Varieties of characters and knot symmetries (I)

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INdAM Meeting
Geometric Topology in Cortona

Organization

Day 1 Character Varieties.
Definition and computation

Day 2 Distinguished curves for hyperbolic knots
(and of hyperbolic 3-manifolds of finite volume).

Day 3 Knot symmetries (joint with Luisa Paoluzzi)
(Distinguish in the variety of characters whether a knot symmetry has fixed points or not)

Motivation

- $\hom(\pi_1(M^3), (P)\mathrm{SL}_2\mathbb{C})/\mathrm{PSL}_2\mathbb{C}$ contains the space of hyperbolic structures on M^3, as $\mathrm{PSL}_2(\mathbb{C}) \cong \mathrm{Isom}(\mathbb{H}^3)$.
- Used in the proof of the hyperbolic Dehn filling theorem, hyperbolic cone manifolds, also for degeneration of structures, and many other situations...

- Culler-Shalen theory of surfaces associated to ideal points.
- A-polynomial.

- Representations have been used to distinguish knots, in Casson's invariant, ...

- Study the variety of characters as an algebraic object.
- Dynamics of group actions
- ...

Variety of representations

- $\Gamma = \langle \gamma_1, \ldots, \gamma_n \mid (r_j)_{j \in J} \rangle$ finitely generated group
- $\mathrm{SL}_2\mathbb{C} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \; a, b, c, d \in \mathbb{C}, \; ad - bc = 1 \right\}$

Def: $R(\Gamma) = \hom(\Gamma, \mathrm{SL}_2\mathbb{C})$

- It is an <u>affine algebraic set</u> (zero set of polynomials in $\mathbb{C}^{4n}$)

$$\begin{array}{rcl} R(\Gamma) & \to & \mathrm{SL}_2\mathbb{C} \times \overset{(n)}{\cdots} \times \mathrm{SL}_2\mathbb{C} \subset \mathbb{C}^{4n} \\ \rho & \mapsto & (\rho(\gamma_1), \ldots, \rho(\gamma_n)) \end{array}$$

The algebraic structure is independent of the presentation.

- The action by conjugation is algebraic

$$\begin{array}{rcl} \mathrm{SL}_2\mathbb{C} \times R(\Gamma) & \to & R(\Gamma) \\ A, \; \rho & \mapsto & \gamma \mapsto A\rho(\gamma)A^{-1} \end{array}$$

but the quotient $R(\Gamma)/\mathrm{PSL}_2(\mathbb{C})$ may be not Hausdorff.

Quotient in the "algebraic category"

- $\Gamma = \langle \gamma_1, \ldots, \gamma_n \mid (r_j)_{j \in J} \rangle$
  $R(\Gamma) = \hom(\Gamma, \mathrm{SL}_2\mathbb{C}) \subset \mathbb{C}^{4n}$
  $\mathbb{C}$-algebra of functions: $\mathbb{C}[R(\Gamma)] = \mathbb{C}[x_1, \ldots, x_{4n}]/I$
  where $I = \{p \in \mathbb{C}[x_1, \ldots, x_{4n}] \mid p(R(\Gamma)) = 0\}$
- Look for functions invariant by conjugation: $\mathbb{C}[R(\Gamma)]^{\mathrm{SL}_2\mathbb{C}}$
  For $\gamma \in \Gamma$, the <u>trace function</u> is

$$\begin{array}{rcl} \tau_\gamma \colon R(\Gamma) & \to & \mathbb{C} \\ \rho & \mapsto & \mathrm{tr}(\rho(\gamma)) \end{array}$$

Thm: (Procesi) $\exists \gamma_1, \ldots, \gamma_N \in \Gamma$ such that

$$\mathbb{C}[R(\Gamma)]^{\mathrm{SL}_2\mathbb{C}} = \langle \tau_{\gamma_1}, \ldots, \tau_{\gamma_N} \rangle$$

- Setting $y_j = \tau_{\gamma_j}$, $\mathbb{C}[R(\Gamma)]^{\mathrm{SL}_2\mathbb{C}} \cong \mathbb{C}[y_1, \ldots, y_N]/J$
  hence the zero set of J in $\mathbb{C}^N$ is the algebraic quotient.

Variety of characters

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trace functions	characters
$\begin{array}{rcl} \tau_\gamma \colon R(\Gamma) & \to & \mathbb{C} \\ \rho & \mapsto & \mathrm{tr}(\rho(\gamma)) \end{array}$	$\begin{array}{rcl} \chi_\rho \colon \Gamma & \to & \mathbb{C} \\ \gamma & \mapsto & \mathrm{tr}(\rho(\gamma)) \end{array}$

Def: The variety of characters is $X(\Gamma) = \{\chi_\rho \mid \rho \in R(\Gamma)\}$

- Since $\mathbb{C}[R(\Gamma)]^{\mathrm{SL}_2\mathbb{C}} = \langle \tau_{\gamma_1}, \ldots, \tau_{\gamma_N} \rangle$, $X(\Gamma)$ has a natural algebraic structure so that $\mathbb{C}[X(\Gamma)] = \mathbb{C}[R(\Gamma)]^{\mathrm{SL}_2\mathbb{C}}$.
- For $f \colon R(\Gamma) \to \mathbb{C}$ polynomial and $\mathrm{SL}_2\mathbb{C}$-invariant, there exists a unique $\tilde{f} \colon X(\Gamma) \to \mathbb{C}$ such that

$$\begin{tikzcd} R(\Gamma) \arrow[r, "f"] \arrow[d] & \mathbb{C} \\ X(\Gamma) \arrow[ur, dashed, "\tilde{f}"'] & \end{tikzcd}$$

- Conjugate representations have the same character:

$$\begin{array}{rcccl} R(\Gamma) & \to & R(\Gamma)/\mathrm{PSL}_2\mathbb{C} & \to & X(\Gamma) \\ \rho & \mapsto & [\rho] & \mapsto & \chi_\rho \end{array}$$

- What is the difference between $R(\Gamma)/\mathrm{PSL}_2\mathbb{C}$ and $X(\Gamma)$?

Irreducible representations

Def: $\rho \in R(\Gamma)$ is <u>reducible</u> if $\rho(\Gamma)$ has an invariant line in $\mathbb{C}^2$

equivalently, $\rho \in R(\Gamma)$ is reducible if $\rho(\Gamma) \sim \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$

Lemma: ρ is reducible iff $\operatorname{tr} \rho([\Gamma, \Gamma]) = 2$

Cor: $R^{red}(\Gamma)$ and $X^{red}(\Gamma)$ Zariski closed

Lemma: For $\rho, \rho' \in R(\Gamma)$ *irreducible*, ρ and ρ' are conjugate iff $\chi_\rho = \chi_{\rho'}$,
In fact, $\mathrm{PSL}_2\mathbb{C} \to R^{irr}(\Gamma) \to X^{irr}(\Gamma)$ is a principal bundle

- For reducible characters χ, there is a unique conjugacy class of diagonal representations ρ with $\chi_\rho = \chi$
- For every continuous $f\colon R(\Gamma)/\mathrm{PSL}_2\mathbb{C} \to H$ with H Hausdorff:

$$\begin{tikzcd} R(\Gamma)/\mathrm{PSL}_2\mathbb{C} \arrow[r, "f"] \arrow[d] & H \\ X(\Gamma) \arrow[ur, dashed, "\tilde{f}"'] & \end{tikzcd}$$

Free group of rank two

Lemma: For A, $B \in \mathrm{SL}_2\mathbb{C}$:

(i) $\mathrm{tr}(AB) = \mathrm{tr}(BA)$;
(ii) $\mathrm{tr}(A^{-1}) = \mathrm{tr}(A)$;
(iii) $\mathrm{tr}(AB) + \mathrm{tr}(A^{-1}B) = \mathrm{tr}(A)\,\mathrm{tr}(B)$.

For (iii), $A^2 - \mathrm{tr}(A)A + \mathrm{Id} = 0$ times $A^{-1}B$ and take trace.

Thm: (Fricke-Klein) Let $F_2 = \langle a, b \mid \rangle$, we have an isomorphism

$$(\tau_a, \tau_b, \tau_{ab}) \colon X(F_2) \xrightarrow{\cong} \mathbb{C}^3$$

- Setting $x = \tau_a$, $y = \tau_b$, and $z = \tau_{ab}$,

$$\mathbb{C}[X(F_2)] \cong \mathbb{C}[x, y, z].$$

Cor: $\forall \gamma \in F_2$, τ_γ is a (unique) polynomial on x, y, z

Groups of rank two

- $F_2 = \langle a, b \mid \rangle$, $x = \tau_a, y = \tau_b, z = \tau_{ab}$, $\quad \mathbb{C}[X(F_2)] = \mathbb{C}[x, y, z]$
- $\tau_\gamma = \tau_{\gamma^{-1}}$ • $\tau_{\gamma\mu} = \tau_{\mu\gamma}$ • $\tau_\gamma \tau_\mu = \tau_{\gamma\mu} + \tau_{\gamma^{-1}\mu}$
- Powers:
  - $\tau_{a^2} = \tau_a^2 - \tau_1 = x^2 - 2$
  - $\tau_{a^3} = \tau_a \tau_{a^2} - \tau_a = x^3 - 3x$
  - $\tau_{a^n} = \tau_a \tau_{a^{n-1}} - \tau_{a^{n-2}} =$Chebyshev pol. on x.
- $[a, b] = aba^{-1}b^{-1}$

$$\tau_{aba^{-1}b^{-1}} = \tau_{aba^{-1}}\tau_b - \tau_{aba^{-1}b} = y^2 - \tau_{aba^{-1}b}$$

$$\tau_{aba^{-1}b} = \tau_{ab}\tau_{a^{-1}b} - \tau_{a^2} = z\,\tau_{a^{-1}b} - x^2 + 2$$

$$\tau_{a^{-1}b} = \tau_a\tau_b - \tau_{ab} = xy - z$$

$$\tau_{[a,b]} = x^2 + y^2 + z^2 - xyz - 2$$

- $\mathbb{Z}^2 = F_2 = \langle a, b \mid [a, b] = 1 \rangle$. We have:

$$X(\mathbb{Z}^2) \cong \{(x, y, z) \in \mathbb{C}^3 \mid x^2 + y^2 + z^2 - xyz - 4 = 0\}$$

The trefoil knot exterior

- $\Gamma = \langle a, b \mid aba = bab \rangle$ $\qquad bab^{-1} = a^{-1}ba \Rightarrow \tau_a = \tau_b$
  Choose $x = \tau_a = \tau_b$ and $y = \tau_{ab^{-1}}$
- Write $\tau_{abab^{-1}} - \tau_{ab} = 0$:

$$\tau_{ab} = \tau_a\tau_b - \tau_{ab^{-1}} = x^2 - y$$
$$\tau_{abab^{-1}} = \tau_{ba}\tau_{ab^{-1}} - \tau_{b^2} = (x^2 - y)y - x^2 + 2$$

$$\tau_{abab^{-1}} - \tau_{ab} = x^2y - 2x^2 - y^2 + y + 2 = (y-2)(x^2 - y - 1) = 0$$

We have $X(\Gamma) \cong \{(x, y) \in \mathbb{C}^2 \mid (y-2)(x^2 - y - 1) = 0\}$

- $y = 2$: abelian (and reducible) representations
- $x^2 - y - 1 = 0$:
  $\tau_{ab} = x^2 - y = 1$ and $\tau_{aba} = \tau_a\tau_{ab} - \tau_b = x \cdot 1 - x = 0$
  $\Rightarrow \rho(ab)^3 = \rho(aba)^2 = -\mathrm{Id}$ (set $\alpha = ab$ and $\beta = aba$):

$$\Gamma \cong \langle \alpha, \beta, t \mid \alpha^3 = \beta^2 = t \rangle, \qquad [\alpha, t] = [\beta, t] = 1$$

Irreducible reps map the center to $\pm\,\mathrm{Id}$

The figure eight knot exterior (I)

- $\Gamma = \langle a, b \mid w\,a = b\,w \rangle$ with $w = ab^{-1}a^{-1}b$
  $x = \tau_a = \tau_b$, $y = \tau_{ab^{-1}}$
- $\tau_\omega - \tau_{b^{-1}wa} = (y-2)(y^2 - (x^2-1)y + x^2 - 1) = 0$
  - $y - 2 = 0$ consists of abelian representations
  - $y^2 - (x^2-1)y + x^2 - 1 = 0$ contains the holonomy of the hyperbolic structure and all deformations corresponding to Dehn filling.

  Hence
  $X(\Gamma) \cong \{(x,y) \in \mathbb{C}^2 \mid (y-2)(y^2 - (x^2-1)y + x^2 - 1) = 0\}$

- For a 2-bridge knot exterior in general
  $\Gamma = \langle a, b \mid w\,a = b\,w \rangle$ for some $w = w(a,b)$
  $X(\Gamma)$ is a plane curve.
  For a 2-bridge knot exterior, $X(\Gamma)$ may have arbitrarily many components (Ohtsuki-Riley-Sakuma)

The figure eight knot exterior (II)

- $\Gamma = \langle a, b \mid w\,a = b\,w \rangle$ with $w = ab^{-1}a^{-1}b$

- Write $\rho(a) = \begin{pmatrix} s & 1 \\ 0 & s^{-1} \end{pmatrix}$ and $\rho(b) = \begin{pmatrix} s & 0 \\ 2-y & s^{-1} \end{pmatrix}$
  $s + s^{-1} = \tau_a = \tau_b = x$ and $\tau_{a^{-1}b} = y$

- $\rho(w\,a) - \rho(b\,w) = (y^2 - (x^2-1)y + x^2 - 1)\begin{pmatrix} 0 & -1 \\ y-2 & 0 \end{pmatrix}$

  Component of irreducible representations (Whittemore 1973):

$$y^2 - (x^2-1)y + x^2 - 1 = 0$$

The figure eight knot exterior (III)

- It is a bundle over S^1 with fibre a punctured torus.
  $\Gamma = \langle \alpha, \beta, \mu \mid \mu\alpha\mu^{-1} = \alpha\beta, \mu\beta\mu^{-1} = \beta\alpha\beta \rangle$

$$1 \to F_2 = \langle \alpha, \beta \mid \rangle \to \Gamma \to \mathbb{Z} = \langle \mu | \rangle \to 1$$

  $\phi\colon F_2 \to F_2, \qquad \phi(\alpha) = \alpha\beta,\ \phi(\beta) = \beta\alpha\beta,$

- $X(F_2) \cong \mathbb{C}^3$ with coordinates $x_1 = \tau_\alpha$, $x_2 = \tau_\beta$, $x_3 = \tau_{\alpha\beta}$
  $\phi(x_1, x_2, x_3) = (x_1, x_2, x_3) \Leftrightarrow x_3 = x_1,\ x_1 + x_2 = x_1 x_2$

$$X(F_2)^\phi = \{\chi \mid \chi \circ \phi = \chi\} \cong \{(x_1, x_2) \in \mathbb{C}^2 \mid x_1 + x_2 = x_1 x_2\}$$

- $\mathrm{res}\colon X(\Gamma) \to X(F_2)^\phi$ is surjective and, for $\chi \in X(F_2)^\phi$:

$$\mathrm{res}^{-1}(\chi) = \begin{cases} \text{2 characters} & \text{if } \chi \text{ irreducible} \\ \text{1 irred. character} & \text{if } \chi \text{ reducible non trivial} \\ \text{a line of red. ch.} & \text{if } \chi \text{ trivial} \end{cases}$$

- $X^{irr}(\Gamma, \mathrm{PSL}_2\mathbb{C}) \cong X(F_2)^\phi$

The Whitehead link exterior

- $\Gamma = \langle a, b \mid aw = wa \rangle$ where $w = bab^{-1}a^{-1}b^{-1}ab$.
- Coordinates

$$x = \tau_a, \quad y = \tau_b, \quad z = \tau_{ab},$$

then $X(\Gamma) \cong \{(x, y, z) \in \mathbb{C}^3 \mid pq = 0\}$ with

$$\begin{cases} p = xy - (x^2 + y^2 - 2)z + xyz^2 - z^3 \\ q = x^2 + y^2 + z^2 - xyz - 4 \end{cases}$$

- $\{q = 0\} \cong X(\mathbb{Z}^2)$ abelian/reducible characters
- $\{p = 0\}$ Component containing a lift of the holonomy of the complete hyperbolic structure (and Dehn fillings)

$X(F_n),\ n \geq 3$

- F. González-Acuña and J.M. Montesinos-Amilibia (1993), based on H. Vogt (1889) and W. Magnus (1980)

- $F_3 = \langle a, b, c \mid \rangle$
  $(\tau_a, \tau_b, \tau_c, \tau_{ab}, \tau_{bc}, \tau_{ca})\colon X(F_3) \to \mathbb{C}^6$ is a branched covering

  There is one more coordinate algebraically dependent:

  τ_{abc} and τ_{acb} are the solutions of the equation
  $z^2 - Pz + Q = 0$ with

  $$P = \tau_a\tau_{bc} + \tau_b\tau_{ca} + \tau_c\tau_{ab} - \tau_a\tau_b\tau_c$$
  $$Q = \tau_a^2 + \tau_b^2 + \tau_c^2 + \tau_{ab}^2 + \tau_{bc}^2 + \tau_{ca}^2 + \tau_{ab}\tau_{bc}\tau_{ca} - \tau_a\tau_b\tau_{ab} - \tau_b\tau_c\tau_{bc} - \tau_c\tau_a\tau_{ca} - 4$$

  eg $\tau_{abc} + \tau_{acb} = P$ and $\tau_{abc}\tau_{acb} = Q$.

- $F_4 = \langle a, b, c, d \mid \rangle$
  τ_{abcd} is a polynomial on the traces of words on a, b, c and d of length ≤ 3 with coefficients in $\frac{1}{2}\mathbb{Z}$

$X(\Gamma)$, Γ *of finite type*

- $\Gamma = \langle \gamma_1, \ldots, \gamma_n \mid \{w_j\}_{j \in J} \rangle$

Thm: (González-Acuña and Montesinos-Amilibia)

$$X(\Gamma) \cong \{\chi \in X(F_n) \mid \chi_{\gamma_i \omega_j} = \chi_{\gamma_i},\ i = 1, \ldots, n, j \in J\}$$

- Fico and Montesinos give the explicit description of $X(F_n)$, defined with polynomials with coefficients in $\mathbb{Z}$
- $\forall \gamma \in F_n$, τ_γ is a polynomial in the traces of words of length ≤ 3 on the generators, with coefficients in $\frac{1}{2}\mathbb{Z}$.

$\Rightarrow$ Can take reduction mod p, $p \neq 2$ prime, of $X(\Gamma)$, to compute the variey of characters in an algebraicly closed field of characteristic p.

Varieties of characters and knot symmetries (II)

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Organization

Day 1 Character Varieties.
Definition and computation

Day 2 Canonical component for hyperbolic 3-manifolds of finite volume (...and further components).

Day 3 Knot symmetries (joint with Luisa Paoluzzi)
(Action of knot symmetries on $X(S^3 - K)$)

Manifolds of finite volume

- M^3 orientable & (complete) hyperbolic. Its volume is finite iff
$$\mathrm{Ends}(M^3) \cong (T_1^2 \sqcup \cdots \sqcup T_k^2) \times [0, \infty)$$
with $T_i^2 \cong S^1 \times S^1$, and each end is a cusp.
- By defn, a link $L \subset S^3$ is hyperbolic if $S^3 - L$ is hyperbolic (hence of finite volume).
- If M^3 is hyperbolic, then hol: $\pi_1 M^3 \to \mathrm{Isom}^+(\mathbb{H}^3) \cong \mathrm{PSL}_2\mathbb{C}$ lifts to ρ_0 (Culler, Thurston):

$$\begin{tikzcd} & \mathrm{SL}_2\mathbb{C} \arrow[d] \\ \pi_1(M^3) \arrow[ur, dashed, "\rho_0"] \arrow[r, "\mathrm{hol}"] & \mathrm{PSL}_2\mathbb{C} \end{tikzcd}$$

Use that M^3 is parallelizable, and $SO(3) \to \mathrm{PSL}_2\mathbb{C} \to \mathbb{H}^3$ is the frame bundle on $\mathbb{H}^3$

- Results in this talk work also for $X(M^3, \mathrm{PSL}_2\mathbb{C})$

Canonical component

- Let M^3 be hyperbolic, orientable, of finite volume, and with $k > 0$ ends (that are cusps), $\rho_0 = \widetilde{\mathrm{hol}}\colon \pi_1 M^3 \to \mathrm{SL}_2\mathbb{C}$.

Thm: (Thurston) χ_{ρ_0} is a smooth point of $X(M^3)$, and the component $X_0(M^3)$ that contains χ_{ρ_0} has dimension k.

Def: $X_0(M^3)$ is called the <u>*canonical component*</u> of $X(M^3)$

- Perhaps not unique?
  - If there are several components, then they are isomorphic, because other lifts of hol are $(-1)^\epsilon \rho_0$ for $\epsilon\colon \pi_1 M^3 \to \mathbb{Z}/2\mathbb{Z}$
  - Isometries in $\mathrm{PSL}_2\mathbb{C}$ preserve orientation, so $\chi_{\rho_0} \neq \overline{\chi_{\rho_0}}$.
- Characters close to χ_{ρ_0} are holonomies of incomplete hyperbolic structures (used for hyperbolic Dehn filling).
- For a hyperbolic knot exterior $S^3 - K$,

$$X_0(K) := X_0(S^3 - K) \text{ is a curve.}$$

Furthermore the A-polynomial is nontrivial

Example: cone manifold structures on the fig 8 knot

- The n-cyclic branched covering of S^3 branched on the figure eight knot is:

$$\begin{cases} \text{hyperbolic} & \text{if } n \geq 4 \\ \text{Euclidean} & \text{if } n = 3 \\ \text{spherical} & \text{if } n = 2 \end{cases}$$

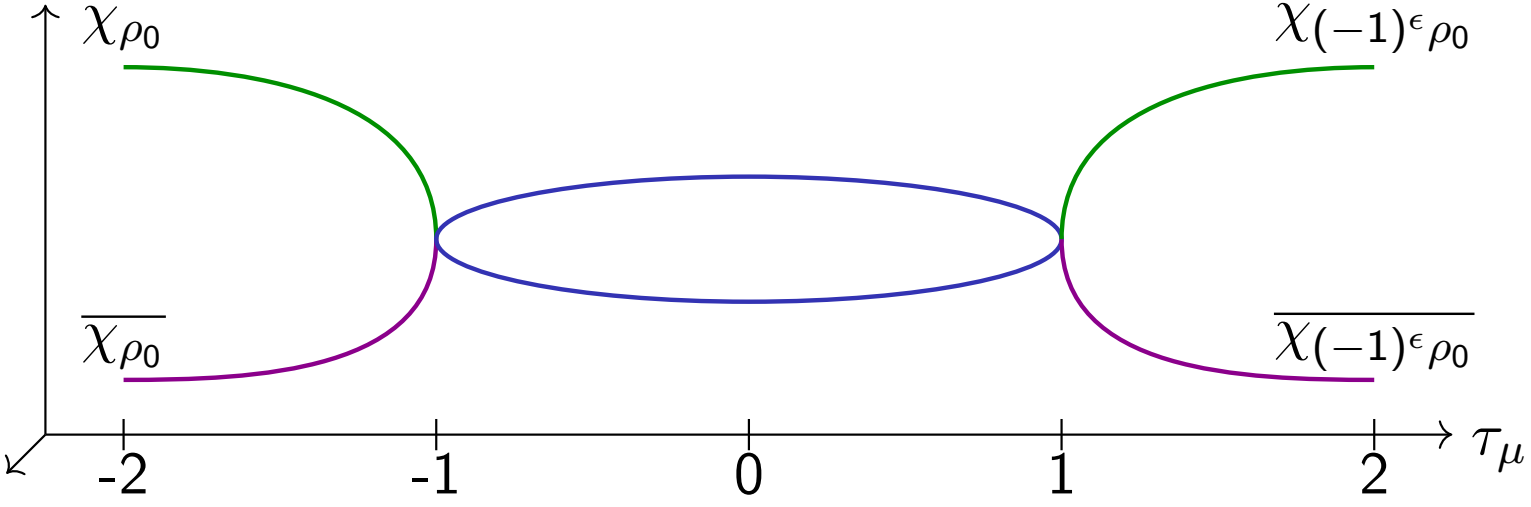
- Setting $\alpha = \frac{2\pi}{n}$, particular case of: There exists a family of cone manifold structures on S^3, with singular locus the figure eight knot, and with cone angle $\alpha \in (0, \frac{4\pi}{3})$ that are

$$\begin{cases} \text{hyperbolic} & \text{if } \alpha < \frac{2\pi}{3} \\ \text{Euclidean} & \text{if } \alpha = \frac{2\pi}{3} \\ \text{spherical} & \text{if } \frac{2\pi}{3} < \alpha < \frac{4\pi}{3} \end{cases}$$

- As the trace of a rotation of angle α is $\pm 2\cos(\frac{\alpha}{2})$, look at characters s.t. $\chi(\mu) \in [-2, 2]$ for a meridian $\mu \in \pi_1(S^3 - K)$

Transition geometry in the character variety

$K \subset S^3$ fig eight knot, $\mu \in \Gamma = \pi_1(S^3 - K)$ a meridian. Since $\chi(\mu) = \pm 2 \cos \frac{\alpha}{2}$, look at the set $\{\chi \in X(\Gamma), \chi(\mu) \in [-2, 2]\}$



- $1 < |\chi(\mu)| < 2$ are holonomies of hyperbolic cone structures (in magenta the complex conjugate, ie opposite orientation)
- The blue curve consists of characters in $SU(2)$.
  Pairs (χ, χ') with $-1 < \chi(\mu) = \chi'(\mu) < 1$ are lifts of spherical holonomies as $\widetilde{SO(4)} \cong SU(2) \times SU(2)$
- $\chi(\mu) = \pm 1$ Euclidean collapse.

Back to the canonical component thm

Thm: If M^3 hyperbolic of finite volume, with one single end, ρ_0 lift of the holonomy, then χ_{ρ_0} is a smooth point of $X(M^3)$, and the component $X_0(M^3)$ that contains χ_{ρ_0} is a curve

Proof based on:

(a) for each component $Y \subset X(M^3)$ containing an irreducible character χ so that $\chi(\pi_1 \partial \overline{M^3}) \not\subset \{\pm 2\}$, $\Rightarrow$ dim $Y \geq 1$

(b) The dimension of the Zariski tangent space at χ_0 is dim $T^{Zar}_{\chi_0} X(\Gamma) = 1$

- (a) is a lower bound of the dimension. It is proved in Thurston notes (sufficient for hyperbolic Dehn filling)
- (b) gives an upper bound of the dimension.
- (a)+(b) $\Rightarrow$ smoothness.
  (the dimension of the Zariski tangent space is an upper bound for the dimension, with equality precisely at smooth points)

Thurston's lower bound (one cusp case)

Thm: Let $\chi_\rho \in X(M^3)$ be irreducible so that $\chi(\pi_1 \partial \overline{M^3}) \not\subset \{\pm 2\}$.
For each component $Y \subset X(M^3)$ containing χ, $\dim Y \geq 1$

Proof: Chose a simple closed curve α with base point in $\partial \overline{M^3}$, s.t.

1. $\mathrm{tr}(\rho(\alpha)) \neq \pm 2$
2. the restriction $\rho|_{\langle \alpha, \pi_1(\partial \overline{M^3})\rangle}$ is irreducible.

Set $M' = \overline{M^3} - \mathcal{N}(\alpha)$. As α is simple:

$$\chi(M') = \tfrac{1}{2}\chi(\partial M') = -1.$$

$\Rightarrow$ M' has the homotopy type of a 2-dim CW-complex with
1 zero-cell, r one-cells and $(r-2)$ two-cells:

$\Rightarrow$ $\pi_1 M'$ has presentation with r generators and $r-2$ relations

$\Rightarrow$ $\dim R(\pi_1 M') \geq 6 \Rightarrow \dim X(\pi_1 M') \geq 3$

- View $\alpha \in \pi_1(\partial M')$ and chose $\beta \in \pi_1(\partial M')$ a meridian around α, so that $[\alpha, \beta]$ bounds a disk in $\partial \overline{M^3}$

Claim: For $\rho' \in R(M')$ in a neighborhood of ρ, the conditions

$$\mathrm{tr}(\rho'(\beta)) = \mathrm{tr}(\rho'([\alpha, \beta])) = 2 \text{ imply } \rho'(\beta) = \mathrm{Id}$$

- The claim and $\dim X(\pi_1 M') \geq 3$ yield the theorem.

Zariski tangent space

Def: For $V = \{x \in \mathbb{C}^N \mid p_1(x) = \cdots = p_r(x) = 0\}$
the Zariski tangent space at $x \in V$ is
$T_x^{Zar} V = \{v \in \mathbb{C}^N \mid p_1(x + \varepsilon v), \ldots, p_r(x + \varepsilon v) \in O(\varepsilon^2)\}$

- $\dim T_x^{Zar} V \geq \dim(\text{component of } V \text{ containing } x)$
  with equality iff x smooth point of V (and one single comp)

Def: Crossed morphisms

$$Z^1 = \{\theta \colon \Gamma \to \mathfrak{sl}_2\mathbb{C} \mid \theta(\gamma\mu) = \theta(\gamma) + \rho(\gamma)\theta(\mu)\rho(\gamma^{-1})\}$$

Inner morphisms:

$$B^1 = \{\theta_a \in Z^1 \mid \theta_a(\gamma) = \rho(\gamma)a\rho(\gamma^{-1}) - a, \forall \gamma \in \Gamma\}$$

$H^1(\Gamma, \mathrm{Ad}\,\rho) = Z^1/B^1$

Thm: (Weil)
$$\begin{array}{ccc} Z^1 & \xrightarrow{\cong} & T_\rho^{Zar} R(\Gamma) \\ \theta & \longmapsto & \gamma \mapsto (\mathrm{Id} + \varepsilon\theta(\gamma))\rho(\gamma) = \rho_\varepsilon(\gamma) \end{array}$$

$B^1 \cong$ orbit by conjugation

Cor: (Weil) For χ_ρ irreducible, $H^1(\Gamma, \mathrm{Ad}\,\rho) \cong T_{\chi_\rho}^{Zar} X(\Gamma)$

Thm: (Garland) M^3 hyperbolic, orient., finite vol, and k cusps.
For $\rho_0 \in R(M^3)$ lift of the holonomy, $H^1(\pi_1 M^3, \mathrm{Ad}\,\rho_0) \cong \mathbb{C}^k$

Cohomology thm

Thm: (Garland) M^3 hyperbolic, orient, finite vol, and k cusps.
For $\rho_0 \in R(M^3)$ lift of the holonomy, $H^1(\pi_1 M^3, \mathrm{Ad}\,\rho_0) \cong \mathbb{C}^k$

- Use de Rham cohomology:
  - Flat bundle $\mathfrak{sl}_2\mathbb{C} \to E_{\mathrm{Ad}\,\rho_0} \to M$ where
    $E_{\mathrm{Ad}\,\rho_0} = \widetilde{M} \times \mathfrak{sl}_2\mathbb{C}/\sim$ with $(x, a) \sim (\gamma x, \mathrm{Ad}_{\rho_0(\gamma)}\, a)$
  - $\Omega^p(M, E_{\mathrm{Ad}\,\rho_0}) = \Gamma(\wedge^p T^*M \otimes E_{\mathrm{Ad}\,\rho_0}) = \text{vector valued p-forms}$
  - $d\colon \Omega^p(M, E_{\mathrm{Ad}\,\rho_0}) \to \Omega^{p+1}(M, E_{\mathrm{Ad}\,\rho_0})$ yields de Rham cohom,

and $H^*_{dR}(M^3, E_{\mathrm{Ad}\,\rho_0}) \cong H^*(\pi_1 M^3, \mathrm{Ad}\,\rho_0)$

Thm: (...? 1960's) closed L^2-forms are exact.

Set $U = M^3 - \{\text{Compact core}\} \cong \bigcup T_i^2 \times (0, +\infty)$
hence

$$H^1(M, U, \mathrm{Ad}\,\rho_0) \xrightarrow{0} H^1(M, \mathrm{Ad}\,\rho_0) \to H^1(U, \mathrm{Ad}\,\rho)$$

and $\dim H^1(M, \mathrm{Ad}\,\rho_0) = \sum_i \dim \frac{1}{2} H^1(T_i^2, \mathrm{Ad}\,\rho_0) = \sum_i 1 = k$.

Galois conjugates

- M^3 hyperbolic, orientable, finite volume,
  $\rho_0 \colon \pi_1 M^3 \to \mathrm{SL}_2\mathbb{C}$ lift of the holonomy.
- $\rho_0(\pi_1 M^3) \subset \mathrm{SL}_2\mathbb{K}$, for $\mathbb{K}$ a number field (Vinberg).
  May assume that the finite extension $\mathbb{K}|\mathbb{Q}$ is Galois.

Rmk: For each $\sigma \in \mathrm{Galois}(\mathbb{K})$,
$\chi_{\rho_0^\sigma}$ is a smooth point of $X(M^3)$ and the unique component that contains $\chi_{\rho_0^\sigma}$ has dimension the number of cusps.

Why? View the space of crossed morphisms as a linear space
$Z^1 = \{\theta \colon \pi_1 M^3 \to \mathfrak{sl}_2\mathbb{C} \mid \theta(\gamma\mu) = \theta(\gamma) + \rho_0(\gamma)\theta(\mu)\rho_0(\gamma^{-1})\}$

From $\Gamma = \pi_1 M^3 = \langle \gamma_1, \ldots, \gamma_n \mid r_1, \ldots, r_m \rangle$, embed

$$\begin{array}{rcl} Z^1 & \hookrightarrow & \mathfrak{sl}_2\mathbb{C} \times \cdots \times \mathfrak{sl}_2\mathbb{C} \\ \theta & \mapsto & (\theta(\gamma_1), \ldots, \theta(\gamma_n)) \end{array}$$

The image is the kernel of a linear map given by $r_1, \ldots, r_m$, with coefficients in $\mathbb{K}$.
So $\dim(Z^1(\pi_1 M^3, \mathrm{Ad}\,\rho_0^\sigma)) = \dim(Z^1(\pi_1 M^3, \mathrm{Ad}\,\rho_0))$.

Further components

- Have shown $X(M^3)$ has the canonical component and the abelian ones. Look for further components.

Thm: (Ohtsuki-Riley-Sakuma) Among 2-bridge knots K and K', the set $\{f : \pi_1(S^3 - K) \to \pi_1(S^3 - K') \text{ epimorphism}\}/\{\text{inner}\}$ may have arbitrarily large cardinality.

Cor: (Ohtsuki-Riley-Sakuma) For 2-bridge knots K, $X(K)$ can have arbitrarily many components

Rmk: For 2-bridge knots K, components of $X(K)$ are plane curves:

- As $\pi_1(S^3 - K) = \langle a, b \mid a\, w(a,b) = w(a,b)\, b \rangle$, use coordiates $x = \tau_a = \tau_b$ and $y = \tau_{ab^{-1}}$. Hence $X(K) \subset \mathbb{C}^2$
- By Thuston's estimate, the dim of components is at least 1.
- Cannot have dim 2.

Thm: (Paoluzzi-P, using an idea of Riley) Given $m \in \mathbb{N}$, $m \geq 2$, for a Montesinos knot K, $X(K)$ may have arbitrarily many components of dim m.

Varieties of characters and knot symmetries (III)

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INdAM Meeting
Geometric Topology in Cortona

Recap

- $R(M) = \hom(\pi_1 M, \mathrm{SL}_2\mathbb{C})$. $\qquad \chi_\rho(\gamma) = \mathrm{tr}(\rho(\gamma))$
  $X(M) = \{\chi_\rho \colon \pi_1 M \to \mathbb{C} \mid \rho \in R(M)\} \cong R(M)//\mathrm{SL}_2\mathbb{C}$

- M^3 hyperbolic, orientable and of finite volume.

  The *canonical component* $X_0(M^3)$
  has dimension the number of cusps.

- $X(M^3)$ may have other irreducible components.

- Fico-Montesinos: coeffs. of the polynomial equations in $\frac{1}{2}\mathbb{Z}$, so that its reduction mod p is the variety of $\mathrm{SL}_2\mathbb{F}$-characters, for $\mathbb{F} =$ algebraically closed field of characteristic $p \neq 2$

- For a knot $K \subset S^3$, $X(K) := X(S^3 - K)$

Knot symmetries

Let K be a knot in S^3 and $\psi\colon (S^3, K) \to (S^3, K)$ a diffeomorphism of finite order p, that preserves the orientation of S^3.
Either $Fix(\psi) = \emptyset$ or $Fix(\psi) \cong S^1$ unknotted in S^3

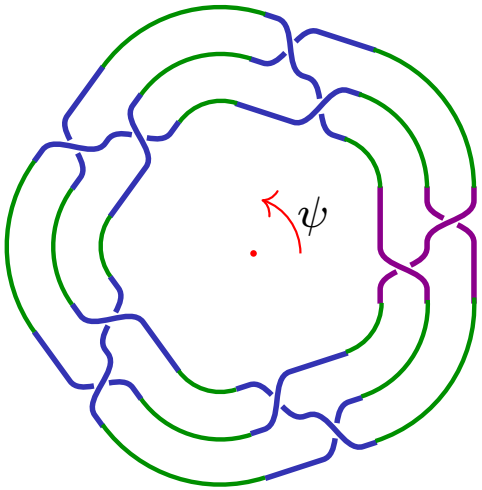
Def: ψ is said to be:

- a *free symmetry* if the group $\langle\psi\rangle \cong \mathbb{Z}/p\mathbb{Z}$ acts freely.
- a *periodic symmetry* if $Fix(\psi)$ is a circle disjoint from K.
- a *strong inversion* if ψ has order 2 and $|Fix(\psi) \cap K| = 2$.
- a *pseudo–periodic symmetry* otherwise.

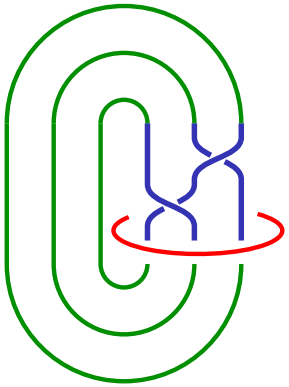
Assume that ψ has order a prime $p \neq 2$.
Hence ψ is either free or periodic.

Goal: show different behavior between periodic and free symmetries in the variety of characters.

Example: periodic (non-free) symmetry



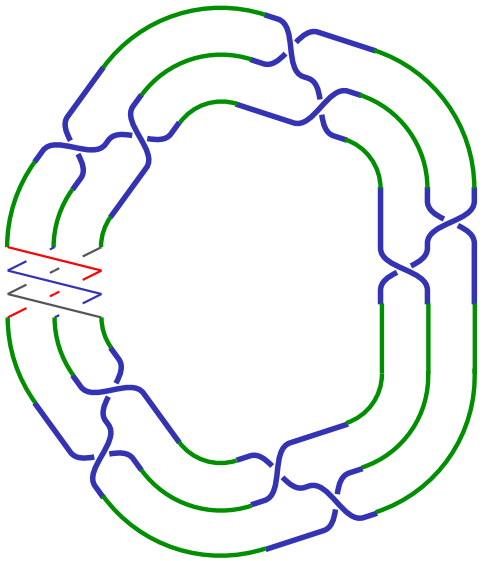
$\mathcal{O} = (S^3 - K)/\psi$



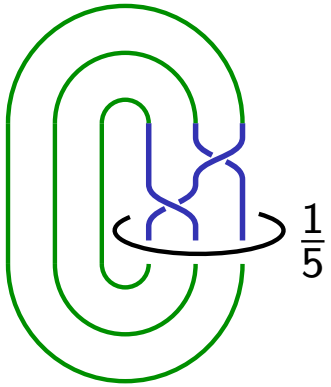
$\Sigma = Fix(\psi)/\psi$

$(K \cup Fix(\psi))/\psi = 6^2_2$

Example: free symmetry



$L(5,1) = S^3/\psi$



Components of the variety of characters

- $K \subset S^3$ hyperbolic knot
- $\psi\colon (S^3, K) \to (S^3, K)$ symmetry of prime order $p \neq 2$.

$$X(K)^\psi = \{\chi \in X(K) \mid \chi \circ \psi_* = \chi\}$$

Thm: (Paoluzzi-P) If ψ is periodic then $X(K)^\psi$ has at least $\frac{p-1}{2}$ components that are also components of $X(K)$.

Rmks:
- *(a)* For each prime $p > 4$ there is a knot K_p with a free symmetry ψ of order p so that $X(K_p)^\psi$ has at most 20 components
- *(b)* When reducing mod p, all the components of the theorem collapse to a single one.
- *(c)* If we look at components of $X(K)$, without ψ-invariance, many further components may appear, for ψ either free or periodic.

Components of the variety of characters

- $K \subset S^3$ hyperbolic knot
- $\psi\colon (S^3, K) \to (S^3, K)$ symmetry of prime order $p \neq 2$.

$$X(K)^\psi = \{\chi \in X(K) \mid \chi \circ \psi_* = \chi\}$$

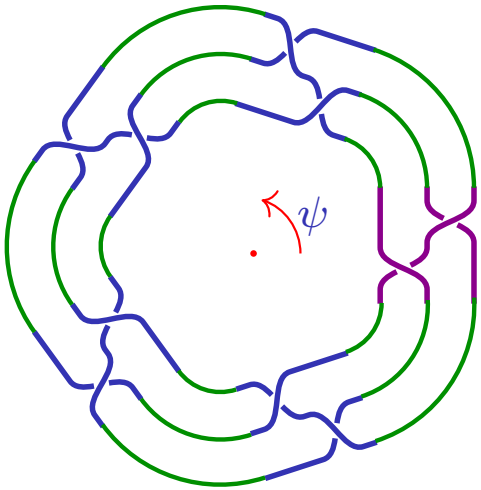
Thm: (Paoluzzi-P) If ψ is periodic then $X(K)^\psi$ has at least $\frac{p-1}{2}$ components that are also components of $X(K)$.

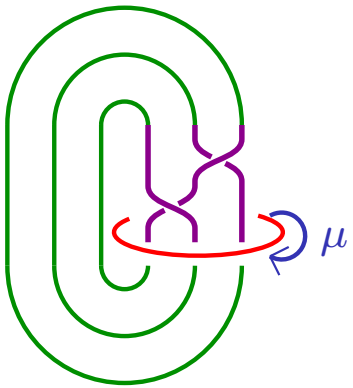
- The proof has 3 steps. Set $M = S^3 - K$, $\mathcal{O} = M/\psi$.
  (1) The restriction map res: $X^{irr}(\mathcal{O}) \to X^{irr}(M)^\psi$ is a bijection
  (2) Find several components for $X(\mathcal{O})$
  (3) Use (1) + (2) to find several components for $X(M)^\psi$

Step 1: Extending ψ-invariant characters

$M = S^3 - K$ $\qquad$ $\mathcal{O} = M/\psi$ $\qquad$ $\pi_1 M \to \pi_1 \mathcal{O} \to \mathbb{Z}/p\mathbb{Z}$

$\mu \leftarrow 1$





$$\begin{aligned} \psi_* \colon \pi_1 M &\to \pi_1 M \\ \gamma &\mapsto \mu\gamma\mu^{-1} \end{aligned}$$

$$\pi_1 \mathcal{O} = \langle \pi_1 M, \mu | \mu^p = 1, \mu\gamma\mu^{-1} = \psi_*\mu, \ \forall \gamma \in \pi_1 M \rangle$$

Lemma The restriction map res: $X^{irr}(\mathcal{O}) \to X^{irr}(M)^\psi$ is a bijection

Proof Want to extend $\chi_\rho \in X^{irr}(M)^\psi$ to $\pi_1\mathcal{O}$ in a unique way.
Since $\chi_{\rho\circ\psi_*} = \chi_\rho \ \Rightarrow \exists A \in \mathrm{SL}_2\mathbb{C}$ s.t. $\rho(\psi_*(\gamma)) = A\rho(\gamma)A^{-1}$.
A is unique up to sign. If furthermore we require $A^p = \mathrm{Id} \Rightarrow$
A exists and unique because p odd $\Rightarrow$ Set $\rho(\mu) = A$.
(need to show also that restriction of irreducible is irreducible)

Step 2: Finding components for $X(\mathcal{O})$

- $\mathcal{O}$ is hyperbolic and hol: $\pi_1\mathcal{O} \to \mathrm{PSL}_2\mathbb{C}$ lifts to $\rho_0 \colon \pi_1\mathcal{O} \to \mathrm{SL}_2\mathbb{C}$ because p is odd.
- $\rho_0(\pi_1\mathcal{O}) \subset \mathrm{SL}_2\mathbb{K}$, for $\mathbb{K}$ a number field (Vinberg). May assume that the finite extension $\mathbb{K}|\mathbb{Q}$ is Galois.
- Since $\mu^p = 1$, then $\mathrm{tr}(\rho_0(\mu)) = -2\cos\frac{\pi}{p}$ and $\{\tau_\mu(\rho_0^\sigma) | \sigma \in \mathrm{Galois}(\mathbb{K})\} = \{-2\cos\frac{\pi r}{p} | r = 1, 3, 5, \ldots, p-2\}$.
- $\tau_\mu = -2\cos\frac{\pi r}{p}$ distinguishes $\frac{p-1}{2}$ components $Y_1, \ldots, Y_{\frac{p-1}{2}}$ of $X(\mathcal{O})$ that contain different Galois conjugates $\chi_{\rho_0^\sigma}$ of χ_{ρ_0}
- By Garland's Thm, since $\mathcal{O}$ has a cusp, $H^1(\pi_1\mathcal{O}, \mathrm{Ad}\,\rho_0) = \mathbb{C}$
  $\Rightarrow H^1(\pi_1\mathcal{O}, \mathrm{Ad}\,\rho_0^\sigma) = \mathbb{C}$, $\forall \sigma \in \mathrm{Galois}(\mathbb{K})$
  $\Rightarrow$ each Y_r is a curve.
- Next step: using that res: $X^{irr}(\mathcal{O}) \to X^{irr}(M)^\psi$ is a bijection, the curves $Y_1, \ldots, Y_{\frac{p-1}{2}}$ yield different components of $X(M)^\psi$.

Step 3: components for $X(M)^\psi$

We know that $\mathrm{res}\colon X^{irr}(\mathcal{O}) \to X^{irr}(M)^\psi$ is a bijection. We have $Y_1, \ldots, Y_{\frac{p-1}{2}}$ components of $\overline{X^{irr}(\mathcal{O})}$ that are curves.

$$\begin{array}{ccccc} X(\mathcal{O}) & \overset{\mathrm{res}}{\to} & X(M)^\psi & \subset & X(M) \\ Y_1 & & & & \\ \vdots & & W_i = \overline{\mathrm{res}(Y_i)} & & \\ Y_{\frac{p-1}{2}} & & & & \end{array}$$

- Since $Y_i = Y_i^{irr} \cup$(a finite set), $W_i = \mathrm{res}(Y_i) \cup$(a finite set) and $W_1, \ldots, W_{\frac{p-1}{2}}$ are different components of $X(M)^\psi$
- As $\dim H^1(\pi_1 M, \mathrm{Ad}\,\rho_0^\sigma) = \dim H^1(\pi_1 M, \mathrm{Ad}\,\rho_0) = 1$, $W_1, \ldots, W_{\frac{p-1}{2}}$ are also components of $X(M)$. $\square$

Characteristic p

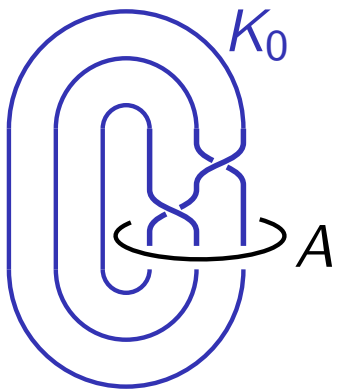
Rmk: Let $\mathbb{F}$ be an algebraically closed field of characteristic p.
As an element of $\mathrm{SL}_2\mathbb{F}$ of order p is conjugate to

$$\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix},$$

all these components $W_1, \ldots, W_{\frac{p-1}{2}}$ in the theorem collapse to a single one mod p.

Free symmetries: a family of examples

Link 6^2_2



Consider $\frac{q}{p}$-Dehn filling on A and take the covering $(S^3, K_{\frac{q}{p}}) \to (L(p,q), K_0)$ where $p > 4$ prime and p, q coprime $\psi\colon (S^3, K_{\frac{q}{p}}) \to (S^3, K_{\frac{q}{p}})$ free symmetry of order p

- $X(6^2_2)$ has two components: $X_0(6^2_2)$ and $X^{ab}(6^2_2)$.
  - The map $X_0(6^2_2) \to X(\partial\mathcal{N}(A))$ is dominant and its generic fibre is finite.
  - $\forall \gamma \in \pi_1\partial\mathcal{N}(A)$ primitive, $\{\tau_\gamma = 2\}$ is a line in $X(\partial\mathcal{N}(A))$,

$\Rightarrow X(L(p,q) - K_0)$ has at most C components, C uniform

$\Rightarrow X(S^3 - K_{\frac{q}{p}})^\psi$ has at most C components (C uniform) and ψ has order p (that can be any prime > 4).

Thanks for your attention!