Varieties of characters
and knot symmetries (I)

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Day 1  Character Varieties.
       Definition and computation

Day 2  Distinguished curves for hyperbolic knots
       (and of hyperbolic 3-manifolds of finite volume).

Day 3  Knot symmetries (joint with Luisa Paoluzzi)
       (Distinguish in the variety of characters whether a knot
        symmetry has fixed points or not)
Motivation

- $\text{hom}(\pi_1(M^3), (P)\text{SL}_2(\mathbb{C})/\text{PSL}_2(\mathbb{C})$ contains the space of hyperbolic structures on $M^3$, as $\text{PSL}_2(\mathbb{C}) \cong \text{Isom}(\mathbb{H}^3)$.
- Used in the proof of the hyperbolic Dehn filling theorem, hyperbolic cone manifolds, also for degeneration of structures, and many other situations...

- Culler-Shalen theory of surfaces associated to ideal points.
- A-polynomial.

- Representations have been used to distinguish knots, in Casson’s invariant, ...

- Study the variety of characters as an algebraic object.
- Dynamics of group actions
- ...
Variety of representations

- $\Gamma = \langle \gamma_1, \ldots, \gamma_n \mid (r_j)_{j \in J} \rangle$ finitely generated group
- $\text{SL}_2 \mathbb{C} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \bigg| a, b, c, d \in \mathbb{C}, \ ad - bc = 1 \right\}$

Def: $R(\Gamma) = \text{hom}(\Gamma, \text{SL}_2 \mathbb{C})$

- It is an affine algebraic set (zero set of polynomials in $\mathbb{C}^{4n}$)

\[ R(\Gamma) \rightarrow \text{SL}_2 \mathbb{C} \times \cdots \times \text{SL}_2 \mathbb{C} \subset \mathbb{C}^{4n} \]
\[ \rho \mapsto (\rho(\gamma_1), \ldots, \rho(\gamma_n)) \]

The algebraic structure is independent of the presentation.

- The action by conjugation is algebraic

\[ \text{SL}_2 \mathbb{C} \times R(\Gamma) \rightarrow R(\Gamma) \]
\[ A, \rho \mapsto \gamma \mapsto A \rho(\gamma) A^{-1} \]

but the quotient $R(\Gamma)/\text{PSL}_2(\mathbb{C})$ may be not Hausdorff.
Quotient in the “algebraic category”

- $\Gamma = \langle \gamma_1, \ldots, \gamma_n \mid (r_j)_{j \in J} \rangle$
  
  $R(\Gamma) = \text{hom}(\Gamma, \text{SL}_2\mathbb{C}) \subset \mathbb{C}^{4n}$

  $\mathbb{C}$-algebra of functions: $\mathbb{C}[R(\Gamma)] = \mathbb{C}[x_1, \ldots, x_{4n}]/I$

  where $I = \{p \in \mathbb{C}[x_1, \ldots, x_{4n}] \mid p(R(\Gamma)) = 0\}$

- Look for functions invariant by conjugation: $\mathbb{C}[R(\Gamma)]^{\text{SL}_2\mathbb{C}}$

  For $\gamma \in \Gamma$, the trace function is

  $\tau_\gamma: R(\Gamma) \to \mathbb{C}$

  $\rho \mapsto \text{tr}(\rho(\gamma))$

  Thm: (Procesi) $\exists \gamma_1, \ldots, \gamma_N \in \Gamma$ such that

  $\mathbb{C}[R(\Gamma)]^{\text{SL}_2\mathbb{C}} = \langle \tau_{\gamma_1}, \ldots, \tau_{\gamma_N} \rangle$

- Setting $y_j = \tau_{\gamma_j}$, $\mathbb{C}[R(\Gamma)]^{\text{SL}_2\mathbb{C}} \cong \mathbb{C}[y_1, \ldots, y_N]/J$

  hence the zero set of $J$ in $\mathbb{C}^N$ is the algebraic quotient.
Variety of characters

<table>
<thead>
<tr>
<th>trace functions</th>
<th>characters</th>
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<tbody>
<tr>
<td>$\tau_\gamma: R(\Gamma) \to \mathbb{C}$</td>
<td>$\chi_\rho: \Gamma \to \mathbb{C}$</td>
</tr>
<tr>
<td>$\rho \mapsto \text{tr}(\rho(\gamma))$</td>
<td>$\gamma \mapsto \text{tr}(\rho(\gamma))$</td>
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Def: The variety of characters is $X(\Gamma) = \{ \chi_\rho \mid \rho \in R(\Gamma) \}$

- Since $\mathbb{C}[R(\Gamma)]^{\text{SL}_2\mathbb{C}} = \langle \tau_{\gamma_1}, \ldots, \tau_{\gamma_N} \rangle$, $X(\Gamma)$ has a natural algebraic structure so that $\mathbb{C}[X(\Gamma)] = \mathbb{C}[R(\Gamma)]^{\text{SL}_2\mathbb{C}}$.
- For $f: R(\Gamma) \to \mathbb{C}$ polynomial and $\text{SL}_2\mathbb{C}$-invariant, there exists a unique $\tilde{f}: X(\Gamma) \to \mathbb{C}$ such that

\[
\begin{array}{ccc}
R(\Gamma) & \xrightarrow{f} & \mathbb{C} \\
\rho & \mapsto & \tilde{f}
\end{array}
\]

- Conjugate representations have the same character:

\[
R(\Gamma) \to R(\Gamma)/\text{PSL}_2\mathbb{C} \to X(\Gamma) \\
\rho \mapsto [\rho] \mapsto \chi_\rho
\]

- What is the difference between $R(\Gamma)/\text{PSL}_2\mathbb{C}$ and $X(\Gamma)$?
**Irreducible representations**

**Def:** \( \rho \in R(\Gamma) \) is **reducible** if \( \rho(\Gamma) \) has an invariant line in \( \mathbb{C}^2 \)  
equiv, \( \rho \in R(\Gamma) \) is reducible if \( \rho(\Gamma) \sim \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \)

**Lemma:** \( \rho \) is reducible iff \( \text{tr} \rho([\Gamma, \Gamma]) = 2 \)

**Cor:** \( R^{red}(\Gamma) \) and \( X^{red}(\Gamma) \) Zariski closed

**Lemma:** For \( \rho, \rho' \in R(\Gamma) \) **irreducible**, \( \rho \) and \( \rho' \) are conjugate iff \( \chi_\rho = \chi_{\rho'} \). In fact, \( \text{PSL}_2 \mathbb{C} \to R^{irr}(\Gamma) \to X^{irr}(\Gamma) \) is a principal bundle

- For reducible characters \( \chi \), there is a unique conjugacy class of diagonal representations \( \rho \) with \( \chi_\rho = \chi \)
- For every continuous \( f : R(\Gamma)/\text{PSL}_2 \mathbb{C} \to H \) with \( H \) Hausdorff:

\[
\begin{array}{ccc}
R(\Gamma)/\text{PSL}_2 \mathbb{C} & \xrightarrow{f} & H \\
\downarrow & & \downarrow \\
X(\Gamma) & \xrightarrow{\tilde{f}} & H
\end{array}
\]
Free group of rank two

**Lemma:** For $A, B \in \text{SL}_2 \mathbb{C}$:

(i) $\text{tr}(AB) = \text{tr}(BA)$;

(ii) $\text{tr}(A^{-1}) = \text{tr}(A)$;

(iii) $\text{tr}(AB) + \text{tr}(A^{-1}B) = \text{tr}(A) \text{tr}(B)$.

For (iii), $A^2 - \text{tr}(A)A + \text{Id} = 0$ times $A^{-1}B$ and take trace.

**Thm:** *(Fricke-Klein)* Let $F_2 = \langle a, b \mid \rangle$, we have an isomorphism

$$(\tau_a, \tau_b, \tau_{ab}) : X(F_2) \overset{\cong}{\longrightarrow} \mathbb{C}^3$$

- Setting $x = \tau_a$, $y = \tau_b$, and $z = \tau_{ab}$,

$$\mathbb{C}[X(F_2)] \cong \mathbb{C}[x, y, z].$$

**Cor:** $\forall \gamma \in F_2, \tau_\gamma$ is a (unique) polynomial on $x, y, z$
Groups of rank two

- \(F_2 = \langle a, b \mid \rangle, x = \tau_a, y = \tau_b, z = \tau_{ab} \), \( \mathbb{C}[X(F_2)] = \mathbb{C}[x, y, z] \)
- \(\tau_{\gamma} = \tau_{\gamma^{-1}} \quad \tau_{\gamma\mu} = \tau_{\mu\gamma} \quad \tau_{\gamma\tau_{\mu}} = \tau_{\gamma\mu} + \tau_{\gamma^{-1}\mu} \)
- Powers:
  - \(\tau_{a^2} = \tau_a^2 - \tau_1 = x^2 - 2 \)
  - \(\tau_{a^3} = \tau_a \tau_{a^2} - \tau_a = x^3 - 3x \)
  - \(\tau_{a^n} = \tau_a \tau_{a^{n-1}} - \tau_{a^{n-2}} = \text{Chebyshev pol. on } x \).
- \([a, b] = aba^{-1}b^{-1} \)
  \[\tau_{aba^{-1}b^{-1}} = \tau_{aba^{-1}} \tau_b - \tau_{aba^{-1}b} = y^2 - \tau_{aba^{-1}b} \]
  \[\tau_{aba^{-1}b} = \tau_{ab} \tau_{a^{-1}b} - \tau_a = z \tau_{a^{-1}b} - x^2 + 2 \]
  \[\tau_{a^{-1}b} = \tau_a \tau_b - \tau_{ab} = xy - z \]
  \(\tau_{[a,b]} = x^2 + y^2 + z^2 - xyz - 2 \)
- \(\mathbb{Z}^2 = F_2 = \langle a, b \mid [a, b] = 1 \rangle \). We have:
  \(X(\mathbb{Z}^2) \cong \{(x, y, z) \in \mathbb{C}^3 \mid x^2 + y^2 + z^2 - xyz - 4 = 0 \}\)
The trefoil knot exterior

- $\Gamma = \langle a, b | aba = bab \rangle$ \quad $bab^{-1} = a^{-1}ba \Rightarrow \tau_a = \tau_b$
  
  Choose $x = \tau_a = \tau_b$ and $y = \tau_{ab^{-1}}$

- Write $\tau_{abab^{-1}} - \tau_{ab} = 0$:

  \[
  \tau_{ab} = \tau_a \tau_b - \tau_{ab^{-1}} = x^2 - y
  \]

  \[
  \tau_{abab^{-1}} = \tau_{ba} \tau_{ab^{-1}} - \tau_{b^2} = (x^2 - y)y - x^2 + 2
  \]

  $\tau_{abab^{-1}} - \tau_{ab} = x^2y - 2x^2 - y^2 + y + 2 = (y - 2)(x^2 - y - 1) = 0$

  We have $X(\Gamma) \cong \{(x, y) \in \mathbb{C}^2 \mid (y - 2)(x^2 - y - 1) = 0\}$

- $y = 2$: abelian (and reducible) representations

- $x^2 - y - 1 = 0$:

  $\tau_{ab} = x^2 - y = 1$ \quad and \quad $\tau_{aba} = \tau_a \tau_{ab} - \tau_b = x \cdot 1 - x = 0$

  $\Rightarrow \rho(ab)^3 = \rho(aba)^2 = -\text{Id}$ \quad (set $\alpha = ab$ and $\beta = aba$):

  $\Gamma \cong \langle \alpha, \beta, t \mid \alpha^3 = \beta^2 = t \rangle$, \quad $[\alpha, t] = [\beta, t] = 1$

  Irreducible reps map the center to $\pm \text{Id}$
The figure eight knot exterior (I)

- $\Gamma = \langle a, b \mid w a = b w \rangle$ with $w = ab^{-1}a^{-1}b$
  
  \begin{align*}
  x &= \tau_a = \tau_b, \quad y = \tau_{ab^{-1}} \\
  \tau_\omega - \tau_{b^{-1}wa} &= (y - 2)(y^2 - (x^2 - 1)y + x^2 - 1) = 0
  \end{align*}

  - $y - 2 = 0$ consists of abelian representations
  - $y^2 - (x^2 - 1)y + x^2 - 1 = 0$ contains the holonomy of the hyperbolic structure and all deformations corresponding to Dehn filling.

  Hence
  \[ X(\Gamma) \cong \{(x, y) \in \mathbb{C}^2 \mid (y - 2)(y^2 - (x^2 - 1)y + x^2 - 1) = 0 \} \]

- For a 2-bridge knot exterior in general
  
  $\Gamma = \langle a, b \mid w a = b w \rangle$ for some $w = w(a, b)$

  $X(\Gamma)$ is a plane curve.

  For a 2-bridge knot exterior, $X(\Gamma)$ may have arbitrarily many components (Ohtsuki-Riley-Sakuma)
The figure eight knot exterior (II)

1. \( \Gamma = \langle a, b \mid w a = b w \rangle \) with \( w = ab^{-1}a^{-1}b \)

2. Write \( \rho(a) = \begin{pmatrix} s & 1 \\ 0 & s^{-1} \end{pmatrix} \) and \( \rho(b) = \begin{pmatrix} s & 0 \\ 2 - y & s^{-1} \end{pmatrix} \)

   \[ s + s^{-1} = \tau_a = \tau_b = x \text{ and } \tau_{a^{-1}b} = y \]

3. \( \rho(wa) - \rho(bw) = (y^2 - (x^2 - 1)y + x^2 - 1) \begin{pmatrix} 0 & -1 \\ y - 2 & 0 \end{pmatrix} \)

Component of irreducible representations (Whittemore 1973):

\[ y^2 - (x^2 - 1)y + x^2 - 1 = 0 \]
The figure eight knot exterior (III)

- It is a bundle over $S^1$ with fibre a punctured torus.
  \[ \Gamma = \langle \alpha, \beta, \mu \mid \mu \alpha \mu^{-1} = \alpha \beta, \mu \beta \mu^{-1} = \beta \alpha \beta \rangle \]
  \[ 1 \to F_2 = \langle \alpha, \beta \mid \rangle \to \Gamma \to \mathbb{Z} = \langle \mu \rangle \to 1 \]
  \[ \phi : F_2 \to F_2, \quad \phi(\alpha) = \alpha \beta, \phi(\beta) = \beta \alpha \beta, \]

- $X(F_2) \cong \mathbb{C}^3$ with coordinates $x_1 = \tau_\alpha, x_2 = \tau_\beta, x_3 = \tau_{\alpha \beta}$
  \[ \phi(x_1, x_2, x_3) = (x_1, x_2, x_3) \iff x_3 = x_1, x_1 + x_2 = x_1x_2 \]
  \[ X(F_2)^\phi = \{ \chi \mid \chi \circ \phi = \chi \} \cong \{ (x_1, x_2) \in \mathbb{C}^2 \mid x_1 + x_2 = x_1x_2 \} \]

- res: $X(\Gamma) \to X(F_2)^\phi$ is surjective and, for $\chi \in X(F_2)^\phi$:
  \[ \text{res}^{-1}(\chi) = \begin{cases} 
    2 \text{ characters} & \text{if } \chi \text{ irreducible} \\
    1 \text{ irred. character} & \text{if } \chi \text{ reducible non trivial} \\
    \text{a line of red. ch.} & \text{if } \chi \text{ trivial}
  \end{cases} \]

- $X^{irr}(\Gamma, \text{PSL}_2 \mathbb{C}) \cong X(F_2)^\phi$
The Whitehead link exterior

- $\Gamma = \langle a, b \mid aw = wa \rangle$ where $w = bab^{-1}a^{-1}b^{-1}ab$.
- Coordinates
  
  \[ x = \tau_a, \quad y = \tau_b, \quad z = \tau_{ab}, \]

  then $X(\Gamma) \cong \{ (x, y, z) \in \mathbb{C}^3 \mid pq = 0 \}$ with

  \[
  \begin{cases}
  p = xy - (x^2 + y^2 - 2)z + xyz^2 - z^3 \\
  q = x^2 + y^2 + z^2 - xyz - 4
  \end{cases}
  \]

- $\{ q = 0 \} \cong X(\mathbb{Z}^2)$ abelian/reducible characters
- $\{ p = 0 \}$ Component containing a lift of the holonomy of the complete hyperbolic structure (and Dehn fillings)
$X(F_n), \ n \geq 3$


- $F_3 = \langle a, b, c | \rangle$
  \[
  (\tau_a, \tau_b, \tau_c, \tau_{ab}, \tau_{bc}, \tau_{ca}): X(F_3) \to \mathbb{C}^6 \text{ is a branched covering}
  \]

  There is one more coordinate algebraically dependent:
  
  $\tau_{abc}$ and $\tau_{acb}$ are the solutions of the equation
  
  \[z^2 - Pz + Q = 0\] with
  
  \[P = \tau_a \tau_{bc} + \tau_b \tau_{ca} + \tau_c \tau_{ab} - \tau_a \tau_b \tau_c\]
  \[Q = \tau_a^2 + \tau_b^2 + \tau_c^2 + \tau_{ab}^2 + \tau_{bc}^2 + \tau_{ca}^2 + \tau_{ab} \tau_{bc} \tau_{ca} - \tau_a \tau_b \tau_{ab} - \tau_b \tau_c \tau_{bc} - \tau_c \tau_a \tau_{ca} - 4\]
  \[\text{eg } \tau_{abc} + \tau_{acb} = P \text{ and } \tau_{abc} \tau_{acb} = Q.\]

- $F_4 = \langle a, b, c, d | \rangle$
  \[\tau_{abcd} \text{ is a polynomial on the traces of words on } a, \ b, \ c \text{ and } d\]
  \[\text{of length } \leq 3 \text{ with coefficients in } \frac{1}{2}\mathbb{Z}\]
\( X(\Gamma), \Gamma \text{ of finite type} \)

- \( \Gamma = \langle \gamma_1, \ldots, \gamma_n \mid \{w_j\}_{j \in J} \rangle \)

**Thm:** (González-Acuña and Montesinos-Amilibia)

\[
X(\Gamma) \cong \{ \chi \in X(F_n) \mid \chi_{\gamma_i \omega_j} = \chi_{\gamma_i}, \ i = 1, \ldots, n, j \in J \}
\]

- Fico and Montesinos give the explicit description of \( X(F_n) \), defined with polynomials with coefficients in \( \mathbb{Z} \)
- \( \forall \gamma \in F_n, \tau_\gamma \) is a polynomial in the traces of words of length \( \leq 3 \) on the generators, with coefficients in \( \frac{1}{2} \mathbb{Z} \).

\( \Rightarrow \) Can take reduction \( \text{mod} \ p, \ p \neq 2 \text{ prime}, \) of \( X(\Gamma) \), to compute the variety of characters in an algebraically closed field of characteristic \( p \).
Varieties of characters
and knot symmetries (II)

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INDAM Meeting
Geometric Topology in Cortona
Day 1 Character Varieties.
Definition and computation

Day 2 Canonical component for hyperbolic 3-manifolds of finite volume (...and further components).

Day 3 Knot symmetries (joint with Luisa Paoluzzi)
(Action of knot symmetries on $X(S^3 - K)$)
Manifolds of finite volume

- $M^3$ orientable & (complete) hyperbolic. Its volume is finite iff
  $\text{Ends}(M^3) \cong (T_1^2 \sqcup \cdots \sqcup T_k^2) \times [0, \infty)$
  with $T_i^2 \cong S^1 \times S^1$, and each end is a cusp.
- By defn, a link $L \subset S^3$ is hyperbolic if $S^3 - L$ is hyperbolic
  (hence of finite volume).
- If $M^3$ is hyperbolic, then $\text{hol}: \pi_1 M^3 \to \text{Isom}^+(\mathbb{H}^3) \cong \text{PSL}_2 \mathbb{C}$
  lifts to $\rho_0$ (Culler, Thurston):

$$
\begin{array}{ccc}
\pi_1(M^3) & \xrightarrow{\text{hol}} & \text{PSL}_2 \mathbb{C} \\
\downarrow & & \downarrow \\
\text{SL}_2 \mathbb{C} & \xrightarrow{\rho_0} & \text{PSL}_2 \mathbb{C}
\end{array}
$$

Use that $M^3$ is parallelizable, and $SO(3) \to \text{PSL}_2 \mathbb{C} \to \mathbb{H}^3$ is the frame bundle on $\mathbb{H}^3$
- Results in this talk work also for $X(M^3, \text{PSL}_2 \mathbb{C})$
Canonical component

- Let $M^3$ be hyperbolic, orientable, of finite volume, and with $k > 0$ ends (that are cusps), $\rho_0 = \tilde{\text{hol}}: \pi_1 M^3 \to \SL_2\mathbb{C}$.

**Thm:** (Thurston) $\chi_{\rho_0}$ is a smooth point of $X(M^3)$, and the component $X_0(M^3)$ that contains $\chi_{\rho_0}$ has dimension $k$.

**Def:** $X_0(M^3)$ is called the **canonical component** of $X(M^3)$

- Perhaps not unique?
  - If there are several components, then they are isomorphic, because other lifts of $\text{hol}$ are $(-1)^\epsilon \rho_0$ for $\epsilon: \pi_1 M^3 \to \mathbb{Z}/2\mathbb{Z}$
  - Isometries in $\PSL_2\mathbb{C}$ preserve orientation, so $\chi_{\rho_0} \neq \overline{\chi_{\rho_0}}$.

- Characters close to $\chi_{\rho_0}$ are holonomies of incomplete hyperbolic structures (used for hyperbolic Dehn filling).

- For a hyperbolic knot exterior $S^3 - K$, $X_0(K) := X_0(S^3 - K)$ is a curve.

  Furthermore the A-polynomial is nontrivial
Example: cone manifold structures on the fig 8 knot

- The $n$-cyclic branched covering of $S^3$ branched on the figure eight knot is:

$$
\begin{cases}
\text{hyperbolic} & \text{if } n \geq 4 \\
\text{Euclidean} & \text{if } n = 3 \\
\text{spherical} & \text{if } n = 2
\end{cases}
$$

- Setting $\alpha = \frac{2\pi}{n}$, particular case of: There exists a family of cone manifold structures on $S^3$, with singular locus the figure eight knot, and with cone angle $\alpha \in (0, \frac{4\pi}{3})$ that are

$$
\begin{cases}
\text{hyperbolic} & \text{if } \alpha < \frac{2\pi}{3} \\
\text{Euclidean} & \text{if } \alpha = \frac{2\pi}{3} \\
\text{spherical} & \text{if } \frac{2\pi}{3} < \alpha < \frac{4\pi}{3}
\end{cases}
$$

- As the trace of a rotation of angle $\alpha$ is $\pm 2 \cos(\frac{\alpha}{2})$, look at characters s.t. $\chi(\mu) \in [-2, 2]$ for a meridian $\mu \in \pi_1(S^3 - K)$
Transition geometry in the character variety

$K \subset S^3$ fig eight knot, $\mu \in \Gamma = \pi_1(S^3 - K)$ a meridian. Since $\chi(\mu) = \pm 2 \cos \frac{\alpha}{2}$, look at the set \{\$x \in X(\Gamma), \chi(\mu) \in [-2, 2]\}$

- $1 < |\chi(\mu)| < 2$ are holonomies of hyperbolic cone structures (in magenta the complex conjugate, ie opposite orientation)
- The blue curve consists of characters in $SU(2)$. Pairs $(\chi, \chi')$ with $-1 < \chi(\mu) = \chi'(\mu) < 1$ are lifts of spherical holonomies as $SO(4) \cong SU(2) \times SU(2)$
- $\chi(\mu) = \pm 1$ Euclidean collapse.
Back to the canonical component thm

**Thm:** If $M^3$ hyperbolic of finite volume, with one single end, $ho_0$ lift of the holonomy, then $\chi_{\rho_0}$ is a smooth point of $X(M^3)$, and the component $X_0(M^3)$ that contains $\chi_{\rho_0}$ is a curve.

**Proof** based on:

(a) for each component $Y \subset X(M^3)$ containing an irreducible character $\chi$ so that $\chi(\pi_1 \partial M^3) \not\subset \{\pm 2\}$, $\Rightarrow \dim Y \geq 1$

(b) The dimension of the Zariski tangent space at $\chi_0$ is

$$\dim T_{\chi_0}^{\text{Zar}} X(\Gamma) = 1$$

- (a) is a lower bound of the dimension. It is proved in Thurston notes (sufficient for hyperbolic Dehn filling)
- (b) gives an upper bound of the dimension.
- (a)+(b) $\Rightarrow$ smoothness.
  (the dimension of the Zariski tangent space is an upper bound for the dimension, with equality precisely at smooth points)
Thurston’s lower bound (one cusp case)

**Thm:** Let $\chi_\rho \in X(M^3)$ be irreducible so that $\chi(\pi_1 \partial \overline{M^3}) \not\subset \{\pm 2\}$. For each component $Y \subset X(M^3)$ containing $\chi$, $\dim Y \geq 1$

**Proof:** Chose a simple closed curve $\alpha$ with base point in $\partial \overline{M^3}$, s.t.

1. $\text{tr}(\rho(\alpha)) \neq \pm 2$
2. the restriction $\rho|_{\langle \alpha, \pi_1(\partial \overline{M^3}) \rangle}$ is irreducible.

Set $M' = \overline{M^3} - N'(\alpha)$. As $\alpha$ is simple:

$$\chi(M') = \frac{1}{2}\chi(\partial M') = -1.$$

$\Rightarrow$ $M'$ has the homotopy type of a 2-dim CW-complex with

1 zero-cell, $r$ one-cells and $(r - 2)$ two-cells:

$\Rightarrow \pi_1 M'$ has presentation with $r$ generators and $r - 2$ relations

$\Rightarrow \dim R(\pi_1 M') \geq 6 \Rightarrow \dim X(\pi_1 M') \geq 3$

- View $\alpha \in \pi_1(\partial M')$ and chose $\beta \in \pi_1(\partial M')$ a meridian around $\alpha$, so that $[\alpha, \beta]$ bounds a disk in $\partial \overline{M^3}$

**Claim:** For $\rho' \in R(M')$ in a neighborhood of $\rho$, the conditions

$$\text{tr}(\rho'(\beta)) = \text{tr}(\rho'([\alpha, \beta])) = 2$$

imply $\rho'(\beta) = \text{Id}$

- The claim and $\dim X(\pi_1 M') \geq 3$ yield the theorem.
Zariski tangent space

**Def:** For $V = \{ x \in \mathbb{C}^N \mid p_1(x) = \cdots = p_r(x) = 0 \}$ the Zariski tangent space at $x \in V$ is
\[ T^\text{Zar}_x V = \{ v \in \mathbb{C}^N \mid p_1(x + \varepsilon v), \ldots, p_r(x + \varepsilon v) \in O(\varepsilon^2) \} \]

- $\dim T^\text{Zar}_x V \geq \dim(\text{component of } V \text{ containing } x)$ with equality iff $x$ smooth point of $V$ (and one single comp)

**Def:** Crossed morphisms
\[ Z^1 = \{ \theta : \Gamma \to \mathfrak{sl}_2 \mathbb{C} \mid \theta(\gamma \mu) = \theta(\gamma) + \rho(\gamma) \theta(\mu) \rho(\gamma^{-1}) \} \]

 Inner morphisms:
\[ B^1 = \{ \theta_a \in Z^1 \mid \theta_a(\gamma) = \rho(\gamma) a \rho(\gamma^{-1}) - a, \forall \gamma \in \Gamma \} \]
\[ H^1(\Gamma, \text{Ad } \rho) = Z^1 / B^1 \]

**Thm:** (Weil) \[ Z^1 \xrightarrow{\cong} T^\text{Zar}_\rho R(\Gamma) \]
\[ \theta \quad \longmapsto \quad \gamma \mapsto (\text{Id} + \varepsilon \theta(\gamma)) \rho(\gamma) = \rho_\varepsilon(\gamma) \]

$B^1 \cong$ orbit by conjugation

**Cor:** (Weil) For $\chi_\rho$ irreducible, \[ H^1(\Gamma, \text{Ad } \rho) \cong T^\text{Zar}_\chi X(\Gamma) \]

**Thm:** (Garland) $M^3$ hyperbolic, orient., finite vol, and $k$ cusps.
For $\rho_0 \in R(M^3)$ lift of the holonomy, \[ H^1(\pi_1 M^3, \text{Ad } \rho_0) \cong \mathbb{C}^k \]
Cohomology thm

Thm: (Garland) $M^3$ hyperbolic, orient, finite vol, and $k$ cusps. For $\rho_0 \in R(M^3)$ lift of the holonomy, $H^1(\pi_1M^3, \text{Ad} \rho_0) \cong \mathbb{C}^k$

- Use de Rham cohomology:
  - Flat bundle $\mathfrak{sl}_2\mathbb{C} \rightarrow E_{\text{Ad} \rho_0} \rightarrow M$ where $E_{\text{Ad} \rho_0} = \tilde{M} \times \mathfrak{sl}_2\mathbb{C}/\sim$ with $(x, a) \sim (\gamma x, \text{Ad}_{\rho_0(\gamma)} a)$
  - $\Omega^p(M, E_{\text{Ad} \rho_0}) = \Gamma(\wedge^p T^*M \otimes E_{\text{Ad} \rho_0}) = \text{vector valued p-forms}$
  - $d: \Omega^p(M, E_{\text{Ad} \rho_0}) \rightarrow \Omega^{p+1}(M, E_{\text{Ad} \rho_0})$ yields de Rham cohom,

and $H^*_d(M^3, E_{\text{Ad} \rho_0}) \cong H^*(\pi_1M^3, \text{Ad} \rho_0)$

Thm: (...? 1960’s) closed $L^2$-forms are exact.

Set $U = M^3 - \{\text{Compact core}\} \cong \bigcup T_i^2 \times (0, +\infty)$ hence

$H^1(M, U, \text{Ad} \rho_0) \xrightarrow{0} H^1(M, \text{Ad} \rho_0) \rightarrow H^1(U, \text{Ad} \rho)$

and $\dim H^1(M, \text{Ad} \rho_0) = \sum_i \dim \frac{1}{2} H^1(T_i^2, \text{Ad} \rho_0) = \sum_i 1 = k$. 
Galois conjugates

- \( M^3 \) hyperbolic, orientable, finite volume, \( \rho_0 : \pi_1 M^3 \to SL_2 \mathbb{C} \) lift of the holonomy.

- \( \rho_0(\pi_1 M^3) \subset SL_2 \mathbb{K} \), for \( \mathbb{K} \) a number field (Vinberg).
  May assume that the finite extension \( \mathbb{K}|\mathbb{Q} \) is Galois.

**Rmk:** For each \( \sigma \in \text{Galois} (\mathbb{K}) \),

\( \chi_{\rho_0^\sigma} \) is a smooth point of \( X(M^3) \) and the unique component that contains \( \chi_{\rho_0^\sigma} \) has dimension the number of cusps.

**Why?** View the space of crossed morphisms as a linear space

\[
Z^1 = \{ \theta : \pi_1 M^3 \to \mathfrak{sl}_2 \mathbb{C} \mid \theta(\gamma \mu) = \theta(\gamma) + \rho_0(\gamma)\theta(\mu)\rho_0(\gamma^{-1}) \}
\]

From \( \Gamma = \pi_1 M^3 = \langle \gamma_1, \ldots, \gamma_n \mid r_1, \ldots, r_m \rangle \), embed

\[
Z^1 \hookrightarrow \mathfrak{sl}_2 \mathbb{C} \times \cdots \times \mathfrak{sl}_2 \mathbb{C}
\]

\[
\theta \mapsto (\theta(\gamma_1), \ldots, \theta(\gamma_n))
\]

The image is the kernel of a linear map given by \( r_1, \ldots, r_m \),

with coefficients in \( \mathbb{K} \).

So \( \dim(Z^1(\pi_1 M^3, \text{Ad } \rho_0^\sigma)) = \dim(Z^1(\pi_1 M^3, \text{Ad } \rho_0)) \).
**Further components**

- Have shown $X(M^3)$ has the canonical component and the abelian ones. Look for further components.

*Thm:* (Ohtsuki-Riley-Sakuma) Among 2-bridge knots $K$ and $K'$, the set $\{ f : \pi_1(S^3 - K) \to \pi_1(S^3 - K') \text{ epimorphism} \}/\{\text{inner}\}$ may have arbitrarily large cardinality.

*Cor:* (Ohtsuki-Riley-Sakuma) For 2-bridge knots $K$, $X(K)$ can have arbitrarily many components

*Rmk:* For 2-bridge knots $K$, components of $X(K)$ are plane curves:
- As $\pi_1(S^3 - K) = \langle a, b \mid a w(a, b) = w(a, b) b \rangle$, use coordinates $x = \tau_a = \tau_b$ and $y = \tau_{ab^{-1}}$. Hence $X(K) \subset \mathbb{C}^2$
- By Thuston's estimate, the dim of components is at least 1.
- Cannot have dim 2.

*Thm:* (Paoluzzi-P, using an idea of Riley) Given $m \in \mathbb{N}$, $m \geq 2$, for a Montesinos knot $K$, $X(K)$ may have arbitrarily many components of dim $m$. 
Varieties of characters
and knot symmetries (III)

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Recap

• $R(M) = \text{hom}(\pi_1 M, \text{SL}_2 \mathbb{C})$. \quad $\chi_\rho(\gamma) = \text{tr}(\rho(\gamma))$
  
  $X(M) = \{ \chi_\rho: \pi_1 M \to \mathbb{C} \mid \rho \in R(M) \} \cong R(M)//\text{SL}_2 \mathbb{C}$

• $M^3$ hyperbolic, orientable and of finite volume.
  The \textit{canonical component} $X_0(M^3)$
  has dimension the number of cusps.

• $X(M^3)$ may have other irreducible components.

• Fico-Montesinos: coeffs. of the polynomial equations in $\frac{1}{2} \mathbb{Z}$, so that its reduction \textit{mod} $p$ is the variety of $\text{SL}_2 \mathbb{F}$-characters, for $\mathbb{F} = \text{algebraically closed field of characteristic } p \neq 2$

• For a knot $K \subset S^3$, $X(K) := X(S^3 - K)$
**Knot symmetries**

Let $K$ be a knot in $S^3$ and $\psi: (S^3, K) \to (S^3, K)$ a diffeomorphism of finite order $p$, that preserves the orientation of $S^3$. Either $\text{Fix}(\psi) = \emptyset$ or $\text{Fix}(\psi) \cong S^1$ unknotted in $S^3$.

**Def:** $\psi$ is said to be:

- a **free symmetry** if the group $\langle \psi \rangle \cong \mathbb{Z}/p\mathbb{Z}$ acts freely.
- a **periodic symmetry** if $\text{Fix}(\psi)$ is a circle disjoint from $K$.
- a **strong inversion** if $\psi$ has order 2 and $|\text{Fix}(\psi) \cap K| = 2$.
- a **pseudo–periodic symmetry** otherwise.

**Assume** that $\psi$ has order a prime $p \neq 2$.
Hence $\psi$ is either free or periodic.

**Goal:** show different behavior between periodic and free symmetries in the variety of characters.
Example: periodic (non-free) symmetry

\[ \mathcal{O} = (S^3 - K) / \psi \]

\[ \Sigma = \text{Fix}(\psi) / \psi \]

\[ (K \cup \text{Fix}(\psi)) / \psi = 6_2^2 \]
Example: free symmetry

\[ L(5, 1) = S^3 / \psi \]
Components of the variety of characters

- $K \subset S^3$ hyperbolic knot
- $\psi: (S^3, K) \rightarrow (S^3, K)$ symmetry of prime order $p \neq 2$.

\[ X(K)^\psi = \{ \chi \in X(K) \mid \chi \circ \psi_* = \chi \} \]

Thm: (Paoluzzi-P) If $\psi$ is periodic then $X(K)^\psi$ has at least $\frac{p-1}{2}$ components that are also components of $X(K)$.

Rmks: (a) For each prime $p > 4$ there is a knot $K_p$ with a free symmetry $\psi$ of order $p$ so that $X(K_p)^\psi$ has at most 20 components
(b) When reducing mod $p$, all the components of the theorem collapse to a single one.
(c) If we look at components of $X(K)$, without $\psi$-invariance, many further components may appear, for $\psi$ either free or periodic.
Components of the variety of characters

- $K \subset S^3$ hyperbolic knot
- $\psi: (S^3, K) \rightarrow (S^3, K)$ symmetry of prime order $p \neq 2$.

$$X(K)^\psi = \{ \chi \in X(K) \mid \chi \circ \psi_\ast = \chi \}$$

Thm: (Paoluzzi-P) If $\psi$ is periodic then $X(K)^\psi$ has at least $\frac{p-1}{2}$ components that are also components of $X(K)$.

- The proof has 3 steps. Set $M = S^3 - K$, $O = M/\psi$.
  1. The restriction map $\text{res}: X^{\text{irr}}(O) \rightarrow X^{\text{irr}}(M)^\psi$ is a bijection
  2. Find several components for $X(O)$
  3. Use (1) + (2) to find several components for $X(M)^\psi$
Step 1: Extending $\psi$-invariant characters

$M = S^3 - K$ \hspace{1cm} $O = M/\psi$ \hspace{1cm} $\pi_1 M \rightarrow \pi_1 O \rightarrow \mathbb{Z}/p\mathbb{Z}$

$\mu \leftarrow 1$

$\psi : \pi_1 M \rightarrow \pi_1 M$

$\gamma \mapsto \mu \gamma \mu^{-1}$

$\pi_1 O = \langle \pi_1 M, \mu | \mu^p = 1, \mu \gamma \mu^{-1} = \psi_* \mu, \forall \gamma \in \pi_1 M \rangle$

**Lemma**  The restriction map $\text{res} : X^{irr}(O) \rightarrow X^{irr}(M)^\psi$ is a bijection

**Proof**  Want to extend $\chi_\rho \in X^{irr}(M)^\psi$ to $\pi_1 O$ in a unique way.

Since $\chi_{\rho \circ \psi_*} = \chi_\rho \Rightarrow \exists A \in \text{SL}_2 \mathbb{C}$ s.t. $\rho(\psi_*(\gamma)) = A \rho(\gamma) A^{-1}$. $A$ is unique up to sign. If furthermore we require $A^p = \text{Id} \Rightarrow A$ exists and unique because $p$ odd $\Rightarrow$ Set $\rho(\mu) = A$.

(need to show also that restriction of irreducible is irreducible)
Step 2: Finding components for $X(\mathcal{O})$

- $\mathcal{O}$ is hyperbolic and $\text{hol}: \pi_1\mathcal{O} \to \text{PSL}_2\mathbb{C}$ lifts to $\rho_0: \pi_1\mathcal{O} \to \text{SL}_2\mathbb{C}$ because $p$ is odd.
- $\rho_0(\pi_1\mathcal{O}) \subset \text{SL}_2\mathbb{K}$, for $\mathbb{K}$ a number field (Vinberg). May assume that the finite extension $\mathbb{K}|\mathbb{Q}$ is Galois.
- Since $\mu^p = 1$, then $\text{tr}(\rho_0(\mu)) = -2 \cos \frac{\pi}{p}$ and
  \[
  \{\tau_\mu(\rho_0^\sigma)|\sigma \in \text{Galois}(\mathbb{K})\} = \{-2 \cos \frac{\pi r}{p}|r = 1, 3, 5, \ldots, p-2\}.
  \]
- $\tau_\mu = -2 \cos \frac{\pi r}{p}$ distinguishes $\frac{p-1}{2}$ components $Y_1, \ldots, Y_{\frac{p-1}{2}}$ of $X(\mathcal{O})$ that contain different Galois conjugates $\chi_{\rho_0^\sigma}$ of $\chi_{\rho_0}$.
- By Garland’s Thm, since $\mathcal{O}$ has a cusp, $H^1(\pi_1\mathcal{O}, \text{Ad} \rho_0) = \mathbb{C}$
  \[
  \Rightarrow H^1(\pi_1\mathcal{O}, \text{Ad} \rho_0^\sigma) = \mathbb{C}, \forall \sigma \in \text{Galois}(\mathbb{K})
  \]
  \[
  \Rightarrow \text{each } Y_r \text{ is a curve.}
  \]
- Next step: using that $\text{res}: X^{\text{irr}}(\mathcal{O}) \to X^{\text{irr}}(M)^\psi$ is a bijection, the curves $Y_1, \ldots, Y_{\frac{p-1}{2}}$ yield different components of $X(M)^\psi$. 

Step 3: components for $X(M)\psi$

We know that $\text{res}: X^{\text{irr}}(O) \to X^{\text{irr}}(M)\psi$ is a bijection. We have $Y_1, \ldots, Y_{\frac{p-1}{2}}$ components of $X^{\text{irr}}(O)$ that are curves.

\[
\begin{align*}
X(O) & \xrightarrow{\text{res}} X(M)\psi \subset X(M) \\
Y_1 & \\
\vdots & \quad W_i = \text{res}(Y_i) \\
Y_{\frac{p-1}{2}} &
\end{align*}
\]

- Since $Y_i = Y_i^{\text{irr}} \cup (\text{a finite set})$, $W_i = \text{res}(Y_i) \cup (\text{a finite set})$ and $W_1, \ldots, W_{\frac{p-1}{2}}$ are different components of $X(M)\psi$
- As $\dim H^1(\pi_1 M, \text{Ad } \rho_0^\sigma) = \dim H^1(\pi_1 M, \text{Ad } \rho_0) = 1$, $W_1, \ldots, W_{\frac{p-1}{2}}$ are also components of $X(M)$. \qed
Rmk: Let $\mathbb{F}$ be an algebraically closed field of characteristic $p$. As an element of $\text{SL}_2 \mathbb{F}$ of order $p$ is conjugate to

$$\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix},$$

all these components $W_1, \ldots, W_{\frac{p-1}{2}}$ in the theorem collapse to a single one mod $p$. 
Free symmetries: a family of examples

Consider $\frac{q}{p}$-Dehn filling on $A$ and take the covering $(S^3, K_{\frac{q}{p}}) \to (L(p, q), K_0)$ where $p > 4$ prime and $p, q$ coprime.

$\psi: (S^3, K_{\frac{q}{p}}) \to (S^3, K_{\frac{q}{p}})$ free symmetry of order $p$

- $X(6_2^2)$ has two components: $X_0(6_2^2)$ and $X^{ab}(6_2^2)$.
  - The map $X_0(6_2^2) \to X(\partial \mathcal{N}(A))$ is dominant and its generic fibre is finite.
  - $\forall \gamma \in \pi_1 \partial \mathcal{N}(A)$ primitive, $\{\tau_\gamma = 2\}$ is a line in $X(\partial \mathcal{N}(A))$.
  - $\Rightarrow X(L(p, q) - K_0)$ has at most $C$ components, $C$ uniform

$\Rightarrow X(S^3 - K_{\frac{q}{p}})\psi$ has at most $C$ components (uniform) and $\psi$ has order $p$ (that can be any prime $> 4$).
Thanks for your attention!