Comparison theorems between crystalline and étale cohomology: a short introduction.

Xavier Xarles

Preliminary Version

Introduction

The p-adic comparison theorems (or the p-adic periods isomorphisms) are isomorphisms, analog to the "complex periods isomorphism"

$$H^i_{dR}(X/\mathbb{C}) \cong H^i(X(\mathbb{C}), \mathbb{Q}) \otimes \mathbb{C}$$

for a smooth and projective variety over $\mathbb{C}$, between the p-adic cohomology and the de Rham cohomology (plus some additional structure) of smooth and projective varieties over a finite extension of $\mathbb{Q}_p$.

The theory started with the work of Tate [Ta] and Fontaine [Fo1] on abelian varieties and p-divisible groups, and continues with the work of Fontaine, Bloch, Kato, Messing, Faltings, Hyodo, Tsuji, Breuil and others.

1 Brief review of Hodge theory

We will consider only algebraic varieties that are smooth an projective; of course, Hodge theory can be used for all algebraic varieties (and also for more general complex varieties). Consider X an smooth and projective variety over the complex numbers $\mathbb{C}$. We have then the so called singular or Betti cohomology $H^i(X(\mathbb{Z}), \mathbb{Z})$, which are finitely generated $\mathbb{Z}$-modules, and equal to zero for $i < 0$ and $i > 2\dim(X) = 2d$. On the other hand, we have de Rham cohomology $H^i_{dR}(X(\mathbb{C})/\mathbb{C})$, which are complex vector spaces and, as before, equal to zero for $i < 0$ and $i > 2d$. What is the relation between this to objects? The answer is given by de Rham theorem.

Theorem 1 *(de Rham) There exist a natural and canonical isomorphism*

$$H^i(X(\mathbb{C}), \mathbb{C}) := H^i(X(\mathbb{C}), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \cong H^i_{dR}(X(\mathbb{C})/\mathbb{C})$$

In fact, the isomorphism goes as follows: By Poincare duality, we have that

$$\mathrm{Hom}_{\mathbb{C}}(H_{2d-i}(X(\mathbb{C}),\mathbb{C}),\mathbb{C}) \cong H^i(X(\mathbb{C}),\mathbb{C}).$$

Now, we have a periods pairing defined by

$$H^i_{dR}(X(\mathbb{C})/\mathbb{C}) \times H_{2d-i}(X(\mathbb{C}),\mathbb{C}) \longrightarrow \mathbb{C}$$

$$(\omega,\Gamma) \mapsto \int_\Gamma \omega$$

where Γ is a cicle, ω a differential form and $\int_\Gamma \omega$ means the integral along Γ.

Now, the next step in Hodge theory is the Hodge theorem. Recall that de Rham cohomology $H^i_{dR}(X(\mathbb{C})/\mathbb{C})$ is formed by classes of forms, and that forms can be of type (p,q) with $p+q=i$. Denote by $H^{p,q}(X)$ the subspace of (p,q)-forms.

Theorem 2 *(Hodge) We have a canonical decomposition*

$$H^i_{dR}(X(\mathbb{C})/\mathbb{C}) = \bigoplus_{p+q=i} H^{p,q}(X),$$

and $\overline{H^{p,q}(X)} = H^{q,p}(X)$, *where* $\overline{H}$ *means complex conjugation.*

And have also Dolbeault's theorem.

Theorem 3 *(Dolbeault) We have a canonical and functorial isomorphism*

$$H^{p,q}(X) \cong H^p(X(\mathbb{C}),\Omega^q)$$

where $\Omega^q = \bigwedge^q \Omega^1$, *and* Ω^1 *is the sheaf of holomorphic differential forms.*

Now, by using the GAGA theorems, one can show that

$$H^i_{dR}(X/\mathbb{C}) \cong H^i_{dR}(X(\mathbb{C})/\mathbb{C}) \text{ and } H^p(X,\Omega^q) \cong H^p(X(\mathbb{C}),\Omega^q),$$

where Ω^1 is the sheaf of algebraic differential forms, $\Omega^i := \bigwedge^i \Omega^1$, $\Omega^\bullet$ is the usual de Rham complex and $H^i_{dR}(X/\mathbb{C})$ is the hypercohomology of the de Rham complex.

All this results can be then be write as:

$$H^i(X,\mathbb{C}) \overset{(1)}{\cong} H^i_{dR}(X/\mathbb{C}) \overset{(2)}{\cong} \bigoplus_{p+q=i} H^p(X,\Omega^q). \quad (*)$$

Our first question is now: at what extend can this results be true on other type of fields K?

Observe that the second isomorphism (2) in (*) can be extended to other fields in the following form. Recall that (algebraic) de Rham cohomology is the hypercohomology of the de Rham complex $\Omega^{\bullet}_{X/K}$, which can be defined for any variety over a field (or, more generally, over any scheme). So, we have a natural spectral sequence (the so-called second spectral sequence of hypercohomology)

$$E_1^{q,p} = H^p(X, \Omega^q_{X/K}) \Longrightarrow H^{p+q}_{dR}(X/K).$$

Now, this spectral sequence gives us a way to find a relation between $H^i_{dR}(X/K)$ and $H^p(X, \Omega^q_{X/K})$.

Theorem 4 *Let K be a field of characteristic zero. Let X be a smooth and projective variety over K. Then the spectral sequence*

$$E_1^{q,p} = H^p(X, \Omega^q_{X/K}) \Longrightarrow H^{p+q}_{dR}(X/K).$$

degenerates at E_1, and thus we have a natural filtration (so called Hodge filtration) $Fil^q(H_{dR}(X/K))$ with graduate quotients

$$Fil^q(H^i_{dR}(X/K))/Fil^{q+1}(H^i_{dR}(X/K)) \cong H^{q-i}(X, \Omega^q_{X/K})$$

One can think that this result gives us a non-canonical isomorphism

$$H^i_{dR}(X/K) \cong \bigoplus_{p+q=i} H^p(X, \Omega^q_{X/K}) =: H^i_{Hod}(X/K).$$

The idea of the proof is to reduce to the case of a finitely generated extension of $\mathbb{Q}$, and then reduce to the case of complex numbers.

There is also a purely "algebraic" proof of this result by Deligne and Illusie using cristalline cohomology techniques (see [De-Il]).

Observe also that this result is not true in general in characteristic $p > 0$, see for example [De-Il].

Now, to obtain an analog (if exists) of the first isomorphism, we need to know what's the precise analog of the Betti cohomology. Of course, this cannot be done over a general field. So, we restrict ourselves to p-adic fields.

2 p-adic cohomology over p-adic fields.

From now on, let K be a finite extension of $\mathbb{Q}_p$, and X be an smooth and projective variety over K. Consider also $\overline{K}$ an algebraic closure of K and

$C := \widehat{\overline{K}}$ the completion of $\overline{K}$; C is then algebraically closed and complete with respect to an absolute value that extents the p-adic absolute value. Denote also by $\mathcal{O}_K$ the ring of integers of K and by k the residue field.

The field C can be think as an analog of $\mathbb{C}$; in fact, it does not depend of the field K, just depends on p. Some people call it $\mathbb{C}_p$.

To obtain analogs of the classical Hodge theory, we need a cohomology that behaves like the Betti cohomology (with coeficients in $\mathbb{C}$), but with coefficients in our field. The natural one to consider is the p-adic cohomology:

$$H^i(\overline{X}, \mathbb{Q}_p) := (\lim_{\leftarrow} H^i(\overline{X}_{\text{ét}}, \mathbb{Z}/p^n\mathbb{Z})) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

where $\overline{X}$ denotes as usual $X \otimes_K \overline{K}$,

Then, we can obtain, by tensoring with K or with C, K- or C-vector spaces. Observe that this cohomology has the "right" dimension, since, if we choose any embedding σ of K into $\mathbb{C}$ (there are always such embeddings), we have a comparison theorem (Grothendieck)

$$H^i(\sigma(X)(\mathbb{C}), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}_p \cong H^i(\overline{X}, \mathbb{Q}_p)$$

as $\mathbb{Q}_p$-vector spaces.

But we have now a new ingredient: this vector spaces have a natural action of the absolute Galois group G_K of K. Our objective will be to compare this cohomology groups (with the G_K-action) with the de Rham cohomology groups (plus some additional structures). Observe that both cohomology groups have the same dimension as K-vector spaces (use the comparison theorem I just wrote), so they are isomorphic (as vector spaces); but we would like to have a "canonical" isomorphism; and we would like also that the Galois action plays some role.

3 The Hodge-Tate comparison theorem

The first step to obtain a p-adic Hodge theory was made by J. Tate in 1967 [Ta]. Tate showed, by working with p-divisible groups, that, if A is an abelian variety over K which have good reduction over K, then we have a natural isomorphism

$$H^1(\overline{A}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} C \cong (H^1(A, \Omega^0_{A/K}) \otimes_K C(-1)) \oplus (H^0(A, \Omega^1_{A/K}) \otimes_K C)$$

respecting the Galois action of G_K, where $C(-1) = C \otimes Q_p(-1)$ is the Tate twist, and the Galois group G_K acts on C in the natural way.

Tate formulated the conjecture that this result should be true for all smooth and projective varieties. This conjecture is now a theorem.

Theorem 5 *(Faltings) For all varieties X smooth and projective over K we have a canonical isomorphism*

$$H^i(\overline{X}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} C \cong \bigoplus_{p+q=i} \Big(H^p(X, \Omega^q_{X/K}) \otimes_K C(-p) \Big)$$

respecting the action of G_K.

This theorem was proved by S. Bloch and K. Kato in the case when X has ordinary good reduction and is of dimension $< p$, and by G. Faltings ([Fa1]) in full generality, and it is also a consequence of the results we are going to explain.

One way to write this theorem, which resembles the way this result has been generalized, is by saying that there is a natural isomorphism

$$H^i(\overline{X}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{HT} \cong H^i_{Hod}(X/K) \otimes_K B_{HT},$$

where B_{HT} (Hodge-Tate) is the ring

$$B_{HT} := \oplus_{j \in \mathbb{Z}} C(j),$$

with the natural action of G_K (acting on both sides of $C(j) := C \otimes_{\mathbb{Z}_p} \mathbb{Z}_p)(j)$).

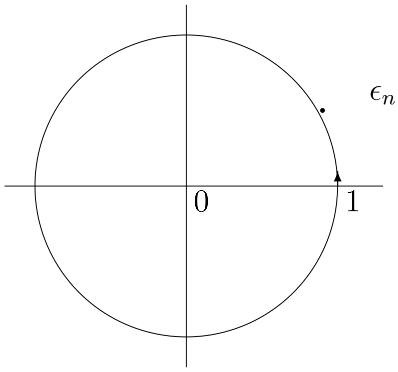
Observe that we have gone directly from the first term in $(*)$ to the last term in $(*)$ (plus the fact that we have a Galois action).

4 The de Rham comparison theorem: B_{dR}

It is natural to ask now if we have a canonical isomorphism of $H^i(\overline{X}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} C$ with the de Rham cohomology over C. Of course, as I said before, we have an isomorphism because both are vector spaces over the same field and with the same dimension. But we want it canonical and we like to have some information of the Galois action. The fact is that we have not! And the reason is that we do not have sufficiently many "periods". Let's see this in the following example.

Example 6 *Consider the variety $X := \mathbb{G}_m = Spec(K[Z, 1/Z])$; this is an smooth variety, but not projective. So is not an example of the type of varieties we are interested in, but still we can see some idea of what's going on. Fix p a prime number.*

Let's consider the periods pairing over $\mathbb{C}$, that we will compute in a particular way, suitable to be generalized to p-adic numbers. It works as follows:



given the element $\frac{dZ}{Z} \in H^0(X, \Omega^1_{X/\mathbb{C}})$ *(which generates the invariant differentials) and a generator* $\gamma \in H_1(X, \mathbb{Z}) \cong Z$ *(the circle of radius 1 around 0), we associate*

$$\int_\gamma w = \int_\gamma \frac{dZ}{Z} = p^n \int_1^{\epsilon_n} \frac{dZ}{Z} = p^n \int_0^{2\pi/p^n} \frac{de^{i\theta}}{e^{i\theta}} = 2i\pi$$

where $\epsilon_n := e^{\frac{2\pi i}{p^n}}$ *is a* p^n*-root of unity.*

Now, over K*, the natural substitute for* $H_1(X, \mathbb{Z})$ *is the Tate module* $T_p(\mathbb{G}_m) = \mathbb{Z}_p(1)$*, generated by an element of the form* $\widetilde{\epsilon} := (\epsilon_n)_n$*, where* $\epsilon_n \in \overline{K}$ *verifies* $\epsilon_n^p = \epsilon_{n-1}$ *for all* $n > 1$*,* $\epsilon_1^p = 1$ *and* $\epsilon_1 \neq 1$*. Now*

$$\int_1^{\widetilde{\epsilon}} \frac{dZ}{Z} = p^n \int_1^{\epsilon_n} \frac{dZ}{Z} = p^n \log_p(\epsilon_n) = \log_p(\epsilon_n^{p^n}) = \log_p(1) = 0,$$

for any "natural" definition of the integral (for example, for the integral defined by Coleman [C-I] or Colmez [Co]).

You can think that the problem is that the p-adic logarithm is not multivalued. Or you can think that we want an analog of $2\pi i$: $\log(\widehat{\epsilon})$.

The following naive idea does not work completely but it goes in the good direction: consider the field of fractions of the formal powers series of $\log(\widehat{\epsilon})$ with coefficients in C: $C((\log(\widehat{\epsilon})))$.

We have a natural action of G_K: we can define the action to be the natural one on C and the cyclotomic on $\log(\widehat{\epsilon})$, i.e. $g(\log(\widehat{\epsilon})) = \epsilon(g)\log(\widehat{\epsilon})$, where $\varepsilon : G_K \to \mathbb{Z}^\times$ is the p-adic cyclotomic character. Then you get B_{HT} and we know it works only with the Hodge cohomology. In some sense, what we need is a ring where the filtration is not trivially split.

The good idea goes in a different direction. Let $\mathcal{O}$ be the ring of integers of C: is the ring of elements with absolute value less than or equal to 1; it is a local ring with residue field isomorphic to an algebraic closure of k.

Consider now the category of "formal p-adic pro-infinitesimal coverings" of $\mathcal{O}$; it is formed by objects of the form $\{\theta : V \longrightarrow \mathcal{O}\}$ where V is an $\mathcal{O}$-module, θ a surjective morphism of $\mathcal{O}$-modules and V is complete with respect to $\mathrm{Ker}(\theta)$ and with respect to p.

Now, this category has an initial object, that is a universal formal p-adic pro-infinitesimal covering, called A_{inf}.

Claim 7 *Consider the ring*

$$R_{\mathcal{O}} := \lim_{\leftarrow}(\mathcal{O}/p\mathcal{O} \leftarrow \mathcal{O}/p\mathcal{O} \leftarrow \cdots),$$

where the maps $\mathcal{O}/p\mathcal{O} \to \mathcal{O}/p\mathcal{O}$ are $a \mapsto a^p$; it is a ring of characteristic p, so we can apply the Witt construction to get a ring of characteristic zero. Then

$$A_{\mathrm{inf}} = W\left(R_{\mathcal{O}}\right).$$

For a proof, see [Fo2] 1.2.

Now, we have a natural map $\theta : A_{inf} \to \mathcal{O}$, hence a natural map

$$\theta_K : A_{inf} \otimes_{\mathcal{O}} K \longrightarrow C$$

Consider $J_K := \mathrm{Ker}(\theta_K)$ and define B^+_{dR} as the completion of $A_{inf} \otimes_{\mathcal{O}} K$ with respect to the ideal J_K. Finally, let B_{dR} be the field of fractions of B^+_{dR}. We have then the following properties (see [Fo2]).

Properties 8

1. *B^+_{dR} is a complete discrete valuation ring, with maximal ideal J_K and residue field C.*

2. *B^+_{dR} has a natural continue action of G_K, and $B_{dR}^{G_K} = K$.*

3. *Consider an element $\widetilde{\epsilon} = (\epsilon_n)_n$ of $\mathbb{Z}_p(1)$ as before. Consider $a_n \in A_{inf}$ such that $\theta(a_n) = \epsilon_n$. Then*

$$\nu(\epsilon) := \lim_{n\to\infty} a_n^{p^n}$$

exists and $\theta(\nu(\epsilon)) = 1$, so $\nu(\epsilon) - 1 \in J_K$. Hence, we have

$$\log(\nu(\epsilon)) := \sum_{m\geq 1} \frac{(\nu(\epsilon)-1)^m}{m} \in B^+_{dR},$$

which is a uniformizer of B_{dR}^{+} if $\widetilde{\epsilon}$ is a basis and gives an inclusion respecting the action of G_K

$$\log \circ \nu : \mathbb{Z}_p(1) \hookrightarrow B_{dR}^{+}.$$

We will denote by t this uniformizer in B_{dR}^{+}.

4. *B_{dR} has a natural filtration given by the valuation*

$$Fil^m B_{dR} := \{ \ b \in B_{dR} \ \| \ v(b) \geq m \ \}$$

and the graduate quocients are isomorphic to $C(-m)$:

$$gr^m B_{dR} = Fil^m B_{dR} / Fil^{m+1} B_{dR} \cong C(-m).$$

Now, the p-adic de Rham comparison theorem says that this field is the one we need for the comparison with de Rham cohomology.

Theorem 9 *Let X be a smooth and projective algebraic variety over a p-adic field K. Then, we have a canonical isomorphism*

$$C_{dR} : H^i(\overline{X}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{dR} \cong H^i_{dR}(X/K) \otimes_K B_{dR},$$

verifying that

1. *It is compatible with the action of G_K given at the left hand side by $\sigma \otimes \sigma$ and on the right hand side by $id \otimes \sigma$.*

2. *It is compatible with the filtration given at the left hand side by $id \otimes Fil^q$ and on the right hand side by $\sum_{a+b=q} Fil^a \otimes Fil^b$ (the convolution filtration). (recall that the filtration on the de Rham cohomology is the de Rham filtration given by the Hodge spectral sequence.)*

3. *It is compatible with Poincaré duality, Künneth formula, cicle map and Chern class maps.*

This theorem was proved by Faltings in 1988 ([Fa2]), and it is also a consequence of more general theorems, like C_{st} (Tsuji's theorem [Ts]) plus the theory of de Jong alterations (see section 8).

As a consequence, we can recover $H^i_{dR}(X/K)$ with it's filtration from the p-adic cohomology, since we have that

$$\left(H^i(\overline{X}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{dR}\right)^{G_K} \cong H^n_{dR}(X/K).$$

Now, it is natural to ask if it is possible to recover the p-adic cohomology of X from the de Rham cohomology of X plus some data; this is what we are going to explain in the next sections. This problem in fact has its origins in the so-called "mysterious functor".

5 The Grothendieck mysterious functor

This history begins probably with J. Tate's paper that we already mentioned [Ta]: Tate studies there abelian varieties with good reduction and also p-divisible groups over $\mathcal{O}_K$. He proves that the p-divisible group associated to an abelian scheme $\mathcal{A}$ (the smooth and proper model over $\mathcal{O}_K$ of an abelian variety A over K of good reduction) it is determined modulo isogeny by one of this two objects:

1. The Tate module $V_p(A) := T_p(A) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ of A, which is a $\mathbb{Q}_p$-vector space of dimension $2\dim(A)$ with an action of G_K.

2. The Dieudonné module $D(\mathcal{A}_k)$ of the reduction of $\mathcal{A}$, which is a K_0-vector space of dimension $2\dim(A)$ with a σ-linear isomorphism φ and a two steps filtration.

(K_0 means as usual the field of fractions of the Witt ring $W(k)$ of the residue field k.)

Grothendieck suggested then the following question: Is there an "algebraic" functor $\mathcal{D}$ that associates to every p-adic representation a K_0-vector space endowed with a σ-linear isomorphism, and such that, for any good reduction abelian variety A,

$$\mathcal{D}(T_p(A)) \cong D(\mathcal{A}_k) \ ?$$

And, viceversa, is there a functor on the other direction?

First thing we can do before answering this question is to generalize it to all varieties X with good reduction. In this case we have the p-adic cohomology in one side and the crystalline cohomology of the reduction on the other side.

Given a good reduction variety (smooth and projective) X over K, fix $\mathcal{X}$ a smooth and proper model of X over $\mathcal{O}_K$ and denote by Y its reduction to k. The variety Y is smooth and proper, so we can construct its crystalline cohomology $H^i_{cris}(Y/W)$ and

$$H^i_{cris}(Y/K_0) := H^i_{cris}(Y/W) \otimes_W K_0.$$

It is a K_0-vector space endowed with a σ-linear isomorphism φ, and such that we have a canonical isomorphism

$$H^i_{cris}(Y/K_0) \otimes_{K_0} K \cong H^i_{dR}(Y/K).$$

So, we can think that the crystalline cohomology is a way to define a Frobenius in the de Rham cohomology (plus a K_0-vector space structure).

So the question of Grothendieck can be generalize to the conjecture: The p-adic cohomology of a good reduction variety determines the crystalline cohomology of the reduction (plus the filtration when tensored by K) and viceversa.

This conjecture was made precise by Fontaine in [Fo1] by the construction of a ring of p-adic periods (or Barsotti-Tate ring) similar to the B_{dR}: B_{cris}.

6 The crystalline comparison theorem: B_{cris}

Fontaine's idea was to construct a field B_{cris} such that has all the structures we want: it is a $\mathbb{Q}_p$-vector space with an action of G_K such that $B_{cris}^{G_K} = K_0$ and it is a K_0-vector space (K_0-structure given by $B_{cris}^{G_K} = K_0$) with a σ-linear endomorphism

$$\varphi : B_{cris} \longrightarrow B_{cris}$$

and having a natural K-morphism

$$B_{cris} \otimes_{K_0} K \hookrightarrow B_{dR}$$

and hence endowing $B_{cris} \otimes_{K_0} K$ with a filtration, verifying that

$$\mathbb{Q}_p \cong B_{cris}^{\varphi=1} \cap Fil^0 \left(B_{cris} \otimes_{K_0} K\right).$$

Then, the functor we are looking for is, for any p-adic representation V,

$$D_{cris}(V) := (V \otimes_{\mathbb{Q}_p} B_{cris})^{G_K}$$

with quasi-inverse

$$V_{cris}(D) := (D \otimes_{K_0} B_{cris})^{\varphi=1} \cap Fil^0 \left((D \otimes_{K_0} K) \otimes (B_{cris} \otimes_{K_0} K)\right)$$

for any K_0-vector space D with a Frobenius and a filtration when tensored by K (called a filtered φ-module).

How this B_{cris} can be defined? There are two equivalent ways:

1. As before with the B_{dR}, but using the "universal p-adic formal DP-covering": i.e. $A_{cris} := W^{DP}(R_{\mathcal{O}})$ (the p-adic completion of the divided power envelope of $W(R_{\mathcal{O}})$), $B_{cris}^{+} := A_{cris}[1/p]$, and B_{cris} field of fractions of B_{cris}^{+}.

2. By using crystalline cohomology:

$$B_{cris}^{+} := \left(\lim_{\leftarrow} H^0_{cris}(\mathrm{Spec}(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}})/W_n)\right) \otimes_W K_0$$

The Frobenius element comes then from the Frobenius in the crystalline cohomology; the action of G_K comes from $\mathcal{O}_{\overline{K}}$; and the map to B_{dR} from the comparison between the de Rham and crystalline cohomology.

We have also that the element $t := \log(\widehat{\epsilon})$ belongs naturally to B^+_{cris}, so we have an inclusion $\mathbb{Q}_p(1) \hookrightarrow B^+_{cris}$. Then $B_{cris} = B_{cris}[1/t]$.

Now, the "Grothendieck mysterious functor problem" is made precise by the following theorem.

Theorem 10 *(Fontaine, Messing, Faltings, Kato, Tsuji) Let X be a smooth and projective variety over K with good reduction. Let Y be the reduction of (a model of) X. Then, there is a natural isomorphism*

$$C_{cris} : H^i(\overline{X}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{cris} \cong H^i_{cris}(Y/K_0) \otimes_{K_0} B_{cris}$$

compatible with the action of G_K, the Frobenius, the Filtration (and Poincaré duality, Künneth formula, cycle and Chern class maps).

Corollary 11 *We have natural isomorphisms*

$$D_{cris}(H^i(\overline{X}, \mathbb{Q}_p)) \cong H^i_{cris}(Y/K_0) \quad \textit{and} \quad V_{cris}(H^i_{cris}(Y/K_0)) \cong H^i(\overline{X}, \mathbb{Q}_p)$$

This result implies that the p-adic cohomology of a good reduction variety is "crystalline", in the sense that

$$\dim_{\mathbb{Q}_p}(V) = \dim_{K_0}(D_{cris}(V)).$$

This theorem was proved by Faltings in [Fa2], by Fontaine and Messing in [F-M] when the dimension of X is strictly less than p and $K = K_0$, by Kato and Messing in [K-M] when $2\dim(X) < p - 1$, and it is a consequence of the work of Tsuji [Ts].

7 Some words around the proof of C_{cris}

7.1 Faltings approach

Faltings proof [Fa2] of the crystalline comparison theorem uses his theory of almost étale extensions. He defines for every $\mathcal{X}$ over $\mathcal{O}_K$ a situs $\widetilde{X}$ and sheaves $\mathbb{Z}/p^m \otimes B^+/I^{[n]}$ for all n, m (B^+ should be think as a sheafication of B^+_{cris}). He defines

$$H^i(\widehat{X}, B^\wedge) := \left(\varprojlim_n \varprojlim_m H^i(\widetilde{X}, \mathbb{Z}/p^m \otimes B^+/I^{[n]}) \right) [1/p, 1/t]$$

(where t is a special element there, which corresponds to "our" $t := \log(\widehat{\epsilon})$)), and he proves that there is an isomorphism

$$H^i(\overline{X}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{cris} \cong H^i(\widehat{X}, B^\wedge)$$

compatible with all the structures.

After this, he constructs a natural transformation

$$H^i_{cris}(Y/K_0) \otimes_{K_0} B_{cris} \longrightarrow H^i(\widehat{X}, B^\wedge)$$

also compatible with all the structures.

From this one gets a map

$$H^i_{cris}(Y/K_0) \otimes_{K_0} B_{cris} \longrightarrow H^i(\overline{X}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{cris}$$

compatible with all the structures. One shows that this map is also compatible with Poincaré duality and with cycle class maps and Chern class maps. Once this is shown, the isomorphism is deduced easily.

7.2 Fontaine-Kato-Messing approach

Fontaine and Messing proof of the theorem [F-M] is by using the so-called syntomic cohomology. They construct complexes of sheaves for the syntomic topology $S^r_{n,X}$ for every r and n, as well as complex of sheaves $s_{n,X}(r)$ in the derived category $D(X_{et}, \mathbb{Z}/p^n\mathbb{Z})$ if $0 \leq r < p$. Then one shows that

$$H^i(\mathcal{X} \times_{\mathcal{O}_K} \mathcal{O}_{\overline{K}}, s_n(r)) \cong H^i(\overline{X}, \mathbb{Z}/p^n\mathbb{Z})$$

if $i \leq r < p$, by a delicate study of the p-adic vanishing cycles.

Now, by using the crystalline interpretation of B_{cris}, one gets maps

$$H^i(\mathcal{X} \times_{\mathcal{O}_K} \mathcal{O}_{\overline{K}}, s_n(r)) \longrightarrow H^i((\mathcal{X} \times_{\mathcal{O}_K} \mathcal{O}_{\overline{K}}/p^n)/W_n)^{\varphi=p^r}$$

By taking $\mathbb{Q} \otimes \varprojlim_n$, one finally get maps

$$H^i(\overline{X}, \mathbb{Q}_p) \longrightarrow (H^i_{cris}(Y/K_0) \otimes_{K_0} B_{cris})^{\varphi=1}$$

for any $i < p-1$ (and independent of the auxiliary r, $i \leq r < p-1$).

One shows that this maps are compatible with the Chern class of line bundles and the isomorphism when $2d < p-1$ is obtained. To obtain the proof in general one can do a "modification" like in Tsuji's proof (see section 9).

8 Semi-stable reduction and log-crystalline cohomology

Suppose now we have an smooth and projective variety X which does not have good reduction, but it has semi-stable reduction; that is, there exists a proper model $\mathcal{X}$ over $\mathcal{O}_K$ such that locally for the étale topology is of the form $\mathcal{O}_K[t_1, \ldots, t_n]/(t_1 \ldots t_r - \pi)$, where as usual π denotes a uniformizer of $\mathcal{O}_K$.

In this case Jannsen conjectured (by analogy with the ℓ-adic cohomology, $\ell \neq p$) the existence of K_0-estructure on the de Rham cohomology of X, with a Frobenius and a nilpotent map N, the monodromy. This map should be think also as an analog of the residue of the Gauss-Manin connection. The existence of this structure was shown by Hyodo and Kato (see [H-K]) by constructing the log-crystalline cohomology, which is some sort of crystalline cohomology with logarithmic poles (i.e. for log-schemes). We will denote by $H^i_{\text{log-cris}}(\mathcal{X})$ the log-crystalline cohomology of the reduction of $\mathcal{X}$ as log-scheme with the canonical log-structure, and with respect to the canonical log-structure on k (sometimes called Hyodo-Kato cohomology); it is a K_0-vector space, with a σ-linear isomorphism φ (the Frobenius), a nilpotent endomorphism N (the monodromy) verifying $N\varphi = p\varphi N$, and with an isomorphism

$$\rho_\pi : H^i_{\text{log-cris}}(\mathcal{X}) \otimes_{K_0} K \cong H^i_{dR}(X/K)$$

that depends on the choice of a uniformizer π of $\mathcal{O}_K$ (see [H-K]). So $H^i_{\text{log-cris}}(\mathcal{X})$ is a filtered (φ, N)-module.

On the other hand, he and Fontaine conjectured the existence of a ring of periods B_{st} giving a comparison isomorphism between this log-crystalline cohomology and the p-adic cohomology. Before stating the theorem, let's describe an example.

Example 12 *Let E be the Tate elliptic curve $\mathbb{G}_m/\pi^{\mathbb{Z}}$, where π is the uniformizer of $\mathcal{O}_K$. Now, E has semi-stable reduction and there is a map (as rigid-analytic varieties) $\mathbb{G}_m \to E$. The invariant differential dZ/Z gives then a differential $\omega \in H^1_{dR}(E/K)$.*

On the other hand, the Tate module $T_p(E)$ is generated (as $\mathbb{Z}_p$-module) by two elements: $\epsilon = (\epsilon_n)_n$ as before and $g = (g_n)_n$, where $g_n^p = g_{n-1}$ for all $n > 0$ and $g_0 = \pi$.

Now, the de Rham periods paring

$$\langle\ ,\ \rangle : H^1_{dR}(E/K) \times T_p(E) \longrightarrow B_{dR}$$

is done by

$$\langle \omega, \epsilon \rangle = \log(\nu(\epsilon))$$

and

$$\langle \omega, g \rangle = \log(b)$$

where

$$b := \lim_{n\to\infty} b_n^{p^n} \quad \textit{and} \quad \theta(b_n) = g_n$$

($\log$ *is the unique extension of the logarithm that verifies* $\log(\pi) = 0$*).*

See the paper [C-I] of Coleman and Iovita for details.

So, since we want the "periods" coming from semi-stable varieties, we need this element $u := \log(b)$. Define

$$B_{st} := B_{cris}[u]$$

(where u is a variable), with maps $N : B_{st} \to B_{st}$ the B_{cris}-derivation which is 0 on B_{cris} and $N(u) = 1$, and $\varphi : B_{st} \to B_{st}$ which is the Frobenius on B_{cris} and $\varphi(u) = pu$. Finally define a map

$$K \otimes_{K_0} B_{st} \longrightarrow B_{dR}$$

given by sending u to $\log(b)$. Observe that this last map depends on π!

Theorem 13 *(Kato,Tsuji,Faltings) Let X be a smooth projective variety over K with semi-stable reduction. There exist a natural isomorphism*

$$C_{st} : B_{st} \otimes_{\mathbb{Q}_p} H^i(\overline{X}, \mathbb{Q}_p) \cong B_{st} \otimes_{K_0} H^i_{\textit{log-cris}}(\mathcal{X})$$

compatible with φ, N and G_K, and compatible with filtrations after $B_{dR} \otimes_{B_{st}} -$.

Furthermore, the isomorphism is compatible with Poincaré duality, cycle map and Chern class maps.

This theorem was proved by K. Kato in [Ka] in the case that $2d < p - 1$, by T. Tsuji [Ts] in general and by G. Faltings in [Fa3] including the case of non-constant coeficients; also C. Breuil has a partial result in the cases of p-torsion étale cohomology [Br].

Before giving some ideas of the proof, we will explain some consequences of this result.

Corollary 14 *$H^i(\overline{X}, \mathbb{Q}_p)$ is a semi-stable representation, an one has that*

$$D_{st}(H^i(\overline{X}, \mathbb{Q}_p))) = \left(B_{st} \otimes_{K_0} H^i_{\textit{log-cris}}(\mathcal{X})\right)$$

Corollary 15 *Let X be a proper and smooth variety over K. Then $H^i(\overline{X}, \mathbb{Q}_p)$ is a potentially semi-stable representation of G_K.*

Proof. (Idea) We can suppose that X is geometrically irreducible. Then there exist a finite extension K'/K and an alteration $X' \to X_{K'}$ with X' semi-stable over $\mathcal{O}_{K'}$. Now $H^i(\overline{X}, \mathbb{Q}_p) \subseteq H^i(\overline{X'}, \mathbb{Q}_p)$ is a direct summand, so it is semi-stable as $G_{K'}$-representation.

There is also some generalizations. For example, T. Tsuji has generalized this result to simplicials schemes, hence deducing the result for any proper variety (which may have singularities).

Theorem 16 *Let X be a proper scheme over K. Then $H^i(\overline{X}, \mathbb{Q}_p)$ is potentially semi-stable.*

The proof of this result is publish in [Ts2].

9 Some words around the proof of C_{st}

The approach of Tsuji works similarily that for the proof of C_{cris}, but replacing the syntomic cohomology by the log-syntomic cohomology. Essentially, the idea is to use the "cohomology of log-syntomic complexes" as a link between $H^i(\overline{X}, \mathbb{Q}_p)$ and $H^i_{\text{log-cris}}(\mathcal{X})$. Suppose we have

$$H^i(\overline{X}, \mathbb{Q}_p) \xleftarrow{\sim} \mathbb{Q}_p(-r) \otimes H^i(\overline{\mathcal{X}}, s_{\mathbb{Q}_p}(r)) \longrightarrow (B_{st} \otimes_{K_0} H^i_{\text{log-cris}}(\mathcal{X})^{N=0,\varphi=1}.$$

One gets then by adjuntion a map

$$B_{st} \otimes_{\mathbb{Q}_p} H^i(\overline{X}, \mathbb{Q}_p) \longrightarrow B_{st} \otimes_{K_0} H^i_{\text{log-cris}}(\mathcal{X}).$$

If one shows that this map is compatible with Poincaré duality and Chern class maps, then this map is an isomorphism. Finally, one only need to verify the compatibility with the filtrations.

Tsuji's idea was, instead of considering the syntomic complexes of sheaves $S_n^{log}(r)$ (the logarithmic versions of Fontaine and Messing approach), to consider some "modifications" $\widetilde{S}_n(r)$ and then define

$$H^i(\overline{\mathcal{X}}, s_{\mathbb{Q}_p}(r)) := \mathbb{Q}_p \otimes \left(\lim_{\leftarrow} H^i(\overline{\mathcal{X}_n}, \widetilde{S}_n(r)) \right).$$

Now, by the log-crystalline interpretation of B_{st} in Kato's paper [Ka], one has a natural map

$$H^i(\overline{\mathcal{X}}, s_{\mathbb{Q}_p}(r)) \longrightarrow \left(B_{st} \otimes_{K_0} H^i_{log-crys}(\mathcal{X}) \right)^{N=0,\varphi=p^r}.$$

(it is a kind of Künneth formula).

Next, if $0 \leq i \leq r$, Tsuji defines maps

$$\mathcal{H}^i(\widetilde{S}_n(r)) \longrightarrow \bar{i}^* R^i \bar{j}_*(\mathbb{Z}/p^n\mathbb{Z})'(r)$$

where $\bar{i} : \overline{Y} \to \overline{\mathcal{X}}$ and $\bar{j} : \overline{X} \to \overline{\mathcal{X}}$ as usual, and

$$(\mathbb{Z}/p^n\mathbb{Z})' = \frac{1}{p^a a!}\mathbb{Z}_p(r) \otimes \mathbb{Z}/p^n\mathbb{Z},$$

where $a := \lceil \frac{r}{p-1} \rceil$.

By studying in detail the vanishing cycles (done by Bloch and Kato (in the good reduction case) and by Hyodo) and using some results of Kurihara, one shows that this maps have kernel and cokernel killed by some power of p, independent of n. One gets then the desired isomorphism

$$H^i(\overline{X}, \mathbb{Q}_p(r)) \xleftarrow{\sim} H^i(\overline{\mathcal{X}}, s_{\mathbb{Q}_p}(r))$$

for $i \leq r$, by using the proper base change theorem.

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