Math. Ann. 306, 459–486 (1996)

Mathematische Annalen

Component groups of Néron models via rigid uniformization

Siegfried Bosch[1], **Xavier Xarles**[2,*]

[1] Mathematisches Institut der Universität, Einsteinstrasse 62, D-48149 Münster, Germany
[2] Departament de Matemàtiques, Universitat Autònoma de Barcelona, E-08193 Bellaterra, Barcelona, Spain

Received: 16 January 1995 / Revised version: 26 October 1995

Mathematics Subject Classification (1991): 11Gxx, 14Gxx, 14Kxx, 14Lxx

Let R be a discrete valuation ring with field of fractions K, and let j be the inclusion of $\operatorname{Spec} K$ into $\operatorname{Spec} R$. Then, for any smooth algebraic K-group scheme E_K, the associated Néron model E, if it exists, may be viewed as a smooth R-group scheme representing the sheaf j_*E_K on the small smooth site over R. The functor j_* is left-exact and its right derived functors can be defined so that, for any exact sequence

$$0 \longrightarrow E'_K \longrightarrow E_K \longrightarrow E''_K \longrightarrow 0,$$

the sheaf j_*E_K can be accessed in terms of the associated long cohomology sequence. This is less than we need in order to actually describe the Néron model E of E_K as an R-group scheme, since there are non-exact sequences of smooth R-group schemes which become exact in the setting of sheaves on the small smooth site over R. However, if we are only interested in the group ϕ_E of components of E, the approach works well, since ϕ_E is totally determined by its structure as a sheaf on the small étale site over R. In addition, we may assume R to be complete, since Néron models behave well with respect to completion of K and since this process does not change associated component groups.

Using the above point of view, the second author has been able to describe certain aspects of component groups of tori and of semi-abelian varieties in cohomological terms; see [27]. In the present paper we look at the case of an abelian variety A_K and describe the component group ϕ_A in terms of the uniformization of A_K, the latter being meant in the sense of rigid geometry. Two problems arise. First, we have to show that the rigid uniformization of A_K, which exists after replacing K by a finite separable extension L (cf. [24] or [3], [4]), descends from L to K. We do this by means of Galois descent for rigid groups.

* Partially supported by grant PB93-0862 from DGICYT

Thereby we obtain a semi-abelian K-group scheme E_K, being an extension of an abelian variety with potentially good reduction by a torus, and a (not necessarily split) lattice $M_K \subset E_K$ of maximal rank such that, in terms of rigid K-groups, A_K is isomorphic to the quotient E_K/M_K; see Theorem 1.2. The second problem becomes apparent, when we want to apply the functor j_*, as introduced above, to the exact sequence

$$0 \longrightarrow M_K \longrightarrow E_K \longrightarrow A_K \longrightarrow 0.$$

Namely, the morphism $E_K \longrightarrow A_K$ exists only on the level of rigid K-groups, not on the level of K-schemes. To fix this we switch from ordinary Néron models to the notion of formal Néron models of rigid groups, see [9]. In this setting we interpret j_* as the functor, which assigns to any sheaf $\mathscr{F}_K$ on the small rigid smooth site over K its "direct image sheaf" $j_*\mathscr{F}_K$ given by $U \longmapsto \mathscr{F}_K(U_K)$ on the small formal smooth site over R; here we mean by U_K the rigid space (or "generic fibre") associated to the formal scheme U, as explained in [5]. No problems occur, since for an algebraic K-group like E_K, the formal Néron model of the associated rigid K-group is just the formal completion of the ordinary Néron model of E_K; cf. [9], 6.2. In particular, the component group ϕ_E remains untouched this way. Using the fact that $E_K \longrightarrow A_K$ gives rise to an isomorphism $E^0 \xrightarrow{\sim} A^0$ between identity components of the associated formal Néron models, see 2.3, we can construct an exact sequence of type

$$0 \longrightarrow \phi_M \longrightarrow \phi_E \longrightarrow \phi_A \longrightarrow H^1(I, M_K) \longrightarrow \dots ,$$

which is basic for investigating ϕ_A; I is the inertia subgroup of the absolute Galois group of K. More precise information on ϕ_A can be obtained by investing what is known about ϕ_E.

Working within the framework of Tate modules, quite precise information on the prime-to-p part $\phi_A^{(p)}$ of ϕ_A (where p is the residue characteristic of K) has been obtained by Lorenzini [19]. He constructed a filtration

$$\phi_A^{(p)} \supset \Sigma_1 \supset \Sigma_2 \supset \Sigma_3 \supset 0,$$

as well as a dual one, and showed that successive quotients satisfy certain bounds; the bounds were further improved by Edixhoven [13]. Since our methods do not force us to restrict ourselves to prime-to-p parts, we are able to construct such filtrations for the whole group ϕ_A and, furthermore, to give a geometric meaning to the members of the filtrations. As an illustration, we show how to obtain the bounds of [13] for prime-to-p parts. Although it seems difficult to decide if these bounds will extend to p-parts, we can prove such a fact for weaker bounds in the style of Lenstra and Oort [18]; see 5.10.

The notion of Néron models extends naturally to sheaves on the small rigid smooth site over K or even to complexes in the derived category over this site. As we can define identity components as well as groups of connected components for Néron models already on the level of sheaves, we have worked out exact sequences as mentioned above in this more general setting; cf. Sect. 4. Thus, apart from looking at uniformizations of abelian varieties, further applications are possible.

Finally, we would like to thank Bas Edixhoven for pointing out an error in an earlier version of this manuscript and the referee for his critical remarks, leading to a substantial improvement of Sect. 4.

1 Uniformization of abelian varieties

Let K be a field which is complete under a height 1 valuation, and consider an abelian variety A_K over K. Then there is a finite separable extension L of K such that $A_L = A_K \otimes_K L$ admits a uniformization in the sense of rigid geometry; cf. [24], [3], [4]. To be more precise, we will talk about a *split* uniformization. This means, there is a semi-abelian L-group scheme E_L, say given by an exact sequence

$$(*) \qquad 0 \longrightarrow T_L \longrightarrow E_L \longrightarrow B_L \longrightarrow 0$$

with $T_L = \mathbb{G}^d_{\mathrm{m},L}$ a split torus and B_L an abelian variety with good reduction, such that, in the sense of rigid geometry, A_L is isomorphic to the quotient E_L/M_L by a split lattice $M_L \hookrightarrow E_L$ of rank d. A split lattice of rank d in E_L is a closed analytic subgroup $M_L \subset E_L$, which is a constant L-group isomorphic to $\mathbb{Z}^d$.

The notion of a split lattice $M_L \hookrightarrow E_L$ can be described in more concrete terms. In the above situation, E_L may be viewed as a T_L-torsor over B_L. As any such torsor, E_L is locally trivial over B_L and, thus, the map $E_L \longrightarrow B_L$ admits local sections with respect to the Zariski topology. In particular, E_L is a product of d primitive line bundles over B_L. Since B_L has good reduction, it extends to an abelian scheme B over the valuation ring R of L, and the primitive line bundles defining E_L extend to primitive line bundles on B. This means that the above sequence $(*)$ is endowed with an integral structure and, by means of formal completion, one arrives at an exact sequence of formal R-schemes

$$(**) \qquad 0 \longrightarrow \overline{T}_L \longrightarrow \overline{E}_L \longrightarrow \overline{B}_L \longrightarrow 0,$$

where $\overline{T}_L = \overline{\mathbb{G}}^d_{\mathrm{m},R}$ is a formal torus over R. More precisely, viewing $(*)$ as a sequence of rigid K-groups and identifying the smooth formal R-group schemes of $(**)$ with their associated rigid K-groups (cf. [9], 2.3 and 2.4), we can say that $(*)$ is the push-out of $(**)$ with respect to the inclusion $\overline{T}_L \hookrightarrow T_L$. Furthermore, there is a canonical isomorphism $T_L(L)/\overline{T}_L(L) \xrightarrow{\sim} E_L(L)/\overline{E}_L(L)$ and, since a basis of the group of characters of T_L gives rise to an isomorphism $T_L(L)/\overline{T}_L(L) \simeq |L^*|^d$, we get an "absolute value"

$$\rho \colon E_L(L) \longrightarrow |L^*|^d \xrightarrow{-\log} \mathbb{R}^d.$$

Now to say that a monomorphism $M_L \longrightarrow E_L$ with $M_L \simeq \mathbb{Z}^d$ defines M_L as a split lattice in E_L is the same as saying that the intersection of $\overline{E}_L$ with M_L

restricts to the unit section or, equivalently, that ρ maps M_L bijectively onto a lattice of rank d in $\mathbb{R}^d$.

Having described the notion of a split uniformization, we want to introduce now uniformizations of more general type. To do this, we call a closed analytic subgroup M_K of a semi-abelian K-group scheme E_K a *lattice* (of maximal rank) if there is a finite separable extension L/K such that M_L is a split lattice (of maximal rank) in E_L; i. e., a constant L-group isomorphic to $\mathbb{Z}^d$, where d is the dimension of the torus part of E_K.

Definition 1.1. *Let A_K be an abelian variety over K. A rigid uniformization of A_K consists of the following data:*
(i) *a closed immersion of rigid K-groups $M_K \hookrightarrow E_K$, where E_K is (the analytification of) a semi-abelian K-group scheme whose abelian part has potentially good reduction and where M_K is a lattice (of maximal rank) in E_K.*
(ii) *a faithfully flat morphism $p\colon E_K \longrightarrow A_K$ of rigid K-groups with $\ker p = M_K$.*

In the situation of the definition, A_K is the quotient of E_K by the lattice M_K. We can make this plausible. Namely, if we choose a finite separable extension L/K such that M_L as well as the torus part of E_L are split and the abelian part of E_L has good reduction, then p induces a faithfully flat morphism of rigid L-groups $\rho\colon E_L/M_L \longrightarrow A_L$ having trivial kernel. In particular, ρ is a faithfully flat monomorphism and, thus, by [6], 5.4 (a), an isomorphism. So we can say that a morphism of rigid K-groups $E_K \longrightarrow A_K$ with E_K coming from a semi-abelian K-group scheme gives rise to a rigid uniformization of A_K if and only if there is a finite separable extension L/K such that the corresponding morphism $E_L \longrightarrow A_L$ gives rise to a split uniformization of A_L.

Theorem 1.2. *Let A_K be an abelian variety over K. Then A_K admits a rigid uniformization.*

As method of proof, we will use the existence of split uniformizations and then apply Galois descent. This type of descent works in the rigid setting just as it does in the case of schemes. Namely, consider a finite Galois extension L/K with Galois group G and an action of G on some affinoid L-space X_L such that the action is compatible with the one of G on L. Taking invariants under G in $\Gamma(X_L, \mathcal{O}_{X_L})$, the L-space X_L descends to the affinoid K-space $X_K = \operatorname{Sp}\Gamma(X_L, \mathcal{O}_{X_L})^G$; use [2], 6.3.3/3, and the fact that, on affinoid K-algebras, the complete tensor product over K with a finite extension like L, coincides with the usual tensor product. Using the universal property of affinoid subdomains, it is easily checked that any affinoid subdomain $U_L \subset X_L$ which is invariant under the action of G descends to an affinoid subdomain $U_K \subset X_K$. From this one deduces the usual criterion for the effectiveness of descent. Consider a global rigid L-space X_L with a G-action which is compatible with the G-action on L. Then X_L descends to a rigid K-space X_K if and only if there is an admissible affinoid covering $\mathfrak{U} = (U_i)_{i\in I}$ of X_L such that the action of G restricts to an action on U_i for each i. We call $\mathfrak{U}$ an (affinoid) G-invariant admissible covering of X.

Concerning the descent of morphisms, there is no problem for morphisms between affinoid spaces. However, in the general case an additional argument is necessary.

Lemma 1.3. *Let X_K, Y_K be rigid K-spaces, and consider an L-morphism $f_L\colon X_L \longrightarrow Y_L$ which is compatible with the canonical G-action on X_L and Y_L. Then f_L descends to a K-morphism $f_K\colon X_K \longrightarrow Y_K$.*

Proof. We have to show the following: there exist affinoid G-invariant admissible open coverings $(X_i)_{i\in I}$ of X_L and $(Y_i)_{i\in I}$ of Y_L such that $f_L(X_i) \subset Y_i$ for all $i \in I$. To construct such coverings, start out from an affinoid G-invariant admissible open covering $(Y_j)_{j\in J}$ of Y_L. Then $\mathfrak{Z} = (f_L^{-1}(Y_j))_{j\in J}$ is a G-invariant admissible open covering of X_L. Thus, it is enough to show that any G-invariant admissible open covering $\mathfrak{Z} = (Z_j)_{j\in J}$ of X_L admits an admissible refinement $\mathfrak{X} = (X_i)_{i\in I}$ with *affinoid* G-invariant subspaces $X_i \subset X_L$. To construct such a covering, choose an arbitrary G-invariant affinoid admissible open covering $\mathfrak{X} = (X_i)_{i\in I}$ of X_L. Then, for each $i \in I$, the restriction $\mathfrak{Z}_i = (X_i \cap Z_j)_{j\in J}$ is a G-invariant admissible covering of the affinoid L-space X_i, and we can choose a finite admissible refinement $\mathfrak{Z}_i' = (X_{i\lambda}')_\lambda$ of $\mathfrak{Z}_i$ consisting of affinoid subdomains $X_{i\lambda}' \subset X_i$. Let $j(i,\lambda) \in J$ be an index satisfying $X_{i\lambda}' \subset Z_{j(i,\lambda)}$.

Now look at the projection $q\colon X_L \longrightarrow X_K$, which is a flat morphism. For each $i \in I$, it restricts to a flat morphism of affinoid spaces $X_i \longrightarrow q(X_i)$. Using the openness of flat morphisms in the sense of [6], 5.11, it follows that $q(X_{i\lambda}')$ is a finite union of affinoid subdomains in $q(X_i)$. Taking inverses of these with respect to $X_i \longrightarrow q(X_i)$, we see that each $X_{i\lambda}'$ is contained in a finite union of G-invariant affinoid subdomains $X_{i\lambda\mu}'' \subset X_i$. This union is the G-saturation of $X_{i\lambda}'$ and, thus, contained in $Z_{j(i,\lambda)}$. It follows that, as required, the $X_{i\lambda\mu}''$ form a G-invariant affinoid admissible covering of X_L, which refines $\mathfrak{Z} = (Z_j)_{j\in J}$. □

Now we come to the *proof of Theorem* 1.2. Choose a finite Galois extension L/K such that $A_L = A_K \otimes_K L$ admits a split uniformization. The latter is given by a faithfully flat projection $p_L\colon E_L \longrightarrow A_L$, where E_L corresponds to a semi-abelian L-group scheme with a split torus part and an abelian part with good reduction, and where $M_L = \ker p_L$ is a split lattice (of maximal rank) in E_L. Let G be the Galois group of L/K and consider an automorphism $\sigma \in G$. For any rigid L-space X_L, we will write X_L^σ for the rigid L-space obtained from X_L by composing the structural morphism $X_L \longrightarrow \operatorname{Sp} L$ with the morphism $\operatorname{Sp} L \longrightarrow \operatorname{Sp} L$ associated to σ. Then the K-isomorphisms $f\colon X_L \longrightarrow X_L$ which are compatible with σ in the sense that the diagram

$$\begin{array}{ccc} X_L & \xrightarrow{\ f\ } & X_L \\ \downarrow & & \downarrow \\ \operatorname{Sp} L & \xrightarrow{\ \sigma\ } & \operatorname{Sp} L \end{array}$$

is commutative, correspond bijectively to the L-isomorphisms $X_L^\sigma \longrightarrow X_L$. Furthermore, if X_L is an L-group space, the same is canonically the case for X_L^σ, and it is easy to see that the K-isomorphisms which are compatible with σ and with the group structure of X_L correspond bijectively to the isomorphisms of L-group spaces $X_L^\sigma \longrightarrow X_L$.

For each $\sigma \in G$ the morphism p_L gives rise to a split uniformization $p_L^\sigma : E_L^\sigma \longrightarrow A_K^\sigma$. Considering the canonical action of G on A_L, any $\sigma \in G$ gives rise to an L-isomorphism $A_L^\sigma \xrightarrow{\sim} A_L$ and, thus, using [4], 6.10 (a), to an L-isomorphism $E_L^\sigma \xrightarrow{\sim} E_L$ which is compatible with p_L^σ and p_L, and which restricts to an L-isomorphism $M_L^\sigma \xrightarrow{\sim} M_L$. Hence, G acts on the uniformization of A_L and, in particular, on E_L. If

$$0 \longrightarrow T_L \longrightarrow E_L \longrightarrow B_L \longrightarrow 0$$

is the associated exact sequence with T_L a split torus and B_L an abelian variety with good reduction, we see as above that G acts on this sequence. The action of G on B_L is algebraic due to algebraization theorems; see [17]. Similarly, the action on T_L is algebraic, since G acts on the group of characters of T_L. From this it follows that the action of G on E_L is algebraic. Thus, using a quasi-projectivity argument and thinking in terms of algebraic K-groups, E_L descends to a semi-abelian K-group scheme with an abelian part which has potentially good reduction. The same holds then in the setting of rigid K-groups. Furthermore, it is clear that the descent is effective on M_L. Using 1.3, we see that the uniformization of A_L descends and thus gives rise to a uniformization of A_K. □

2 Néron models in the rigid setting

In the following, let K be the field of fractions of a discrete valuation ring R. Similarly as for smooth K-schemes, see [8], 1.2/1 and 10.1/1, there is the notion of a Néron model of a smooth rigid K-space; cf. [9], 1.1.

Definition 2.1. *Let X_K be a smooth rigid K-space. A (formal) Néron model of X_K is a smooth formal R-model U of some open rigid subspace $U_K \subset X_K$ such that the usual Néron mapping property is satisfied:*

If Z is a smooth formal R-scheme and $f_K : Z_K \longrightarrow X_K$ a morphism of rigid K-spaces, the latter extends uniquely to a morphism of formal R-schemes $f : Z \longrightarrow U$.

It follows that the formal Néron model of a smooth rigid K-group, if it exists, is a (separated) formal R-group scheme. A general existence theorem for (quasi-compact) formal Néron models was shown in [9], 1.2; see [8], 1.3/1, for the classical analogue. For us it is important that Néron models in the algebraic and rigid context are canonically related, at least, if we restrict ourselves to commutative groups.

Proposition 2.2. ([9], 6.2) *Let $\mathfrak{X}_K$ be a smooth algebraic K-group scheme with Néron model $\mathfrak{X}$ such that $\mathfrak{X}$ is quasi-compact or $\mathfrak{X}_K$ is commutative. Then, writing $\overline{X}$ for the formal completion of $\mathfrak{X}$ and $\overline{X}_K$ for the associated rigid K-group, as* *well as X_K for the rigid K-group associated to $\mathfrak{X}_K$, the canonical map $\overline{X}_K \longrightarrow X_K$ is an open immersion of rigid K-groups, and $\overline{X}$ is a formal Néron model of X_K.*

In particular, since special fibres are invariant under formal completion, the component group of the Néron model of $\mathfrak{X}_K$ coincides with the component group of the formal Néron model of X_K.

In the following we will work exclusively in the rigid setting. In particular, as we have done already in Sect. 1, we will make no notational difference between a K-group scheme of finite type and its associated rigid K-group. Also, using [9], 2.3 and 2.4, we will make no difference between smooth formal R-group schemes and their associated rigid K-groups.

Theorem 2.3. *Let $p_K : E_K \longrightarrow A_K$ be a rigid uniformization of an abelian variety A_K. Then p_K extends uniquely to a morphism of formal R-schemes $p : E \longrightarrow A$ between associated formal Néron models, and p induces an isomorphism $E^0 \xrightarrow{\sim} A^0$ between identity components.*

Proof. It follows from [8], 10.2/2, that E_K (and A_K) admit Néron models in the algebraic context. Hence, using 2.2, the formal Néron models E of E_K and A of A_K exist. Then, due to the Néron mapping property, we see that p_K extends uniquely to a morphism $p : E \longrightarrow A$ and, thus, to a morphism $p^0 : E^0 \longrightarrow A^0$. To show that p^0 is, in fact, an isomorphism, choose a finite separable extension L/K such that the uniformization p_K splits over L. Then, as we have shown in [9], 6.3, the formal R-group scheme $\overline{E}_L$, as introduced in Sect. 1, is the identity component of the formal Néron model of E_L, and the uniformization map p_L restricts to an open immersion $\overline{E}_L \hookrightarrow A_L$; it maps $\overline{E}_L$ onto the identity component of the formal Néron model $\overline{A}_L$ of A_L. In particular, the assertion of the proposition holds if p_K is a split uniformization.

In the general case, the situation is more complicated. Writing G for the Galois group of L/K, we have seen in Sect. 1 that G acts on the sequence

$$(*) \qquad 0 \longrightarrow T_L \longrightarrow E_L \longrightarrow B_L \longrightarrow 0,$$

which describes the structure of E_L as a semi-abelian group scheme. Since G acts on the character group of T_L, it follows that the action of G on the sequence $(*)$ restricts to an action on the sequence

$$(**) \qquad 0 \longrightarrow \overline{T}_L \longrightarrow \overline{E}_L \longrightarrow \overline{B}_L \longrightarrow 0.$$

Viewing the latter as a sequence of rigid L-groups, we claim that Galois descent works well in this situation and that $(**)$ descends to an exact sequence of *rigid K-groups*

$$0 \longrightarrow \overline{T}_K \longrightarrow \overline{E}_K \longrightarrow \overline{B}_K \longrightarrow 0.$$

For example, when we want to do the descent of $\overline{E}_L$, we use the fact that the Galois group $G = \mathrm{Gal}(L/K)$ acts on $\overline{E}_L$ as a formal group scheme. Due to the quasi-projectivity of the special fibre of $\overline{E}_L$ (or by a more elementary argument), there is a finite affine open covering of $\overline{E}_L$, which is invariant under the action of G. Now switching to the category of rigid spaces, we see that, as a rigid L-group, $\overline{E}_L$ admits an admissible affinoid covering which is invariant under the action of G. This is enough for doing Galois descent with respect to G. However, note that the resulting K-groups $\overline{T}_K$, $\overline{E}_K$, and $\overline{B}_K$ will not necessarily admit formal R-models any more, since we are using Galois descent in the setting of rigid spaces, not in the setting of formal schemes.

In particular, it follows that the open immersion $\overline{E}_L \xrightarrow{\sim} \overline{A}_L \subset A_L$ descends to an open immersion $\overline{E}_K \xrightarrow{\sim} \overline{A}_K \subset A_K$, and we see that $p_K : E_K \longrightarrow A_K$ restricts to the isomorphism $\overline{E}_K \xrightarrow{\sim} \overline{A}_K$. We claim that, as a rigid K-group, the identity component E^0 of the formal Néron model of E_K is an open subgroup in $\overline{E}_K$. To see this, consider the morphism of rigid L-groups $E^0 \otimes_K L \longrightarrow E_L$. It factors through the formal Néron model of E_L and, since E^0 is connected, through the identity component $\overline{E}_L$. Thereby we see $E^0 \subset \overline{E}_K$. In a similar way, one shows $A^0 \subset \overline{A}_K$. Now, using the Néron mapping property, p_K must map E into A and, hence, E^0 into A^0 so that $p_K(E^0) \subset A^0$. On the other hand we have $A^0 \subset \overline{A}_K \simeq \overline{E}_K \subset E_K$. So, again by the Néron mapping property, A^0 must correspond to a subgroup of $p_K(E^0)$, and therefore we must have $A^0 = p_K(E^0)$, as claimed. □

3 Topologies

Let $\mathfrak{T}$ be the usual Grothendieck topology on the category of rigid K-spaces, which gives rise to the structural topology on any rigid K-space; cf. [2], 9.3.1/4. In analogy to the scheme case we want to consider the *smooth topology* $\mathfrak{T}_{\text{sm}}$ as well as the *étale topology* $\mathfrak{T}_{\text{et}}$ on the category C of *quasi-separated* rigid K-spaces. To explain the general procedure, consider a class E of *flat* morphisms in C such that

(i) all open immersions are in E,

(ii) the composite of two morphisms in E is in E, and

(iii) any base change of a morphism in E is in E.

For example, E can be the class of all flat morphisms, of smooth morphisms, of étale morphisms, or the class of all open immersions. Accordingly, we will talk about the flat, smooth or étale topology, whereas in the case, where E consists of all open immersions, we reproduce the Grothendieck topology $\mathfrak{T}$. To define the E-topology $\mathfrak{T}_E$ on C, we proceed as follows. For any object S of C, its $\mathfrak{T}_E$-open sets consist of all quasi-compact E-morphisms $S' \longrightarrow S$ in C. Furthermore, a $\mathfrak{T}_E$-covering of S is given by a family $(\varphi_i : S_i \longrightarrow S)_{i \in I}$ of $\mathfrak{T}_E$-open sets of S with the property that the covering $S = \bigcup_{i \in I} \operatorname{im}\varphi_i$ can be refined by a $\mathfrak{T}$-covering of S. It is not necessarily true that $\operatorname{im}\varphi_i$ is $\mathfrak{T}$-open in S. However, due to the flatness of the φ_i, the latter is the case if S_i is quasi-compact; cf. [6], 5.11.

Lemma 3.1. *Let $(\varphi_i : S_i \longrightarrow S)_{i \in I}$ be a $\mathfrak{T}_E$-covering of a quasi-separated rigid K-space S. For each $i \in I$ let $S_{ij} \subset S_i$, $j \in J_i$, be $\mathfrak{T}$-open subsets giving rise to a $\mathfrak{T}$-covering of S_i. Then, writing $\varphi_{ij} = \varphi_i|_{S_{ij}}$, the family*

$$(\varphi_{ij} : S_{ij} \longrightarrow S)_{i \in I, j \in J_i}$$

is a $\mathfrak{T}_E$*-covering of* S.

In particular, choosing the S_{ij} *quasi-compact, the sets* $\varphi_{ij}(S_{ij})$ *are* $\mathfrak{T}$*-open and, hence, form a* $\mathfrak{T}$*-covering of* S.

Proof. Let $(V_{i\mu})_{i \in I, \mu \in M_i}$ be a $\mathfrak{T}$-covering of S with $V_{i\mu} \subset \varphi_i(S_i)$ so that it refines the covering $(\varphi_i(S_i))_{i \in I}$. We may assume that each $V_{i\mu}$ and each S_{ij} is quasi-compact. Fixing i and μ and using the quasi-compactness of φ_i, the inverse image $\varphi_i^{-1}(V_{i\mu})$ is quasi-compact and, hence, covered by finitely many of the S_{ij}. Likewise, $V_{i\mu}$ is covered by finitely many of the sets $\varphi_i(S_{ij}) \subset S$; the latter are quasi-compact and $\mathfrak{T}$-open, due to the flatness of φ_i. From this it follows that the covering $\varphi_{ij}(S_{ij})_{i \in I, j \in J_i}$ restricts to a $\mathfrak{T}$-covering on each $V_{i\mu}$ and, hence, is a $\mathfrak{T}$-covering of S. □

Proposition 3.2. *The* $\mathfrak{T}_E$*-open sets and the* $\mathfrak{T}_E$*-coverings define a Grothendieck topology on the category* C *of quasi-separated rigid* K*-spaces.*

Proof. We verify the conditions in [1], Chap. I, 0.1. Let S be an object of C. Clearly any isomorphism $S' \longrightarrow S$ is a $\mathfrak{T}_E$-covering. Next, consider a $\mathfrak{T}_E$-covering $(\varphi_i : S_i \longrightarrow S)_{i \in I}$ of S and a $\mathfrak{T}_E$-covering $(\psi_{ij} : S_{ij} \longrightarrow S_i)_{j \in J_i}$ of S_i for each $i \in I$. Then $(\varphi_i \circ \psi_{ij} : S_{ij} \longrightarrow S)_{i \in I, j \in J_i}$ is a family of quasi-compact E-morphisms. To show that it is a $\mathfrak{T}_E$-covering of S, consider for all indices $i \in I, j \in J_i$, a quasi-compact $\mathfrak{T}$-covering $(S_{ij\mu})_{\mu \in M_{ij}}$ of S_{ij}. Then we see from 3.1 that $(\varphi_i \circ \psi_{ij}(S_{ij\mu}))_{i \in I, j \in J_i, \mu \in M_{ij}}$ is a quasi-compact $\mathfrak{T}$-covering refining the covering $(\varphi_i \circ \psi_{ij}(S_{ij}))_{i \in I, j \in J_i}$ of S. Finally it is easily checked that $\mathfrak{T}_E$-coverings are preserved under base change with morphisms in C. □

As usual one defines presheaves and sheaves with respect to the E-topology $\mathfrak{T}_E$ on C. A presheaf is just a contravariant functor $\mathscr{F} : C \longrightarrow$ Sets, and the latter is a sheaf if for each $\mathfrak{T}_E$-covering $(\varphi_i : S_i \longrightarrow S)_{i \in I}$ in C, the sequence

$$\mathscr{F}(S) \to \prod_{i \in I} \mathscr{F}(S_i) \rightrightarrows \prod_{i,j \in I} \mathscr{F}(S_i \times_S S_j)$$

is exact. One uses the usual argumentation, as contained in [21], Chap. II, 1.5, to show that a presheaf $\mathscr{F}$ is a sheaf with respect to the E-topology if and only if, first, it is a sheaf with respect to the restriction of $\mathfrak{T}$ to C and, second, if for any surjective E-morphism $S' \longrightarrow S$ between quasi-compact objects in C, the sequence

$$\mathscr{F}(S) \to \mathscr{F}(S') \rightrightarrows \mathscr{F}(S' \times_S S')$$

is exact.

Dealing with the category C of all quasi-separated rigid K-spaces, we are considering the E-topology on the *big rigid E-site* over K. Alternatively, we can restrict ourselves to the full subcategory of C consisting of all objects X whose structural morphism $X \longrightarrow \operatorname{Sp} K$ is an E-morphism, thereby obtaining the *small rigid E-site* over K. Of course, one can also replace $\operatorname{Sp} K$ by a more general base space and consider the big or small E-site over it.

Proposition 3.3. *For any quasi-separated rigid K-space X, the presheaf*

$$S \longmapsto \mathrm{Hom}_K(S, X)$$

is a sheaf with respect to the E-topology $\mathfrak{T}_E$.

Proof. Since a morphism of rigid K-spaces $S \longrightarrow X$ can be defined locally with respect to the restriction of $\mathfrak{T}$ to S, we have only to show that for any surjective E-morphism $\varphi: S' \longrightarrow S$ between quasi-compact objects in C, the sequence

$$\mathrm{Hom}_K(S, X) \to \mathrm{Hom}_K(S', X) \rightrightarrows \mathrm{Hom}_K(S' \times_S S', X)$$

is exact. To do this, we may assume that X is quasi-compact, too. Dealing with formal R-schemes of topologically finite type, faithfully flat and quasi-compact descent works well. So, on this level, the corresponding sequence is exact for a faithfully flat morphism of formal R-schemes of topologically finite type in place of $S' \longrightarrow S$. We want to reduce to this case.

To show that the first map of the above sequence is injective, consider two morphisms $f, g \in \mathrm{Hom}_K(S, X)$ satisfying $f \circ \varphi = g \circ \varphi$ and choose flat formal R-models $\hat{S}$, $\hat{S}'$, and $\hat{X}$ for S, S', X; cf. [5], 4.1. Blowing up $\hat{S}$, we may assume that f, g admit formal R-models $\hat{f}, \hat{g}: \hat{S} \longrightarrow \hat{X}$. In the same way we can blow up $\hat{S}'$ and assume that φ has a formal R-model $\hat{\varphi}: \hat{S}' \longrightarrow \hat{S}$. Since φ is flat, $\hat{\varphi}$ becomes flat and, hence, faithfully flat after blowing up $\hat{S}$; cf. [6], 5.2. Now, from $f \circ \varphi = g \circ \varphi$ one concludes $\hat{f} \circ \hat{\varphi} = \hat{g} \circ \hat{\varphi}$ (see the assertion (b) in the proof of [5], 4.1) and, thus, $\hat{f} = \hat{g}$, which implies $f = g$. In a similar way one shows that the image of the first map of the above sequence equals the kernel of the second. □

In the classical case, we can interpret a Néron model of a smooth K-scheme X_K as the direct image $j_* X_K$ with respect to the smooth topology, where $j: \mathrm{Spec}\, K \longrightarrow \mathrm{Spec}\, R$ is the canonical immersion. The same is possible for formal Néron models. Namely, let X_K be a smooth and quasi-separated rigid K-space and, using 3.3, view X_K as a sheaf with respect to the smooth topology on the small rigid smooth site over K. Then we can define a sheaf $j_* X_K$ with respect to the smooth topology on the category of smooth formal R-schemes (the small formal smooth site over R) by setting

$$U \longmapsto \mathrm{Hom}_K(U_K, X_K).$$

If X_K admits a formal Néron model (or Néron quasi-model in the sense of [9], 6.4), the latter represents the sheaf $j_* X_K$.

In the next sections, we will look at sheaves of type $j_* X_K$ where X_K is a smooth commutative rigid K-group or, more generally, an abelian sheaf on the small rigid smooth site over K. In particular, we will work in the derived category and use the right derived functors of j_*. One may consult [1], Chap. II and [16] as a reference for this machinery. For example, that the category of abelian sheaves on quasi-separated rigid K-spaces has enough injectives, follows from [1], Chap. II, 1.8.

4 Néron models in terms of sheaves

As before, let K be the field of fractions of a complete discrete valuation ring R, with residual field k and residue characteristic $p \geq 0$. Denote by K_{sm} the small rigid smooth site over K and by R_{sm} the small formal smooth site over R. Furthermore, consider the category of complexes of abelian sheaves on K_{sm}, which are bounded below, and let $D^+(K_{\mathrm{sm}})$ be its localization with respect to quasi-isomorphisms. In the same way, let us introduce the category $D^+(R_{\mathrm{sm}})$. Then the functor j_*, as defined in the preceding section, induces a right derived functor

$$Rj_* \colon D^+(K_{\mathrm{sm}}) \longrightarrow D^+(R_{\mathrm{sm}}),$$

which might be called the *Néron functor.*

Definition 4.1. *For any complex $\mathscr{C}_K \in D^+(K_{\mathrm{sm}})$, the sheaf $\mathscr{C} = R^0 j_* \mathscr{C}_K$ is called the Néron model of $\mathscr{C}_K$.*

Clearly, if G_K is a smooth rigid K-group with formal Néron model G, then, viewing G_K as a complex $\mathscr{C}_K \in D^+(K_{\mathrm{sm}})$, concentrated at level 0, its Néron model $\mathscr{C}$ in the above sense coincides with G, since $R^0 j_* G_K = j_* G_K$. Furthermore, if $\mathscr{F}_K' \longrightarrow \mathscr{F}_K$ is a monomorphism of sheaves on K_{sm}, we can view it as a complex of $D^+(K_{\mathrm{sm}})$ in degrees -1 and 0, and we see that the Néron model of the cokernel $\mathscr{F}_K / \mathscr{F}_K'$ coincides with the one of the complex $\mathscr{F}_K' \longrightarrow \mathscr{F}_K$. For example, we can look at the 1-motive $M_K \hookrightarrow E_K$ given by the uniformization of an abelian variety A_K. Then its Néron model coincides with the formal Néron model A of A_K.

As any short exact sequence $0 \longrightarrow \mathscr{F}_K' \longrightarrow \mathscr{F}_K \longrightarrow \mathscr{F}_K'' \longrightarrow 0$ of abelian sheaves on K_{sm} with Néron models $\mathscr{F}', \mathscr{F}, \mathscr{F}''$ gives rise to a long exact cohomology sequence

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow R^1 j_* \mathscr{F}_K' \longrightarrow \cdots$$

on R_{sm}, it is clear that information on $R^1 j_*$ is desirable in order to have an estimate for the defect of exactness of the functor $R^0 j_*$. We begin by looking at the multiplicative group $\mathbb{G}_{\mathrm{m},K}$.

Lemma 4.2. *Let T_K be an algebraic torus over K, which is viewed as a sheaf on K_{sm}. Then $R^1 j_* T_K = 0$ if T_K splits over an unramified extension of K or if the residue field of K is perfect.*

Proof. We have to show that the sheaf, which is associated to the presheaf $U \longmapsto H^1(U_K, T_K)$, vanishes on R_{sm}, the H^1 being computed with respect to the rigid smooth topology on U_K. Let us first assume that T_K splits over a finite unramified extension K' of K. Then, since the valuation ring of K' is étale over R, we may just as well assume that T_K is split over K, and it is enough to consider the case where $T_K = \mathbb{G}_{\mathrm{m},K}$. Using O. Gabber's version of faithfully flat descent for coherent modules on rigid spaces, see [23], 1.9, we need only to compute the H^1 for the rigid topology. Now consider a (quasi-compact) smooth formal

R-scheme U and a class $\mathscr{L}_K \in H^1(U_K, \mathbb{G}_{m,K})$, which we might interpret as an invertible sheaf on U_K. As we can extend U via some étale cover $U' \longrightarrow U$, we can apply a base change $R \longrightarrow R'$ to U with R' being a valuation ring which is étale over R. In particular, we can assume that U is smooth and geometrically connected over R'. But then, by [20], 2.9, $\mathscr{L}_K$ extends to a line bundle $\mathscr{L}$ on U, and we see that $\mathscr{L}_K$ is trivial on local parts $V_K \subset U_K$ coming from small enough open parts $V \subset U$. It follows that $R^1 j_* \mathbb{G}_{m,K}$ vanishes.

To settle the assertion of 4.2, it remains to look at the case where the residue field of K is perfect and where T_K splits over a totally ramified extension K' of K. Let $\pi_K : \operatorname{Sp} K' \longrightarrow \operatorname{Sp} K$ be the canonical map, and look at the exact sequence

$$(*) \qquad 0 \longrightarrow T_K \longrightarrow \pi_{K*}(T_{K'}) \longrightarrow \tilde{T}_K \longrightarrow 0$$

of sheaves on K_{sm}, where $T_{K'}$ is the K'-extension of T_K and $\tilde{T}_K$ the quotient of $\pi_{K*}(T_{K'})$ by T_K. We claim that the associated sequence

$$(**) \qquad 0 \longrightarrow j_* T_K \longrightarrow j_*(\pi_{K*}(T_{K'})) \longrightarrow j_* \tilde{T}_K \longrightarrow 0$$

of sheaves on R_{sm} is exact or, equivalently, that the map $j_*(\pi_{K*}(T_{K'})) \longrightarrow j_* \tilde{T}_K$ admits sections locally with respect to the smooth topology on $j_* \tilde{T}_K$. To verify this, we switch to the small algebraic smooth site over R and show the existence of such sections in the corresponding algebraic setting. Using the facts that, in our situation, π_{K*} commutes with the rigid analytification functor and that formal Néron models of tori are obtained from the corresponding algebraic ones via formal completion, cf. [9], 6.2, we then get sections in the formal setting via formal completion.

Thus, let us look at the above sequences $(*)$ and $(**)$, thinking in terms of ordinary group schemes. Let $\mathscr{O}_\zeta$ be the local ring of $j_* \tilde{T}_K$ at a generic point ζ of the special fibre; it is a discrete valuation ring, which is of ramification index 1 over R. Let L be its field of fractions, and apply the base change L/K to the exact sequence $(*)$, thus getting an exact sequence of L-group schemes. Its associated sequence of Néron models, which we denote by $(**)_{\mathscr{O}_\zeta}$, is obtained from $(**)$ by applying the base change $\mathscr{O}_\zeta / R$; cf. [8], 10.1/3. Writing j again for the canonical map $\operatorname{Spec} L \longmapsto \operatorname{Spec} \mathscr{O}_\zeta$ and working with respect to the étale topology, we know from [27], 2.3, that the sequence $(**)_{\mathscr{O}_\zeta}$ is exact as a sequence of sheaves on the small étale site over $\mathscr{O}_\zeta$. Applying this fact to the tautological point $\operatorname{Spec} \mathscr{O}_\zeta \longrightarrow j_* \tilde{T}_K \otimes_R \mathscr{O}_\zeta$, there is a discrete valuation ring $\mathscr{O}'_\zeta$, étale over $\mathscr{O}_\zeta$, such that the resulting morphism $\operatorname{Spec} \mathscr{O}'_\zeta \longrightarrow j_* \tilde{T}_K \otimes_R \mathscr{O}_\zeta$ factors through $j_*(\pi_{K*}(T_{K'})) \otimes_R \mathscr{O}_\zeta$. Then we construct a scheme Z of finite type over R which admits $\mathscr{O}'_\zeta$ as local ring at a generic point of its special fibre. Choosing Z small enough, we can assume that the morphism $\operatorname{Spec} \mathscr{O}'_\zeta \longrightarrow j_* \tilde{T}_K$ extends to Z, and that $Z \longrightarrow j_* \tilde{T}_K$ is étale and factors through $j_*(\pi_{K*}(T_{K'}))$. Thereby we see that $j_*(\pi_{K*}(T_{K'})) \longrightarrow j_* \tilde{T}_K$ admits a section with respect to the étale topology over an open neighborhood of ζ in $j_* \tilde{T}_K$. Finally, using étale base change on R, there are enough translations on $j_* \tilde{T}_K$ to construct such a section over all of $j_* \tilde{T}_K$. In particular, we see that the above sequence (**) is exact in terms of sheaves on the small algebraic smooth site over R.

We go back now to the small formal smooth site over R. Knowing the exactness of the sequence (**), we will use the exactness of the extended sequence

$$0 \longrightarrow j_*T_K \longrightarrow j_*(\pi_{K*}(T_{K'})) \longrightarrow j_*\tilde{T}_K \longrightarrow R^1j_*T_K \longrightarrow R^1j_*(\pi_{K*}(T_{K'}))$$

in order to conclude $R^1j_*T_K = 0$ from $R^1j_*(\pi_{K*}(T_{K'})) = 0$. To see that the latter sheaf vanishes, write R' for the valuation ring of K' and $\pi\colon \operatorname{Spf} R' \longrightarrow \operatorname{Spf} R$ for the canonical projection. We have $j_* \circ \pi_{K*} = \pi_* \circ j'_*$ with j'_* being the analogue of j_* over R', and a spectral sequence argument shows

$$R^1(\pi_* \circ j'_*)(T_{K'}) = R^1(j_* \circ \pi_{K*})(T_{K'}) = j_*R^1\pi_{K*}(T_{K'}) \oplus R^1j_*(\pi_{K*}(T_{K'})).$$

Hence, it is enough to verify that $R^1(\pi_* \circ j'_*)(T_{K'})$ vanishes, and we can do this by showing that π_* is exact, of course, using the fact that $R^1j'_*T_{K'}$ vanishes since $T_{K'}$ is split. However, the exactness of π_* is trivial since, due to our assumption on the extension K'/K, the map $U \otimes_R R' \longrightarrow U$ is bijective (on special fibres) for any formal R-scheme U. $\square$

As a consequence, we can compute the functor R^1j_* on R_{sm} for certain constant sheaves.

Corollary 4.3. (i) $R^1j_*\mathbb{Z}^d = 0$ *for the constant group* $M_K = \mathbb{Z}^d$,
(ii) $R^1j_*(\mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$ *for the constant group* $M_K = \mathbb{Z}/n\mathbb{Z}$, *provided* n *is not divisible by* p.

Proof. To verify assertion (i), look at a Tate elliptic curve C_K with uniformization given by the short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{G}_{\mathrm{m},K} \longrightarrow C_K \longrightarrow 0,$$

as well as its associated long cohomology sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow G \longrightarrow C \longrightarrow R^1j_*\mathbb{Z} \longrightarrow R^1j_*\mathbb{G}_{\mathrm{m},K} \longrightarrow \cdots,$$

where G and C are the formal Néron models of $\mathbb{G}_{\mathrm{m},K}$ and C_K. It is well-known that the morphism $G \longrightarrow C$ is an epimorphism; cf. [9], Example 6.3. Since $R^1j_*\mathbb{G}_{\mathrm{m},K}$ vanishes by 4.2, the same is true for $R^1j_*\mathbb{Z}$.

In case (ii) we do a similar reasoning, looking at the short exact sequence

$$0 \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{G}_{\mathrm{m},K} \longrightarrow \mathbb{G}_{\mathrm{m},K} \longrightarrow 0$$

induced from multiplication by n on $\mathbb{G}_{\mathrm{m},K}$. On the level of Néron models, the cokernel of this morphism is just $\mathbb{Z}/n\mathbb{Z}$, since multiplication by n is an epimorphism on identity components. But then an evaluation of the associated long cohomology sequence yields $R^1j_*(\mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$, as desired. $\square$

Using the language of Galois modules, we want to look more closely at étale K-group schemes and compute the corresponding sheaves R^0j_* as well as R^1j_* on R_{sm}. All occurring group schemes or rigid groups will be assumed to be *commutative*.

Lemma 4.4. *Let M_K be an étale K-group scheme. Write M_K also for the associated Galois module. Let I be the inertia subgroup of the absolute Galois group G_K of K, and $G_k = G_K/I$ the absolute Galois group of the residue field k of K. Then:*
(i) $R^0 j_* M_K = H^0(I, M_K) = M_K^I$,
(ii) $R^1 j_* M_K = H^1(I, M_K)$ *if M_K is finitely generated and torsion-free; in particular, $R^1 j_* M_K = 0$ if the action of I on M_K is trivial.*

In these formulas, right-hand sides have to be interpreted as the étale (formal) R-group schemes associated to the corresponding G_k-modules, these schemes, of course, being viewed as sheaves on the small formal smooth site R_{sm}.

Proof. To verify assertion (i), we have to show that $j_* M_K$ is represented by the étale formal R-group scheme associated to the G_k-module $H^0(I, M_K)$ consisting of all I-invariants of M_K. Let us write M_K^I for the corresponding étale K-group scheme and M^I for its R-version; later we will forget about this notational difference. We have to verify that, for each smooth formal R-scheme Z and each morphism $f_K : Z_K \longrightarrow M_K$ of rigid K-spaces, f_K factors through M_K^I and the resulting morphism extends uniquely to a morphism of formal schemes $f : Z \longrightarrow M^I$. We can suppose that Z and, hence, Z_K are connected; then f_K maps Z_K to a connected component P_K of M_K. Considering points with values in the field of fractions K^{sh} of a strict henselization of R, it is easy to see that P_K must be in M_K^I. Now, using étale descent on R, we are reduced to the case where M^I and, hence, M_K^I are constant. But then, clearly, $M_K^I(Z_K) = M^I(Z)$.

To show the second fact, observe first that, as in the scheme case, $R^1 j_* M_K$ is the sheaf associated to the presheaf that assigns to each smooth formal R-scheme U the group $H^1_{\mathrm{sm}}(U_K, M_K)$. Denote by G the Galois group of the extension L/K where M_K is constant, and consider the induced G-module $\mathrm{Ind}^G(M_L)$; see [26], Chap. VII, Sect. 1. As G is finite, the induced module is isomorphic to the coinduced module of M_L and, thus, corresponds to the étale group scheme representing the sheaf $\pi_* M_L$, where π is the morphism $\mathrm{Sp}\, L \longrightarrow \mathrm{Sp}\, K$. Now, for every smooth rigid K-space U_K, we have an injection $H^1(U_K, \pi_* M_L) \hookrightarrow H^1(U_L, M_L)$ because of the Leray spectral sequence. Furthermore, if U_K admits a smooth formal R-model which is quasi-compact, the corresponding fact is also true for U_L over L, and we obtain $H^1(U_L, M_L) = 0$ from [25], Corollary 4.4.7 in Sect. 4.2, since M_L is constant free. Thus, $R^1 j_*(\mathrm{Ind}^G(M_L)) = 0$. On the other hand, writing I/I' for the subgroup of G induced from I, we get $H^1(I/I', \mathrm{Ind}^G(M_L)) = 0$, using [26], Chap. IX, Sect. 3, and this implies $H^1(I, \mathrm{Ind}^G(M_L)) = 0$, since $\mathrm{Ind}^G(M_L)$ is torsion-free; cf. [26], Chap. VII, Sect. 3 and Sect. 6. To conclude our argumentation, there is a natural injection $M_K \hookrightarrow \mathrm{Ind}^G(M_L)$ which has as a cokernel an étale group scheme, too. Applying the functor j_* to this exact sequence and using $j_* M_K = M_K^I$, we can easily check that $R^1 j_* M_K \cong H^1(I, M_K)$. □

As a consequence of 4.3 (ii) (which we may read on the small formal smooth site or, alternatively, on the small formal étale site over K) and of 4.4, we can express the long exact cohomology sequence associated to a short exact sequence of sheaves on K_{sm} in more precise terms.

Proposition 4.5. *Let M_K be an étale K-group scheme which, as a G_K-module, is of one of the following types: torsion-free and finitely generated, or isomorphic to $\mathbb{Z}/n\mathbb{Z}$ with trivial action and n prime to p, or isomorphic to a finite product of such modules. Then, for any exact sequence*

$$0 \longrightarrow M_K \longrightarrow \mathscr{F}_K \longrightarrow \mathscr{F}_K' \longrightarrow 0$$

of sheaves on K_{sm}, the exact long cohomology sequence of sheaves on R_{sm}, which is associated to the derived functor of j_*, starts as

$$0 \longrightarrow M \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}' \longrightarrow H^1(I, M_K) \longrightarrow \cdots,$$

where M, $\mathscr{F}$, and $\mathscr{F}'$ are the Néron models of M_K, $\mathscr{F}_K$, and $\mathscr{F}_K'$, respectively. Furthermore, $M = H^0(I, M_K)$.

Remark 4.6. If $0 \longrightarrow M_K \longrightarrow E_K \longrightarrow A_K \longrightarrow 0$ is a sequence of smooth rigid K-groups, where $M_K \longrightarrow E_K$ is a closed immersion and $E_K \longrightarrow A_K$ is faithfully flat with kernel M_K, then this sequence is an exact sequence of sheaves on K_{sm}. Namely, in this situation, $E_K \longrightarrow A_K$ is smooth due to the fibre criterion of smoothness; see [7], 2.9. So $E_K \longrightarrow A_K$ admits a section locally with respect to the smooth topology, which shows that $E_K \longrightarrow A_K$ is an epimorphism of sheaves on K_{sm}.

We will, of course, use the assertion of 4.5 in the case where A_K as in 4.6 is an abelian variety and $M_K \longrightarrow E_K$ is the 1-motive given by its uniformization. One can show then for the corresponding Néron models that A contains E/M as a well-defined open and closed formal subgroup; use 2.3. On the other hand, the consideration of more general 1-motives is not excluded.

Next, we want to concentrate our efforts on the study of component groups of Néron models or, more generally, of sheaves on R_{sm}. Let $\mathscr{F}$ be an abelian sheaf on R_{sm}. To define its identity component as a subsheaf $\mathscr{F}^0 \subset \mathscr{F}$, we will use the notion of *étale point* of a formal R-scheme T. Thereby we mean a point $\operatorname{Spf} A \longrightarrow T$ with values in a valuation ring A which is étale over R. To facilitate our language, sections of $\mathscr{F}$ over T will frequently be denoted in the form of arrows $T \longrightarrow \mathscr{F}$.

Definition 4.7. *As before, let $\mathscr{F}$ be an abelian sheaf on R_{sm}. For any smooth formal R-scheme T, let $\mathscr{F}^0(T)$ consist of all sections $f \in \mathscr{F}(T)$ satisfying the following property: For each étale point $u : \operatorname{Spf} A \longrightarrow T$ there exist*

(i) *a valuation ring R', which is étale over A and, thus, over R,*

(ii) *a section $g: T' \longrightarrow \mathscr{F}$, where T' is a geometrically connected smooth formal R'-scheme,*

(iii) *R'-valued points $u_0', u_1' : \operatorname{Spf} R' \longrightarrow T'$ such that $g|_{u_0'}$ is the zero section and $g|_{u_1'}$ factors through $f|_u$.*

It is easy to verify that $\mathscr{F}^0$ is a well-defined abelian subsheaf of $\mathscr{F}$. For example, to show that the sum $f+\tilde{f}$ of two sections $f,\tilde{f} \in \mathscr{F}^0(T)$ belongs again to $\mathscr{F}^0(T)$, we proceed as follows. Fixing an étale point u of T, we can choose R' in condition (i) big enough such that it works for f and $\tilde{f}$ simultaneously. For f we have a section $g\colon T' \longrightarrow \mathscr{F}$ and R'-valued points u_0', u_1' as asserted in conditions (ii) and (iii). Let us denote the corresponding objects for $\tilde{f}$ by $\tilde{g}\colon \tilde{T}' \longrightarrow \mathscr{F}$ and $\tilde{u}_0', \tilde{u}_1'$. Then we can consider $g+\tilde{g}$ as a section $T' \times_{R'} \tilde{T}' \longrightarrow \mathscr{F}$, with restriction $f|_u + \tilde{f}|_u$ at the point $(u_1', \tilde{u}_1')$ as well as the zero section at $(u_0', \tilde{u}_0')$. This verifies our claim.

We want to show that $\mathscr{F}^0$ coincides with the usual identity component if $\mathscr{F}$ is representable by a smooth formal R-group scheme G. To see this, look at a morphism of smooth formal R-schemes $f\colon T \longrightarrow G$, and assume that f corresponds to a section $T \longrightarrow \mathscr{F}^0$. Then, using the notation of the above definition, the image of $g\colon T' \longrightarrow G$ meets the identity component G^0 and, hence, since T' is connected, g must map T' to G^0. But then f maps u to G^0. Thus, all étale points of T are mapped to G^0. Since the étale points are dense in T, it follows that f maps T to G^0 and, hence, that $\mathscr{F}^0$ is a subsheaf of G^0. The converse is trivial. Namely, given a morphism $f\colon T \longrightarrow G^0$ of smooth formal R-schemes and an étale point $u\colon \operatorname{Spf} A \longrightarrow T$, we may set $R' = A$, $T' = G^0 \otimes_R R'$, and define $g\colon T' \longrightarrow G$ as the canonical map. Then, writing u_0' for the R'-point of T' given by the zero section of G^0 as well as u_1' for the R'-point of T' induced from u, we have $g|_{u_1'} = f|_u$ as well as $g|_{u_0'} = 0$, and we see that f is a section of $\mathscr{F}^0$.

Proposition 4.8. *Let $\varphi\colon \mathscr{F}' \longrightarrow \mathscr{F}$ be an epimorphism of abelian sheaves on R_{sm}. Then the induced morphism $\mathscr{F}'^0 \longrightarrow \mathscr{F}^0$ is an epimorphism, too.*

Proof. Let us consider a section $f\colon T \longrightarrow \mathscr{F}$ giving rise to a section of $\mathscr{F}^0$. There is a commutative diagram

$$\begin{array}{ccc} T' & \xrightarrow{\alpha} & T \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathscr{F}' & \xrightarrow{\varphi} & \mathscr{F}, \end{array}$$

where α is a smooth covering of T. We have to show that we can choose f' in such a way that it maps T' to $\mathscr{F}'^0$. To do this we may extend the base ring R and replace it by a valuation ring being étale over R, which we will denote by R again. In particular, instead of looking at general étale points of T or T', it is enough to look at R-valued points only.

Clearly, we may assume that T is connected and, in fact, by extending R, geometrically connected. In addition, let us first assume that T' is geometrically connected, too, and that there is an R-valued point $u\colon \operatorname{Spf} R \longrightarrow T$ satisfying $f|_u = 0$. Since α is smooth, we can find an étale point u' of T' belonging to the fibre $\alpha^{-1}(u)$. Extending R, we may assume u' to be an R-valued point of T'. By construction, $f'|_{u'}$ belongs to the kernel of φ. Writing $f'_{u'}$ for the section induced from $f'|_{u'}$ on T', we can subtract it from f'. As a result, T' contains now an

R-valued point u' satisfying $f'|_{u'} = 0$. But then, by definition, f' is a section of $\mathscr{F}'^0$, as required.

Next, let us consider a situation as before, but where T' is not necessarily geometrically connected any more. Extending R, we may suppose, however, that all connected components of T' are geometrically connected. Since $\alpha: T' \longrightarrow T$ is a covering, there is a connected component T'_0 of T' such that its image $T_0 = \alpha(T'_0)$, which is an open formal subscheme of T, contains the R-valued point u of T satisfying $f|_u = 0$. As we have seen, we can assume that f' restricts to a section of $\mathscr{F}'^0$ on T'_0. Now look at a second connected component T'_1 of T' and write T_1 for its image under α. There is an étale point v'_1 of T'_1 such that $v = \alpha(v'_1)$ belongs to the intersection $T_0 \cap T_1$. Let v'_0 be an étale point of T'_0, which is an inverse image of v. Extending R, we can assume that v, v'_0, v'_1 are R-valued points. Clearly, $f'|_{v'_1} - f'|_{v'_0}$ belongs to the kernel of φ. Consequently, extending this section to T'_1 and subtracting it from f' on T'_1, we can assume $f'|_{v'_1} = f'|_{v'_0}$. In particular, $f'|_{v'_1}$ is now a section of $\mathscr{F}'^0$. Thus, if we subtract $f'|_{v'_1}$ from f' on T'_1, we get a section which, restricted to T'_1, belongs to $\mathscr{F}'^0$. But then the section f' itself must have this property. Proceeding like this with each connected component of T', we can modify f' in such a way that, finally, f' belongs to $\mathscr{F}'^0$.

It remains to consider the general case. Again, we may assume that T is geometrically connected. Furthermore, we replace T' by one of its connected components. Therby we can assume that also T' is geometrically connected, although we might loose the property that $\alpha: T' \longrightarrow T$ is a covering. Now consider an étale point $u': \operatorname{Spf} A \longrightarrow T'$; we may assume $A = R$. Let u be the projection of u' onto T. By definition, up to base change on R, there exists a section $g: \tilde{T} \longrightarrow \mathscr{F}$, where $\tilde{T}$ is a geometrically connected smooth formal R-scheme, together with R-valued points u_0, u_1 of $\tilde{T}$ such that $g|_{u_1} = f|_u$ as well as $g|_{u_0} = 0$. We have just seen that we can find a smooth cover $\tilde{\alpha}: \tilde{T}' \longrightarrow \tilde{T}$ such that the section $g \circ \tilde{\alpha}$ factors via a section $g': \tilde{T}' \longrightarrow \mathscr{F}'^0$. Extending R if necessary, we may assume that u_0, u_1 lift to sections $u'_0, u'_1: \operatorname{Spf} R \longrightarrow \tilde{T}'$. Since $f'|_{u'} - g'|_{u'_1}$ belongs to the kernel of φ, we may subtract this section from f' to obtain $f'|_{u'} = g'|_{u'_1}$. But then $f'|_{u'}$ as well as its extension $f'_{u'}$ on T' belong to $\mathscr{F}'^0$. Thus, by definition, $f' - f'_{u'}$ is a section of $\mathscr{F}'^0$, and the same is true for f'. □

Having introduced the identity component $\mathscr{F}^0$ of a sheaf $\mathscr{F}$ on R_{sm}, we define, of course, $\phi_{\mathscr{F}} = \mathscr{F}/\mathscr{F}^0$ as the *component group* of $\mathscr{F}$. As $\mathscr{F}^0$ gives the right object in the case where $\mathscr{F}$ is representable by a smooth R-group scheme, the same is true for $\phi_{\mathscr{F}}$.

Corollary 4.9. *Let*

$$0 \longrightarrow \mathscr{F}'_K \longrightarrow \mathscr{F}_K \longrightarrow \mathscr{F}''_K \longrightarrow 0$$

be an exact sequence of sheaves on K_{sm}, *and assume that the identity component of* $R^1 j_* \mathscr{F}'_K$ *is trivial (for example, this will be the case if* $R^1 j_* \mathscr{F}'_K$ *is represented by an étale* R*-group scheme). Then the associated long cohomology sequence*

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow R^1 j_* \mathscr{F}' \longrightarrow \cdots$$

starting with the Néron models $\mathscr{F}', \mathscr{F}$, and $\mathscr{F}''$ of $\mathscr{F}'_K, \mathscr{F}_K$, and $\mathscr{F}''_K$ induces an exact sequence

$$\phi_{\mathscr{F}'} \longrightarrow \phi_{\mathscr{F}} \longrightarrow \phi_{\mathscr{F}''} \longrightarrow R^1 j_* \mathscr{F}'$$

of sheaves on R_{sm}.

Proof. Since the identity component of $R^1 j_* \mathscr{F}'_K$ is trivial, it follows that $\mathscr{F}''^0$ belongs to the kernel of $\mathscr{F}'' \longrightarrow R^1 j_* \mathscr{F}'_K$, thus also to the image of $\mathscr{F} \longrightarrow \mathscr{F}''$. Then, by 4.8, $\mathscr{F}^0$ is mapped onto $\mathscr{F}''^0$, and the sequence of our assertion is exact at $\phi_{\mathscr{F}''}$. Since $\mathscr{F}'^0$ is mapped into $\mathscr{F}^0$, the exactness at $\phi_{\mathscr{F}}$ is trivial. □

The same argument of proof shows:

Corollary 4.10. *The functor $\mathscr{F} \longmapsto \phi_{\mathscr{F}}$ from sheaves on R_{sm} to associated component groups is right exact.*

We want to apply the assertion of 4.9 to the situation of semi-abelian varieties and, after this, to uniformizations of abelian varieties. We begin with a discussion about tori. For a torus T_K, we denote by $T_{K,I}$ the maximal subtorus with multiplicative reduction; cf. [22], 1.2. Writing X_K for the character group of T_K and G_K for the absolute Galois group of K, it is the torus, whose group of characters is the maximal $\mathbb{Z}$-free G_K-module quotient $X_{K,I}$ of X_K which is fixed by I.

Theorem 4.11. *Consider an exact sequence*

$$0 \longrightarrow T_K \longrightarrow E_K \longrightarrow B_K \longrightarrow 0$$

of sheaves on K_{sm}, where T_K is a torus and B_K an abelian variety.
(i) *The associated morphism of component groups $\phi_T \longrightarrow \phi_E$ has finite kernel and finite cokernel. In particular, it induces a monomorphism between the free parts of ϕ_T and ϕ_E. Furthermore,* $\mathrm{rank}\phi_T = \mathrm{rank}\phi_E = \dim T_{K,I}$.
(ii) *If T_K splits over an unramified extension of K, or if the residue field of K is perfect, the associated sequence of component groups*

$$\phi_T \longrightarrow \phi_E \longrightarrow \phi_B \longrightarrow 0.$$

is exact.

Proof. We start by showing that $\phi_T \longrightarrow \phi_E$ induces a monomorphism on free parts. Let T, E be the Néron models of T_K, E_K, and write $T_{\mathrm{rig}}, E_{\mathrm{rig}}$ for their associated rigid K-groups. Then the morphism $T_K \longrightarrow E_K$ gives rise to a commutative diagram of rigid K-groups

$$\begin{array}{ccc} T_{\mathrm{rig}} & \longrightarrow & E_{\mathrm{rig}} \\ \downarrow & & \downarrow \\ T_K & \xrightarrow{\alpha_K} & E_K, \end{array}$$

where the vertical maps are open immersions. As α_K is a closed immersion, a fact which might not be true for the corresponding morphism $\alpha: T \longrightarrow E$ of Néron models, we see that the quasi-compactness of E^0_{rig} implies the one of $\alpha_K^{-1}(E^0_{\text{rig}})$. As the latter group admits $\alpha^{-1}(E^0)$ as Néron model and since Néron models of quasi-compact rigid K-groups are quasi-compact by [9], 1.2, it follows that the kernel of $\phi_T \longrightarrow \phi_E$ is finite. Thus, this map must be injective on free parts.

Next, write $T'_K = T_K/T_{K,I}$ and consider the exact sequence

$$0 \longrightarrow T_{K,I} \longrightarrow T_K \longrightarrow T'_K \longrightarrow 0.$$

Then the associated sequence

$$0 \longrightarrow \phi_{T_I} \longrightarrow \phi_T \longrightarrow \phi_{T'} \longrightarrow 0$$

is exact. In fact, it is right exact due to 4.2, 4.9 and left exact due to a consideration as above, since ϕ_{T_I} does not have torsion. Furthermore, if K^{sh} is the field of fractions of a strict henselisation of the valuation ring of K, we see that $T'_K \otimes_K K^{\text{sh}}$ cannot contain a subgroup of type $\mathbb{G}_{\text{m}}$. Then $\phi_{T'}$ is finite by [8], 10.2/1, and it follows $\text{rank}\phi_T = \dim T_{K,I}$. A similar reasoning shows $\text{rank}\phi_E = \dim T_{K,I}$ and, thus, that the cokernel of $\phi_T \longrightarrow \phi_E$ is finite.

Finally, assertion (ii) is a consequence of 4.2 and 4.9. □

Theorem 4.12. *Consider an abelian variety A_K and the exact sequence*

$$0 \longrightarrow M_K \longrightarrow E_K \longrightarrow A_K \longrightarrow 0$$

given by its uniformization. Then the associated sequence

$$0 \longrightarrow \phi_M \longrightarrow \phi_E \longrightarrow \phi_A \longrightarrow H^1(I, M_K)$$

is exact.

Proof. As $R^1 j_* M_K$ coincides with $H^1(I, M_K)$ by 4.4 and, thus, is an étale R-group scheme, the assertion of 4.9 applies. Alternatively, we can use 2.3. A similar reasoning as in the proof of 4.11 shows that the kernel of $\phi_M \longrightarrow \phi_E$ is finite and, hence, trivial since ϕ_M is torsion-free. □

5 Applications to abelian varieties

In the following, let A_K be an abelian variety, and let

$$(*) \qquad 0 \longrightarrow M_K \longrightarrow E_K \longrightarrow A_K \longrightarrow 0$$

be the exact sequence of rigid K-groups associated to its uniformization. Furthermore, let T_K be the toric and B_K the abelian part of E_K so that there is an exact sequence

$$(**) \qquad 0 \longrightarrow T_K \longrightarrow E_K \longrightarrow B_K \longrightarrow 0;$$

let X_K be the group of characters of T_K. We want to look more closely at the associated exact sequences of component groups, as given in 4.11 and 4.12.

First we assume that the Néron model A of A_K has semi-abelian reduction. This case is well understood; we include it here for completeness and to show how our methods work in this simple situation.

Proposition 5.1. *Assuming that B_K has good reduction, the following are equivalent*:
(i) *M_K is invariant under the inertia group I.*
(ii) *X_K is invariant under the inertia group I.*
(iii) *The Néron model of T_K has multiplicative reduction.*
(iv) *The identity component of the Néron model of A_K has semi-abelian reduction.*

Proof. Looking at the exact sequence in 4.12, we see that the natural morphism of G_k-modules between M_K^I and the free part of ϕ_E is injective with finite cokernel. Thus, both modules have the same rank. On the other hand, X_K^I has rank equal to the rank of $X_{I,K}$, the maximal quotient of X_K which is fixed by I so that $\mathrm{rank} X_K^I = \mathrm{rank}\phi_E$ by 4.11. This shows the equivalence between (i) and (ii). Furthermore, the equivalence between (ii) and (iii) follows from the fact that X_K is invariant under I if and only if $X_K = X_{I,K}$.

It remains to show that (iii) is equivalent to (iv). To do this, look at the sequence

$$0 \longrightarrow T_k \longrightarrow E_k \longrightarrow B_k \longrightarrow 0$$

of special fibres of Néron models associated to the sequence (**) above. In general, it will not be exact. We want to show that the toric rank $t(T_k)$ equals the toric rank $t(E_k)$. Using the infinitesimal lifting property of tori [11], exp. IX, 3.6, together with the Néron mapping property, we see that any subtorus of E_k factors through T_k. Thus, $t(E_k) \leq t(T_k)$. On the other hand, looking at ℓ-division points for primes ℓ different from $p = \mathrm{char} k$ and using the fact that $T \longrightarrow E$ is a monomorphism of sheaves on R_{sm}, we can see that $T_k \longrightarrow E_k$ is injective, when restricted to the toric part of T_k. Therefore, $t(E_k) \geq t(T_k)$, and both ranks must coincide.

Now assume that condition (iii) is satisfied. Then we have $R^1 j_* T_K = 0$ by 4.2. It follows that, in particular, $E_k \longrightarrow B_k$ is surjective, and we see that the identity component of E_k is semi-abelian. Conversely, assume that the latter is the case. Then, by the above, $t(T_k)$ must coincide with $\dim T_K$, and we see that T_K has multiplicative reduction. □

In the situation of 5.1, we can evaluate characters of T_K and, thus, points of X_K at points of the lattice M_K, taking values in the Poincaré bundle P_K over $B_K \times B_K'$, where B_K' is the dual of B_K; cf. [4], 3.2. Since P_K extends to a formal line bundle on the associated product of Néron models $B \times B'$, we obtain a non–degenerate $\mathbb{Z}$-bilinear pairing

$$\sigma : M_K \times X_K \longrightarrow P_K \xrightarrow{-\log|\ \ |} \mathbb{Z},$$

which we may interpret as an injection $i: M_K \hookrightarrow X_K^\vee$ into the linear dual of X_K. Using this terminology, we can formulate the following result, also contained in [10], III, 8.2, and [9], 6.3.

Proposition 5.2. *In the situation of* 5.1, *the component group ϕ_A is isomorphic to $X_K^\vee/i(M_K)$. In particular, assuming that X_K and M_K are split, the number of elements of ϕ_A is given by the determinant of the pairing σ.*

Proof. Applying 4.11 and using the fact that ϕ_T cannot have torsion (cf. [27], 2.19), we see that the monomorphism $T_K \longrightarrow E_K$ induces an isomorphism $\phi_T \xrightarrow{\sim} \phi_E$, since ϕ_B is trivial. Next, observe that the evaluation of characters in X_K on the toric part T_K of E_K as well as on E_K itself yields a commutative diagram of sheaves on the small formal étale site over R

$$\begin{array}{ccc} T & \longrightarrow & E \\ {\scriptstyle \sigma_T}\downarrow & & \downarrow{\scriptstyle \sigma_E} \\ X_K^\vee & = & X_K^\vee \,, \end{array}$$

where T and E are the Néron models of T_K and E_K. It is well-known that σ_T identifies $X_K^\vee$ with the component group ϕ_T of T; cf. [27], 2.19. Since σ_E is obviously an epimorphism and the natural map $\phi_T \longrightarrow \phi_E$ is an isomorphism, it follows that also σ_E identifies the component group ϕ_E of E with $X_K^\vee$. But then, using the fact that $H^1(I, M_K)$ vanishes due to 4.4 and 5.1, the assertion follows from the exact sequence

$$0 \longrightarrow \phi_M \longrightarrow \phi_E \longrightarrow \phi_A \longrightarrow 0$$

of 4.12. □

Let us look at another special case, which can easily be handled.

Proposition 5.3. *Assume that the abelian part B_K of E_K is trivial and that $E_K = T_K$ has totally unipotent reduction. Then, if the residue field of K is perfect, there is a natural exact sequence*

$$0 \longrightarrow H^1(I, X_K)^* \longrightarrow \phi_A \longrightarrow H^1(I, M_K) \longrightarrow 0.$$

Proof. It follows from [22], 1.3, that X_K^I is trivial and from [27], 2.19, that ϕ_T is finite and coincides with $H^1(I, X_K)^*$. Thus, since ϕ_M does not admit torsion, 4.2 and 4.12 yield an exact sequence

$$0 \longrightarrow \phi_T \longrightarrow \phi_A \longrightarrow H^1(I, M_K) \longrightarrow 0$$

which is the one we are looking for. □

In the following we want to show how the above results can be extended to the general case. We start with a generalization of 5.2. Similarly, as we have done in the context of 4.11, where we have dealt with the group of characters X_K of the torus T_K, we write $M_{K,J}$ for the maximal $\mathbb{Z}$-free quotient of M_K which is fixed by the inertia subgroup I of the absolute Galois group G_K. Then we have an exact sequence

$$0 \longrightarrow M'_K \longrightarrow M_K \longrightarrow M_{K,I} \longrightarrow 0,$$

and we can set $\tilde{E}_K = E_K/M'_K$, where the latter group is viewed as a sheaf on the small rigid smooth site over K.

Proposition 5.4. *The exact sequence*

$$0 \longrightarrow M_{K,I} \longrightarrow \tilde{E}_K \longrightarrow A_K \longrightarrow 0$$

induced from the exact sequence (∗) *at the beginning of this section yields an exact sequence of associated Néron models*

$$0 \longrightarrow M_{K,I} \longrightarrow \tilde{E} \longrightarrow A \longrightarrow 0$$

and also an exact sequence of component groups

$$0 \longrightarrow M_{K,I} \longrightarrow \phi_{\tilde{E}} \longrightarrow \phi_A \longrightarrow 0.$$

Proof. The sequence between Néron models is exact, since we have $R^1 j_* M_{K,I} = 0$ due to 4.3. Furthermore, that the sequence between component groups is exact follows from 4.10 and the fact that $M_{K,I}$ does not have torsion; use the argument given in the proof of 4.11. □

As in the context of 4.11, let $T_{K,I} \subset T_K$ be the maximal subtorus with multiplicative reduction; its group of characters equals the maximal $\mathbb{Z}$-free G_K-module quotient $X_{K,I}$ of X_K which is fixed under I. Then we have canonical maps

$$T_{K,I} \longrightarrow T_K \longrightarrow E_K \longrightarrow \tilde{E}_K$$

as well as the corresponding maps of associated Néron models

$$T_I \longrightarrow T \longrightarrow E \longrightarrow \tilde{E}.$$

Switching to component groups, we get the following diagram:

$$\begin{array}{ccccccc} \phi_{T_I} & \longrightarrow & \phi_T & \longrightarrow & \phi_E & \longrightarrow & \phi_{\tilde{E}} & & \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & \phi_{T,\mathrm{tor}} & \longrightarrow & \phi_{E,\mathrm{tor}} & \longrightarrow & \phi_{\tilde{E},\mathrm{tor}} & \longrightarrow & \phi_{\tilde{E}} \end{array}$$

Now let us take images in ϕ_A. Recalling the fact that $\phi_{\tilde{E}}$ maps surjectively onto ϕ_A, we obtain the following diagram of subgroups of ϕ_A:

$$\begin{array}{ccccccc} \Sigma_3 & \longrightarrow & \Sigma_2 & \longrightarrow & \Sigma_1 & \longrightarrow & \phi_A & & \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & \Theta_3 & \longrightarrow & \Theta_2 & \longrightarrow & \Theta_1 & \longrightarrow & \phi_A \end{array}$$

Thus, we have filtrations

$$\phi_A \supset \Sigma_1 \supset \Sigma_2 \supset \Sigma_3 \supset 0, \qquad \phi_A \supset \Theta_1 \supset \Theta_2 \supset \Theta_3 \supset 0$$

as considered for prime–to–p parts of ϕ_A by Lorenzini [19] and, as far as the first one is concerned, also by Edixhoven [13]; of course, p is the residue characteristic of K. Let us write t_K for the toric rank of the Néron model of A_K; it coincides with $\dim T_{K,I} = \text{rank}(X_{K,I})$. Furthermore, set $T'_K = T_K/T_{K,I}$ so that $X'_K = \ker(X_K \longrightarrow X_{K,I})$ is the group of characters of T'_K.

We want to state some common properties of the above filtrations. Specific bounds for successive quotients will be given later. As a general assumption, we require that the residue field of K is *perfect*.

Theorem 5.5. *The filtration* $\phi_A \supset \Sigma_1 \supset \Sigma_2 \supset \Sigma_3 \supset 0$ *has the following properties*:
(i) $\Sigma_1 \cong \phi_E/\phi_M$, *and there is a canonical injection* $\phi_A/\Sigma_1 \hookrightarrow H^1(I, M_K)$.
(ii) *There is a canonical injection* $\Sigma_1/\Sigma_2 \hookrightarrow \phi_B$.
(iii) Σ_2/Σ_3 *is a quotient of* $H^1(I, X'_K)^*$; *by the latter we mean the Pontrjagin dual of* $H^1(I, X'_K)$.
(iv) Σ_3 *can be generated by* t_K *elements.*

Theorem 5.6. *The filtration* $\phi_A \supset \Theta_1 \supset \Theta_2 \supset \Theta_3 \supset 0$ *has the following properties*:
(i) *The canonical map* $\phi_{\bar{E},\text{tor}} \longrightarrow \Theta_1$ *is an isomorphism, and the quotient* ϕ_A/Θ_1 *can be generated by* t_K *elements.*
(ii) *The canonical map* $\phi_{E,\text{tor}} \longrightarrow \Theta_2$ *is an isomorphism, and there is a canonical injection* $\Theta_1/\Theta_2 \hookrightarrow H^1(I, M'_K)$.
(iii) *There is a canonical injection* $\Theta_2/\Theta_3 \hookrightarrow \phi_B$.
(iv) Θ_3 *is a quotient of* $\phi_{T,\text{tor}} \cong H^1(I, X_K)^*$.

Remark 5.7. (i) As we have inclusions $\Theta_3 \subset \Sigma_2$ and $\Theta_2 \subset \Sigma_1$, we can construct 2 additional filtrations of ϕ_A from the subgroups Θ_i and Σ_i. They have similar properties as the ones dealt with in 5.5 and 5.6.
(ii) In discussions with D. Lorenzini we found out that the filtration of 5.5 coincides on prime–to–p parts precisely with his filtration as given in [19]. The same turned out to be true for the filtration of 5.6, up to a question of duality, as far as Θ_1 is concerned.
(iii) Writing $A_K^\vee$ for the dual abelian variety of A_K, we can consider the canonical pairing $\phi_A \times \phi_{A^\vee} \longrightarrow \mathbb{Q}/\mathbb{Z}$ of [15], exp. IX,1. In cases where this pairing is perfect, the filtrations considered above induce corresponding filtrations on $\phi_{A^\vee}$. It seems most likely that the filtrations considered in 5.5 and 5.6 are orthogonal to each other.

To show the assertions of 5.5 and 5.6, we need some information on component groups of tori.

Lemma 5.8. (i) *There exists a natural exact sequence*

$$0 \longrightarrow H^1(I, X_K)^* \longrightarrow \phi_T \longrightarrow (X_K^I)^\vee \longrightarrow 0,$$

identifying $H^1(I, X_K)^*$ *with the torsion part and* $(X_K^I)^\vee$ *with the free part of* ϕ_T.
(ii) *The exact sequence*

$$0 \longrightarrow T_{K,I} \longrightarrow T_K \longrightarrow T'_K \longrightarrow 0$$

induces an exact sequence

$$0 \longrightarrow \phi_{T_I} \longrightarrow \phi_T \longrightarrow \phi_{T'} \longrightarrow 0,$$

where $\phi_{T_I} \cong (X_{K,I})^\vee$ *and* $\phi_{T'} \cong H^1(I, X'_K)^*$.

Proof. For the first exact sequence see [27], 2.18. To exhibit the second one, we apply 4.2 and 4.9; that $\phi_{T_I} \longrightarrow \phi_T$ is a monomorphism follows as in the proof of 4.11 from the fact that ϕ_{T_I} is torsion–free. Since $T_{K,I}$ has multiplicative reduction, there is a natural isomorphism of G_K/I-modules between $(X_{K,I})^\vee$ and ϕ_{T_I}. On the other hand, we have $(X'_K)^I = 0$ and, hence, $\phi_{T'} \cong H^1(I, X'_K)^*$. □

Proof of 5.5. Assertion (i) is immediately deduced from 4.12. Since we have a monomorphism $\phi_E/\phi_T \hookrightarrow \phi_B$ by 4.11, assertion (ii) is clear. Next, 5.8 yields an isomorphism $\phi_T/\phi_{T_I} \cong H^1(I, X'_K)^*$, thus a natural isomorphism from $H^1(I, X'_K)^*$ to the quotient Σ_2/Σ_3. This verifies (iii). Finally, assertion (iv) is due the fact that ϕ_{T_I} is isomorphic to $(X_{K,I})^\vee$ and, thus, is free of rank t_K. □

Proof of 5.6. We have $\phi_{\tilde{E}}/M_{K,I} \cong \phi_A$ by 5.4. Since $M_{K,I}$ is free, the map $\phi_{\tilde{E},\mathrm{tor}} \longrightarrow \phi_A$ must be injective, and, thus, induces an isomorphism onto Θ_1. Furthermore, using a reasoning as in 4.12, the exact sequence

$$0 \longrightarrow M'_K \longrightarrow E_K \longrightarrow \tilde{E}_K \longrightarrow 0$$

gives rise to an exact sequence

$$0 \longrightarrow \phi_E \longrightarrow \phi_{\tilde{E}} \longrightarrow H^1(I, M'_K).$$

Since $H^1(I, M'_K)$ is finite, ϕ_E and $\phi_{\tilde{E}}$ have the same rank, namely t_K; cf. 4.11. But then the quotient of $\phi_{\tilde{E}}$ by its torsion subgroup is generated by t_K elements, and the same is true for ϕ_A/Θ_1. This verifies assertion (i).

Assertion (ii) is easily deduced from the above exact sequence of component groups, and the remaining assertions follow from 4.11, using 5.8. □

In the remainder of the present section we want to work out precise bounds for the quotients of the filtrations considered in 5.5 and 5.6. Given a finite abelian group Σ and a prime ℓ, let Σ_ℓ be the ℓ-primary part of Σ. Write Σ_ℓ as a product of cyclic groups

$$\Sigma_\ell := \prod_{i=1}^{s(\ell)} \mathbb{Z}/\ell^{a_i}\mathbb{Z}, \quad \text{with } a_1 \geq \cdots \geq a_{s(\ell)},$$

and define

$$\delta_\ell(\Sigma_\ell) := \sum_{i=1}^{s(\ell)} (\ell^{a_i} - 1),$$

as well as

$$\delta(\Sigma) := \sum_{\ell \text{ prime}} \delta_\ell(\Sigma_\ell).$$

As usual, $\Sigma^{(p)}$ will be the prime–to–p part of Σ.

Now, let us consider the Galois extension L/K, which is minimal with the property that the unipotent rank of the Néron model of A_L is zero; see [12], 5.15. Let t_K be the toric rank of the special fibre of A (as before), and t_L the corresponding rank for the Néron model of A_L. In the same way, we write a_K and a_L for the abelian rank of the special fibre of the Néron model of A_K and A_L.

Corollary 5.9. *Consider the situations of* 5.5 *and* 5.6.
(i) *Let Σ be any of the groups*

$$\phi_A/\Sigma_1, \quad \Sigma_2/\Sigma_3, \quad \Theta_1/\Theta_2, \quad \Theta_3.$$

Then Σ is killed by $[L:K]$ and can be generated by $t_L - t_K$ elements. Moreover, $\delta(\Sigma^{(p)}) \le t_L - t_K$.
(ii) *Let Σ be any of the quotients Σ_1/Σ_2 and Θ_2/Θ_3. Then $\Sigma^{(p)}$ is killed by $[L:K]$ and satisfies $\delta(\Sigma^{(p)}) \le 2(a_L - a_K)$.*

Before giving the proof, let us mention that, as far as the filtration of 5.5 is concerned, the assertions of 5.9 were proved for prime–to–p parts in [13], Corollary 3.4, as well as in a weaker form in [19], Theorem 2.15. It should be observed that the "delta" we use in 5.9 is different from Lorenzini's "delta"; in fact, it corresponds to the δ' of Remark 2.16 in [19] and coincides also with the one in Edixhoven's paper [13].

Of course, one would like to know if the estimates of 5.9 extend to the p-parts of the quotients under consideration, at least for some modified δ. We can read from 5.9 that the p-parts of the groups in 5.9 (i) are bounded in terms of $t_L - t_K$ and of the p-part of $[L:K]$, since the quotients we are considering are killed by $[L:K]$ and since the number of generators is bounded by $t_L - t_K$. To formulate a more precise statement, let us set

$$\tilde{\delta}(\Sigma) = \sum_{\ell \text{ prime}} \mathrm{ord}_\ell(|\Sigma|)(\ell - 1)$$

for finite abelian groups Σ with cardinality $|\Sigma|$. Using this bound, which was introduced by Lenstra and Oort in [18], we will derive the following estimates from 5.9:

Corollary 5.10. *Consider the situation of* 5.9, *and let Σ be any of the groups*

$$\phi_A/\Sigma_1, \quad \Sigma_2/\Sigma_3, \quad \Theta_1/\Theta_2, \quad \Theta_3.$$

Then $\tilde{\delta}(\Sigma) \le t_L - t_K$ and, in particular, $|\Sigma| \le 2^{t_L - t_K}$.

Furthermore, we can see that any prime ℓ dividing the order of Σ as above must satisfy $\ell \leq t_L - t_K$.

We begin with the *proof of Corollary* 5.9. Part (i) is a consequence of 5.5, 5.6, and 5.11 below, using the fact that the prime–to–p part of the inertia subgroup I of a finite extension of local fields is a cyclic group, and that there is an injection $H^1(I,M)^{(p)} \hookrightarrow H^1(I^{(p)},M)$ for every I-module M.

For the remaining assertion (ii) of 5.9, we look at 5.5 (ii) as well as at 5.6 (iii). Then it is enough to know that, for any abelian variety B_K with potentially good reduction, $\phi_B^{(p)}$ is killed by the degree of the minimal extension L/K, over which B_K acquires good reduction, and that

$$\sum_{\ell \neq p} \delta((\phi_B)_\ell) \leq 2(b_L - b_K);$$

b_K resp. b_L is the abelian rank of the Néron model of B_K, resp. B_L. To see this, it seems that one has to go back to methods of Tate modules, and we refer to Theorem 2.15 in [19], or Corollary 3.4. in [13].

To end the proof of 5.9, we still have to prove the proposition which was referred to above.

Proposition 5.11. *Let M be a free abelian group of finite rank r, and let G be a finite group of order m acting on M. Then the finite abelian group $H^1(G,M)$ can be generated by $r - \mathrm{rank}_{\mathbb{Z}}(H^0(G,M))$ elements and is killed by m. If, in addition, G is a cyclic group, then*

$$\delta(H^1(G,M)) \leq r - \mathrm{rank}_{\mathbb{Z}}(H^0(G,M)).$$

Proof. It is a general fact that $H^1(G,M)$ is killed by the order of G. Furthermore, note that we may assume $H^0(G,M) = 0$. In effect, suppose we know the above assertions for any G-module without fixed elements. Then, if M is an arbitrary G-module, we consider the G-module $N := M/H^0(G,M)$. One can easily show that N is torsion–free and that $H^0(G,N) = 0$. Thus, we have an injection

$$H^1(G,M) \hookrightarrow H^1(G,N).$$

Therefore, if we use the assertions for the G-module N, we obtain the following inequalities:

$$\begin{aligned} \#(\text{generators of } H^1(G,M)) &\leq \#(\text{generators of } H^1(G,N)) \\ &\leq \mathrm{rank} N = r - \mathrm{rank}_{\mathbb{Z}}(H^0(G,M)), \end{aligned}$$

and in the case that G is cyclic,

$$\delta(H^1(G,M)) \leq \delta(H^1(G,N) \leq \mathrm{rank} N = r - \mathrm{rank}_{\mathbb{Z}}(H^0(G,M)).$$

From now on, we suppose that M has no G-fixed elements. The induced G-module $\mathrm{Ind}^G(M)$ satisfies $H^0(G,\mathrm{Ind}^G(M)) \cong M$ (as abelian groups). On the other hand, one has $H^1(G,\mathrm{Ind}^G(M)) = 0$. Consider the natural inclusion

i of M into $\mathrm{Ind}^G(M)$; the quotient $Q := \mathrm{Ind}^G(M)/i(M)$ is torsion–free. Since $H^1(G, \mathrm{Ind}^G(M))$ is trivial, $H^1(G, M)$ is isomorphic to the quotient of $H^0(G, Q)$ by the image of the composition $M \cong H^0(G, \mathrm{Ind}^G(M)) \to H^0(G, N)$, where this last morphism is the natural one induced by the projection. Since this quotient is finite and $H^0(G, N)$ is torsion–free, the number of generators of $H^1(G, M)$ is less or equal to the rank of $H^0(N, G)$, which is r.

We prove now the second part of the assertion. First of all, observe that we only need to show the following: if G is a cyclic p-group for some prime p, then

$$\delta_p(H^1(G, M)) \leq r - \mathrm{rank}_{\mathbb{Z}}(H^0(G, M)).$$

In effect, suppose that we know this result, and take G a finite cyclic group. Let H be the p-Sylow subgroup of G. We have an exact sequence

$$0 \to H^1(G/H, M^H) \to H^1(G, M) \to H^1(H, M).$$

Since $H^1(G/H, M^H)$ and $H^1(H, M)$ are killed by coprime numbers, the sequence is necessarily split; thus,

$$\begin{aligned}\delta(H^1(G, M)) &\leq \delta(H^1(H, M)) + \delta(H^1(G/H, M^H))\\ &\leq r - \mathrm{rank}(M^H) + \delta(H^1(G/H, M^H)).\end{aligned}$$

The proof can then be finished by induction on the number of primes dividing the order of G.

Finally, let G be a cyclic p-group, generated by σ. Since it is well-known that $H^1(G, M) = M/(\sigma - \mathrm{id})M$, we can do a computation as in [13], Lemma 4.5, or just apply [13], Corollary 4.7, which says $\delta_p(M/(\sigma - \mathrm{id})M) \leq \mathrm{rank}(M)$. □

To *prove the assertion of* 5.10, we need only show the following substitute of 5.11:

If G is a finite solvable group acting on a free abelian group M of finite rank r, then

$$\tilde{\delta}(H^1(G, M)) \leq r - \mathrm{rank}_{\mathbb{Z}} H^0(G, M).$$

This estimate is easy to obtain. In fact, if G is cyclic, we can deduce it from 5.11, since $\tilde{\delta}$ is majorized by δ. Furthermore, using the fact that $\tilde{\delta}$ is additive in the sense that we have $\tilde{\delta}(H) = \tilde{\delta}(H') + \tilde{\delta}(H'')$ for any exact sequence of finite abelian groups

$$0 \longrightarrow H' \longrightarrow H \longrightarrow H'' \longrightarrow 0,$$

the case where G is solvable can be reduced to the one where G is cyclic. □

Let us mention that, in the above proof, $\tilde{\delta}$ cannot be replaced by δ, since δ is far from being additive; cf. [13], 4.1 and 4.3.

References

1. Artin, M.: Grothendieck topologies. Notes on a seminar by M. Artin, Harvard University (1962)
2. Bosch, S., Güntzer, U., Remmert, R.: Non-Archimedean Analysis. Springer, Grundlehren Bd. 261, Berlin, Heidelberg, New York (1984)
3. Bosch, S., Lütkebohmert, W.: Stable reduction and uniformization of abelian varieties II. Invent. Math. 78, 257–297 (1984)
4. Bosch, S., Lütkebohmert, W.: Degenerating abelian varieties. Topology 30, 653–698 (1991)
5. Bosch, S., Lütkebohmert, W.: Formal and rigid geometry I. Rigid spaces. Math. Ann.295, 291–317 (1993)
6. Bosch, S., Lütkebohmert, W.: Formal and rigid geometry II. Flattening techniques. Math. Ann. 296, 403–429 (1993)
7. Bosch, S., Lütkebohmert, W., Raynaud, M.: Formal and rigid geometry III. The relative maximum principle. Math. Ann. 302, 1–29 (1995)
8. Bosch, S., Lütkebohmert, W., Raynaud, M.: Néron Models. Ergebnisse der Math., 3. Folge, Bd. 21, Springer (1990)
9. Bosch, S., Schlöter, K.: Néron models in the setting of formal and rigid geometry. Math. Ann. 301, 339–362 (1995)
10. Chai, C.L., Faltings, G.: Degeneration of Abelian Varieties. Ergebnisse der Math., 3. Folge, Bd. 21, Springer (1990)
11. Demazure, M., Grothendieck, A.: SGA 3_{II}, Schémas en Groupes. Lecture Notes in Mathematics 152, Springer (1970)
12. Deschamps, D.: Réduction semistable. In "Séminaire sur les pinceaux de courbes de genre au moins deux", L. Szpiro, Ed., 1–34 (1981)
13. Edixhoven, B.: On the prime to p-part of the groups of connected components of Néron models. Preprint (1994)
14. Edixhoven, B., Liu, Q., Lorenzini, D.: The p-part of the group of components. Preprint (1994)
15. Grothendieck, A.: SGA 7_I, Groupes de Monodromie en Géométrie Algébrique. Lecture Notes in Mathematics 288, Springer (1972)
16. Hartshorne, R.: Residues and Duality. Lect. Notes in Math. 20, Springer (1966)
17. Köpf, U.: Über eigentliche Familien algebraischer Varietäten über affinoiden Räumen. Schriftenreihe Math. Inst. Univ. Münster, 2. Serie. Heft 7 (1974)
18. Lenstra, H., Oort, F.: Abelian varieties having purely additive reduction. J. Pure Appl. Alg. 36, 281–298 (1985)
19. Lorenzini, D.: On the group of components of a Néron model. J. reine angew. Math. 445, 109–160 (1993)
20. Lütkebohmert, W.: Formal-algebraic and rigid-analytic geometry. Math. Ann. 286, 341–371 (1990)
21. Milne, J. S.: Etale cohomology. Princeton Math. Series 33, Princeton University Press, Princeton (1980)
22. Nart, E., Xarles, X.: Additive reduction of algebraic tori. Arch. Math. 57, 460–466 (1991)
23. Ogus, A.: F-isocrystals and de Rham cohomology II: Convergent isocrystals. Duke Math. Journal 51, 765–850 (1984)
24. Raynaud, M.: Variétés abéliennes et géométrie rigide. Actes du congrès international de Nice 1970, tome 1, 473–477
25. Raynaud, M.: 1-motifs et monodromie géométrique. Astérisque 223, Exp. VII, 295–319 (1994)
26. Serre, J.-P.: Corps Locaux. Hermann, Paris (1962)
27. Xarles, X.: The scheme of connected components of the Néron model of an algebraic torus. J. reine angew. Math. 437, 167–179 (1993)