} \otimes_{\mathbf{Z}} \mathbf{Q}$$

by taking elements $\alpha \in T^1_1 \otimes_{\mathbf{Z}} \mathbf{Q}, \beta \in T^{3\vee}_0 \otimes_{\mathbf{Z}} \mathbf{Q}$, tensoring the subgroups they generate by $\mathbf{Q}_\ell$ and $\mathbf{Q}_\ell(1)$, respectively, pulling back the extension above by the first and pushing it out by the second. Taking the product for all ℓ and using Proposition 6 above, we see that we get a $K^* \otimes_{\mathbf{Z}} \mathbf{Q}$-valued pairing and hence a map:

$$N_1 : T^1_1 \otimes_{\mathbf{Z}} \mathbf{Q} \to Hom(T^{3\vee}_0 \otimes_{\mathbf{Z}} \mathbf{Q}, K^* \otimes_{\mathbf{Z}} \mathbf{Q}),$$

that we call the *enriched monodromy operator*. See [R2] for more on enriched monodromy operators and their relation with the generalized Hodge-Tate conjecture. We denote also by

$$N'_1 : T^{-1}_2 \otimes_{\mathbf{Z}} \mathbf{Q} \to Hom(T^{3\vee}_0 \otimes_{\mathbf{Z}} \mathbf{Q}, K^* \otimes_{\mathbf{Z}} \mathbf{Q}),$$

the composition of the map $N : T^{-1}_2 \to T^1_1$ (tensored with $\mathbf{Q}$) with the map N_1 just defined.

4.4. **The generalized Hodge-Tate conjecture for H^3.** Recall that if Z and X are two smooth projective varieties over a field, then a correspondence between them is an element η of $CH^r(Z \times X)$; such a cycle induces a Galois equivariant map:

$$\eta_* : H^i(\overline{Z}, \mathbf{Q}_\ell(1)) \to H^{i+2r}(\overline{X}, \mathbf{Q}_\ell(1+r))$$

defined by $\eta_*(\alpha) = pr_{2*}(pr_1^*(\alpha) \cup [\eta])$. Here $[\eta]$ is the cohomology class of η in $H^{2r}(\overline{Z} \times \overline{X}, \mathbf{Q}_\ell(r))$. Let $N^rH^i(\overline{X}, \mathbf{Q}_\ell(1+r))$ be the subspace spanned by the images of such maps η_*, where Z is smooth and projective of dimension $d-r$, where

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$d = \dim(X)$. This is called the *coniveau filtration* on X. Note the superscripts, as we shall also use subscripts on N for the monodromy filtration.

Lemma 20. *Let A be an abelian variety contained in $J_a^2(X)$. Write $V_l(A)$ as an extension*

$$(*) \qquad 0 \to S_0^1 \otimes \mathbf{Q}_\ell(1) \to V_l(A) \to S_1^{-1} \otimes \mathbf{Q}_\ell \to 0,$$

where S_j^i are finitely generated abelian groups (see the proof below for why this is possible). Then for some finite extension L/K, we have:

$$S_0^1 \otimes_{\mathbf{Z}} \mathbf{Q} \subseteq \ker N_{1,L} : \, T_1^1 \otimes_{\mathbf{Z}} \mathbf{Q} \to Hom(T_0^{3\vee} \otimes_{\mathbf{Z}} \mathbf{Q}, L^* \otimes_{\mathbf{Z}} \mathbf{Q})$$

and

$$S_1^{-1} \otimes_{\mathbf{Z}} \mathbf{Q} \subseteq \ker N'_{1,L} : \, T_2^{-1} \otimes_{\mathbf{Z}} \mathbf{Q} \to Hom(T_0^{3\vee} \otimes_{\mathbf{Z}} \mathbf{Q}, L^* \otimes_{\mathbf{Z}} \mathbf{Q}).$$

Proof. Since A is contained in $J^2(X)$, it is an analytic torus. Denote by S_0^1 and by S_1^{-1} the corresponding lattices of A, such that $A(L) \cong Hom(S_0^1, L^*)/S_1^{-1}$. Now the inclusion map from A to $J^2(X)$ gives us monomorphisms $i\colon S_0^1 \to T_1^1$ and $i'\colon S_1^{-1} \to T_2^{-1}$. We must show that $N_1 \circ (i \otimes_{\mathbf{Z}} \mathbf{Q}) = 0$ and $N'_1 \circ (i' \otimes_{\mathbf{Z}} \mathbf{Q}) = 0$ for some finite extension L/K. By the definition of the enriched monodromy operator, to prove this it is sufficient to show that for any prime ℓ, the composition

$$S_0^1 \otimes_{\mathbf{Z}} \mathbf{Q}_\ell \to T_1^1 \otimes_{\mathbf{Z}} \mathbf{Q}_\ell \xrightarrow{N_{L,\ell}} Hom(T_0^{3\vee} \otimes_{\mathbf{Z}} \mathbf{Q}_\ell, L^{*(\ell)} \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell)$$

is trivial.

There exists a divisor D on X with desingularization D' and a correspondence $\eta \in CH^2(X \times D')$ such that $V_\ell(A)$ is the image of η_* in $H^3(\overline{X}, \mathbf{Q}_\ell(2))$. Take L a finite extension of K such that D' has semi-stable reduction. Then the cohomology $H^1(\overline{D'}, \mathbf{Q}\ell(1))$ is semi-stable.

First consider the case $\ell \neq p$. Denoting by $M_\bullet$ the monodromy filtration on the cohomology, we have that the correspondence η gives a map respecting this filtration. This gives us a map

$$Gr_{-1}^M(H^1(\overline{D'}, \mathbf{Q}_p(1))) \to S_0^1 \otimes_{\mathbf{Z}} \mathbf{Q}_\ell \to T_1^1 \otimes_{\mathbf{Z}} \mathbf{Q}_\ell.$$

We have, on the other hand, a commutative diagram of monodromy operators

$$\begin{array}{ccc} T_1^1 \otimes_{\mathbf{Z}} \mathbf{Q}_\ell & \overset{N_{1,L}}{\to} & Hom(T_0^{3\vee} \otimes_{\mathbf{Z}} \mathbf{Q}_\ell, L^{*(\ell)} \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell) \\ \uparrow & & \uparrow \\ Gr_{-1}^M(H^1(\overline{D'}, \mathbf{Q}_p(1))) & \overset{N_{1,L}}{\to} & Hom(Gr_{-3}^M(H^1(\overline{D'}, \mathbf{Q}_p(1)))^\vee(1), L^{*(\ell)} \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell). \end{array}$$

But $Gr_{-3}^M(H^1(\overline{D'}, \mathbf{Q}_p(1))) = 0$, since there is no graded piece of weight ≤ -3 in the monodromy filtration on $H^1(\overline{D'}, \mathbf{Q}_p(1))$, and hence the image of $S_{-1}^1 \otimes_{\mathbf{Z}} \mathbf{Q}_\ell$ in $T_1^1 \otimes_{\mathbf{Z}} \mathbf{Q}_\ell$ is contained in $\ker N_L$, as claimed.

Now, to show the case $\ell = p$, we use instead of the p-adic cohomology of D' the log-crystalline cohomology of a suitable model of D'. We have then, by applying Tsuji's theorem, if necessary, that the correspondence η gives us a map between the respective log-crystalline cohomologies, respecting the monodromy filtration. Then the same argument applies as we have just given for $\ell \neq p$. A similar argument applies also to the other claimed inclusion.

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Conversely, we have the

Conjecture 21 (p-adic Generalized Hodge conjecture for H^3)**.** *For X with totally degenerate reduction, let the enriched monodromy operators $N_{1,L}$ and $N'_{1,L}$ be defined as in §4.3. Then we have*

$$\begin{aligned} &Gr^{-1}N^1H^3(\overline{X},\mathbf{Q}_p) \\ &= \sum_{[L:K]<\infty} [\ker N_{1,L} : T_1^1 \otimes_{\mathbf{Z}} \mathbf{Q} \to Hom(T_0^{3\vee} \otimes_{\mathbf{Z}} \mathbf{Q}, L^* \otimes_{\mathbf{Z}} \mathbf{Q})] \otimes_{\mathbf{Q}} \mathbf{Q}_p(-1)\end{aligned}$$

and

$$\begin{aligned} &Gr^M_{-1}N^1H^3(\overline{X},\mathbf{Q}_p) \\ &= \sum_{[L:K]<\infty} [\ker N'_{1,L} : T_2^{-1} \otimes_{\mathbf{Z}} \mathbf{Q} \to Hom(T_0^{3\vee} \otimes_{\mathbf{Z}} \mathbf{Q}, L^* \otimes_{\mathbf{Z}} \mathbf{Q})] \otimes_{\mathbf{Q}} \mathbf{Q}_p(-2).\end{aligned}$$

Note that these spaces are of the same dimension, since the monodromy operator $N : T_2^{-1} \to T_1^1$ is an isogeny. This is as it should be, since $N^1H^3(\overline{X},\mathbf{Q}_p(2))$ is analogous to a pure Hodge structure of odd weight, and so should be even dimensional.

Remark 22. To explain the motivation for phrasing this conjecture as we have, consider the extension:

$$0 \to M_{-3} \to M_{-1} \to M_{-1}/M_{-3} \to 0,$$

where $M = V_p(E_1) \otimes V_p(E_2) \otimes V_p(E_3)(-1)$.

Taking H-cohomology for an open subgroup H of finite index in G, we get an exact sequence:

$$0 \to M_{-1}^H \to [M_{-1}/M_{-3}]^H \xrightarrow{\partial_H} H^1(H, M_{-3}).$$

The space in the middle is isomorphic to $(T_1^1 \otimes \mathbf{Q}_p)^H$. Arguing as in §4.1, we see that the space M_{-1}/M_{-3} is three dimensional, and for sufficiently small H, the map ∂_H takes a vector (a_1,a_2,a_3) to $\prod q_i^{a_i}$, viewed as an element of $L^{*\,(p)}$. Thus $\ker\,\partial_H$ corresponds to relations between the q_i in $L^{*\,(p)}$ *with p-adic exponents.* There is no reason why these have to be relations with rational integer exponents, as they are for the case of two q_i's. For example, suppose we choose a uniformizing parameter π of K and a logarithm with $\log(\pi) = 0$. Then finding p-adic numbers a_1,a_2,a_3 with

$$\prod_{i=1}^3 q_i^{a_i} = 1$$

is equivalent to solving the simultaneous equations:

$$\sum_{i=1}^3 a_i \log(q_i) = 0$$

and

$$\sum_{i=1}^3 a_i v_i = 0$$

in $\mathbf{Q}_p$, where $v_i = v(q_i)$. This is a system of two equations in three unknowns, so there should always be a solution other than $(0,0,0)$. But in general, there is no

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solution to these equations with the a_i rational integers. In other words, the kernel of the enriched monodromy operator:

$$T_1^1 \to Hom(T_0^{3\vee}, K^*)$$

tensored with $\mathbf{Z}_p$, may be strictly smaller than the kernel of the map:

$$T_1^1 \otimes \mathbf{Z}_p \to Hom(T_0^{3\vee}, K^{*\,(p)}).$$

Note that this map is the completion of the first map with respect to subgroups of finite index, not the tensor product of this map with $\mathbf{Z}_p$. This is why its kernel can be bigger, since the natural map

$$K^* \otimes_{\mathbf{Z}} \mathbf{Z}_p \to K^{*\,(p)}$$

is surjective but not injective. In [R2], this difference between relations with p-adic integer exponents and with rational integer exponents is conjecturally explained via the difference between two types of coniveau filtrations on étale cohomology. One filtration takes $\mathbf{Q}_p$-coefficients throughout, and the other, which is in general bigger, takes the inverse limit of the coniveau filtrations with $\mathbf{Z}/p^n\mathbf{Z}$-coefficients and tensors with $\mathbf{Q}_p$.

In the following, we shall prove Conjecture 21 in some cases when X is the product of three Tate elliptic curves. Complex analogues of these results are well-known (see e.g. [Li], p. 1197). The strategy is simple: by Lemma 20, for H a sufficiently small open subgroup of G with fixed field L, the dimension of $\ker N_{1,L}$ provides an upper bound for the dimension of $Gr^{-1}N^1H^3(\overline{X}, \mathbf{Q}_p)$. For X as above, we can easily compute this bound, and in some cases we can explicitly produce divisorial correspondences η such that the sum of the images of η_* is of the required dimension.

Conjecture 23 (Compare 7.5 on p. 1197 of [Li]). *Let $X = E_1 \times E_2 \times E_3$ be a product of three Tate elliptic curves with parameters q_i $(i = 1, 2, 3)$, and let r be the rank of the space of triples of integers (n_1, n_2, n_3) with*

$$\prod_{i=1}^{3} q_i^{n_i} = 1.$$

Then

$$dim J_a^2(X) = 6 + r.$$

For example, for "generic" q_i (no multiplicative relations), the dimension is 6, and if all of the q_i are equal, the dimension is 8. In particular, for X the product of three Tate elliptic curves, the restriction of the Abel-Jacobi map to $CH^2(X)_{alg}$ is never surjective onto $J^2(X)$.

Proposition 24. *The conjecture is true if there are two independent multiplicative relations between the q_i, if there is one relation and two of the three curves are isogenous, or if there are no nontrivial multiplicative relations between the q_i.*

Proof. The dimension is at least 6 since $J^2(X)$ contains two copies of X, via algebraic cohomology classes of the type

$$\alpha \otimes \beta \otimes \gamma \in H^0(\overline{E}_i) \otimes H^1(\overline{E}_j) \otimes H^2(\overline{E}_k),$$

for each permutation (i, j, k) of $(1, 2, 3)$.

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To determine the dimension of $J_a^2(X)$, assume first that the rank of the space of relations is two. Since the valuations of the q_i are all positive, all of the 2 by 2 minors of the two vectors must be nonzero. We can now reduce a chosen basis vector for the relation exponents so that each contains exactly one 0, and these are in different coordinate positions. But then by Theorem 18 in §4.1, we have two relations between two of the q_i's, which gives isogenies between the respective E_i. Then it is easy to see that the dimension of $J_a^2(X)$ is 8, because $[M_{-1}/M_{-3}]^G$ is then spanned by three cohomology classes of the type:

$$\alpha_{ij} \otimes \beta \in H^1(\overline{E}_i) \otimes H^1(\overline{E}_j) \otimes H^1(\overline{E}_k),$$

where α_{ij} is the class of the graph of a nonzero isogeny between E_i and E_j. These classes cannot all lift to M_{-1}^G, as then the map:

$$[M_{-1}/M_{-3}]^G \to H^1(G, M_{-3})$$

would be zero, which can't be, since there are linear combinations of the valuations of the q_i that are nonzero.

If there is, say, a multiplicative relation between q_1 and q_2, and these have no multiplicative relations with q_3, then we can produce another cohomology class supported on the graph of the isogeny, as above.

If there are no relations, then the dimension is at most six, and so is exactly six. This completes the proof of the proposition. $\square$

Remark 25. i) When there is just one multiplicative relation which does not arise from an isogeny between two of the curves, there is no obvious way to produce the abelian variety of the expected dimension in $J^2(X)$, but Conjecture 21 predicts that it should exist.

ii) In a sequel to this paper [R2], we formulate a p-adic generalized Tate conjecture, which predicts the dimension of the coniveau filtration $N^iH^j(\overline{X}, \mathbf{Q}_p)$ for any i, j in terms of the kernels of enriched monodromy operators.

iii) Since the image of the Abel-Jacobi mapping restricted to $CH^2(X)_{alg}$ is never all of $J^2(X)$ when X is the product of three Tate elliptic curves, we hope to be able to use the quotient of $J^2(X)$ by this image to detect codimension two cycles that are homologically equivalent to zero, but not algebraically equivalent to zero. A regular proper model for such products has been constructed by Gross-Schoen ([GS], §6, especially Props. 6.1.1 and 6.3, and Cor. 6.4), but we do not believe it is strictly semi-stable. Further blowing-up should make it such, however. See also the paper of Hartl [Ha], where he constructs strictly semi-stable regular models of ramified base changes of varieties with strictly semi-stable reduction, and of products of such.

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