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GEOMETRIZATION OF 3-ORBIFOLDS OF CYCLIC TYPE

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Abstract. — We prove the orbifold theorem in the cyclic case: If $\mathcal{O}$ is a compact oriented irreducible atoroidal 3-orbifold whose ramification locus is a non-empty submanifold, then $\mathcal{O}$ is geometric, i.e. it has a hyperbolic, a Euclidean or a Seifert fibered structure. This theorem implies Thurston's geometrization conjecture for compact orientable irreducible three-manifolds having a non-free symmetry.

Résumé **(Géométrisation des orbi-variétés tridimensionnelles de type cyclique)**
Nous démontrons le théorème des orbi-variétés de Thurston dans le cas cyclique : une orbi-variété tridimensionelle, compacte, orientable, irréductible, atoroïdale et dont le lieu de ramification est une sous-variété non vide, admet soit une structure hyperbolique ou Euclidienne, soit une fibration de Seifert. Ce théorème implique qu'une variété tridimensionelle, compacte, irréductible et possédant une symétrie non libre, vérifie la conjecture de géométrisation de Thurston.

CONTENTS

INTRODUCTION

A 3-dimensional orbifold is a metrizable space with coherent local models given by quotients of $\mathbb{R}^3$ by finite subgroups of $O(3)$. For example, the quotient of a 3-manifold by a properly discontinuous group action naturally inherits a structure of a 3-orbifold. When the group action is finite, such an orbifold is said to be *very good*. For a general background about orbifolds see [**BS1**], [**BS2**], [**DaM**], [**Kap**, Chap. 7], [**Sc3**], [**Tak1**] and [**Thu1**, Chap. 13].

The purpose of this monograph is to give a complete proof of Thurston's orbifold theorem in the case *where all local isotropy groups are cyclic subgroups of* $SO(3)$. Following [**DaM**], we say that such an orbifold is of *cyclic type* when in addition the ramification locus is non-empty. Hence a 3-orbifold $\mathcal{O}$ is of cyclic type iff its ramification locus Σ is a non-empty 1-dimensional submanifold of the underlying manifold $|\mathcal{O}|$, which is transverse to the boundary $\partial|\mathcal{O}| = |\partial\mathcal{O}|$. The first result presented here is the following version of Thurston's Orbifold Theorem:

Theorem 1. — *Let $\mathcal{O}$ be a compact, connected, orientable, irreducible, and ∂-incompressible 3-orbifold of cyclic type. If $\mathcal{O}$ is very good, topologically atoroidal and acylindrical, then $\mathcal{O}$ is geometric (i.e. $\mathcal{O}$ admits either a hyperbolic, a Euclidean, or a Seifert fibered structure).*

Remark. — When $\partial\mathcal{O}$ is a union of toric 2-suborbifolds, the hypothesis that $\mathcal{O}$ is acylindrical is not needed.

If $\partial\mathcal{O} \neq \emptyset$ and $\mathcal{O}$ is not I-fibered, then $\mathcal{O}$ admits a hyperbolic structure of finite volume with totally geodesic boundary and cusps.

We only consider *smooth* orbifolds, so that the local isotropy groups are always orthogonal. We recall that an orbifold is said to be *good* if it has a covering which is a manifold. Moreover if this covering is finite then the orbifold is said to be *very good*.

A general compact orientable irreducible and atoroidal 3-orbifold (which is not a priori very good) can be canonically split along a maximal (perhaps empty) collection of disjoint and non-parallel hyperbolic turnovers (i.e. a 2-orbifold with underlying space a 2-sphere and with three branching points) into either *small* or *Haken* 3-suborbifolds.

An orientable compact 3-orbifold $\mathcal{O}$ is *small* if it is irreducible, its boundary $\partial\mathcal{O}$ is a (perhaps empty) collection of turnovers, and $\mathcal{O}$ does not contain any essential orientable 2-suborbifold.

Using Theorem 1, we are able to geometrize such small 3-orbifolds, and hence to show that they are in fact very good.

Theorem 2. — *Let $\mathcal{O}$ be a compact, orientable, connected, small 3-orbifold of cyclic type. Then $\mathcal{O}$ is geometric.*

Therefore, to get a complete picture (avoiding the very good hypothesis), it remains to geometrize the Haken atoroidal pieces.

An orientable compact 3-orbifold $\mathcal{O}$ is *Haken* if:

- $\mathcal{O}$ is irreducible,
- every embedded turnover is parallel to the boundary
- and $\mathcal{O}$ contains an embedded orientable incompressible 2-suborbifold different from a turnover.

The geometrization of Haken atoroidal 3-orbifolds relies on the following extension of Thurston's hyperbolization theorem (for Haken 3-manifolds):

***Theorem 3* (Thurston's hyperbolization theorem).** — *Let $\mathcal{O}$ be a compact orientable connected irreducible Haken 3-orbifold. If $\mathcal{O}$ is topologically atoroidal and not Seifert fibered, nor Euclidean, then $\mathcal{O}$ is hyperbolic.*

It is a result of W. Dunbar [**Dun2**] that an orientable Haken 3-orbifold can be decomposed into either discal 3-orbifolds or thick turnovers (i.e $\{\text{turnovers}\}\times[0,1]$) by repeated cutting along 2-sided properly embedded essential 2-suborbifolds.

Due to this fact, the proof of Theorem 3 follows exactly the scheme of the proof for Haken 3-manifolds [**Thu2, Thu3, Thu5**], [**McM1**], [**Kap**], [**Ot1, Ot2**]. We will not give a detailed proof of it here, but we only present the main steps to take in consideration and indicate shortly how to handle them in Chapter 8.

Since hyperbolic turnovers are rigid, Theorem 2 and Theorem 3 imply Thurston's orbifold theorem in the cyclic type case:

Thurston's Orbifold Theorem. — *Let $\mathcal{O}$ be a compact, connected, orientable, irreducible, 3-orbifold of cyclic type. If $\mathcal{O}$ is topologically atoroidal, then $\mathcal{O}$ is geometric.*

In late 1981, Thurston [**Thu2, Thu6**] announced the Geometrization theorem for 3-orbifolds with non-empty ramification set (without the assumption of cyclic type), and lectured about it. Since 1986, useful notes about Thurston's proof (by Soma, Ohshika and Kojima [**SOK**] and by Hodgson [**Ho1**]) have been circulating. In addition, in 1989 much more details appeared in Zhou's thesis [**Zh1, Zh2**] in the cyclic case. However no complete written proof was available (cf. [**Kir**, Prob. 3.46]).

Recently we have obtained with B. Leeb a proof of Thurston's orbifold theorem in the case where the singular locus has vertices. A complete written version of this proof can be found in [**BLP1, BLP2**]. This proof, in particular for orbifolds with all singular vertices of dihedral type, relies on the proof of the cyclic case presented here. However the methods used in [**BLP2**] to study the geometry of cone 3-manifolds are quite different from the ones used here.

A different proof, more in the spirit of Thurston's original approach, has been announced by D. Cooper, C. Hodgson and S. Kerckhoff in [**CHK**].

In this monograph we work in the category of orbifolds. For the basic definitions in this category, including map, homotopy, isotopy, covering and fundamental group, we refer mainly to Chapter 13 of Thurston's notes [**Thu1**], to the books by Bridson and Haefliger [**BrH**] and by Kapovich [**Kap**], as well as to the articles by Bonahon and Siebenmann [**BS1, BS2**], by Davis and Morgan [**DaM**] and by Takeuchi [**Tak1**].

In the case of good orbifolds, these notions are defined as the corresponding equivariant notions in the universal covering, which is a manifold.

According to [**BS1, BS2**] and [**Thu1**, Ch. 13], we use the following terminology.

Definitions. — We say that a compact 2-orbifold F^2 is respectively *spherical*, *discal*, *toric* or *annular* if it is the quotient by a finite smooth group action of respectively the 2-sphere S^2, the 2-disk D^2, the 2-torus T^2 or the annulus $S^1 \times [0,1]$.

A compact 2-orbifold is *bad* if it is not good. Such a 2-orbifold is the union of two non-isomorphic discal 2-orbifolds along their boundaries.

A compact 3-orbifold $\mathcal{O}$ is *irreducible* if it does not contain any bad 2-suborbifold and if every orientable spherical 2-suborbifold bounds in $\mathcal{O}$ a discal 3-suborbifold, where a *discal* 3-orbifold is a finite quotient of the 3-ball by an orthogonal action.

A connected 2-suborbifold F^2 in an orientable 3-orbifold $\mathcal{O}$ is *compressible* if either F^2 bounds a discal 3-suborbifold in $\mathcal{O}$ or there is a discal 2-suborbifold Δ^2 which intersects transversally F^2 in $\partial\Delta^2 = \Delta^2 \cap F^2$ and such that $\partial\Delta^2$ does not bound a discal 2-suborbifold in F^2.

A 2-suborbifold F^2 in an orientable 3-orbifold $\mathcal{O}$ is *incompressible* if no connected component of F^2 is compressible in $\mathcal{O}$. The compact 3-orbifold $\mathcal{O}$ is *∂-incompressible* if $\partial\mathcal{O}$ is empty or incompressible in $\mathcal{O}$.

A properly embedded 2-suborbifold $(F, \partial F) \hookrightarrow (\mathcal{O}, \partial\mathcal{O})$ is *∂-compressible* if:

- either $(F, \partial F)$ is a discal 2-suborbifold $(D^2, \partial D^2)$ which is ∂-parallel,
- or there is a discal 2-suborbifold $\Delta \subset \mathcal{O}$ such that $\partial\Delta \cap F$ is a simple arc α, $\Delta \cap \partial M$ is a simple arc β, with $\partial\Delta = \alpha \cup \beta$ and $\alpha \cap \beta = \partial\alpha = \partial\beta$

An orientable properly embedded 2-suborbifold F^2 is *∂-parallel* if it belongs to the frontier of a collar neighborhood $F^2 \times [0,1] \subset \mathcal{O}$ of a boundary component $F^2 \subset \partial\mathcal{O}$.

A properly embedded 2-suborbifold F^2 is *essential* in a compact orientable irreducible 3-orbifold, if it is incompressible, ∂-incompressible and not boundary parallel.

A compact 3-orbifold is *topologically atoroidal* if it does not contain any embedded essential orientable toric 2-suborbifold. It is *topologically acylindrical* if every properly embedded orientable annular 2-suborbifold is boundary parallel.

A *turnover* is a 2-orbifold with underlying space a 2-sphere and with three branching points. In an irreducible orientable orbifold an embedded turnover either bounds a discal 3-suborbifold or is incompressible and of non-positive Euler characteristic.

According to [**Thu1**, Ch. 13], the *fundamental group* of an orbifold $\mathcal{O}$, denoted by $\pi_1(\mathcal{O})$, is defined as the Deck transformation group of its universal cover.

A *Seifert fibration* on a 3-orbifold $\mathcal{O}$ is a partition of $\mathcal{O}$ into closed 1-suborbifolds (circles or intervals with silvered boundary) called fibers, such that each fiber has a saturated neighborhood diffeomorphic to $S^1 \times D^2/G$, where G is a finite group which acts smoothly, preserves both factors, and acts orthogonally on each factor and effectively on D^2; moreover the fibers of the saturated neighborhood correspond to the quotients of the circles $S^1 \times \{*\}$. On the boundary $\partial\mathcal{O}$, the local model of the Seifert fibration is $S^1 \times D^2_+/G$, where D^2_+ is a half disk.

A 3-orbifold that admits a Seifert fibration is called Seifert fibered. Every good Seifert fibered 3-orbifold is geometric (cf. [**Sc3**], [**Thu7**]). Seifert fibered 3-orbifolds have been classified in [**BS2**].

A compact orientable 3-orbifold $\mathcal{O}$ is *hyperbolic* if its interior is orbifold-diffeomorphic to the quotient of the hyperbolic space $\mathbb{H}^3$ by a non-elementary discrete group of isometries. In particular I-bundles over hyperbolic 2-orbifolds are hyperbolic, since their interiors are quotients of $\mathbb{H}^3$ by non-elementary Fuchsian groups. In Theorem 1, except for I-bundles, we prove that when $\mathcal{O}$ is hyperbolic, if we remove the toric components of the boundary $\partial_T\mathcal{O} \subset \partial\mathcal{O}$, then $\mathcal{O} - \partial_T\mathcal{O}$ has a hyperbolic structure with finite volume and geodesic boundary. This implies the existence of a complete hyperbolic structure on the interior of $\mathcal{O}$.

We say that a compact orientable 3-orbifold is *Euclidean* if its interior has a complete Euclidean structure. Thus, if a compact orientable and ∂-incompressible 3-orbifold $\mathcal{O}$ is Euclidean, then either $\mathcal{O}$ is a I-bundle over a 2-dimensional Euclidean closed orbifold or $\mathcal{O}$ is closed.

We say that a compact orientable 3-orbifold is *spherical* when it is the quotient of $\mathbb{S}^3$ by the orthogonal action of a finite subgroup of $SO(4)$. A spherical orbifold of cyclic type is always Seifert fibered ([**Dun1**], [**DaM**]).

Thurston's conjecture asserts that the interior of a compact irreducible orientable 3-orbifold can be decomposed along a canonical family of incompressible toric 2-suborbifolds into geometric 3-suborbifolds.

The existence of the canonical family of incompressible toric 2-suborbifolds has been established by Jaco-Shalen [**JS**] and Johannson [**Joh**] for 3-manifolds and by Bonahon-Siebenmann [**BS2**] in the case of 3-orbifolds.

Recall that the eight 3-dimensional geometries involved in Thurston's conjecture are $\mathbb{H}^3$, $\mathbb{E}^3$, $\mathbb{S}^3$, $\mathbb{H}^2\times\mathbb{R}$, $\mathbb{S}^2\times\mathbb{R}$, $\widetilde{SL_2(\mathbb{R})}$, Nil and *Sol*. The non Seifert fibered orbifolds require a constant curvature geometry ($\mathbb{H}^3$, $\mathbb{E}^3$ and $\mathbb{S}^3$) or *Sol*. Compact orbifolds with *Sol* geometry are fibered over a closed 1-dimensional orbifold with toric fiber and thus are not atoroidal (cf. [**Dun3**]).

Thurston has proved his conjecture for Haken 3-manifolds [**Thu1, Thu2, Thu3, Thu4, Thu5**] (cf. [**McM1, McM2**], [**Kap**], [**Ot1, Ot2**]). More generally, his proof works for Haken 3-orbifolds (cf. Chapter 8).

Here is a straightforward application of Thurston's orbifold theorem to the geometrization of 3-manifolds with non-trivial symmetries:

Corollary 1. — *Let M be a compact orientable irreducible and ∂-incompressible 3-manifold. Let G be a finite group of orientation preserving diffeomorphisms acting on M with non-trivial and cyclic stabilizers. Then there exists a (possibly empty) G-invariant family of disjoint essential tori and annuli which splits M into G-invariant geometric pieces.*

Using the fact that 3-orbifolds with a geometric decomposition are very good by [**McCMi**], one obtains the following immediate corollary:

Corollary 2. — *Every compact orientable irreducible 3-orbifold of cyclic type is very good.*

Thurston's hyperbolization theorem for Haken 3-manifolds ([**Thu1, Thu2, Thu3, Thu4, Thu5**], [**McM1, McM2**], [**Kap**], [**Ot1, Ot2**]) and a standard argument of doubling $\mathcal{O}$ along its boundary components, allow to reduce the proof of Theorem 1 reduces to the following theorem, which is one of the main results of this monograph.

Theorem 4. — *Let $\mathcal{O}$ be a closed orientable connected irreducible very good 3-orbifold of cyclic type. Assume that the complement $\mathcal{O}-\Sigma$ of the ramification locus admits a complete hyperbolic structure. Then there exists a non-empty compact essential 3-suborbifold $\mathcal{O}'\subseteq\mathcal{O}$ which is not a product and which is either Euclidean, Seifert fibered, Sol or complete hyperbolic with finite volume. In particular $\partial\mathcal{O}'$ is either empty or a union of toric 2-orbifolds.*

A compact 3-suborbifold $\mathcal{O}'\subseteq\mathcal{O}$ is *essential* in $\mathcal{O}$ if the 2-suborbifold $\partial\mathcal{O}'$ is either empty or incompressible in $\mathcal{O}$.

Remark. — If the orbifold $\mathcal{O}$ is topologically atoroidal, then $\mathcal{O}=\mathcal{O}'$ is geometric.

The following strong version of the Smith conjecture for a knot in S^3 is a straightforward corollary of Theorem 4 and of the classification of the orientable closed Euclidean 3-orbifolds (cf. [**BS1**], [**Dun1**]).

Corollary 3. — *Let $K \subset S^3$ be a hyperbolic knot. Then, for $p \geq 3$, any p-fold cyclic covering of S^3 branched along K admits a hyperbolic structure, except when $p = 3$ and K is the figure-eight knot. In this case the 3-fold cyclic branched covering has an Euclidean structure. Moreover, in all cases the covering transformation group acts by isometries.*

The proof of Theorem 4 follows Thurston's original approach. His idea was to deform the complete hyperbolic structure as far as possible on $O - \Sigma$ into structures whose completion is topologically the underlying manifold $|\mathcal{O}|$ and has cone singularities along Σ. These completions are called hyperbolic cone structures on the pair $(|\mathcal{O}|, \Sigma)$ and their singularities are (locally) described by cone angles. Such structures with small cone angles are provided by Thurston's hyperbolic Dehn filling theorem ([**Thu1**, Chap. 5], cf. Appendix B). The goal is then to study the limit of hyperbolicity when these cone angles increase. Note that a hyperbolic structure on $\mathcal{O}$ induces a hyperbolic cone structure on the pair $(|\mathcal{O}|, \Sigma)$ with cone angles determined by the ramification indices. Hence, if these cone angles can be reached in the space of hyperbolic cone manifold structures, then $\mathcal{O}$ is hyperbolic. Otherwise, the study of the possible "collapses" occurring at the limit of hyperbolicity shows the existence of a non-empty compact essential geometric 3-suborbifold $\mathcal{O}' \subset \mathcal{O}$ which is different from $\mathcal{O}$ when $\mathcal{O}'$ is hyperbolic.

Our main contribution takes place in the analysis of the so called "collapsing cases". There we use the notion of simplicial volume due to Gromov and a cone manifold version of his isolation theorem [**Gro**, Sec. 3.4]. This gives a simpler combinatorial approach to collapses than Thurston's original one. In particular, it spares us the difficult task of establishing a suitable Cheeger-Gromov theory for collapses of cone manifolds.

When some of the branching indices are 2, our proof of Theorem 1 uses in a crucial way the results of Meeks and Scott [**MS**]. This could be avoided by using the extension of Thurston's hyperbolization theorem to Haken 3-orbifolds (cf. Theorem 3). Since we are not giving here a detailed proof of this extension, we have decided to make the proofs of our main results (Theorems 1, 2 and 4) totally independent of it.

Here is a plan of the monograph.

In Chapter 1 we introduce the notion of cone 3-manifold and state the theorems that are the main ingredient in the proof of of Thurston's orbifold theorem.

In Chapter 2 we prove Theorem 4 from the results quoted in Chapter 1. Then we deduce Theorem 1 from Theorem 4.

In Chapter 3 we prove the compactness theorem, which is a cone 3-manifold version of Gromov's compactness theorem for Riemannian manifolds of pinched sectional curvature.

In Chapter 4 we prove the local soul theorem, which gives a bilipschitz approximation of the metric structure of the neighborhood of points with small cone-injectivity radius.

By using the compactness and the local soul theorem, in Chapters 5 and 6 we study sequences of closed hyperbolic cone 3-manifolds with a fixed topological type.

In Chapter 5 we prove Theorem A, which deals with the case where the cone angles are bounded above, uniformly away from π.

In Chapter 6, we prove Theorem B which deals with the case where the cone angles converge to the orbifold angles.

In Chapter 7, we uniformize small 3-orbifolds of cyclic type by proving Theorem 2.

In Chapter 8, we first deduce the complete version of Thurston's orbifold theorem from Theorem 2 and Theorem 3. Then we give a detailed overview of the proof of Theorem 3.

In Chapter 9, we present explicit examples of collapses of hyperbolic cone structures to other geometric structures. These are the difficult phenomena that cannot be avoided in the proof of Theorem 4.

In Appendix A, M. Heusener and the second named author complete the results presented here by showing the following result: if a sequence of hyperbolic cone structures on a pair $(|\mathcal{O}|, \Sigma)$ collapses at angle π, then the closed orientable 3-orbifold $\mathcal{O}$ is not spherical.

In Appendix B, for completeness we give a detailled proof of Thurston's hyperbolic Dehn filling theorem for manifolds and orbifolds.

Acknowledgments. — It is clear that we are greatly indebted to William Thurston: without his seminal work and his generosity in sharing his ideas this monograph would not exist.

We wish to thank Bernhard Leeb and Peter Shalen for their interest in our work and many useful discussions. In particular Peter Shalen kindly pointed to us Theorem 3, in order to prove Thurston's orbifold theorem without the assumption that the orbifold is very good.

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CHAPTER 1

CONE MANIFOLDS

Cone 3-manifolds play a central role in the proof of Thurston's orbifold theorem. Thurston has shown that they appear naturally as a generalization of hyperbolic Dehn filling on cusped hyperbolic 3-manifolds. Thurston's hyperbolic Dehn filling theorem provides a family of cone 3-manifolds with small cone angles and the proof of Thurston's orbifold theorem analyzes the accidents that can occur when we increase the cone angles in order to reach the hyperbolic metric on the orbifold.

This chapter has two sections. In the first one we give the basic definitions for cone 3-manifolds (of non-positive curvature). In the second one we state some theorems about sequences of hyperbolic cone 3-manifolds, which are the key steps in the proof of Thurston's orbifold theorem.

1.1. Basic Definitions

In this monograph we only consider cone 3-manifolds of non-positive constant curvature. Moreover, we also restrict our attention to cone 3-manifolds whose singular set is a link and whose cone angles are less than 2π.

To fix notation, let $\mathbb{H}^3_K$ be the simply connected three-dimensional space of constant sectional curvature $K \leq 0$. Thus $\mathbb{H}^3_{-1} \cong \mathbb{H}^3$ is the usual hyperbolic space and $\mathbb{H}^3_0 \cong \mathbb{E}^3$ is the Euclidean space.

For $\alpha \in (0, 2\pi)$, let $\mathbb{H}^3_K(\alpha)$ be the cone manifold of constant curvature $K \leq 0$ with a singular line of cone angle α, constructed as follows. Consider in $\mathbb{H}^3_K$ a solid angular sector S_α obtained by taking the intersection of two half spaces, such that the dihedral angle at the (infinite) edge Δ is α. The cone manifold $\mathbb{H}^3_K(\alpha)$ is the length space obtained when we identify the faces of S_α by a rotation around Δ. The image of Δ in the quotient gives the singular line $\Sigma \subset \mathbb{H}^3_K(\alpha)$. The induced metric on $\mathbb{H}^3_K(\alpha) - \Sigma$ is a non-singular, incomplete Riemannian metric of constant curvature, whose completion is precisely $\mathbb{H}^3_K(\alpha)$.

In cylindrical or Fermi coordinates, the metric on $\mathbb{H}^3_K(\alpha) - \Sigma$ is:

$$ds^2_K = \begin{cases} dr^2 + \left(\frac{\alpha}{2\pi}\frac{\sinh(\sqrt{-K}r)}{\sqrt{-K}}\right)^2 d\theta^2 + \cosh^2(\sqrt{-K}r)dh^2 & \text{for } K < 0 \\ dr^2 + \left(\frac{\alpha}{2\pi}r\right)^2 d\theta^2 + dh^2 & \text{for } K = 0 \end{cases}$$

where $r \in (0, +\infty)$ is the distance from Σ, $\theta \in [0, 2\pi)$ is the rescaled angle parameter around Σ and $h \in \mathbb{R}$ is the distance along Σ.

Having described the local models, we can now define a cone 3-manifold.

Definition 1.1.1. — A *cone manifold* of dimension three and of constant curvature $K \leq 0$ is a smooth 3-manifold C equipped with a distance so that it is a complete length space locally isometric to $\mathbb{H}^3_K$ or $\mathbb{H}^3_K(\alpha)$ for some $\alpha \in (0, 2\pi)$.

The *singular locus* $\Sigma \subset C$ is the set of points modeled on the singular line of some model $\mathbb{H}^3_K(\alpha)$, and α is called the *cone angle* at a singular point modeled on this singular line. According to our definition, Σ is a submanifold of codimension two and the cone angle is constant along each connected component.

The topological pair (C, Σ) is called the *topological type* of the cone 3-manifold.

The induced metric on $C - \Sigma$ is a Riemannian metric of constant curvature, which is incomplete (unless $\Sigma = \emptyset$), and whose completion is precisely the cone 3-manifold.

By the *developing map of a cone 3-manifold* C with topological type (C, Σ), we mean the developing map of the induced metric on $C - \Sigma$:

$$D : \widetilde{C - \Sigma} \to \mathbb{H}^3_K,$$

where $\widetilde{C - \Sigma}$ is the universal covering of $C - \Sigma$. The associated holonomy representation

$$\rho : \pi_1(C - \Sigma) \to \operatorname{Isom}(\mathbb{H}^3_K)$$

is called the *holonomy representation of* C. If $\mu \in \pi_1(C - \Sigma)$ is represented by a meridian loop around a component Σ_0 of Σ, then $\rho(\mu)$ is a rotation of angle equal to the cone angle of this component.

Thurston's hyperbolic Dehn filling theorem provides many structures on a hyperbolic cusped 3-manifold whose completions are precisely cone 3-manifolds. The cone angles of these cone 3-manifolds are not necessarily less than 2π. The complete cusped structure on $C - \Sigma$ is the limit of these hyperbolic cone structures when the cone angles approach zero. We adopt therefore the standard convention that the cone angle at a component Σ_0 of Σ is zero when the end of $C - \Sigma$ corresponding to Σ_0 is a cusp, of rank 2 or 1 according to whether Σ_0 is compact or not.

We still need two more definitions.

A *standard ball* in a cone 3-manifold C is a ball isometric to either a metric non-singular ball in $\mathbb{H}^3_K$ or a metric singular ball in $\mathbb{H}^3_K(\alpha)$ whose center belongs to the singular axis.

We define the *cone-injectivity radius* at a point $x \in C$ as

$$\operatorname{inj}(x) = \sup\{\delta > 0 \text{ such that } B(x,\delta) \text{ is contained in a standard ball in } C\}.$$

We remark that the standard ball does not need to be centered at the point x. With this definition, regular points close to the singular locus do not have arbitrarily small cone-injectivity radius.

1.2. Sequences of hyperbolic cone 3-manifolds

Let $\mathcal{O}$ be an orbifold as in Theorem 4: a closed orientable irreducible very good 3-orbifold of cyclic type, such that the complement $\mathcal{O} - \Sigma$ of the ramification locus admits a complete hyperbolic structure of finite volume.

Thurston's hyperbolic Dehn filling theorem provides a one-parameter family of hyperbolic cone 3-manifolds with topological type $(|\mathcal{O}|, \Sigma)$. Consider the exterior $X = \mathcal{O} - \operatorname{int}(\mathcal{N}(\Sigma))$ of Σ and for each component $\Sigma_i \subset \Sigma$ choose a meridian curve $\mu_i \subset \partial\mathcal{N}(\Sigma_i)$ and another simple closed curve $\lambda_i \subset \partial\mathcal{N}(\Sigma_i)$ intersecting μ_i in one point; hence μ_i, λ_i generate $\pi_1(\partial\mathcal{N}(\Sigma_i))$.

According to Thurston's hyperbolic Dehn filling theorem, there exists a space of deformations of hyperbolic structures on $\operatorname{int}(X)$ parametrized by generalized Dehn coefficients (p_i, q_i), $1 \leq i \leq k$, in an open neighborhood $U \subset (\mathbb{R}^2 \cup \{\infty\})^k \cong (S^2)^k$ of $(\infty, \dots, \infty)$, where k is the number of connected components of ∂X (and of Σ). The structure at the i-th component of ∂X is described by the Dehn parameters as follows:

- When $(p_i, q_i) = \infty$, the structure at the corresponding cusp remains complete.
- When $p_i, q_i \in \mathbb{Z}$ are coprime, the completion $X((p_1, q_1), \dots, (p_k, q_k))$ is a hyperbolic 3-manifold, obtained by genuine Dehn filling with meridian curves $p_i\mu_i + q_i\lambda_i$, $i = 1, \dots, k$.
- When $p_i/q_i \in \mathbb{Q} \cup \{\infty\}$, let $r_i, s_i \in \mathbb{Z}$ be coprime integers so that $p_i/q_i = r_i/s_i$. Then the completion $X((p_1, q_1), \dots, (p_k, q_k))$ is a hyperbolic cone 3-manifold obtained by gluing solid tori with possibly singular cores. The underlying space is the 3-manifold $X((r_1, s_1), \dots, (r_k, s_k))$ and the cone angle of the i-th singular core is $2\pi r_i/p_i$, $i = 1, \dots, k$.

Here we are interested in the coefficients of the form $(p_i, q_i) = (n_i/t, 0)$, where $t \in [0, 1]$ and n_i is the branching index of the orbifold $\mathcal{O}$ along the i-th component $\Sigma_i \subset \Sigma$, for $i = 1, \dots, k$. Thurston's hyperbolic Dehn filling theorem implies the existence of a real number $\varepsilon_0 > 0$ such that, for any $t \in [0, \varepsilon_0]$, there is a deformation of the complete hyperbolic structure on $\operatorname{int}(X)$ whose completion $X\left((\frac{n_1}{t}, 0), \dots, (\frac{n_k}{t}, 0)\right)$ is a hyperbolic cone 3-manifold with topological type $(|\mathcal{O}|, \Sigma)$ and cone angles $\frac{2\pi}{n_1}t, \dots, \frac{2\pi}{n_k}t$.

The proof of Theorem 4 consists in studying the behavior of the hyperbolic cone 3-manifold $X\left((\frac{n_1}{t}, 0), \dots, (\frac{n_k}{t}, 0)\right)$ while increasing the parameter $t \in [0, 1]$. If the parameter $t = 1$ can be reached so that the cone 3-manifold $X((n_1, 0), \dots, (n_k, 0))$ remains hyperbolic, then the orbifold $\mathcal{O}$ itself is hyperbolic. Otherwise, there is a limit of hyperbolicity $t_\infty \in [0,1]$. Then, since the space of hyperbolic cone structures with topological type $(|\mathcal{O}|, \Sigma)$ and cone angles $\leq 2\pi$ is open, one has to analyze sequences of hyperbolic cone 3-manifolds $X\left((\frac{n_1}{t_n}, 0), \dots, (\frac{n_k}{t_n}, 0)\right)$ where $(t_n)_{n\in\mathbb{N}}$ is an increasing sequence in $[0, t_\infty)$ approaching t_∞. This analysis will be carried out in detail in Chapter 2, by using Theorems A and B below, which are central for the proof of Theorem 4, and should be of independent interest. Their proofs are given respectively in Chapter 5 and 6.

Theorem A is used when $t_\infty < 1$, while Theorem B is used when $t_\infty = 1$.

Theorem A. — *Let $(C_n)_{n\in\mathbb{N}}$ be a sequence of closed hyperbolic cone 3-manifolds with fixed topological type (C, Σ) such that the cone angles increase and are contained in $[\omega_0, \omega_1]$, with $0 < \omega_0 < \omega_1 < \pi$. Then there exists a subsequence $(C_{n_k})_{k\in\mathbb{N}}$ such that one of the following occurs:*

1) *The sequence $(C_{n_k})_{k\in\mathbb{N}}$ converges geometrically to a hyperbolic cone 3-manifold with topological type (C, Σ) whose cone angles are the limit of the cone angles of C_{n_k}.*
2) *For every k, C_{n_k} contains an embedded 2-sphere $S^2_{n_k} \subset C_{n_k}$ that intersects Σ in three points, and the sum of the three cone angles at $S^2_{n_k} \cap \Sigma$ converges to 2π.*
3) *There is a sequence of positive reals λ_k approaching 0 such that the subsequence of rescaled cone 3-manifolds $(\lambda_k^{-1} C_{n_k})_{k\in\mathbb{N}}$ converges geometrically to a Euclidean cone 3-manifold of topological type (C, Σ) and whose cone angles are the limit of the cone angles of C_{n_k}.*

Theorem B. — *Let $\mathcal{O}$ be a closed orientable connected irreducible very good 3-orbifold with topological type $(|\mathcal{O}|, \Sigma)$ and ramification indices $n_1, \dots, n_k$. Assume that there exists a sequence of hyperbolic cone 3-manifolds $(C_n)_{n\in\mathbb{N}}$ with the same topological type $(|\mathcal{O}|, \Sigma)$ and such that, for each component of Σ, the cone angles form an increasing sequence that converges to $2\pi/n_i$ when n approaches ∞.*

Then $\mathcal{O}$ contains a non-empty compact essential 3-suborbifold $\mathcal{O}' \subseteq \mathcal{O}$, which is not a product and which is either complete hyperbolic of finite volume, Euclidean, Seifert fibered or Sol.

As stated, these two theorems deal with geometric convergence of cone 3-manifolds. Up to minor modifications, the term geometric convergence stands for the pointed bilipschitz convergence introduced by Gromov [**GLP**]. The following compactness theorem plays a central role in the proofs of Theorems A and B. It is a cone manifold version of Gromov's compactness theorem for Riemannian manifolds with pinched sectional curvature (cf. [**GLP**] and [**Pe**]). The proof of this theorem is the main content of Chapter 3.

Compactness Theorem. — *Given $a > 0$ and $\omega \in (0, \pi]$, if $(C_n, x_n)_{n\in\mathbb{N}}$ is a sequence of pointed cone 3-manifolds with constant curvature in $[-1, 0]$, cone angles in $[\omega, \pi]$, and such that* $\operatorname{inj}(x_n) \geq a$, *then $(C_n, x_n)_{n\in\mathbb{N}}$ has a subsequence that converges geometrically to a pointed cone 3-manifold (C_∞, x_∞).*

The compactness theorem is used to analyze sequences of hyperbolic cone 3-manifolds which do not collapse. In the collapsing case, we need to rescale the metric in order to apply the compactness theorem. In fact we need a more precise result in the collapsing case, analogous to the "Local Approximation Proposition" of Cheeger and Gromov [**CGv**, Prop. 3.4], which furnishes a description of the (non-trivial) topology of neighborhoods of points with small cone-injectivity radius. This is the local soul theorem, which is the content of Chapter 4.

Local Soul Theorem. — *Given $\omega \in (0, \pi)$, $\varepsilon > 0$ and $D > 1$ there exist $\delta = \delta(\omega, \varepsilon, D) > 0$ and $R = R(\omega, \varepsilon, D) > D > 1$ such that, if C is an oriented hyperbolic cone 3-manifold with cone angles in $[\omega, \pi]$ and if $x \in C$ satisfies* $\operatorname{inj}(x) < \delta$, *then:*

- either C is $(1 + \varepsilon)$-bilipchitz homeomorphic to a compact Euclidean cone 3-manifold E of diameter $\operatorname{diam}(E) \leq R \operatorname{inj}(x)$;

- or there exists $0 < \nu < 1$, depending on x, such that x has an open neighborhood $U_x \subset C$ which is $(1 + \varepsilon)$-bilipschitz homeomorphic to the normal cone fiber bundle $\mathcal{N}_\nu(S)$, of radius ν, of the soul S of a non-compact orientable Euclidean cone 3-manifold with cone angles in $[\omega, \pi]$. In addition, according to $\dim(S)$, *the Euclidean non-compact cone 3-manifold belongs to the following list:*

I) (when $\dim(S) = 1$*), $S^1 \ltimes \mathbb{R}^2$, $S^1 \ltimes$ (open cone disk) and the solid pillow (see Figure 1 of Chapter 4), where $\ltimes$ denotes the metrically twisted product;*

II) (when $\dim(S) = 2$*)*

i) a product $T^2 \times \mathbb{R}$; $S^2(\alpha, \beta, \gamma) \times \mathbb{R}$, with $\alpha + \beta + \gamma = 2\pi$ (the thick turnover); $S^2(\pi, \pi, \pi, \pi) \times \mathbb{R}$ (the thick pillow);

ii) the orientable twisted line bundle over the Klein bottle $K^2 \tilde{\times} \mathbb{R}$ or over the projective plane with two silvered points $\mathbb{P}^2(\pi, \pi) \tilde{\times} \mathbb{R}$;

iii) a quotient by an involution of either $S^2(\pi, \pi, \pi, \pi) \times \mathbb{R}$, $T^2 \times \mathbb{R}$ or $K^2 \tilde{\times} \mathbb{R}$, that gives an orientable bundle respectively over either $D^2(\pi, \pi)$, an annulus, or a Möbius strip, with silvered boundary in the three cases (see Figure 2 of Chapter 4).

In addition, the $(1 + \varepsilon)$-bilipschitz homeomorphism $f : U_x \to \mathcal{N}_\nu(S)$ satisfies the inequality

$$\max\big(\operatorname{inj}(x), d(f(x), S), \operatorname{diam}(S)\big) \leq \nu / D.$$

These two theorems, compactness theorem and local soul theorem, are the main ingredients in the proofs of Theorems A and B. The assumption that cone angles are bounded above by π is crucial for the proof of both theorems (and cannot be removed). Geometrically this property is related to convexity: the Dirichlet polyhedron of a cone 3-manifold with cone angles bounded above by π is convex; moreover convex subsets of such cone 3-manifolds have nice properties.

One more ingredient for the study of the collapsing case is a cone manifold version of Gromov's isolation theorem [**Gro**, Sec. 3.4] (cf. Proposition 5.2.5). This involves the notion of simplicial volume due to Gromov [**Gro**].

A strengthening of Theorem A leads to the following Margulis type result (cf. Chapter 5):

Proposition 1. — *Given $0 < \omega_0 < \omega_1 < \pi$, there exists a positive constant $\delta_0 = \delta_0(\omega_0, \omega_1) > 0$ such that every oriented closed hyperbolic cone 3-manifold with cone angles in $[\omega_0, \omega_1]$ and diameter > 1 contains a point x with* $\mathrm{inj}(x) \geq \delta_0 > 0$.

Stronger thickness results, for general hyperbolic cone 3-manifolds (not assuming any more, the singular locus to be a link) can be found in [**BLP2**]. This is a part of the complete proof of Thurston's orbifold theorem, including the case where the singular locus has vertices, that we have written with B. Leeb in [**BLP1, BLP2**].

For cone angles bounded away $2\pi/3$, using Hamilton's theorem (cf. [**Zh2**, Thm 3.2]) we can get ride of the lower bound on the diameter:

Proposition 2. — *Given $0 < \omega_0 < \omega_1 < 2\pi/3$, there exists a positive constant $\delta_1 = \delta_1(\omega_0, \omega_1) > 0$ such that every oriented closed hyperbolic cone 3-manifold with cone angles in $[\omega_0, \omega_1]$ contains a point x with* $\mathrm{inj}(x) \geq \delta_1 > 0$.

CHAPTER 2

PROOF OF THURSTON'S ORBIFOLD THEOREM FOR VERY GOOD 3-ORBIFOLDS

In this chapter we prove Theorem 4, assuming Theorems A and B. Then we deduce Theorem 1 from it.

2.1. Generalized Hyperbolic Dehn Filling

Let M be a compact 3-manifold with non-empty boundary $\partial M = T_1^2 \cup \cdots \cup T_k^2$ a union of tori, whose interior is hyperbolic (complete with finite volume). Thurston's hyperbolic Dehn filling theorem provides a parametrization of a space of hyperbolic deformations of this structure on $\mathrm{int}(M)$.

To describe the deformations on the ends of $\mathrm{int}(M)$, we fix two simple closed curves μ_i and λ_i on each torus T_i^2 of the boundary, which generate $H_1(T_i^2, \mathbb{Z})$. The structure around the i-th end of $\mathrm{int}(M)$ is described by the generalized Dehn filling coefficients $(p_i, q_i) \in \mathbb{R}^2 \cup \{\infty\} = S^2$, such that the structure at the i-th end is complete iff $(p_i, q_i) = \infty$. The interpretation of the coefficients $(p_i, q_i) \in \mathbb{R}^2$ is the following:

- If $p_i, q_i \in \mathbb{Z}$ are coprime, then the completion at the i-th torus is a non-singular hyperbolic 3-manifold, which topologically is the Dehn filling with surgery meridian $p_i\mu_i + q_i\lambda_i$.
- When $p_i/q_i \in \mathbb{Q} \cup \{\infty\}$, let $r_i, s_i \in \mathbb{Z}$ be coprime integers such that $p_i/q_i = r_i/s_i$. The completion is a cone 3-manifold obtained by gluing a torus with singular core. The surgery meridian is $r_i\mu_i + s_i\lambda_i$ and the cone angle of the singular component is $2\pi|r_i/p_i|$.
- When $p_i/q_i \in \mathbb{R} - \mathbb{Q}$, then the completion (by equivalence classes of Cauchy sequences) is not topologically a manifold. These singularities are called of Dehn type, cf. [**Ho2**].

Theorem 2.1.1 **(Thurston's Hyperbolic Dehn Filling [Thu1])**

There exists a neighborhood $U \subset S^2 \times \cdots \times S^2$ of $\{\infty, \cdots, \infty\}$ such that the complete hyperbolic structure on $\mathrm{int}(M)$ *has a space of hyperbolic deformations parametrized by U via generalized Dehn filling coefficients.*

The proof yields not only the existence of a one parameter family of cone 3-manifold structures but also gives a path of corresponding holonomies in the variety $R(M)$ of representations of $\pi_1(M)$ into $SL_2(\mathbb{C})$. The holonomy of the complete structure on $\mathrm{int}(M)$ is a representation of $\pi_1(M)$ into $PSL_2(\mathbb{C})$ that can be lifted to $SL_2(\mathbb{C})$. A corollary of the proof of Thurston's hyperbolic Dehn filling theorem is the following:

Corollary 2.1.2. — *For any real numbers $\alpha_1, \ldots, \alpha_k \geq 0$ there exist $\varepsilon > 0$ and a path $\gamma : [0, \varepsilon) \to R(M)$, such that, for every $t \in [0, \varepsilon)$, $\gamma(t)$ is a lift of the holonomy of a hyperbolic structure on M corresponding to the generalized Dehn filling coefficients*

$$\big((p_1, q_1), \ldots, (p_k, q_k)\big) = \big((2\pi/(\alpha_1 t), 0), \ldots, (2\pi/(\alpha_k t), 0)\big).$$

When $\alpha_i t = 0$, the structure at the i-th cusp is complete; otherwise its completion is a cone 3-manifold obtained by adding to T_i^2 a solid torus with meridian curve μ_i and singular core with cone angle $\alpha_i t$.

2.2. The space of hyperbolic cone structures

Let $\mathcal{O}$ be an irreducible orientable connected closed 3-orbifold of cyclic type, with ramification locus Σ, and such that $\mathcal{O} - \Sigma$ admits a complete hyperbolic structure of finite volume. In this section we study the space of hyperbolic cone structures with topological type $(\mathcal{O}, \Sigma)$. The main result of this section, Proposition 2.2.4, can be deduced from Thurston's Hyperbolic Dehn filling theorem (Theorem 2.1.1) and Hodgson-Kerckhoff rigidity theorem [**HK**]. Nevertheless we present here an elementary proof, based only on Thurston's Dehn filling theorem, but independent of Hodgson-Kerckhoff rigidity theorem.

Notation 2.2.1. — Let $m_1, \ldots, m_q$ be the ramification indices of $\mathcal{O}$ along Σ. We set

$$\alpha = (\alpha_1, \ldots, \alpha_q) = (\frac{2\pi}{m_1}, \ldots, \frac{2\pi}{m_q}).$$

For $t \in [0, 1]$, let $C(t\alpha)$ denote the hyperbolic cone 3-manifold having the same topological type as the orbifold $\mathcal{O}$ and cone angles $t\alpha = (t\alpha_1, \ldots, t\alpha_q)$ (the ordering of the components of Σ is fixed throughout this section). With this notation, $C(0)$ is the complete hyperbolic structure of finite volume on $\mathcal{O} - \Sigma$.

Thurston's hyperbolic Dehn filling theorem (Corollary 2.1.2) means that for small values of $t > 0$ the hyperbolic cone 3-manifold $C(t\alpha)$ exists. Thurston's idea is to increase t whilst keeping $C(t\alpha)$ hyperbolic and to study the limit of hyperbolicity.

More precisely, consider the variety of representations of $\pi_1(\mathcal{O}-\Sigma)$ into $SL_2(\mathbb{C})$,

$$R := \mathrm{Hom}(\pi_1(\mathcal{O}-\Sigma), SL_2(\mathbb{C})).$$

Since $\pi_1(\mathcal{O}-\Sigma)$ is finitely generated, R is an affine algebraic subset of $\mathbb{C}^N$ (it is not necessarily irreducible). The holonomy representation of the complete hyperbolic structure on $\mathcal{O}-\Sigma$ lifts to a representation ρ_0 into $SL_2(\mathbb{C})$, which is a point of R. Let R_0 be an irreducible component of R containing ρ_0.

Definition 2.2.2. — Define the subinterval $J \subseteq [0,1]$ to be:

$$J := \left\{ t \in [0,1] \;\middle|\; \begin{array}{l} \text{there exists a path } \gamma : [0,t] \to R_0 \\ \text{such that, for every } s \in [0,t], \\ \gamma(s) \text{ is a lift of the holonomy} \\ \text{of a hyperbolic cone 3-manifold } C(s\alpha) \end{array} \right\}$$

Remark 2.2.3. — We say *"a"* hyperbolic cone 3-manifold $C(s\alpha)$, since we do not use the uniqueness of the hyperbolic cone structure for $s>0$, proved in [**Koj**].

By hypothesis, $J \neq \emptyset$ because $0 \in J$ (i.e. $\mathcal{O}-\Sigma$ has a complete hyperbolic structure).

Proposition 2.2.4. — *The interval J is open in $[0,1]$.*

Proof. — The fact that J is open at the origin is a consequence of Thurston's hyperbolic Dehn filling theorem, as seen in Corollary 2.1.2.

Let $\mu = (\mu_1, \dots, \mu_q)$ be the meridians of Σ. That is, $\mu_i \in \pi_1(\mathcal{O}-\Sigma)$ represents a meridian of the i-th component of Σ, for $i = 1, \dots, q$. Note that μ_i is not unique, only the conjugacy class of $\mu_i^{\pm 1}$ is unique. We consider the regular map:

$$\begin{array}{rcl} Tr_\mu : R_0 & \to & \mathbb{C}^q \\ \rho & \mapsto & (\mathrm{trace}(\rho(\mu_1)), \dots, \mathrm{trace}(\rho(\mu_q))). \end{array}$$

Claim 2.2.5. — *There exists a unique affine irreducible curve $\mathcal{C} \subset \mathbb{C}^q$ such that, for any $t \in J$, $Tr_\mu(\gamma([0,t])) \subset \mathcal{C}$.*

Proof of the claim. — For $n \in \mathbb{N}$, consider the Chebyshev-like polynomial $p_n(x) = 2\cos(n \arccos(x/2))$. It is related to the classical Chebyshev polynomial by a linear change of variable. It can also be defined inductively by the rule

$$\begin{cases} p_0(x) = 2, \qquad p_1(x) = x, \\ p_n(x) = x\, p_{n-1}(x) - p_{n-2}(x), \text{ for } n \in \mathbb{N}, n > 1. \end{cases}$$

We are interested in the following property of polynomials p_n:

$$\mathrm{trace}(M^n) = p_n(\mathrm{trace}(M)), \qquad \forall M \in SL_2(\mathbb{C}),\ \forall n \in \mathbb{N}.$$

Let $\rho_0 = \gamma(0)$ be the lift of the holonomy corresponding to the complete finite volume hyperbolic structure on $\mathcal{O}-\Sigma$. Since ρ_0 applied to a meridian is parabolic, $Tr_\mu(\rho_0) = Tr_\mu(\gamma(0)) = (\epsilon_1 2, \dots, \epsilon_q 2)$, with $\epsilon_1, \dots, \epsilon_q \in \{\pm 1\}$.

We take $\mathcal{C}$ to be the irreducible component of the algebraic set

$$\{z \in \mathbb{C}^q \mid p_{m_1}(\epsilon_1 z_1) = \cdots = p_{m_q}(\epsilon_q z_q)\}$$

that contains $Tr_\mu(\rho_0)$.

To show that the component $\mathcal{C}$ is well defined and is a curve, we use the following identity:

$$p_n'(2) = n^2, \qquad \forall n \in \mathbb{N}.$$

It follows from this formula that $Tr_\mu(\rho_0)$ is a smooth point of $\{p_{m_1}(\epsilon_1 z_1) = \cdots = p_{m_q}(\epsilon_q z_q)\}$ of local dimension 1. Thus $\mathcal{C}$ is the only irreducible component containing $Tr_\mu(\rho_0)$ and it is a curve.

Finally to prove that $Tr_\mu(\gamma([0,t])) \subset \mathcal{C}$, we consider the analytic map:

$$\begin{array}{rcl} \Theta : \mathbb{C} & \to & \mathbb{C}^q \\ w & \to & (\epsilon_1 2\cos(w\pi/m_1), \ldots, \epsilon_q 2\cos(w\pi/m_q)) \end{array}$$

Since $\gamma(s)(\mu_i)$ is a rotation of angle $s\pi/m_i$ and $Tr_\mu(\gamma(0)) = (\epsilon_1 2, \ldots, \epsilon_q 2)$, it is clear that $Tr_\mu(\gamma([0,t])) \subset \Theta(\mathbb{C})$. By construction,

$$\Theta(\mathbb{C}) \subset \{p_{m_1}(\epsilon_1 z_1) = \cdots = p_{m_q}(\epsilon_q z_q)\}.$$

Since analytic irreducibility implies algebraic irreducibility, $\Theta(\mathbb{C}) \subset \mathcal{C}$, and the claim is proved. $\square$

Claim 2.2.6. — *For every $t \in J$, there exists an affine curve $\mathcal{D} \subset R_0$ containing $\gamma(t)$ and such that the restricted map $Tr_\mu : \mathcal{D} \to \mathcal{C}$ is dominant.*

Proof. — We distinguish two cases, according to whether $t > 0$ or $t = 0$.

When $t > 0$, we take an irreducible component $\mathcal{Z}$ of $Tr_\mu^{-1}(\mathcal{C})$ that contains the path $\gamma([t-\varepsilon, t])$, for some $\varepsilon > 0$. Since

$$Tr_\mu(\gamma(s)) = (\epsilon_1 2\cos(s\pi/m_1), \ldots, \epsilon_q 2\cos(s\pi/m_q)),$$

the rational map $Tr_\mu : \mathcal{Z} \to \mathcal{C}$ is not constant, hence dominant. By considering generic intersection with hyperplanes we can find the curve $\mathcal{D}$ of the claim. More precisely, we intersect $\mathcal{Z}$ with a generic hyperplane H passing through $\gamma(t)$ and such that it does not contain $Tr_\mu^{-1}(Tr_\mu(\gamma(t))) \cap \mathcal{Z}$. Such a hyperplane H exists because $Tr_\mu : \mathcal{Z} \to \mathcal{C}$ is dominant. By construction $Tr_\mu : \mathcal{Z} \cap H \to \mathcal{C}$ is not constant and the dimension of $\mathcal{Z} \cap H$ is less than the dimension of $\mathcal{Z}$. By induction we obtain a curve $\mathcal{D}$.

When $t = 0$, we consider again an irreducible component $\mathcal{Z}$ of $Tr_\mu^{-1}(\mathcal{C})$ that contains ρ_0. In this case Thurston's hyperbolic Dehn filling theorem implies that the restriction $Tr_\mu : X \to \mathcal{C}$ is not constant. By considering intersection with generic hyperplanes as before, we obtain the curve $\mathcal{D}$ of the claim. $\square$

We now conclude the proof of Proposition 2.2.4. Given $t \in J$, let $\mathcal{C}$ and $\mathcal{D}$ be as in Claims 2.2.5 and 2.2.6. Since $\mathcal{D}$ and $\mathcal{C}$ are curves, for some $\varepsilon > 0$, the path

$$\begin{array}{rcl} g : [t, t+\varepsilon) & \to & \mathcal{C} \subset \mathbb{C}^q \\ s & \mapsto & (\epsilon_1 2\cos(s\pi/m_1), \ldots, \epsilon_q 2\cos(s\pi/m_q)) \end{array}$$

can be lifted through $Tr_\mu : \mathcal{D} \to \mathcal{C}$ to a map $\tilde{g} : [t, t+\varepsilon) \to \mathcal{D}$. This map $\tilde{g}$ is a continuation of γ.

It remains to check that this algebraic continuation of γ corresponds to the holonomy representations of hyperbolic cone 3-manifolds. To show this, we use Lemma 1.7.2 of Canary, Epstein and Green's Notes on notes of Thurston [**CEG**]. This lemma (called "Holonomy induces structure") proves that, for $s \in [t, t+\varepsilon)$, $\tilde{g}(s)$ is the holonomy of a hyperbolic structure on the complement $C(t\alpha) - \mathcal{N}_r(\Sigma)$ of a tubular neighborhood of the singular set, with arbitrarily small radius $r > 0$. The construction of the hyperbolic structure in a tubular neighborhood of the singular set needs some careful but elementary analysis. In [**Po2**] this is done in the case where the structure is deformed from Euclidean to hyperbolic geometry, and the constant curvature case is somewhat simpler. This finishes the proof of Proposition 2.2.4. $\square$

Remark 2.2.7. — It follows from the proof that, for every $t \in J$, the path $\gamma : [0, t] \to R_0$ of holonomies of hyperbolic cone structures is piecewise analytic. This is useful for applying Schläfli's formula [**Po2**, Prop. 4.2]).

The following technical lemma will be used in the next section.

Lemma 2.2.8. — *The dimension of $Tr_\mu^{-1}(\mathcal{C})$ is 4.*

Proof. — The proof of Thurston's hyperbolic Dehn filling theorem uses the fact that the local dimension of $Tr_\mu^{-1}(Tr_\mu(\rho_0))$ at ρ_0 is 3, because of Weil's local rigidity theorem. Moreover, the dimension of the preimage of any point in $\mathcal{C}$ is at least 3, because this is the dimension of $SL_2(\mathbb{C})$. Since $Tr_\mu : Tr_\mu^{-1}(\mathcal{C}) \to \mathcal{C}$ is dominant, the dimension of $Tr_\mu^{-1}(\mathcal{C})$ is 4. $\square$

2.3. Proof of Theorem 4 from Theorems A and B

Theorem 4. — *Let $\mathcal{O}$ be a closed orientable connected irreducible very good 3-orbifold of cyclic type. Assume that the complement $\mathcal{O} - \Sigma$ of the branching locus admits a complete hyperbolic structure. Then $\mathcal{O}$ contains a non-empty compact essential 3-suborbifold $\mathcal{O}' \subseteq \mathcal{O}$, which is not a product and which is either complete hyperbolic of finite volume, Euclidean, Seifert fibered or Sol.*

Proof of Theorem 4 from Theorems A and B. — We start with the subinterval $J \subseteq [0, 1]$ as in Section 2.2. Recall that J is the set of real numbers $t \in [0, 1]$ such that there is a path $\gamma : [0, t] \to R_0$ with the property that, for every $s \in [0, t]$, $\gamma(s)$

is the holonomy of a hyperbolic cone 3-manifold $C(s\alpha)$. The hyperbolic cone 3-manifold $C(s\alpha)$ has the same topological type as $\mathcal{O}$ and its cone angles are $s\alpha = (s\,2\pi/m_1, \dots, s\,2\pi/m_q)$.

By Proposition 2.2.4, J is open in $[0,1]$, and moreover $0 \in J$ by hypothesis. So there are three possibilities: either $J=[0,1]$, $J=[0,1)$, or $J=[0,t)$ with $0<t<1$.

If $J=[0,1]$ then $\mathcal{O}$ is hyperbolic. Propositions 2.3.1 and 2.3.7 deal with the cases where $J=[0,t)$ with $0<t<1$ and $J=[0,1)$ respectively.

Proposition 2.3.1. — *If $J=[0,t)$ with $0<t<1$, then $\mathcal{O}$ is a spherical 3-orbifold.*

Proof. — Fix $(t_n)_{n\in\mathbb{N}}$ an increasing sequence in $J=[0,t)$ converging to t and consider the corresponding sequence of cone 3-manifolds $C_n = C(t_n\alpha)$. The cone 3-manifolds C_n have the same topological type as $\mathcal{O}$, and the cone angles are contained in some interval $[\omega_0,\omega_1]$, with $0<\omega_0<\omega_1<\pi$, because $0<t<1$. Thus we can apply Theorem A to the sequence $(C_n)_{n\in\mathbb{N}}$, and, after perhaps passing to a subsequence, we have three possibilities:

i) the sequence $(C_n)_{n\in\mathbb{N}}$ converges geometrically to a hyperbolic cone 3-manifold with the same topological type;
ii) each C_n contains an embedded sphere S_n which intersects Σ in 3 points and the sum of the cone angles at these points converges to 2π;
iii) there is a sequence of positive reals $\lambda_n \to 0$ such that $\left(\frac{1}{\lambda_n}C_n\right)_{n\in\mathbb{N}}$ converges geometrically to a Euclidean cone 3-manifold with the same topologica

We want to show that only the last possibility occurs.

If case i) happens, we claim that $t\in J$; this would contradict the hypothesis $J=[0,t)$. Let C_∞ be the limit of the sequence C_n. Since the convergence is geometric, the cone angles of C_∞ are precisely $t\alpha$. Therefore $C(t\alpha)$ is hyperbolic and it remains to show the existence of a path of holonomy representations from 0 to t. To show that, we take a path γ_n for each ρ_n and we prove that the sequence of paths γ_n has a convergent subsequence.

Lemma 2.3.2. — *The sequence of paths γ_n has a subsequence converging to a path γ_∞. Moreover, up to conjugation, for n sufficiently large, γ_∞ is a continuation of γ_n.*

Proof of Lemma 2.3.2. — Consider the algebraic affine set $V = Tr_\mu^{-1}(\mathcal{C})$ and its quotient by conjugation $X = V/PSL_2(\mathbb{C})$. The space X may not be Hausdorff, but since the holonomy of a closed hyperbolic cone 3-manifold is irreducible [**Po1**, Prop. 5.4], the points we are interested in (conjugacy classes of holonomy representations) have neighborhoods that are analytic (see for instance [**CS**] or [**Po1**, Prop. 3.4]). If we remove all reducible representations, then the quotient is analytic, even affine [**CS**], call it X^{irr}. By lemma 2.2.8, X^{irr} is a curve. Moreover, the holonomies of hyperbolic cone structures are contained in a real curve of X^{irr}, because the traces of the meridians are real. Hence, up to conjugation, the paths γ_n are contained in a real analytic curve. Thus, ρ_n converges to ρ_∞, the sequence γ_n has a convergent subsequence, and the limit γ_∞ is a continuation of γ_n. □

It follows from this lemma that the limit γ_∞ is a path of holonomy representations of hyperbolic cone structures. Hence $t \in J$ and we obtain a contradiction.

Next we suppose that case ii) occurs. That is, for each $n \in \mathbb{N}$, $S^2_n \subset C_n$ is an embedded 2-sphere which intersects Σ in three points and the sum of the cone angles at these points converges to 2π. Since Σ has a finite number of components, after passing to a subsequence, we can suppose that S^2_n intersects always the same components of Σ. Let m_1, m_2 and m_3 be the branching indices of the components of Σ which intersect S^2_n, counted with multiplicity if one component of Σ intersects S^2_n more than once. Since the sum of the cone angles at these points converges to 2π, we have:

$$2\pi \leq t\left(\frac{2\pi}{m_1} + \frac{2\pi}{m_2} + \frac{2\pi}{m_3}\right) < 2\pi\left(\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3}\right),$$

because $t < 1$. If we view S^2_n as a 2-suborbifold $F \subset \mathcal{O}$, then F is spherical, because its underlying space is $|F| \cong S^2$ and it has three points of ramification with branching indices m_1, m_2 and m_3, where $\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} > 1$. The suborbifold F cannot bound a discal 3-orbifold, since we assume that the ramification set of $\mathcal{O}$ is a link. This contradicts the irreducibility of the 3-orbifold $\mathcal{O}$, so case ii) cannot happen.

So far we have eliminated cases i) and ii). Now we prove from case iii) that $\mathcal{O}$ is a spherical 3-orbifold.

Case iii) implies that there is a Euclidean cone 3-manifold $C(t\alpha)$ with the same topological type as $\mathcal{O}$ and with cone angles $t\alpha = (t\,2\pi/m_1, \ldots, t\,2\pi/m_q)$, where $0 < t < 1$. We first prove that in this case the 3-orbifold $\mathcal{O}$ is very good.

Lemma 2.3.3. — *If there is a Euclidean cone 3-manifold $C(t\alpha)$ with the same topological type as the orbifold $\mathcal{O}$ and with cone angles $t\alpha = (t\,2\pi/m_1, \ldots, t\,2\pi/m_q)$, where $0 < t < 1$, then $\mathcal{O}$ is a very good 3-orbifold.*

Proof of Lemma 2.3.3. — First we deform the singular Euclidean metric induced by $C(t\alpha)$ on the underlying manifold $|\mathcal{O}|$ of $\mathcal{O}$ to a Riemannian metric with non-negative sectional curvature (cf. [**Jon**], [**GT**], [**Zh1, Zh2**]).

Let $\Sigma \subset |\mathcal{O}|$ be the singular locus of this Euclidean metric, which is also the ramification locus of the orbifold $\mathcal{O}$. We deform the metric on its tubular neighborhood $\mathcal{N}_{r_0}(\Sigma)$ of radius r_0, for some $r_0 > 0$ sufficiently small. Around Σ, the local expression of the singular Euclidean metric in Fermi (cylindrical) coordinates is:

$$ds^2 = dr^2 + t^2 r^2 d\theta^2 + dh^2,$$

where $r \in (0, r_0)$ is the distance from Σ, h is the length parameter along Σ, and $\theta \in (0, 2\pi)$ is the rescaled angle parameter.

The deformation we are introducing depends only on the parameter r. This deformation consists of replacing the above metric by a metric of the form

$$ds^2 = dr^2 + f^2(r)d\theta^2 + dh^2,$$

where $f : [0, r_0 - \varepsilon) \to [0, +\infty)$ is a smooth function that satisfies, for some $\varepsilon > 0$ sufficiently small:

1) $f(r) = r$, for all $r \in [0, t\,\varepsilon)$;
2) $f(r) = t\,r + t\,\varepsilon$, for all $r \in (r_0/2, r_0 - \varepsilon)$,
3) f is concave: $f''(r) \leq 0$, for all $r \in [0, r_0 - \varepsilon)$.

Such a function f exists because $0 < t < 1$. The first property implies that the new metric is non-singular. Property 2) implies that, after reparametrization, this new non-singular metric fits with the original singular metric at the boundary of the tubular neighborhood $\mathcal{N}_{r_0}(\Sigma)$. A classical computation shows that the sectional curvature of the planes orthogonal to Σ is non-negative, by property 3), and it is even positive at some point. Hence, since the metric is locally a product, it has non-negative sectional curvature.

We show now that the manifold $|\mathcal{O}|$ admits a Riemannian metric of constant positive sectional curvature (i.e. is spherical), by applying the following deep theorem of Hamilton ([**Ha1, Ha2**], see also [**Bou**]).

***Theorem 2.3.4* (Hamilton [Ha1, Ha2]).** — *Let N^3 be a closed 3-manifold which admits a Riemannian metric of non-negative Ricci curvature. Then N^3 admits a metric which is either spherical, flat or modelled on $\mathbb{S}^2 \times \mathbb{R}$.*

Furthermore the deformation is natural, and every isometry of the original metric is also an isometry of the new metric. □

Remark 2.3.5. — It follows from Hamilton's proof [**Ha2**] that the flat case occurs only if the initial metric on N^3 was already flat, and that the case modelled on $\mathbb{S}^2 \times \mathbb{R}$ occurs only if the initial metric had reducible holonomy, contained in $SO(2)$.

We apply Hamilton's Theorem 2.3.4 to the metric on $|\mathcal{O}|$ given by Lemma 2.3.3. According to the remark, the flat case of Hamilton's theorem does not occur because the initial metric was not flat. Moreover, we can also eliminate the case $\mathbb{S}^2 \times \mathbb{R}$, because this case would imply that the singular Euclidean cone structure on $|\mathcal{O}|$ is of Seifert type ($|\mathcal{O}|$ admits a Seifert fibration such that the singular locus is an union of fibers). This follows, for instance, from [**Po2**, Lemma 9.1]. Thus Hamilton's theorem implies that $|\mathcal{O}|$ admits a spherical metric.

Therefore, up to passing to a finite cover, we can assume that the underlying space $|\mathcal{O}|$ of $\mathcal{O}$ is S^3. Since the ramification set is a link, $\mathcal{O}$ is a very good 3-orbifold. More precisely, $\mathcal{O}$ admits a finite abelian regular covering which is a manifold. This proves Lemma 2.3.3. □

Let $M \to \mathcal{O}$ be a regular covering of $\mathcal{O}$ with finite deck transformation group G, such that M is a manifold. Since $t < 1$, the Euclidean cone 3-manifold $C(t\,\alpha)$ induces a G-invariant Euclidean cone manifold structure on M, with singular angles $t\,2\pi < 2\pi$, from which we deduce (cf. [**Jon**], [**GT**], [**Zh1, Zh2**]):

Lemma 2.3.6. — *The manifold M admits a non-singular G-invariant Riemannian metric with constant positive sectional curvature.*

Proof of Lemma 2.3.6. — First, we deform the singular Euclidean metric on M (lifted from the one induced by $C(t\alpha)$ on $|\mathcal{O}|$) in a G-invariant way to a G-invariant Riemannian metric on M, which is not flat and has non-negative sectional curvature.

Let $\Sigma_G \subset M$ be the singular set of this Euclidean metric, which is also the set of points where the action of G is not free. We deform the metric in a tubular neighborhood of the singular set $\mathcal{N}_{r_0}(\Sigma_G)$ of radius r_0, for some $r_0 > 0$ sufficiently small, exactly in the same way as in the proof of Lemma 2.3.3. Since the deformation introduced in Lemma 2.3.3 depends only on the radial parameter r, it is G-invariant. Hence we get a G-invariant Riemannian metric which is non flat and has non-negative sectional curvature.

Now, Hamilton's Theorem 2.3.4 implies that this G-invariant Riemannian metric on M can be deformed to a G-invariant Riemannian metric with positive constant sectional curvature. This is because the initial metric is not flat. Moreover $M - \Sigma_G$ admits a complete hyperbolic structure lifted from the one of $\mathcal{O} - \Sigma$, and hence cannot be Seifert fibered. $\square$

This G-invariant spherical metric on M induces a spherical metric on the 3-orbifold $\mathcal{O}$. This concludes the proof of Proposition 2.3.1 $\square$

We complete now the proof of Theorem 4 by dealing with the case where $J = [0, 1)$:

Proposition 2.3.7. — *If $J = [0, 1)$ then $\mathcal{O}$ contains a non-empty compact essential 3-suborbifold which is not a product and which is geometric.*

Proof Proposition 2.3.7. — Let $(t_n)_{n\in\mathbb{N}}$ be a sequence in $[0, 1)$ converging to 1. We apply Theorem B to the corresponding sequence of hyperbolic cone 3-manifolds $(C(t_n\alpha))_{n\in\mathbb{N}}$ whose cone angles form an increasing sequence that converges to $2\pi/n_i$, $i = 1, \dots, k$, when n goes to ∞. By Theorem B, $\mathcal{O}$ contains a non-empty compact essential 3-suborbifold $\mathcal{O}' \subseteq \mathcal{O}$ which is either Euclidean, Seifert fibered, *Sol*, or hyperbolic of finite volume, and which is not a product. $\square$

Since Propositions 2.3.1 and 2.3.7 are proved, the proof of Theorem 4 from Theorems A and B is finished. $\square$

2.4. Proof of Theorem 1

Theorem 1. — *Let $\mathcal{O}$ be a compact, connected, orientable, irreducible and boundary-incompressible 3-orbifold of cyclic type. If $\mathcal{O}$ is very good, topologically atoroidal and acylindrical, then $\mathcal{O}$ is geometric (i.e. $\mathcal{O}$ admits either a hyperbolic, a Euclidean, or a Seifert fibered structure).*

Throughout this section we assume that $\mathcal{O}$ is a 3-orbifold which satisfies the hypothesis of Theorem 1.

Let $D\mathcal{O}$ denote the *double* of $\mathcal{O}$ along *some* components of $\partial\mathcal{O}$, which we call *doubling components*. The ramification set of $D\mathcal{O}$ is denoted by $D\Sigma$. If we double along the empty set, then we choose the convention that $D\mathcal{O} = \mathcal{O}$, so that $D\mathcal{O}$ is always connected.

First we give some results about the topology and the geometry of $D\mathcal{O}$ and $D\mathcal{O} - D\Sigma$ in order to reduce the general case to the case where the hypothesis of Theorem 4 are satisfied. Then we deduce Theorem 1 from Theorem 4.

Throughout Lemmas 2.4.1 to 2.4.7 we assume only that $\mathcal{O}$ is a compact orientable irreducible and ∂-incompressible 3-orbifold which is topologically atoroidal and acylindrical. The hypothesis that the 3-orbifold $\mathcal{O}$ is *very good* will be *used only in Proposition 2.4.9*, to reduce the proof of Theorem 1 to the case where $\partial\mathcal{O}$ does not contain any non-singular torus component.

Lemma 2.4.1. — *Let $\mathcal{O}$ be a compact connected orientable irreducible ∂-incompressible 3-orbifold which is topologically atoroidal and acylindrical. For any choice of doubling components,*

i) $D\mathcal{O}$ is irreducible and topologically acylindrical;
ii) every component of $\partial\mathcal{O}$ is incompressible in $D\mathcal{O}$;
iii) every incompressible toric 2-suborbifold of $D\mathcal{O}$ is parallel to $\partial\mathcal{O} \subset D\mathcal{O}$.

In particular $D\mathcal{O}$ is boundary-incompressible. Furthermore, $D\mathcal{O}$ is topologically atoroidal iff every doubling component is a hyperbolic 2-suborbifold.

Proof of Lemma 2.4.1. — Let $S \subset D\mathcal{O}$ be a spherical 2-suborbifold. After isotopy, we can suppose that S is transverse to $\partial\mathcal{O}$ and that the intersection $S \cap \partial\mathcal{O}$ is minimal. We claim that $S \subset \mathcal{O}$. Seeking a contradiction, we suppose $S \not\subset \mathcal{O}$. Since S is a sphere with at most three cone points, at least one component of $S \cap \mathcal{O}$ is a disk Δ^2 with at most one cone point. Since $\mathcal{O}$ is irreducible, $\partial\Delta^2$ is essential in $\partial\mathcal{O}$ by minimality. Hence Δ^2 is a compressing disk for $\partial\mathcal{O}$, and we get a contradiction because $\mathcal{O}$ is boundary-incompressible. Therefore $S \subset \mathcal{O}$ and S bounds a discal 3-orbifold by irreducibility of $\mathcal{O}$. The same argument goes through to show that $D\mathcal{O}$ does not contain any bad 2-suborbifold. Hence $D\mathcal{O}$ is also irreducible.

Let $A \subset D\mathcal{O}$ be a properly embedded annular 2-suborbifold. Again we deform it so that $A \cap \partial\mathcal{O}$ is transverse and minimal. No component of $A \cap \mathcal{O}$ is a discal orbifold, because $\partial\mathcal{O}$ is incompressible and the intersection $A\cap\partial\mathcal{O}$ is minimal. Hence $A = A_1 \cup \cdots \cup A_k$, where each A_i is an annular 2-suborbifold properly embedded in one of the copies of $\mathcal{O}$. If $k > 1$, then, by minimality of the intersection, none of the annuli A_i is parallel to $\partial\mathcal{O}$ nor compressible in $\mathcal{O}$, contradicting the acylindricity of $\mathcal{O}$. Hence $k = 1$ and $A \subset \mathcal{O}$ is not essential. This proves that $D\mathcal{O}$ is topologically acylindrical.

To show that every component of $\partial\mathcal{O}$ is incompressible in $D\mathcal{O}$, suppose that $\partial\mathcal{O}$ has a compressing disk $\Delta^2 \subset D\mathcal{O}$. By making the intersection $\Delta^2 \cap \partial\mathcal{O}$ minimal, every disk component of $\Delta^2 \cap \mathcal{O}$ is a compressing disk for $\partial\mathcal{O}$ in $\mathcal{O}$, thus we obtain a contradiction that proves assertion ii).

Finally, let $F \subset D\mathcal{O}$ be an incompressible toric 2-suborbifold. After an isotopy, we can again make the intersection $F \cap \partial\mathcal{O}$ transverse and minimal. If $F \cap \partial\mathcal{O} \neq \emptyset$, then the minimality of the intersection implies that each component of $F \cap \mathcal{O}$ is an essential annular 2-suborbifold, and we get a contradiction. Thus $F \subset \mathcal{O}$ and F is parallel to $\partial\mathcal{O}$. □

Lemma 2.4.2. — *Let $\mathcal{O}$ be a compact connected orientable irreducible ∂-incompressible 3-orbifold which is topologically atoroidal and acylindrical. If every doubling component is different from a non-singular torus, then the manifold $D\mathcal{O} - D\Sigma$ is irreducible and topologically atoroidal.*

Proof of Lemma 2.4.2. — Let $S \subset D\mathcal{O} - D\Sigma$ be an embedded 2-sphere. It bounds a discal 3-suborbifold Δ^3 in $D\mathcal{O}$, because $D\mathcal{O}$ is irreducible. Since $S \cap D\Sigma = \emptyset$, Δ^3 is a 3-ball and $\Delta^3 \cap D\Sigma = \emptyset$. So $D\mathcal{O} - D\Sigma$ is irreducible.

Let $T \subset D\mathcal{O} - D\Sigma$ be an embedded torus. By Lemma 2.4.1 iii), either T is compressible in $D\mathcal{O}$ or parallel to a component of $\partial\mathcal{O} \subset D\mathcal{O}$. In this last case, since the doubling components are different from tori, T must be boundary parallel in $D\mathcal{O}$, and thus it is also boundary parallel in $D\mathcal{O} - D\Sigma$. If T admits a compressing disk (a discal 2-suborbifold), then the irreducibility of $D\mathcal{O}$ implies that T bounds either a solid torus or a solid torus with ramified core $S^1 \times D^2(*)$. In the former case T is compressible in $D\mathcal{O} - D\Sigma$; in the latter case, T is boundary parallel in $D\mathcal{O} - D\Sigma$. □

Lemma 2.4.3. — *Let $\mathcal{O}$ be a compact connected orientable irreducible ∂-incompressible 3-orbifold.*

i) For any choice of doubling components, if $D\mathcal{O}$ is Euclidean or Seifert fibered then $\mathcal{O}$ is Euclidean, Seifert fibered, or an I-bundle over a 2-orbifold.

ii) If the doubling components are non-empty, then $D\mathcal{O}$ is not Sol.

Proof of Lemma 2.4.3. — First suppose that $D\mathcal{O}$ is Euclidean or Seifert fibered. We can assume that $\partial\mathcal{O} \neq \emptyset$, otherwise the statement is trivial. Moreover $D\mathcal{O}$ admits a finite regular manifold covering N, because $D\mathcal{O}$ is irreducible by Lemma 2.4.1, and therefore it is geometric. The fundamental group $\pi_1(N)$ is infinite, because each component of $\partial\mathcal{O}$ lifts to incompressible surfaces in N, with infinite fundamental group.

If $D\mathcal{O}$ is Seifert fibered, by [**BS1**, Thm. 4], $\partial\mathcal{O}$ is either isotopic to a vertical (i.e. fibered) or to a horizontal (i.e tranverse to the Seifert fibration), 2-suborbifold because it is incompressible (Lemma 2.4.1). Therefore $\mathcal{O}$ is either Seifert fibered or an I-bundle over a 2-orbifold.

If $D\mathcal{O}$ is Euclidean but $\partial D\mathcal{O}$ is not empty, then $D\mathcal{O}$ admits a Seifert fibration, hence, as above, $\mathcal{O}$ is either Seifert fibred or an I-bundle.

To handle the case where $D\mathcal{O}$ is Euclidean and $\partial D\mathcal{O}$ is empty we consider the natural involution $\tau : D\mathcal{O} \to D\mathcal{O}$ obtained by reflection through the doubling components of $\mathcal{O}$. The reflection $\tau : D\mathcal{O} \to D\mathcal{O}$ lifts to a reflection $\tilde{\tau} : N \to N$ which commutes with the deck transformations group of $N \to D\mathcal{O}$. In particular, N is obtained by doubling a finite regular covering $M \to \mathcal{O}$ along the lifts of the doubling components of $D\mathcal{O}$, which are precisely the fixed point set of the involution $\tilde{\tau}$. We can take N to be the 3-torus T^3. Then the fixed point set of $\tilde{\tau}$, which is an incompressible surface, is isotopic to a disjoint union of parallel copies of fibers $T^2 \times \{*\}$ in T^3. In particular, the finite regular covering M of $D\mathcal{O}$ is homeomorphic to the product $T^2 \times I$. By [**BS1**, Prop. 12] or [**MS**], the orbifold $\mathcal{O}$ inherits an I-fibration and is Euclidean. This proves assertion i).

To prove assertion ii), we suppose that $D\mathcal{O}$ is Sol. There is a finite regular covering $N \to D\mathcal{O}$ which is a manifold and fibers over S^1 with fiber T^2 and Anosov monodromy. Let $\tau : D\mathcal{O} \to D\mathcal{O}$ be the natural involution as above, obtained by reflection through the doubling components of $\partial\mathcal{O}$. Since τ is an involution with a non-empty fixed point set, not included in the ramification locus, it lifts to an involution $\tilde{\tau}$ of N whose fixed point set contains a two dimensional submanifold. By Tollefson's theorem about finite order homeomorphisms of fiber bundles [**To2**] (see also [**MS**]), we may assume that $\tilde{\tau}$ preserves the fibration by tori. Then one can easily check that torus bundles with Anosov monodromy cannot admit the reflection $\tilde{\tau}$. □

Remark 2.4.4. — When $\mathcal{O}$ is an I-bundle over a 2-orbifold F^2, the following facts should be noted:

i) The 2-orbifold F^2 is either Euclidean or hyperbolic, because $\mathcal{O}$ is irreducible. In particular, the interior of $\mathcal{O}$ has a complete Euclidean or hyperbolic structure.
ii) Acylindricity of $\mathcal{O}$ restricts the possibilities for F^2.
iii) The manifold $D\mathcal{O} - D\Sigma$ is Seifert fibered.

We decompose the boundary of $\mathcal{O}$ in three parts:

$$\partial\mathcal{O} = \partial_T\mathcal{O} \sqcup \partial_{SE}\mathcal{O} \sqcup \partial_H\mathcal{O},$$

where:

- $\partial_T\mathcal{O}$ is the union of the boundary components homeomorphic to a torus,
- $\partial_{SE}\mathcal{O}$ is the union of the singular Euclidean boundary components,

– $\partial_H \mathcal{O}$ is the union of the hyperbolic boundary components.

In the following we denote $\mathcal{O} - \mathcal{N}(\Sigma)$ by X. Let $P \subseteq \partial X$ be the union of $\partial_T \mathcal{O}$ with the tori corresponding to circle components of Σ and with the annuli corresponding to arcs in Σ. Equivalently:

$$P = \partial_T \mathcal{O} \cup \overline{(\partial \mathcal{N}(\Sigma) \cap \operatorname{int}(\mathcal{O}))}.$$

Then we have the following proposition (see also [**Dun2**, Thm. 10], [**SOK**, §2]):

Lemma 2.4.5. — *Let $\mathcal{O}$ be a compact orientable irreducible and ∂-incompressible* 3-*orbifold which is topologically atoroidal and acylindrical. Then:*

i) either $X - P$ admits a hyperbolic structure with totally geodesic boundary,
ii) or $\mathcal{O}$ is Seifert fibered or an I-bundle over a 2-orbifold F^2.

Proof of Lemma 2.4.5. — Let DX be obtained by doubling X along $\partial X - P$. We take the convention that $DX = X$ if $\partial X = P$, so that DX is always connected. Then ∂DX is an union of tori. In fact $DX = D\mathcal{O} - \mathcal{N}(D\Sigma)$, where $D\mathcal{O}$ is obtained by doubling $\mathcal{O}$ along all the components of $\partial_{SE}\mathcal{O} \sqcup \partial_H \mathcal{O}$. Thus it follows from Lemma 2.4.2 that DX is irreducible and atoroidal. With these notations, we have the following claim:

Claim 2.4.6. — *Either $DX = D\mathcal{O} - \mathcal{N}(D\Sigma)$ has incompressible boundary and is topologically acylindrical, or $\mathcal{O}$ is Seifert fibered or an I-bundle over a 2-orbifold F^2.*

Proof of the claim. — First we prove that if $D\mathcal{O} - \mathcal{N}(D\Sigma)$ has compressible boundary then $\mathcal{O}$ is Seifert fibered. Assuming that $D\mathcal{O} - \mathcal{N}(D\Sigma)$ has compressible boundary, then $D\mathcal{O} - \mathcal{N}(D\Sigma)$ is a solid torus, because it is irreducible and its boundary is a union of tori. Hence the underlying space of $D\mathcal{O}$ is a generalized Lens space and its ramification locus is the core of one of the solid tori of a genus one Heegard splitting. As the 3-orbifold $D\mathcal{O}$ is irreducible, it cannot be the product $S^1 \times S^2(*)$, where $S^2(*)$ is a 2-sphere with one cone point (a bad 2-orbifold). Hence $D\mathcal{O}$ has S^3 as universal covering. By the equivariant Dehn's lemma, $D\mathcal{O}$ cannot contain an incompressible 2-suborbifold. Hence $\partial \mathcal{O} = \emptyset$, and $D\mathcal{O} = \mathcal{O}$ by Lemma 2.4.1 ii). Thus $\mathcal{O}$ is Seifert fibered.

We prove next that if $D\mathcal{O} - \mathcal{N}(D\Sigma)$ contains an essential annulus then $D\mathcal{O}$ is Seifert fibered. This will imply the claim, because when $D\mathcal{O}$ is Seifert fibered, then, by Lemma 2.4.3, $\mathcal{O}$ is either Seifert fibered or an I-bundle over a 2-orbifold.

Suppose that $D\mathcal{O} - \mathcal{N}(D\Sigma)$ contains an essential annulus. By Lemma 2.4.2 and the characteristic submanifold theorem [**JS**], [**Joh**], $D\mathcal{O} - \mathcal{N}(D\Sigma)$ is Seifert fibered. We consider a component Σ_i of Σ and a solid torus neighborhood $\mathcal{N}(\Sigma_i)$. If the fiber of the Seifert fibration of $D\mathcal{O} - \mathcal{N}(D\Sigma)$ is not homotopic to the meridian of Σ_i in the torus $\partial\mathcal{N}(\Sigma_i)$, then this Seifert fibration can be extended to $\mathcal{N}(\Sigma_i)$ so that Σ_i is a fiber.

We suppose now that the fiber of the Seifert fibration of $D\mathcal{O} - \mathcal{N}(D\Sigma)$ is homotopic to the meridian of Σ_i. If the base 2-orbifold of the Seifert fibration in $D\mathcal{O} - \mathcal{N}(D\Sigma)$

is different from a disk or a disk with one cone point, then $D\mathcal{O} - \mathcal{N}(D\Sigma)$ contains an essential annulus which is vertical and its boundary is in $\partial\mathcal{N}(\Sigma_i)$. In particular, the union of this annulus with two meridian disks (with one cone point) of $\mathcal{N}(\Sigma_i)$ gives an incompressible spherical 2-suborbifold in $D\mathcal{O}$, contradicting the irreducibility of $D\mathcal{O}$. Hence $D\mathcal{O} - \mathcal{N}(D\Sigma)$ is a solid torus, and, as we have already shown above, this implies that $\partial\mathcal{O} = \emptyset$ and $D\mathcal{O} = \mathcal{O}$ is Seifert fibered. □

To achieve the proof of Lemma 2.4.5 we apply Thurston's hyperbolization theorem for topologically atoroidal and acylindrical Haken 3-manifolds. We assume that $\mathcal{O}$ is not Seifert fibered, nor an I-bundle over a 2-orbifold F^2. Then it follows from Lemma 2.4.2 and Claim 2.4.6 that DX is a ∂-incompressible Haken 3-manifold, which is topologically atoroidal and acylindrical. Since ∂DX is an union of tori, Thurston's hyperbolization theorem shows that $DX - \partial DX$ admits a complete hyperbolic structure of finite volume.

If $\partial X = P$, then $DX = X$ and the proof is done. So from now on, we assume that $Q = \partial X - P$ is not empty. In particular $\partial_{SE}\mathcal{O} \sqcup \partial_H\mathcal{O}$ is not empty.

Consider the reflection $\tau_0 : DX \to DX$ through the doubling components Q. By Mostow-Prasad rigidity theorem, τ_0 is homotopic to an isometric involution $\tau_1 : DX \to DX$. By Waldhausen's and Tollefson's Theorems ([**Wa1**], [**To1**]), there exists a homeomorphism $h : DX \to DX$ isotopic to the identity, that conjugates τ_0 and τ_1. This implies that the 3-manifold DX is hyperbolic of finite volume and that the components $Q = \partial X - P$ are totally geodesic. □

Lemma 2.4.7. — *If $D\mathcal{O}$ is hyperbolic with finite volume, then $\mathcal{O}$ is hyperbolic and the doubling components are totally geodesic.*

Proof of Lemma 2.4.7. — Note that if $D\mathcal{O}$ is hyperbolic with finite volume, then the doubling components are precisely the hyperbolic pieces $\partial_H\mathcal{O}$ of $\partial\mathcal{O}$. We assume that $\partial_H\mathcal{O}$ is not empty, otherwise $\mathcal{O} = D\mathcal{O}$ and there is nothing to prove.

Consider the reflection $\tau_0 : D\mathcal{O} \to D\mathcal{O}$ through the doubling components. By Mostow-Prasad rigidity theorem, τ_0 is homotopic (in the orbifold sense) to an isometric involution $\tau_1 : D\mathcal{O} \to D\mathcal{O}$. We want to show that these two involutions are in fact conjugate. This will imply that the 3-orbifold $\mathcal{O}$ is hyperbolic and that the hyperbolic components of $\partial\mathcal{O}$ are totally geodesic.

Since both τ_0 and τ_1 preserve the ramification set, each one induces a involution of $D\mathcal{O} - D\Sigma$. By Lemma 2.4.5, since $D\mathcal{O}$ is not Seifert fibered, nor an I-bundle over a 2-orbifold, the manifold $D\mathcal{O} - D\Sigma$ admits a hyperbolic structure of finite volume with cusps and totally geodesic boundary. Its totally geodesic boundary is exactly $(\partial_{SE}\mathcal{O} - \Sigma) \sqcup (\partial_{SE}\mathcal{O} - \Sigma)$.

The following claim shows that the restrictions of τ_0 and τ_1 to $D\mathcal{O} - D\Sigma$ are respectively homotopic to some isometric involutions g_0 and g_1 on $D\mathcal{O} - D\Sigma$.

Claim 2.4.8. — *Let N be a compact orientable irreducible 3-manifold and $P \subset \partial N$ a disjoint union of incompressible tori and annuli, such that $N - P$ admits a hyperbolic structure of finite volume with totally geodesic boundary. Then any diffeomorphism $h : N \to N$ is homotopic to an isometry on $N - P$.*

Proof of Claim 2.4.8. — If $\partial N = P$, then it is a direct consequence of Mostow rigidity theorem. Thus we assume that $Q = \partial N - P$ is not empty.

Let DN be obtained by doubling N along $Q \subset \partial N$. Then ∂DN is an union of incompressible tori and $DN - \partial DN$ admits a complete hyperbolic structure of finite volume. Moreover the reflection $\rho : DN \to DN$ through the doubling components Q is an isomery on $DN - \partial DN$.

The diffeomorphism $h : N \to N$ extends to give a diffeomorphism $\hat{h} : DN \to DN$ which commutes with the involution ρ. By Mostow rigidity theorem $\hat{h}$ is homotopic to an isometry $\hat{h}_1$ on $DN - \partial DN$. Since the two isometric involutions ρ and $\hat{h}_1 \rho \hat{h}_1^{-1}$ are homotopic on $DN - \partial DN$, they must be equal. Hence $\hat{h}_1$ induces an isometry h_1 of $N - P$, which extends to a diffeomorphism of N.

We show now that h_1 and h are homotopic on N. The diffeomorphism $\hat{\phi} = \hat{h}_1 \hat{h}^{-1}$ is homotopic to the identity on DN. Therefore $\hat{\phi}$ induces an inner automorphism of the fundamental group $\pi_1(DN)$, that sends the subgroup $\pi_1(N)$ to itself (the base point is taken on the doubling surface Q). Since N is not homeomorphic to an I-bundle over a component of Q, it follows that $\pi_1(N)$ is its own normalizer in $\pi_1(DN)$ by [**Hei**, Thm. 2]. Therefore $\hat{\phi}$ induces an inner automorphism of $\pi_1(N)$. This automorphism corresponds to the induced action of $\phi = h_1 h^{-1}$ on $\pi_1(N)$. Since N is a $K(\pi, 1)$-space, it follows that the map ϕ is homotopic to the identity on N hence h_1 and h are homotopic on N. □

To finish the proof of Lemma 2.4.7 we use an unpublished argument of Bonahon and Siebenmann [**BS3**]. We apply Claim 2.4.8 to the restrictions of τ_0 and τ_1 to $D\mathcal{O} - D\Sigma$. Let g_0 and g_1 denote the isometric involutions of $D\mathcal{O} - D\Sigma$ which are homotopic to the restrictions of τ_0 and τ_1 respectively. Then, by Waldhausen's and Tollefson's Theorems [**Wa1**], [**To1**], there exist two homeomorphisms $h_0, h_1 : D\mathcal{O} - D\Sigma \to D\mathcal{O} - D\Sigma$ isotopic to the identity such that the restrictions $\tau_0|_{D\mathcal{O}-D\Sigma} = h_0 g_0 h_0^{-1}$ and $\tau_1|_{D\mathcal{O}-D\Sigma} = h_1 g_1 h_1^{-1}$. Therefore, the involutions g_0 and g_1 on $D\mathcal{O} - D\Sigma$ can be extended respectively to involutions $\bar{g}_0, \bar{g}_1 : D\mathcal{O} \to D\mathcal{O}$. It remains to prove that $\bar{g}_0 = \bar{g}_1$ on $D\mathcal{O}$.

The map $f = \bar{g}_0 \bar{g}_1^{-1}$ is homotopic to the identity on $D\mathcal{O}$ in the orbifold sense; moreover f is of finite order, because its restriction to $D\mathcal{O} - D\Sigma$ is an isometry. Since f is homotopic to the identity, it lifts to a homeomorphism $\tilde{f} : \mathbb{H}^3 \to \mathbb{H}^3$ whose extension to the sphere at infinity $\partial \mathbb{H}^3 \cong S^2$ is the identity. Since f is of finite order, so is $\tilde{f}$, because, if n is the order of f, then $(\tilde{f})^n$ is an isometry of $\mathbb{H}^3$ whose extension to the sphere at infinity is the identity. By Smith theory (cf. [**Bre**]), the identity is the only orientation preserving periodic map of the ball which is the identity on the boundary. Thus $\tilde{f}$ must be the identity, and in particular $\bar{g}_0 = \bar{g}_1$. Hence the involutions τ_0 and τ_1 are conjugate on $D\mathcal{O}$. $\square$

In the proof of the following proposition we use in a crucial way the assumption in Theorem 1 that the 3-orbifold $\mathcal{O}$ is very good.

Proposition 2.4.9. — *If Theorem 1 holds when no component of $\partial\mathcal{O}$ is a non-singular torus, then it holds in general.*

Proof of Proposition 2.4.9. — As above we decompose the boundary of $\mathcal{O}$ in three parts:

$$\partial\mathcal{O} = \partial_T\mathcal{O} \sqcup \partial_{SE}\mathcal{O} \sqcup \partial_H\mathcal{O},$$

where:

- $\partial_T\mathcal{O}$ is the union of the boundary components homeomorphic to a torus
- $\partial_{SE}\mathcal{O}$ is the union of the singular Euclidean boundary components
- $\partial_H\mathcal{O}$ is the union of the hyperbolic boundary components

We assume $\partial_T\mathcal{O} \neq \emptyset$. We double along the hyperbolic components $\partial_H\mathcal{O}$:

$$D\mathcal{O} = \mathcal{O} \underset{\partial_H\mathcal{O}}{\cup} \mathcal{O}.$$

Since $D\mathcal{O}$ is very good, we fix a regular covering $p : DM \to D\mathcal{O}$ of finite order which is a manifold, and let G denote its group of deck transformations. Since $D\mathcal{O}$ is irreducible, topologically atoroidal and boundary-incompressible (Lemma 2.4.1), the Equivariant Sphere and Loop Theorems ([**DD**], [**MY1, MY2**], [**JR**]) imply that DM is also irreducible, topologically atoroidal, and boundary-incompressible. Since by hypothesis $\partial(DM) \neq \emptyset$, Thurston's hyperbolization theorem for Haken 3-manifolds implies that DM is either Seifert fibered or hyperbolic ([**Thu1, Thu2, Thu3, Thu4, Thu5**], [**McM1, McM2**], [**Ot1, Ot2**], [**Kap**]). If DM is Seifert fibered, Meeks-Scott theorem [**MS**] implies that $D\mathcal{O}$ is also Seifert fibered, and thus $\mathcal{O}$ is geometric (Lemma 2.4.3). Therefore we can assume that DM is hyperbolic.

Let $\gamma = \{\gamma_1, \dots, \gamma_r\}$ be a family of simple closed curves, one on each torus component of $\partial_T(D\mathcal{O})$. Let $D\mathcal{O}(\gamma)$ denote the 3-orbifold obtained by generalized Dehn filling with meridian curves $\gamma = \{\gamma_1, \dots, \gamma_r\}$. Generalized Dehn filling means that the filling solid tori may have ramified cores. Moreover, we choose the branching indices of these filling cores so that the generalized Dehn filling $D\mathcal{O}(\gamma)$ lifts to a genuine Dehn filling of DM.

We consider a sequence of families of simple closed curves $(\gamma^n)_{n\in\mathbb{N}} = (\{\gamma_1^n, \dots, \gamma_r^n\})_{n\in\mathbb{N}}$ such that, for each $n \in \mathbb{N}$, γ^n gives precisely one curve on each component of $\partial_T D\mathcal{O}$, and for each $i = 1, \dots, r$, the curves of the sequence $(\gamma_i^n)_{n\in\mathbb{N}}$ represent different homotopy classes on the i-th torus boundary component. For $n \in \mathbb{N}$ sufficiently large, the orbifold $D\mathcal{O}(\gamma^n)$ has a regular covering obtained by Dehn filling of DM, which we may assume to be hyperbolic by Thurston's hyperbolic Dehn filling theorem.

Then, by the equivariant sphere theorem ([**DD**], [**JR**], [**MY1, MY2**]) and the proof of the Smith conjecture [**MB**] $D\mathcal{O}(\gamma^n)$ is irreducible for $n \in \mathbb{N}$ sufficiently large. It is also topologically atoroidal by the equivariant loop theorem ([**JR**], [**MY1, MY2**]). Moreover, by construction, no component of $\partial D\mathcal{O}(\gamma^n)$ is a non-singular torus. Hence, for $n \in \mathbb{N}$ sufficiently large, $D\mathcal{O}(\gamma^n)$ is geometric by hypothesis, and so it is hyperbolic.

For each $n \in \mathbb{N}$ sufficiently large, choose a point $x_n \in D\mathcal{O}(\gamma^n)$ so that $\mathrm{inj}(x_n) > \varepsilon_3$, where $\varepsilon_3 > 0$ is the 3-dimensional Margulis constant. By the compactness theorem (Chap. 3) there is a subsequence of the sequence $(D\mathcal{O}(\gamma^n), x_n)$ which converges geometrically to a hyperbolic 3-orbifold. Moreover the limit is non-compact and gives a hyperbolic structure on the interior of the 3-orbifold $D\mathcal{O}$, because the sequence of coverings of $D\mathcal{O}(\gamma^n)$ converges geometrically to the interior of DM by Thurston's hyperbolic Dehn filling theorem.

Since $D\mathcal{O}$ is hyperbolic, Lemma 2.4.7 shows that $\mathcal{O}$ is also hyperbolic. $\square$

Proof of Theorem 1. — We use the previous results of this section to make some reductions of the general case. First, since we assume that $\mathcal{O}$ is very good, by Proposition 2.4.9, we can assume that no component of $\partial\mathcal{O}$ is a non-singular torus.

Let $D\mathcal{O}$ be the double of $\mathcal{O}$ along all boundary components. In particular $\partial(D\mathcal{O}) = \emptyset$. By Lemma 2.4.1, $D\mathcal{O}$ is irreducible; moreover every incompressible Euclidean 2-suborbifold is singular and parallel to a doubling component. By Lemma 2.4.2, $D\mathcal{O} - D\Sigma$ is irreducible and atoroidal. Furthermore, by Lemma 2.4.5, we can assume that $D\mathcal{O} - D\Sigma$ is also topologically acylindrical and has an incompressible boundary. Hence, by Thurston's hyperbolization theorem ([**Thu3, Thu4, Thu5**], [**McM1, McM2**], [**Kap**], [**Ot1, Ot2**]) $D\mathcal{O} - D\Sigma$ has a complete hyperbolic structure of finite volume, and we can apply Theorem 4 to $D\mathcal{O}$.

By Theorem 4, $D\mathcal{O}$ contains a non-empty compact essential 3-suborbifold $\mathcal{O}' \subset D\mathcal{O}$ which is not a product and which is either Euclidean, Seifert fibered, *Sol* or complete hyperbolic with finite volume. We distinguish two cases, according to whether $\mathcal{O}$ is closed or not.

If $\partial\mathcal{O} = \emptyset$, then $\mathcal{O} = \mathcal{O}'$, because the boundary $\partial\mathcal{O}'$ is either empty or a union of incompressible toric 2-suborbifolds, and $\mathcal{O}$ is topologically atoroidal. Thus $\mathcal{O}$ is either Seifert fibered, Euclidean or hyperbolic; it cannot be *Sol* by atoroidality.

Next we suppose $\partial\mathcal{O} \neq \emptyset$. Note that in this case $\mathcal{O}'$ cannot be *Sol* by Lemma 2.4.3 ii). By Lemma 2.4.1, every component of $\partial\mathcal{O}'$ is isotopic to a Euclidean component of $\partial\mathcal{O}$. Therefore $\mathcal{O}'$ is obtained by cutting open $D\mathcal{O}$ along some (perhaps none) component of $\partial\mathcal{O}$. This implies that $\mathcal{O}$ can be isotoped into $\mathcal{O}'$, because $\mathcal{O}'$ is connected and not a product. Moreover after isotopy we can assume that either $\tau(\mathcal{O}') = \mathcal{O}'$ or $\tau(\mathcal{O}') \cap \mathcal{O}' = \emptyset$, where $\tau : D\mathcal{O} \to D\mathcal{O}$ is the reflection through $\partial\mathcal{O}$. There are three possibilities:

- If $\tau(\mathcal{O}') \cap \mathcal{O}' = \emptyset$ then $\mathcal{O} = \mathcal{O}'$ is Euclidean, Seifert or hyperbolic, possibly with cusps.

- If $\tau(\mathcal{O}') = \mathcal{O}'$ and $\mathcal{O}'$ is hyperbolic, then, by atoroidality, $\partial\mathcal{O}$ has hyperbolic components and $\mathcal{O}'$ is the double of $\partial\mathcal{O}$ along the hyperbolic boundary components. By Lemma 2.4.7, $\mathcal{O}$ is hyperbolic, with some boundary components totally geodesic and possibly some boundary components cusped.
- If $\tau(\mathcal{O}') = \mathcal{O}'$ and $\mathcal{O}'$ is Euclidean or Seifert fibered, then $\mathcal{O}'$ is the double of $\mathcal{O}$ along some boundary components. Lemma 2.4.3 i) implies that $\mathcal{O}$ is Seifert fibered, Euclidean or an I-bundle over a 2-orbifold. The I-bundle case is not possible, because it would imply that $D\mathcal{O} - D\Sigma$ is Seifert. Hence $\mathcal{O}$ is Seifert fibered or Euclidean.

This finishes the proof of Theorem 1. $\square$

CHAPTER 3

A COMPACTNESS THEOREM FOR CONE 3-MANIFOLDS WITH CONE ANGLES BOUNDED ABOVE BY π

The purpose of this chapter is to establish a version of Gromov's compactness theorem for sequences of Riemaniann manifolds (cf. [**GLP**] and [**Pe**]) in the context of cone 3-manifolds.

Before stating the main theorem we need some definitions.

Definition 3.0.1. — For $\varepsilon \geq 0$, a map $f : X \to Y$ between two metric spaces is $(1+\varepsilon)$-*bilipschitz* if:

$$\forall x_1, x_2 \in X, \qquad (1+\varepsilon)^{-1} d(x_1, x_2) \leq d(f(x_1), f(x_2)) \leq (1+\varepsilon) d(x_1, x_2).$$

Remark 3.0.2. — A $(1+\varepsilon)$-bilipischitz map is always an embedding. Hence one can also define a $(1+\varepsilon)$-bilipischitz map as an embedding f such that f and f^{-1} have Lipschitz constant $1+\varepsilon$. A map is 1-bilipischitz if and only if it is an isometric embedding.

Definition 3.0.3. — A sequence of pointed cone 3-manifolds $\{(C_n, x_n)\}_{n\in\mathbb{N}}$ *converges geometrically* to a pointed cone 3-manifold (C_∞, x_∞) if, for every $R > 0$ and $\varepsilon > 0$, there exists an integer n_0 such that, for $n > n_0$, there is a $(1+\varepsilon)$-bilipschitz map $f_n : B(x_\infty, R) \to C_n$ satisfying:

i) $d(f_n(x_\infty), x_n) < \varepsilon$,
ii) $B(x_n, R-\varepsilon) \subset f_n(B(x_\infty, R))$, and
iii) $f_n(B(x_\infty, R) \cap \Sigma_\infty) = (f_n(B(x_\infty, R))) \cap \Sigma_n$.

Remark 3.0.4. — By definition, the following inclusion is also satisfied:

$$f_n(B(x_\infty, R)) \subset B(x_n, R(1+\varepsilon)+\varepsilon).$$

Definition 3.0.5. — For a cone 3-manifold C, we define the *cone-injectivity radius* at $x \in C$:

$$\mathrm{inj}(x) = \sup\{\delta > 0 \text{ such that } B(x,\delta) \text{ is contained in a standard ball in } C\}.$$

We recall that a standard ball is isometric to either a non-singular metric ball in $\mathbb{H}^3_K$, or to a singular metric ball in $\mathbb{H}^3_K(\alpha)$. The definition does not assume the ball to be centered at x, in order to avoid cone-injectivity radius close to zero for non-singular points close to the singular locus.

Given $a > 0$ and $\omega \in (0,\pi]$, $\mathcal{C}_{[\omega,\pi],a}$ is the set of pointed cone 3-manifolds (C,x) with constant curvature in $[-1,0]$, cone angles in $[\omega,\pi]$, and such that $\mathrm{inj}(x) \geq a$.

This chapter is devoted to the proof of the following result:

Compactness Theorem. — *Given $a > 0$ and $\omega \in (0,\pi]$, the closure of $\mathcal{C}_{[\omega,\pi],a}$ in $\bigcup_{b>0} \mathcal{C}_{[\omega,\pi],b}$ is compact for the geometric convergence topology.*

This theorem says that any sequence $\{(C_n,x_n)\}_{n\in\mathbb{N}}$ of pointed cone 3-manifolds in $\mathcal{C}_{[\omega,\pi],a}$ admits a subsequence that converges geometrically to a pointed cone 3-manifold in $\mathcal{C}_{[\omega,\pi],b}$ for some $b > 0$.

The proof of the compactness theorem occupies Sections 3.1 to 3.4. In Section 3.5 we give some properties of the geometric convergence.

The main steps in the proof of the compactness theorem are the following ones. First we show that $\mathcal{C}_{[\omega,\pi],a}$ is relatively compact in the space $\mathcal{L}$ of locally compact metric length spaces equipped with the Hausdorff-Gromov topology (Proposition 3.2.4). Next, in Proposition 3.2.6, we show that, for a sequence of pointed cone 3-manifolds of $\mathcal{C}_{[\omega,\pi],a}$ that converges in $\mathcal{L}$, the limit is a pointed cone 3-manifold in $\mathcal{C}_{[\omega,\pi],b}$ for some $b > 0$. Finally we show that Hausdorff-Gromov convergence in $\mathcal{C}_{[\omega,\pi],a}$ implies geometric convergence (Proposition 3.3.1).

The second and third step of the proof rely on the following technical result (Proposition 3.2.5): given a radius $R > 0$, and constants $a > 0$ and $\omega \in (0,\pi]$, for any pointed cone 3-manifold $(C,x) \in \mathcal{C}_{[\omega,\pi],a}$, the cone-injectivity radius of each point in the ball $B(x,R)$ has a positive uniform lower bound, which only depends on the constants R, a and ω. The proof of this result is postponed until Section 3.4.

This chapter is organized as follows. Section 3.1 is devoted to the Dirichlet polyhedron and the Bishop-Gromov inequality. In Section 3.2 we show that every sequence in $\mathcal{C}_{[\omega,\pi],a}$ has a subsequence that converges to a cone 3-manifold for the Hausdorff-Gromov topology, assuming Proposition 3.2.5. In Section 3.3 we show that the convergence is in fact geometric. In Section 3.4 we prove Proposition 3.2.5 using the Dirichlet polyhedron. In Section 3.5 we show some basic properties of the geometric convergence. Finally, in Section 3.6 we extend the compactness theorem to cone 3-manifolds with totally geodesic boundary.

3.1. The Dirichlet polyhedron

In this section we first describe the Dirichlet polyhedron and give some elementary facts about minimizing paths. Then we prove Bishop-Gromov inequality. The

Dirichlet polyhedron for cone 3-manifolds is also considered in [**Sua**] and minimizing paths for cone 3-manifolds are also studied in [**HT**].

Definition 3.1.1. — Let C be a cone 3-manifold of curvature $K \leq 0$ and $x \in C - \Sigma$. We define the *Dirichlet polyhedron centered at x:*

$$D_x = \{y \in C - \Sigma \mid \text{there exists a unique minimizing path between } y \text{ and } x\}.$$

The open set D_x is star shaped with respect to x, hence it can be locally isometrically embedded in the space of constant sectional curvature $\mathbb{H}^3_K$ as a star shaped domain. The following proposition explains why it is called a polyhedron.

Proposition 3.1.2. — *The open domain D_x is the interior of a solid polyhedron $\overline{D}_x$ of $\mathbb{H}^3_K$. Moreover the cone 3-manifold C is isometric to the quotient of $\overline{D}_x$ under some face identifications.*

In order to prove this proposition we need first to understand the minimizing paths from x to points in $C - D_x$. First we recall a well known fact about minimizing paths in cone 3-manifolds with cone angles less than 2π (cf. [**HT**] for a proof).

Lemma 3.1.3. — *Let C be a cone 3-manifold with cone angles less than 2π and singular set Σ. Let σ be a minimizing path between two points in C. If $\sigma \cap \Sigma \neq \emptyset$, then either $\sigma \subset \Sigma$, or $\sigma \cap \Sigma$ is one or both of the end-points of σ.* □

Recall that a subset $A \subset C$ is called *convex* if every minimizing path between two points of A is itself contained in A. For instance, a subset with only one point is convex. The following lemma will be used in the proof of Proposition 3.1.2 in the case where $A = \{x\}$, but it will be used more generally in Section 3.4 and in Chapter 4.

Lemma 3.1.4. — *Let C be a cone 3-manifold of non-positive curvature, $A \subset C$ a convex subset and $y \in C$. Then the following hold:*

i) There exist a finite number of minimizing paths from y to A.
ii) Minimizing paths to A with origin close to y are obtained by perturbation: for every $\varepsilon > 0$ there exists a neighborhood $U \subset C$ of y such that, for every $z \in U$ and every minimizing path σ_z from z to A, there exists a minimizing path σ_y from y to A such that $\sigma_z \subset \mathcal{N}_\varepsilon(\sigma_y)$, where $\mathcal{N}_\varepsilon(\sigma_y)$ is the set of points whose distance to σ_y is less than ε.

Proof of Lemma 3.1.4. — We prove i) by contradiction. We assume that we have an infinite sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ of different minimizing paths between y and A. Since $\mathrm{Length}(\sigma_n)$ is constant, up to taking a subsequence, $\{\sigma_n\}_{n\in\mathbb{N}}$ converges to a path σ_∞. Let $\varepsilon > 0$ be sufficiently small so that the developing map around a tubular neighborhood $\mathcal{N}_\varepsilon(\sigma_\infty)$ is defined. By considering developing maps, we have an infinite sequence of different minimizing paths between one point and a convex subset in $\mathbb{H}^3_K$ or $\mathbb{H}^3_K(\alpha)$, and this is not possible when the curvature is $K \leq 0$.

We also prove ii) by contradiction: we assume that there is $\varepsilon > 0$ and a sequence of points $z_n \in C$ such that $z_n \to y$ and every z_n has a minimizing path σ_n to A not contained in $\mathcal{N}_\varepsilon(\sigma_y)$, for any minimizing path σ_y between y and A. Since $z_n \to y$, there is a subsequence of $\{\sigma_n\}_{n\in\mathbb{N}}$ that converges to a path σ_∞. Moreover, by taking limits in the inequality

$$d(y,A) \geq \text{Length}(\sigma_n) - d(y,z_n),$$

we get that $d(y,A) \geq \text{Length}(\sigma_\infty)$. Therefore σ_∞ is a minimizing path from y to A whose ε-tubular neighborhood $\mathcal{N}_\varepsilon(\sigma_\infty)$ contains infinitely many σ_n, and we get a contradiction. □

Proof of Proposition 3.1.2. — We describe locally $C - D_x$ by using Lemma 3.1.4. Let $y \in C - D_x$, we consider six different cases.

Case 1. Consider first the case where $y \notin \Sigma$ and there are precisely two minimizing paths σ_1 and σ_2 between y and x. Take $\varepsilon > 0$ so that the developing map is defined around the ε-neighborhoods $\mathcal{N}_\varepsilon(\sigma_1)$ and $\mathcal{N}_\varepsilon(\sigma_2)$. By Lemma 3.1.4. there is an open neighborhood $y \in U_y \subset C$ so that all minimizing paths from points $z \in U_y$ to x are in one of these tubular neighborhoods $\mathcal{N}_\varepsilon(\sigma_1)$ or $\mathcal{N}_\varepsilon(\sigma_2)$. Since the set of points in $\mathbb{H}^3_K$ equidistant from two given different points is a plane, by using developing maps we conclude that $(C - D_x) \cap U_y$ is the "bisector" plane between σ_1 and σ_2. This case corresponds to the interior of two faces of $\overline{D}_x$ identified.

Case 2. Next consider the case where $y \notin \Sigma$ and there are $n \geq 3$ minimizing paths $\sigma_1, \dots, \sigma_n$ from y to x satisfying the following property

(1) there exists $v \in T_yC,\ v \neq 0$ such that $\langle \sigma_1'(0), v\rangle = \cdots = \langle \sigma_n'(0), v\rangle$,

where the minimizing paths are parametrized by arc length (in particular $\|\sigma_i'(0)\| = 1$). Property (1) means that the vectors $\sigma_1'(0), \dots, \sigma_n'(0)$ can be ordered in such a way that, if P_i denotes the "bisector" plane between $\sigma_i'(0)$ and $\sigma_{i+1}'(0)$, then the intersection $P_1 \cap \cdots \cap P_n$ is a line (generated by the vector v). See Fig. 1. When $n = 3$ property (1) always holds.

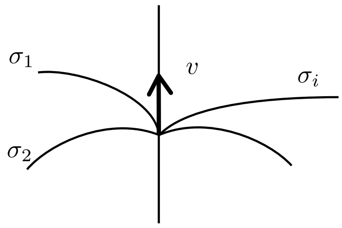

FIGURE 1

An argument similar to case 1 shows that, for some neighborhood U_y of y, $(C - D_x) \cap U_y$ is the union of n half planes bounded by the same line. These are precisely the "bisector" half planes between the n pairs of paths σ_i and σ_{i+1}. This case corresponds to the interior of several edges of $\overline{D}_x$ identified. Note that the dihedral angles are less than π by construction.

Case 3. To finish with the non-singular possibilities, consider the case where $y \notin \Sigma$ and there are $n \geq 4$ minimizing paths $\sigma_1, \ldots, \sigma_n$ from y to x that do not satisfy property (1) above. This case is treated as the previous ones and corresponds to n vertices of $\overline{D}_x$ identified.

Case 4. When $y \in \Sigma$ and y has only one minimizing path σ to x. This case corresponds to the interior of an edge of $\overline{D}_x$, whose dihedral angle equals the cone angle of Σ at y. The two adjacent faces of this edges are identified by a rotation around this edge.

Case 5. When $y \in \Sigma$ and y has $n \geq 2$ minimizing paths $\sigma_1, \ldots, \sigma_n$ to x that satisfy property (1) of case 2. In this case, the vector v of property (1) is necessarily tangent to Σ and it corresponds to n edges of $\overline{D}_x$ that are identified to get a piece of Σ.

Case 6. Finally, consider the case where $y \in \Sigma$ and there are $n \geq 2$ minimizing paths $\sigma_1, \ldots, \sigma_n$ between y and x that do not satisfy property (1). It can be shown that this case corresponds to n vertices of $\overline{D}_x$ identified. □

Corollary 3.1.5. — *If the cone angles of C are less than or equal to π, then for every $x \in C - \Sigma$ the Dirichlet polyhedron D_x is convex.*

Proof. — It suffices to show that the dihedral angles of $\overline{D}_x$ are less than or equal to π. We have seen in the proof of Proposition 3.1.2 that this is true for dihedral angles of non-singular edges. For singular edges, this follows from the hypothesis about cone angles, because dihedral angles are bounded above by cone angles. □

We next define the Dirichlet polyhedron centered at singular points. Recall that $\mathbb{H}^3_K(\alpha)$ denotes the simply connected space of curvature K with a singular axis of cone angle α.

Definition 3.1.6. — Let C be a cone 3-manifold of curvature $K \leq 0$ and $x \in \Sigma \subset C$. We define the *Dirichlet polyhedron centered at x:*

$$D_x = \left\{ y \in C \;\middle|\; \begin{array}{l} \text{there exists a unique minimizing path } \sigma \text{ between } y \text{ and } x, \\ \text{and, in addition, if } y \in \Sigma \text{ then } \sigma \subset \Sigma. \end{array} \right\}$$

As in the non-singular case, D_x is open, star shaped and it can be locally isometrically embedded in $\mathbb{H}^3_K(\alpha)$.

Remark 3.1.7. — It is possible to work in the non-singular space $\mathbb{H}^3_K$ by using the following construction. Let S_α be an infinite sector of $\mathbb{H}^3_K$ of dihedral angle α and consider the quotient map $p : S_\alpha \to \mathbb{H}^3_K(\alpha)$ that identifies the faces of S_α by a rotation around its axis. Then we look at the inverse image $p^{-1}(D_x) \subset S_\alpha \subset \mathbb{H}^3_K$. As in Proposition 3.1.2, the set $\overline{p^{-1}(D_x)}$ is a solid polyhedron and C is the quotient of $\overline{p^{-1}(D_x)}$ by isometric face identifications. The point x is in the boundary of $\overline{p^{-1}(D_x)}$, although the polyhedron is star shaped with respect to x. As in Corollary 3.1.5, if the cone angles of C are bounded above by π, then $\overline{p^{-1}(D_x)}$ is convex (and so is $\overline{D}_x \subset \mathbb{H}^3_K(\alpha)$).

The following lemma will be used in the proof of Lemma 3.4.5.

Lemma 3.1.8. — *Let C be a cone 3-manifold with cone angles less than or equal to π. If x is in a compact component Σ_0 of Σ, then D_x is contained in a region of $\mathbb{H}^3_K(\alpha)$ bounded by two planes orthogonal to the singular axis of $\mathbb{H}^3_K(\alpha)$. Moreover, the distance between these two planes is bounded above by* $\mathrm{Length}(\Sigma_0)$.

Proof. — By convexity, it suffices to study the points $y \in \Sigma_0$ that have at least two minimizing paths to x, one of them contained in Σ_0. When we embed D_x into $\mathbb{H}^3_K(\alpha)$, these are the points that will correspond to the intersection of $\partial\overline{D}_x$ with the singular axis of $\mathbb{H}^3_K(\alpha)$.

As in the proof of Proposition 3.1.2 the local geometry of $\partial\overline{D}_x$ will be given by "bisector" planes between the minimizing paths from y to x. We distinguish two cases.

First we consider the case where there are precisely two minimizing paths σ_1, σ_2 between y and x and $\sigma_1, \sigma_2 \subset \Sigma_0$. In this case $\sigma_1 \cup \sigma_2 = \Sigma_0$ and the point y is obtained by identifying the two points of the intersection of $\partial\overline{D}_x$ with the axis of $\mathbb{H}^3_K(\alpha)$. The bisector plane to σ_1 and σ_2 passing through y is the plane orthogonal to Σ_0, therefore the lemma is clear in this case.

In the second case, among all the minimizing paths between y and x, one of them $\sigma_1 \subset \Sigma_0$ but at least another $\sigma_2 \not\subset \Sigma_0$. Moreover, we may assume that one of the faces of $\partial\overline{D}_x$ is given by the bisector plane between σ_1 and σ_2. In this case, consider the projection $p : S_\alpha \to \mathbb{H}^3_K(\alpha)$ described in the remark above. The pre-image $p^{-1}(\sigma_1)$ is contained in the axis of the sector S_α, and we choose the projection p so that $p^{-1}(\sigma_2)$ is conta.ined in the bisector plane of S_α. That is, $p^{-1}(\sigma_2)$ defines the same angle between both faces of S_α. Since $\alpha \leq \pi$, the region of S_α bounded by the bisector plane between $p^{-1}(\sigma_1)$ and $p^{-1}(\sigma_2)$ is contained in the region of S_α bounded by the plane orthogonal to the axis of S_α passing through $p^{-1}(y) \cap p^{-1}(\sigma_2)$. By convexity, it follows that $p^{-1}(\overline{D}_x)$ is contained in the region of S_α bounded by this orthogonal plane. Therefore, $p^{-1}(\overline{D}_x) \subset S_\alpha$ is contained in a region bounded by two planes orthogonal to the axis of S_α and the distance of these planes is bounded above by $\mathrm{Length}\,\Sigma_0$. This finishes the proof of the lemma. □

Finally, we prove Bishop-Gromov inequality as an application of the Dirichlet polyhedron.

For $r \geq 0$, let $\mathrm{v}_K(r)$ denote the volume of the ball of radius r in $\mathbb{H}^3_K$ the simply connected 3-space of curvature $K \leq 0$.

***Proposition 3.1.9* (Bishop-Gromov inequality).** — *Let C be a cone 3-manifold of curvature $K \leq 0$ and let $x \in C$. If $0 < r \leq R$ then:*

$$\frac{\mathrm{vol}\,(B(x,r))}{\mathrm{v}_K(r)} \geq \frac{\mathrm{vol}\,(B(x,R))}{\mathrm{v}_K(R)}.$$

Proof. — The proof follows from the fact that the Dirichlet polyhedron is star shaped. Namely, if $\mathbb{B}(x,r)$ is the ball of radius r in $\mathbb{H}^3_K$ and D_x is the Dirichlet polyhedron centered at x, then $\mathrm{vol}(B(x,r)) = \mathrm{vol}(\mathbb{B}(x,r) \cap \overline{D}_x)$ and $\mathrm{v}_K(r) = \mathrm{vol}(\mathbb{B}(x,r))$. Since $\overline{D}_x$ is star shaped, the function $r \mapsto \mathrm{vol}(\mathbb{B}(x,r) \cap \overline{D}_x)/\,\mathrm{vol}(\mathbb{B}(x,r))$ is decreasing in r, and the proposition is proved. □

Corollary 3.1.10. — *Let C be a cone 3-manifold of curvature $K \in [-1, 0]$. Given $\varepsilon > 0$ and $R > 0$, the number of disjoint balls of radius $\varepsilon > 0$ that can be contained in a ball of radius R in C has a uniform upper-bound, independent of C.* □

3.2. Hausdorff-Gromov convergence for cone 3-manifolds

We first recall some well known definitions.

Definition 3.2.1. — For $\varepsilon > 0$, an *ε-approximation* between two pointed compact metric spaces (X, x) and (Y, y) is a distance d on the disjoint union $X \sqcup Y$ whose restrictions coincide with the original distances on X and Y, and such that X (resp. Y) belongs to a ε-neighborhood of Y (resp. X) and $d(x, y) \leq \varepsilon$.

Let (X, x) and (Y, y) be two pointed compact metric spaces. The *Hausdorff-Gromov distance* $d_H((X, x), (Y, y))$ is defined as:

$$d_H((X,x),(Y,y)) = \inf\{\varepsilon > 0 \mid \exists \text{ a } \varepsilon\text{-approximation between } (X,x) \text{ and } (Y,y)\}.$$

Remark 3.2.2. — By [**GLP**, Prop. 3.6], two pointed compact metric spaces are isometric by an isometry respecting base points if and only if their Hausdorff-Gromov distance is zero (see also [**BrS**]).

Moreover, since d_H verifies the triangle inequality, it is a distance on the set of pointed compact metric spaces.

A cone 3-manifold is a complete metric length space (cf. Chapter 1): the distance between two points is the infimum of the lengths of paths joining both points.

In the sequel, $\mathcal{L}$ will denote the set of complete locally compact pointed length spaces. Thus we have the inclusion $\mathcal{C}_{[\omega,\pi],a} \subset \mathcal{L}$. In a complete locally compact metric length space, closed balls are compact (see for instance [**GLP**, Thm. 1.10]). Hence the following definition makes sense.

Definition 3.2.3. — A sequence (X_n, x_n) in $\mathcal{L}$ *converges for the Hausdorff-Gromov topology* to $(X_\infty, x_\infty) \in \mathcal{L}$ if for every $R > 0$ the Hausdorff-Gromov distance between the closed balls of radius R $d_H\big(\overline{B(x_n, R)}, \overline{B(x_\infty, R)}\big)$ tends to zero as n goes to infinity.

The following proposition is the first step in the proof of the compactness theorem.

Proposition 3.2.4. — *The space $\mathcal{C}_{[\omega,\pi],a}$ is relatively compact in $\mathcal{L}$ for the Hausdorff-Gromov topology.*

Proof. — It is a consequence of Gromov relative compactness criterion [**GLP**, Prop. 5.2] for sequences of pointed complete locally compact metric spaces and the fact that the space $\mathcal{L}$ is closed for the Hausdorff-Gromov topology [**GLP**, Prop. 3.8 and 5.2].

By Gromov's relative compactness criterion, a sequence (X_n, x_n) in $\mathcal{L}$ has a convergent subsequence if and only if, for every $R > 0$ and for every $\varepsilon > 0$, the number of disjoint balls with radius ε included in the ball $B(x_n, R)$ is uniformly bounded above independently of n. In our case, such a uniform bound follows from Bishop-Gromov inequality for cone 3-manifolds with constant curvature $K \in [-1, 0]$ (Corollary 3.1.10 above, see also [**HT**]). □

We are now stating a key result for the remaining of the proof of the compactness theorem. This result needs the fact that cone angles are bounded above by π, and is not true anymore for cone angles bigger than π.

Proposition 3.2.5 **(Uniform lower bound for cone-injectivity radius)**
Given $R > 0$, $a > 0$ and $\omega \in (0, \pi]$, there exists a uniform constant $b = b(R, a, \omega) > 0$ such that, for every pointed cone 3-manifold $(C, x) \in \mathcal{C}_{[\omega,\pi],a}$, the cone-injectivity radius at any point of $B(x, R) \subset C$ is greater than b.

The proof of this proposition is rather long, so we postpone it to Section 3.4. We will use it in the proof of the following proposition as well as in Section 3.3.

Proposition 3.2.6. — *Let (C_n, x_n) be a sequence of pointed cone 3-manifolds in $\mathcal{C}_{[\omega,\pi],a}$ that converges to (X_∞, x_∞) in $\mathcal{L}$ for the Hausdorff-Gromov topology. Then the limit (X_∞, x_∞) is a pointed cone 3-manifold in $\mathcal{C}_{[\omega,\pi],b}$ for some $b > 0$. Moreover the curvature of X_∞ is the limit of the curvatures of C_n.*

Remark 3.2.7. — The cone-injectivity radius is lower semi-continuous, and it could happen that $\mathrm{inj}(x_\infty) < a$.

Proof of Proposition 3.2.6. — Let (C_n, x_n) be a sequence in $\mathcal{C}_{[\omega,\pi],a}$ that converges to $(X_\infty, x_\infty) \in \mathcal{L}$ for the Hausdorff-Gromov topology. Since X_∞ is a complete, locally compact, metric length space, we have to show that X_∞ is locally isometric to a cone 3-manifold of constant sectional curvature.

Let $y \in X_\infty$; choose $R = d(y, x_\infty) + 1$. From Hausdorff-Gromov convergence, for n large enough we have an ε_n-approximation d_n between the closed balls $\overline{B(x_\infty, R)}$ and $\overline{B(x_n, R)}$, with $\varepsilon_n \to 0$. We take $y_n \in \overline{B(x_n, R)}$ such that $d_n(y, y_n) < \varepsilon_n$.

By Proposition 3.2.5, there is a uniform constant $b > 0$ independent of n such that $\mathrm{inj}(y_n) \geq b$, for every n large enough. Since both C_n and X_n are length spaces,

d_n induces a $3\varepsilon_n$-approximation between the compact balls $\overline{B(y_n, b)}$ and $\overline{B(y, b)}$. By taking a subsequence if necessary, there are three cases to be considered:

Case 1. For every $n \in \mathbb{N}$, $B(y_n, b)$ is a standard non-singular ball (i.e. $B(y_n, b)$ is isometric to a metric ball in the space $\mathbb{H}^3_{K_n}$, where K_n is the curvature of C_n). Since $K_n \in [-1, 0]$, up to a subsequence K_n converges to $K_\infty \in [-1, 0]$. Moreover, since

$$\lim_{n \to \infty} d_H(\overline{B(y_n, b)}, \overline{B(y, b)}) = 0,$$

the uniqueness of the Hausdorff-Gromov limit for compact spaces [**GLP**, Prop. 3.6] shows that the ball $\overline{B(y,b)}$ must be isometric to a metric ball in the space of constant curvature $\mathbb{H}^3_{K_\infty}$.

Case 2. For every $n \in \mathbb{N}$, $B(y_n, b)$ is contained in a standard singular ball, but the distance between y_n and Σ is bounded below, uniformly away from zero. In this case, since the cone angles are also bounded below by $\omega > 0$, there exists a uniform constant $b' > 0$ such that $B(y_n, b') \cap \Sigma = \emptyset$ and $B(y_n, b')$ is isometric to a metric ball in $\mathbb{H}^3_{K_n}$, for every $n \in \mathbb{N}$. Thus we are in the first case and we can conclude that $B(y, b')$ is isometric to a metric ball in $\mathbb{H}^3_{K_\infty}$, where $K_\infty = \lim\limits_{n \to \infty} K_n$.

Case 3. For every $n \in \mathbb{N}$, $B(y_n, b) \cap \Sigma \neq \emptyset$ and the distance between y_n and Σ tends to zero. In this case we replace y_n by $y'_n \in \Sigma$ so that $d(y_n, y'_n) \to 0$. The ball $B(y'_n, b)$ is isometric to a singular ball in the space $\mathbb{H}^3_{K_n}(\alpha_n)$ of constant curvature K_n with a singular axis, where K_n is the curvature of C_n and α_n the cone angle at y'_n. Since $K_n \in [-1, 0]$ and $\alpha_n \in [\omega, \pi]$, up to a subsequence we may assume that $K_n \to K_\infty \in [-1, 0]$ and $\alpha_n \to \alpha_\infty \in [\omega, \pi]$. As in case 1, the fact that the Hausdorff-Gromov distance $d_H(\overline{B(y'_n, b)}, \overline{B(y, b)})$ tends to zero and the uniqueness of the Hausdorff-Gromov limit imply that the ball $B(y, b)$ is isometric to a singular metric ball in $\mathbb{H}^3_{K_\infty}(\alpha_\infty)$ and $y \in \Sigma_\infty$.

This achieves the proof of Proposition 3.2.6. □

The following corollary is a direct consequence of the proof of Proposition 3.2.6 and will be used later in the proof of Proposition 3.3.1.

Corollary 3.2.8. — *Let (C_n, x_n) be a sequence of pointed cone 3-manifolds that converges to the pointed cone 3-manifold (C_∞, x_∞) for the Hausdorff-Gromov topology. Given $y \in C_\infty$, choose $R \geq d(x_\infty, y) + 1$ and d_n an ε_n-approximation between $\overline{B(x_n, R)}$ and $\overline{B(x_\infty, R)}$, with $\varepsilon_n \to 0$. Then there exists a sequence $y_n \in B(x_n, R)$ such that $d_n(y_n, y) \to 0$ as $n \to \infty$, and $y_n \in \Sigma$ if and only if $y \in \Sigma$. Moreover, when $y \in \Sigma$, the sequence of cone angles at y_n converges to the cone angle at y.* □

3.3. Hausdorff-Gromov convergence implies geometric convergence

The goal of this section is to prove the following:

Proposition 3.3.1. — *If a sequence (C_n, x_n) of pointed cone 3-manifolds converges in $\mathcal{C}_{[\omega,\pi],a}$ to a pointed cone 3-manifold $(C_\infty, x_\infty) \in \mathcal{C}_{[\omega,\pi],a}$ for the Hausdorff-Gromov topology, then it also converges geometrically.*

Proof of Proposition 3.3.1. — Fix a radius $R > 0$. Let T be a compact triangulated subset of the underlying space of C_∞ such that $B(x_\infty, 12R) \subset T$. By subdividing the triangulation we may assume that:

i) all simplices are totally geodesic,
ii) $\Sigma_\infty \cap T$ belongs to the 1-skeleton $T^{(1)}$, and
iii) the base point x_∞ is a vertex of $T^{(0)}$.

Let $R' > 0$ be such that $T \subset B(x_\infty, R')$ and let d_n be a ε_n-approximation between the compact balls $\overline{B(x_\infty, R')}$ and $\overline{B(x_n, R')}$, with $\lim_{n\to\infty} \varepsilon_n = 0$.

Let $T^{(0)} = \{z_\infty^0, \dots, z_\infty^r\}$ be the vertices of T, with $z_\infty^0 = x_\infty$. We choose some points $z_n^0, \dots, z_n^r \in B(x_n, R')$ such that $\lim_{n\to\infty} d_n(z_n^i, z_\infty^i) = 0$, for $i = 0, \dots, r$. It follows from Corollary 3.2.8 that one can choose $z_n^i \in \Sigma_n$ if and only if $z_\infty^i \in \Sigma_\infty$.

For a simplex Δ of T, $\mathrm{star}(\Delta)$ denotes the star of Δ, and $\mathrm{star}^*(\Delta)$, the union of simplices of T that intersect Δ but not the singular set Σ. With this notation we have the following:

Lemma 3.3.2. — *It is possible to geodesically subdivide the triangulation T so that any simplex Δ satisfies the following properties:*

i) $\mathrm{star}(\Delta)$ *is included in a standard ball of C_∞.*
ii) Let $\{z_\infty^{i_1}, \dots, z_\infty^{i_s}\}$ be the vertices of $\mathrm{star}(\Delta)$. *For n sufficiently large, $\{z_n^{i_1}, \dots, z_n^{i_s}\}$ belongs to a standard ball in C_n,*
iii) If $\Delta \cap \Sigma_\infty = \emptyset$ then $\mathrm{star}^*(\Delta)$ *is included in a non-singular standard ball.*
iv) If $\Delta \cap \Sigma_\infty = \emptyset$ and $\{z_\infty^{i_1}, \dots, z_\infty^{i_t}\}$ are the vertices of $\mathrm{star}^*(\Delta)$ *then, for n sufficiently large, $\{z_n^{i_1}, \dots, z_n^{i_t}\}$ belongs to a non-singular standard ball in C_n.*

Remark 3.3.3. — It is worthwhile to recall that a standard ball in a cone 3-manifold C with constant sectional curvature K is isometric either to a non-singular metric ball in $\mathbb{H}^3_K$, or to a singular metric ball in $\mathbb{H}^3_K(\alpha)$ whose center lies in the singular axis.

Proof of Lemma 3.3.2. — From Proposition 3.2.5, there are two constants $r_1 > 0$ and $r_2 > 0$ such that for any $y \in B(x_\infty, R')$ or $y_n \in B(x_n, R')$:

a) If $y \in \Sigma_\infty$, then $B(y, r_1)$ is a standard singular ball in C_∞, and if $y_n \in \Sigma_n$, then $B(y_n, r_1)$ is a standard singular ball in C_n.
b) If $d(y, \Sigma_\infty) > \frac{r_1}{8}$, then $B(y, r_2)$ is a non-singular standard ball in C_∞, and if $d(y_n, \Sigma_n) > \frac{r_1}{8}$, then $B(y_n, r_2)$ is a non-singular standard ball C_n.

Using geodesic barycentric subdivision, we first achieve that for each simplex Δ of T, the diameter $\mathrm{diam}(\Delta) \le \frac{1}{8}\inf\{r_1, r_2\}$. Thus $\mathrm{diam}(\mathrm{star}(\Delta)) \le \frac{3}{8}\inf\{r_1, r_2\}$, which implies assertions i) and ii).

To prove iii) and iv) we introduce the constant $r_3(T) > 0$ depending on T:

$$r_3(T) = \inf\{d(\Delta, \Sigma_\infty), \text{ for any simplex } \Delta \subset T \text{ such that } \Delta \cap \Sigma_\infty = \emptyset\}.$$

Proposition 3.2.5 and the fact that cone angles are bounded below by $\omega > 0$ imply the existence of a constant $r_4 = r_4(r_3, \omega) > 0$ depending on r_3 and ω such that for any point $y \in B(x_\infty, R')$, if $d(y, \Sigma_\infty) > r_3/2$ then $B(y, r_4)$ is a non-singular standard ball in C_∞, and for any point $y_n \in B(x_n, R')$, if $d(y_n, \Sigma_n) > r_3/2$ then $B(y_n, r_4)$ is a non-singular standard ball in C_n.

Next we will subdivide T in such a way that the constants r_3 and hence $r_4 = r_4(r_3, \omega)$ do not change, but the diameter of any simplex not meeting Σ_∞ becomes less than $\frac{r_4}{8}$. This will imply properties iii) and iv) of the lemma.

The process for subdividing a simplex Δ of T is the following:

a) When $\Delta \cap \Sigma_\infty = \emptyset$, we apply geodesic barycentric subdivision to Δ.
b) When $\Delta \subset \Sigma_\infty$, we do not subdivide Δ.
c) When $\emptyset \neq \Delta \cap \Sigma_\infty \neq \Delta$, we express Δ as a joint $\Delta = \Delta_0 * \Delta_1$, where $\Delta_0 \subset \Sigma_\infty$ and $\Delta_1 \cap \Sigma_\infty = \emptyset$. Then we apply geodesic barycentric subdivision Δ_1' to Δ_1 and consider $\Delta' = \Delta_0 * \Delta_1'$ (see Figures 2 to 4, which describe this process).

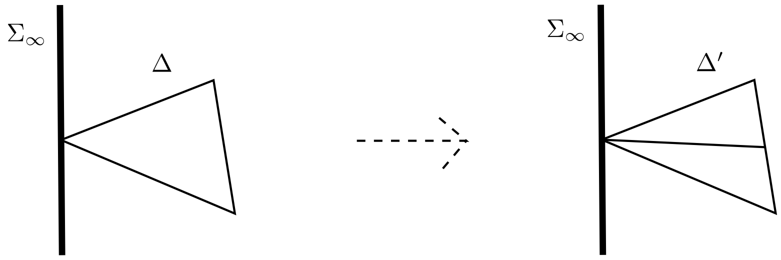

FIGURE 2.

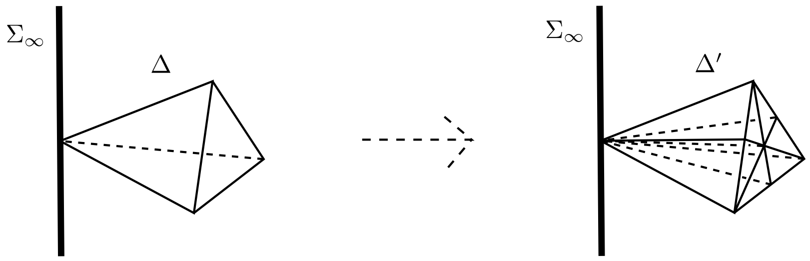

FIGURE 3.

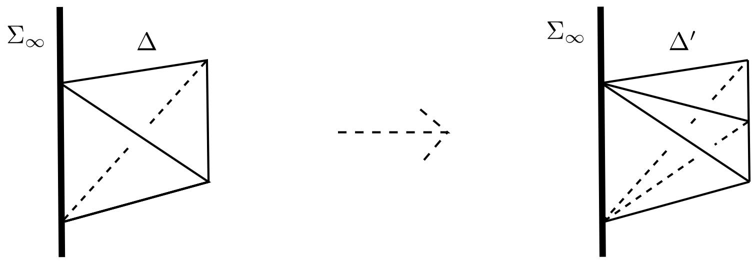

FIGURE 4

This process of subdivision makes the diameter of any simplex disjoint from Σ_∞ arbitrarily small without decreasing its distance to Σ_∞. □

Now we define $g_n : T \to C_n$ by mapping z^i_∞ to z^i_n, $g_n(z^i_\infty) = z^i_n$ for $i = 1, \dots, r$, and we extend g_n piecewise-linearly on each simplex of T. To show that the restriction of g_n to $\overline{B(x_\infty, R)}$ is a $(1 + \varepsilon_n)$-bilipischitz map with $\varepsilon_n \to 0$, we need the following lemma:

Lemma 3.3.4. — *For n sufficiently large, $g_n : T \to C_n$ is a well defined map having the following properties:*

i) $g_n(T)$ is a geodesic polyhedron in C_n, and $g_n(T \cap \Sigma_\infty) = g_n(T) \cap \Sigma_n$;
ii) $\forall x, y \in B(x_\infty, 6R)$, $d(g_n(x), g_n(y)) \leq (1 + \delta_n) d(x, y)$ with $\delta_n \to 0$;
iii) the restriction of g_n induces a homeomorphism from $\mathrm{int}(T)$ *onto its image* $g_n(\mathrm{int}(T))$.

Proof of Lemma 3.3.4. — Let Δ be a 3-simplex in T such that $\Delta \cap \Sigma_\infty = \emptyset$. Up to permutation of indices, let $\{z^1_\infty, z^2_\infty, z^3_\infty, z^4_\infty\}$ denote the vertices of Δ. By Lemma 3.3.2, there exists n_1 such that for $n \geq n_1$, the points $\{z^1_n, z^2_n, z^3_n, z^4_n\}$ are contained in a non-singular standard ball in C_n. By construction, the sequence $\{d(z^i_n, z^j_n)\}_{n \in \mathbb{N}}$ tends to $d(z^i_\infty, z^j_\infty)$ as $n \to \infty$, for any $i, j \in \{1, 2, 3, 4\}$. Moreover, the sectional curvature K_n of C_n tends to K_∞. It follows that the the bijection between $\{z^1_\infty, z^2_\infty, z^3_\infty, z^4_\infty\}$ and $\{z^1_n, z^2_n, z^3_n, z^4_n\}$ extends linearly to a map from the geodesic simplex Δ onto the non-degenerated geodesic simplex in C_n with vertices $\{z^1_n, z^2_n, z^3_n, z^4_n\}$. That is, $g_n(\Delta)$ is a well defined non-degenerated simplex such that the restriction map $g_n|_\Delta : \Delta \to g_n(\Delta)$ is $(1 + \delta_n)$-bilipschitz, with $\delta_n \to 0$.

If $\Delta \cap \Sigma_\infty \neq \emptyset$ then, by Corollary 3.2.8, for n sufficiently large, $z^i_n \in \Sigma_n$ if and only if $z^i_\infty \in \Sigma_\infty$, $i = 1, 2, 3, 4$. By using the same method as in the non-singular case and Lemma 3.3.2, one shows that g_n is well defined on Δ and that $g_n(\Delta)$ is a non-degenerated totally geodesic simplex in C_n such that $g_n(\Delta \cap \Sigma_\infty) = g_n(\Delta) \cap \Sigma_n$. We remark that minimizing paths between two points are not necessarily unique in singular balls, but they are unique if at least one of the points lies in Σ. Hence, g_n is well defined on Δ, because of assertions iii) and iv) of Lemma 3.3.2. Moreover the restriction map $g_n|_\Delta : \Delta \to g_n(\Delta)$ is $(1+\delta_n)$-bilipschitz with $\delta_n \to 0$. This proves property i).

To prove property ii), consider $x, y \in B(x_\infty, 6R)$. Let σ be a minimizing path between x and y. Since $\sigma \subset B(x_\infty, 12R) \subset T$, the inequality $d(g_n(x), g_n(y)) \leq (1+\delta_n)d(x,y)$ follows from the fact that, for any 3-simplex Δ of T, $g_n : \Delta \to g_n(\Delta)$ is a $(1+\delta_n)$-bilipschitz map, with $\delta_n \to 0$. It suffices to break up σ into pieces $\sigma \cap \Delta$ and to use the fact that σ is minimizing.

Finally we prove property iii). Note that the restriction of g_n from $\mathrm{star}(\Delta)$ onto $g_n(\mathrm{star}(\Delta))$ is a homeomorphism. This follows from the construction of g_n by piecewise-linear extension and the fact that, for n sufficiently large and for any simplex Δ of T, $\mathrm{star}(\Delta)$ and $g_n(\mathrm{star}(\Delta))$ are contained in standard balls. Thus it remains to show that the restriction of g_n to $\mathrm{int}(T)$ is injective for n sufficiently large.

Suppose that $x, y \in \mathrm{int}(T)$ are two points such that $g_n(x) = g_n(y)$; we claim that $x = y$. Let Δ_x and Δ_y be the simplices of T containing x and y respectively. We claim first that $\Delta_x \cup \Delta_y$ is contained in a standard ball in C_∞ and that $g_n(\Delta_x \cup \Delta_y)$ is also contained in a standard ball in C_n. Recall that the diameter of the simplices is chosen to be small with respect to the lower bound of the cone-injectivity radius on T. Thus we prove the claim by showing that the diameter of $\Delta_x \cup \Delta_y$ is also small. To show this, we first remark that the diameter of $g_n(\Delta_x \cup \Delta_y)$ is small, because $g_n(\Delta_x) \cap g_n(\Delta_y) \neq \emptyset$ and $\mathrm{diam}(g_n(\Delta)) \leq (1+\delta_n)\,\mathrm{diam}(\Delta)$, with $\delta_n \to 0$. In particular, $g_n(\Delta_x \cup \Delta_y)$ is contained in a standard ball. Moreover, the limit $\lim\limits_{n\to\infty} d(z_n^i, z_n^j) = d(z_\infty^i, z_\infty^j)$ means that the distance between vertices of $g_n(\Delta_x \cup \Delta_y)$ converges to the distance between vertices of $\Delta_x \cup \Delta_y$, therefore $\mathrm{diam}(\Delta_x \cup \Delta_y)$ is small.

Finally we prove that $g_n(\Delta_x \cap \Delta_y) = g_n(\Delta_x) \cap g_n(\Delta_y)$. This follows from the facts that $g_n(\Delta_x \cup \Delta_y)$ and $\Delta_x \cup \Delta_y$ are contained in standard balls, that $d(z_n^i, z_n^j)$ converges to $d(z_\infty^i, z_\infty^j)$, and that Δ_x, Δ_y, $g_n(\Delta_x)$ and $g_n(\Delta_y)$ are the convex hulls of their vertices. Hence, if $g_n(x) = g_n(y)$ then x and y belong to the same simplex and $x = y$. This proves Lemma 3.3.4. □

Lemma 3.3.5. — *For n sufficiently large, two points in $g_n(B(x_\infty, R))$ are joined by a minimizing geodesic contained in $g_n(B(x_\infty, 5R))$.*

Proof. — Since two points in $B(z_n^0, 2R) \subset C_n$ are joined by a minimizing geodesic contained in $B(z_n^0, 4R)$, it suffices to show the following inclusions for n large enough:

a) $g_n(B(x_\infty, R)) \subset B(z_n^0, 2R)$,
b) $B(z_n^0, 4R) \subset g_n(B(x_\infty, 5R))$,

where we recall that $z_n^0 = g_n(x_\infty)$ and $x_\infty = z_\infty^0 \in T^{(0)}$.

Property a) follows from Lemma 3.3.4 ii), which states that for $x, y \in B(x_\infty, 6R)$, $d(g_n(x), g_n(y)) \leq (1+\delta_n)d(x,y)$, with $\delta_n \to 0$.

For n sufficiently large, the restriction $g_n : B(x_\infty, 6R) \to g_n(B(x_\infty, 6R))$ is a homeomorphism and hence $g_n(\partial B(x_\infty, 5R)) = \partial g_n(B(x_\infty, 5R))$. Thus inclusion b) will follow from the inequality $d(z_n^0, g_n(\partial B(x_\infty, 5R))) > 4R$, for n sufficiently large.

Let y be a point in $\partial B(x_\infty, 5R)$, that is $d(x_\infty, y) = 5R$. Set:

$$r_0 = \sup\{\mathrm{diam}(\Delta) \mid \Delta \text{ is a 3-simplex of } T\}.$$

Since $y \in \partial B(x_\infty, 5R) \subset T$, there exists a vertex $z_\infty^i \in T^{(0)}$ such that $d(z_\infty^i, y) \leq r_0$. So we write:

$$d(z_n^0, g_n(y)) \geq d(z_n^0, z_n^i) - d(z_n^i, g_n(y)).$$

By Lemma 3.3.4, $d(z_n^i, g_n(y)) \leq (1+\delta_n) d(z_\infty^i, y) \leq 2r_0$. Moreover,

$$\lim_{n\to\infty} d(z_n^0, z_n^i) = d(z_\infty^0, z_\infty^i) = d(x_\infty, z_\infty^i) \geq d(x_\infty, y) - d(z_\infty^i, y) \geq 5R - r_0.$$

Summarizing these inequalities we conclude that $d(z_n^0, g_n(y)) \geq 5R - 4r_0$ and it suffices to choose $r_0 < R/4$ using the proof of Lemma 3.3.2. This achieves the proof of inclusion b) and of Lemma 3.3.5. $\square$

The following lemma concludes the proof of Proposition 3.3.1.

Lemma 3.3.6. — *For any real $\varepsilon > 0$, there is an integer n_0 such that, for $n \geq n_0$:*

i) The restriction $g_n : B(x_\infty, R) \to C_n$ is $(1+\varepsilon)$-bilipschitz,
ii) $d(g_n(x_\infty), x_n) < \varepsilon/2$,
iii) $B(x_n, R-\varepsilon) \subset g_n(B(x_\infty, R))$.

Proof. — By Lemma 3.3.4 ii), there exists a sequence $\delta_n \to 0$ such that

$$d(g_n(x), g_n(y)) \leq (1+\delta_n) d(x,y) \qquad \forall x, y, \in B(x_\infty, 6R), \forall n \geq n_0.$$

By choosing n sufficiently large we may assume $\delta_n < \varepsilon$, hence property i) will follow from the following inequality

$$(1+\varepsilon)^{-1} d(x,y) \leq d(g_n(x), g_n(y)), \qquad \forall n \geq n_0, \forall x, y \in B(x_\infty, R).$$

To prove this inequality, given $x, y \in B(x_\infty, R)$, we choose a minimizing path σ between $g(x)$ and $g(y)$ that is contained in $g_n(B(x_\infty, 5R))$, by Lemma 3.3.5. Since $g_n : B(x_\infty, 5R) \to g_n(B(x_\infty, 5R))$ is a homeomorphism, $\tilde{\sigma} = g_n^{-1}(\sigma)$ is a path joining x and y. The map g_n is constructed in the proof of Lemma 3.3.4 so that its restriction to each simplex Δ of T is $(1+\delta_n)$-bilipschitz. Then, by breaking $\tilde{\sigma}$ into pieces $\tilde{\sigma} \cap \Delta$ we prove that $(1+\varepsilon)^{-1}\,\mathrm{Length}(\tilde{\sigma}) \leq \mathrm{Length}(\sigma)$, and the claimed inequality follows.

Property ii) follows from the construction, because the Hausdorff-Gromov distance between the pointed balls $(\overline{B(x_n, R')}, x_n)$ and $(\overline{B(x_\infty, R')}, x_\infty)$ goes to zero, and the points x_∞ and $g_n(x_\infty)$ are arbitrarily close in the Hausdorff-Gromov approximations.

Next we prove property iii). Let $y_n \in B(x_n, R-\varepsilon)$. By property ii), $y_n \in B(g_n(x_\infty), R) = B(z_n^0, R)$. Moreover, in the proof of Lemma 3.3.5 (inclusion b)) we have seen that $B(z_n^0, R) \subseteq g_n(B(x_\infty, 5R))$. Hence we can choose a point $y \in B(x_\infty, 5R)$ such that $g_n(y) = y_n$. Then, for n large, we have

$$d(x_\infty, y) \leq (1+\delta_n)\big(d(y_n, x_n) + d(x_n, g_n(x_\infty))\big) \leq (1+\delta_n)(d(x_n, y_n) + \varepsilon/2),$$

with $\delta_n \to 0$. For n sufficiently large so that $\delta_n < \varepsilon/(2R)$, we conclude that $y \in B(x_\infty, R)$. Hence $B(x_n, R-\varepsilon) \subseteq g_n(B(x_\infty, R))$ and the lemma is proved. □

3.4. Uniform lower bound for the cone-injectivity radius

The goal of this section is to prove Proposition 3.2.5. For convenience we recall the statement.

Proposition 3.2.5 **(Uniform lower bound for cone-injectivity radius)**
Given $R > 0$, $a > 0$ and $\omega \in (0, \pi]$, there exists a uniform constant $b = b(R, a, \omega) > 0$ such that, for any pointed cone 3-manifold $(C, x) \in \mathcal{C}_{[\omega,\pi],a}$, the cone-injectivity radius at any point of $B(x, R) \subset C$ is greater than b.

Remark 3.4.1. — We recall the definition of cone-injectivity radius at a point $x \in C$:

$$\mathrm{inj}(x) = \sup\{\delta > 0 \text{ such that } B(x, \delta) \text{ is contained in a standard ball in } C\}.$$

Note that the definition does not assume the ball to be centered at x, otherwise regular points near the singular locus would have arbitrarily small cone-injectivity radius; such points are contained in larger standard balls centered at nearby singular points. Moreover, if $x \in \Sigma$ then the standard ball in the definition can be assumed to be centered at x. Proposition 3.2.5 implies that there is a uniform lower bound for the radius of a tubular neighborhood of the singular locus Σ. In particular the singular locus can not cross itself when the cone angles are $\leq \pi$. The proof of Proposition 3.2.5 is based on volume estimates using the convexity of the Dirichlet polyhedron (Corollary 3.1.5).

The proof of Proposition 3.2.5 is divided in two propositions, the first one deals with the case of singular points, the second one with the case of regular points.

Proposition 3.4.2. — *Given $R > 0$, $a > 0$ and $\omega \in (0, \pi]$, there exist constants $\delta_1 = \delta_1(R, a, \omega) > 0$ and $\delta_2 = \delta_2(R, a, \omega) > 0$ (depending only on R, a and ω) such that any pointed cone 3-manifold $(C, x) \in \mathcal{C}_{[\omega,\pi],a}$ satisfies:*

i) any component Σ_0 of the singular locus $\Sigma \subset C$ that intersects $B(x, R)$ has length $|\Sigma_0| \geq \delta_1$,
ii) $\mathcal{N}_{\delta_2}(\Sigma) \cap B(x, R) = \{y \in B(x, R) \mid d(y, \Sigma) < \delta_2\}$ is a tubular neighborhood of $\Sigma \cap B(x, R)$.

Proposition 3.4.3. — *Given $R > 0$, $a > 0$ and $\omega \in (0, \pi]$, there exists a constant $\delta_3 = \delta_3(R, a, \omega) > 0$ (depending only on R, a and ω) such that for any pointed cone*

3-manifold $(C,x) \in \mathcal{C}_{[\omega,\pi],a}$, *if* $y \in B(x,R) \subset C$ *and* $d(y,\Sigma) > \min(\delta_1,\delta_2)$ *then* $\mathrm{inj}(y) > \delta_3$ *(where* δ_1 *and* δ_2 *are the constants given in Proposition 3.4.2).*

The proof of Proposition 3.4.3 is given in [**Koj**, Prop. 5.1.1]. It may be proved also by perturbing the singular metric on the tubular neighborhood of $\Sigma \cap B(x,R)$ with radius $\min\{\delta_1,\delta_2\}$ to get a Riemannian metric with pinched sectional curvature, with a pinching constant depending only on δ_1 and δ_2; then we are in the case of non-singular Riemannian metrics for which the result is well known (cf. [**GLP**], [**Pe**]). Therefore we only give the proof of Proposition 3.4.2.

Proof of Proposition 3.4.2 i). — The proof follows from the following volume estimations.

Lemma 3.4.4. — *Given* $a > 0$ *and* $\omega \in (0,\pi]$, *there exists a constant* $c_1 = c_1(a,\omega) > 0$ *such that for any pointed cone 3-manifold* $(C,x) \in \mathcal{C}_{[\omega,\pi],a}$, $\mathrm{vol}(B(x,1)) \geq c_1$.

Lemma 3.4.5. — *Given* $R > 0$ *there is a constant* $c_2 = c_2(R) > 0$ *such that, if* C *is a cone 3-manifold of curvature* $K \in [-1,0]$ *and if* Σ_0 *is a component of the singular locus of* C, *then for any* $y \in \Sigma_0$, $\mathrm{vol}(B(y,R+1)) \leq c_2(R)|\Sigma_0|$, *where* $|\Sigma_0|$ *is the length of* Σ_0.

Proof of Proposition 3.4.2 i) from Lemmas 3.4.4 and 3.4.5. — Let Σ_0 be a component of Σ that intersects $B(x,R)$ and $y \in \Sigma_0 \cap B(x,R)$. By Lemmas 3.4.4 and 3.4.5 we have:

$$c_1 = c_1(a,\omega) \leq \mathrm{vol}(B(x,1)) \leq \mathrm{vol}(B(y,R+1)) \leq c_2(R)|\Sigma_0|.$$

Therefore $|\Sigma_0| \geq \delta_1 = c_1/c_2$. □

We now give the proofs of Lemmas 3.4.4 and 3.4.5.

Proof of Lemma 3.4.4. — Let $(C,x) \in \mathcal{C}_{[\omega,\pi],a}$; in particular $\mathrm{inj}(x) > a$. Because of the definition of the cone-injectivity radius, we distinguish two cases, according to whether $B(x,a)$ is contained in a singular standard ball or in a non-singular one.

Non-singular case. When $B(x,a)$ is a non-singular standard ball, by taking $a_0 = \inf\{1,a\}$ we have $\mathrm{vol}(B(x,1)) \geq \mathrm{vol}(B(x,a_0)) \geq \frac{4}{3}\pi a_0^3$, because the curvature $K \leq 0$.

Singular case. When $B(x,a)$ is contained in a standard singular ball, there exists a point $z \in \Sigma$ and $a' \geq a$ such that $B(z,a')$ is a singular standard ball that contains $B(x,a)$. We may assume that $d(x,z) = d(x,\Sigma)$. We distinguish again two sub-cases.

If $d(x,z) \leq 1/2$, then by taking $a_0 = \inf\{1/2,a\}$, we have that $B(z,a_0) \subset B(x,1)$ and thus $\mathrm{vol}(B(x,1)) \geq \mathrm{vol}(B(z,a_0)) \geq \frac{2}{3}\omega a_0^3$, because the cone angles are bounded below by ω and the curvature $K \leq 0$.

If $d(x,z) = d(x,\Sigma) \geq 1/2$, an elementary trigonometric argument shows that we can find a constant $b = b(\omega,a) > 0$ such that $B(x,b)$ is a non-singular standard ball. This constant depends only on ω and a, because the curvature $K \in [-1,0]$. As in the non-singular case, by taking $b_0 = \inf\{b, 1\}$, we have the inclusion $B(x, b_0) \subset B(x, 1)$ and the inequality $\mathrm{vol}(B(x,1)) \geq \frac{4}{3}\pi b_0^3$.

This finishes the proof of Lemma 3.4.4. □

Proof of Lemma 3.4.5. — Let Σ_0 and $y \in \Sigma_0$ be as in the statement of Lemma 3.4.5. Consider D_y the Dirichlet polyhedron centered at y. By Lemma 3.1.8, D_y is contained in the region of $\mathbb{H}^3_K(\alpha)$ bounded by two planes orthogonal to the singular axis of $\mathbb{H}^3_K(\alpha)$ and the distance between them is less that or equal to the length $|\Sigma_0|$. Therefore we have:

$$\mathrm{vol}(D_y \cap B(y, R+1)) \leq 2\pi |\Sigma_0| \sinh_K^2(R+1)$$

where $\sinh_K(r) = \sinh(\sqrt{-K}r)/\sqrt{-K}$ if $K < 0$ and $\sinh_0(r) = r$. Since $K \in [-1, 0]$, $\sinh_K(r) \leq \sinh(r)$ and we conclude:

$$\mathrm{vol}(D_y \cap B(y, R+1)) \leq \left(2\pi \sinh^2(R+1)\right) |\Sigma_0|.$$

This inequality proves Lemma 3.4.5. □

Proof of Proposition 3.4.2 ii). — Let σ be a minimizing arc between two points of $B(x, R) \cap \Sigma$ that is not contained in Σ, in particular $\sigma \cap \Sigma = \partial\sigma$. We assume that σ has minimal length among all such possible arcs. Proposition 3.4.2 will follow from Lemma 3.4.4 and the following:

Lemma 3.4.6. — *There exists a constant $c_3 = c_3(R) > 0$ depending only on R such that* $\mathrm{vol}(\mathcal{N}_{R+1}(\sigma)) < c_3(R)|\sigma|$, *where $\mathcal{N}_{R+1}(\sigma) = \{y \in C \mid d(y, \sigma) \leq R+1\}$ and $|\sigma|$ is the length of σ.*

The proof of Proposition 3.4.2 ii) from Lemmas 3.4.4 and 3.4.6 is similar to the proof of Proposition 3.4.2 i). From the inclusion $B(x, 1) \subset \mathcal{N}_{R+1}(\sigma)$ and the inequalities of Lemmas 3.4.4 and 3.4.6 we conclude that $|\sigma| > c_1/c_3$. Thus it suffices to choose $\delta_2 = \frac{1}{2}c_1/c_3$ in Proposition 3.4.2 ii). □

The remaining of this section is devoted to the proof of Lemma 3.4.6.

Proof of Lemma 3.4.6. — Let D_σ be the open subset of C defined as:

$$D_\sigma = \{y \in C - \Sigma \mid \text{ there is a unique minimizing arc between } y \text{ and } \sigma\}.$$

The open set D_σ is perhaps not convex, but it is star shaped with respect to σ. So D_σ may be isometrically embedded in $\mathbb{H}^3_K$, the space of constant sectional curvature $K \in [-1, 0]$.

Claim 3.4.7. — *The set $C - D_\sigma$ has Lebesgue measure zero.*

Proof. — Since Σ is 1-dimensional, it suffices to show that $C - (\Sigma \cup D_\sigma)$ has measure zero. Given $z \in C - (\Sigma \cup D_\sigma)$ there are several but only a finite number of minimizing paths from z to σ, by Lemma 3.1.4. Moreover, by the same lemma, there is a neighborhood U_z of z such that for every $y \in U_z \cap C - (\Sigma \cup D_\sigma)$ the minimizing paths between y and σ are in tubular neighborhoods of the minimizing paths between z and σ. Therefore, by using developing maps along these tubular neighborhoods and the fact that the set of points in $\mathbb{H}^3_K$ that are equidistant from two geodesics has measure zero, we conclude that $U_z \cap C - (\Sigma \cup D_\sigma)$ has measure zero. This implies in particular that $C - (\Sigma \cup D_\sigma)$ itself has measure zero and the claim is proved. □

This claim implies that $\mathrm{vol}(\mathcal{N}_{R+1}(\sigma)) = \mathrm{vol}(D_\sigma \cap \mathcal{N}_{R+1}(\Sigma))$. We next use the fact that $D_\sigma \cap \mathcal{N}_{R+1}(\Sigma)$ can be isometrically embedded in $\mathbb{H}^3_K$ to get an upper bound for its volume.

Let $\{p, q\} = \sigma \cap \Sigma$ be the end-points of σ. The proof of Lemma 3.4.6 is divided in three cases: Lemmas 3.4.8, 3.4.9 and 3.4.10,

Lemma 3.4.8. — *If σ is orthogonal to Σ at p and q, then Lemma 3.4.6 holds.*

Proof. — Since the cone angles of C are $\leq \pi$, the orthogonality hypothesis implies that D_σ is contained in the subspace of $\mathbb{H}^3_K$ bounded by the two planes orthogonal to σ at its end-points p and q. Therefore, as in the proof of Lemma 3.4.5, we obtain the following inequality:

$$\mathrm{vol}(\mathcal{N}_{R+1}(\Sigma) \cap D_\sigma) \leq 2\pi|\sigma| \sinh^2_K(R+1) \leq 2\pi \sinh^2(R+1)|\sigma|,$$

because $K \in [-1, 0]$. □

It may happen that σ is not orthogonal to Σ at p or q, when p or q belong to the boundary of $B(x, R)$. Let θ and ϕ in $[0, \pi/2]$ be the angles between σ and Σ at p and q respectively.

Lemma 3.4.9. — *If $\max\{\cos(\theta), \cos(\phi)\} \leq 2|\sigma|$ then Lemma 3.4.6 holds true.*

Proof. — By assumption, D_σ is contained in the union $S_\sigma \cup S_p \cup S_q \subset \mathbb{H}^3_K$, where S_σ is the subspace of $\mathbb{H}^3_K$ bounded by the two planes orthogonal to σ at p and q, S_p is a solid angular sector with axis passing through p and dihedral angle $\pi/2 - \theta$ at the axis, and S_q is a solid angular sector with axis passing through q and dihedral angle $\pi/2 - \phi$ at the axis. One of the faces of S_p (and of S_q) is a face of S_σ and the other contains a piece of Σ (cf. Fig. 5).

Since

$$\mathrm{vol}(\mathcal{N}_{R+1}(\Sigma) \cap D_\sigma) = \mathrm{vol}(\mathcal{N}_{R+1}(\Sigma) \cap S_\sigma) + \mathrm{vol}(\mathcal{N}_{R+1}(\Sigma) \cap S_p) + \mathrm{vol}(\mathcal{N}_{R+1}(\Sigma) \cap S_q),$$

to prove Lemma 3.4.6 it suffices to get a suitable upper bound for each one of these three volumes.

For $\mathrm{vol}(\mathcal{N}_{R+1}(\Sigma) \cap S_\sigma)$ the same upper bound as in Lemma 3.4.8 goes through:

$$\mathrm{vol}(\mathcal{N}_{R+1}(\Sigma) \cap S_\sigma) \leq 2\pi \sinh^2(R+1)|\sigma|.$$

For $\mathrm{vol}(\mathcal{N}_{R+1}(\Sigma) \cap S_p)$ we use the volume of the sector of angle $\pi/2 - \theta$:

$$\mathrm{vol}(\mathcal{N}_{R+1}(\Sigma) \cap S_p) \leq \frac{\pi/2 - \theta}{2\pi} \mathrm{v}_K(R+1),$$

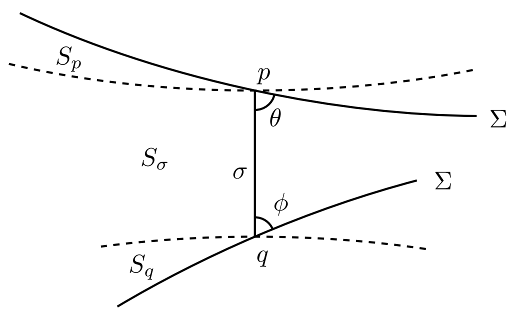

FIGURE 5

where $\mathrm{v}_K(R+1)$ is the volume of the ball of radius $R+1$ in $\mathbb{H}^3_K$. Moreover, since $K \geq -1$, $\mathrm{v}_K(R+1) \leq \mathrm{v}_{-1}(R+1) \leq \pi \sinh(2R+2)$. Therefore:

$$\mathrm{vol}(\mathcal{N}_{R+1}(\Sigma) \cap S_p) \leq (\pi/2 - \theta)\frac{1}{2}\sinh(2R+2).$$

Since $\lim_{\theta\to\pi/2} \cos(\theta)/(\pi/2-\theta) = 1$, there is a constant $\lambda > 0$ such that $\pi/2-\theta \leq \lambda\cos(\theta)$. Therefore the hypothesis $\cos(\theta) \leq 2|\sigma|$ gives:

$$\mathrm{vol}(\mathcal{N}_{R+1}(\Sigma) \cap S_p) \leq \lambda \sinh(2R+2)|\sigma|.$$

The same upper bound can be applied to $\mathrm{vol}(\mathcal{N}_{R+1}(\Sigma) \cap S_q)$. This proves Lemma 3.4.9. $\square$

To achieve the proof of Lemma 3.4.6 we need the following:

Lemma 3.4.10. — *There is a universal constant $\mu > 0$ such that the following holds. If $\max\{\cos\theta, \cos\phi\} > 2|\sigma|$, then one can find a minimizing path σ' between two singular points, that satisfies the following:*

i) $\mathrm{int}(\sigma') \cap \Sigma = \emptyset$,

ii) $|\sigma'| \leq |\sigma|$,

iii) $\sigma \subset \mathcal{N}_\mu(\sigma')$,

iv) if θ' and ϕ' are the angles between σ' and Σ at the end-points of σ' then

$$\max\{\cos\theta', \cos\phi'\} \leq 2|\sigma|$$

(note that we are using $|\sigma|$ instead of $|\sigma'|$).

Assuming Lemma 3.4.10, if the minimizing arc σ does not fulfills the hypothesis of Lemmas 3.4.8 and 3.4.9, then we apply the upper bounds obtained in these lemmas to the region $\mathcal{N}_{R+1+\mu}(\sigma')$ that contains $\mathcal{N}_{R+1}(\sigma)$. Thus we obtain the upper bound

$$\mathrm{vol}(\mathcal{N}_{R+1}(\sigma)) \leq \mathrm{vol}(\mathcal{N}_{R+1+\mu}(\sigma')) \leq c_3(R+\mu)|\sigma|$$

where $c_3(R+\mu) > 0$ depends only on R because μ is universal. This inequality completes the proof of Lemma 3.4.6 (assuming Lemma 3.4.10). $\square$

Proof of Lemma 3.4.10. — By hypothesis, $\max\{\cos\theta, \cos\phi\} > 2|\sigma|$. For $\varepsilon > 0$ sufficiently small there is a homotopy $\{\sigma_t\}_{t\in[0,\varepsilon)}$ of $\sigma = \sigma_0$ such that, for any $t \in [0, \varepsilon)$, σ_t is a geodesic arc between two points of Σ satisfying the following:

1) $\mathrm{int}(\sigma_t) \cap \Sigma = \emptyset$, for all $t \in [0, \varepsilon)$;
2) the length $|\sigma_t|$ is decreasing with t;
3) the angles $\theta_t, \phi_t \in [0, \pi/2]$ between σ_t and Σ are increasing with t.

When we increase the parameter t, we end up with one of the following possibilities.

1) either for some parameter t_0 we reach a path σ_{t_0} that satisfies i) and ii), and moreover $\max\{\cos\theta_{t_0}, \cos\phi_{t_0}\} \leq 2|\sigma|$;
2) or before reaching such a t_0 the homotopy crosses Σ: there is $t_1 > 0$ such that $\mathrm{int}(\sigma_{t_1}) \cap \Sigma \neq \emptyset$ and, for any $t \in [0, t_1)$, $\max\{\cos\theta_t, \cos\phi_t\} > 2|\sigma|$.

Both possibilities happen at bounded distance, because of the following claim:

Claim 3.4.11. — *In both cases, $d(\sigma_t, \sigma) \leq 1$, where $t \leq t_0$ in case a) and $t \leq t_1$ in case b).*

Proof. — By using developing maps, we embed the homotopy $\{\sigma_t\}_{t\in[0,\varepsilon)}$ locally isometrically in $\mathbb{H}^3_K$. In particular, the pieces of Σ and the arcs σ_t are embedded as geodesic arcs.

Up to permutation, we can assume that $\cos\theta \geq \cos\phi$, where θ and ϕ are the angles at p and q respectively. Let $p_t \in \Sigma$ be the end-point of σ_t obtained by moving p along Σ and let p'_t its orthogonal projection to the geodesic of $\mathbb{H}^3_K$ containing σ. This projection p'_t lies between p and q (hence it is contained in σ) by construction of the homotopy (see Fig. 6). In particular $d(p, p'_t) \leq |\sigma|$.

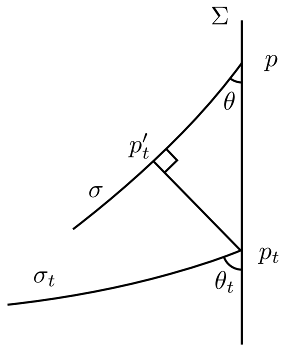

FIGURE 6

Next we consider the right-angle triangle with vertices p, p_t and p'_t (cf. Fig. 6) and we apply the trigonometric formula for $\cos\theta$:

$$\cos\theta = \tanh_K(d(p, p'_t))/\tanh_K(d(p, p_t)) \leq \tanh_K|\sigma|/\tanh_K(d(p, p_t)),$$

where $\tanh_K(r) = \tanh(\sqrt{-K}r)/\sqrt{-K}$ for $K > 0$ and $\tanh_0(r) = r$. We recall that $\tanh_K(|x|) \leq |x|$.

Since $\cos\theta \geq 2|\sigma| \geq 2\tanh_K |\sigma|$, we obtain that $\tanh_K(d(p, p_t)) \leq 1/2$. Moreover $1/2 \leq \tanh_K 1$ because $K \in [-1, 0]$. Thus the monotonicity of $\tanh_K$ implies that $d(p, p_t) \leq 1$, and in particular $d(\sigma, \sigma_t) \leq 1$. This proves the claim. □

From this claim, we deduce that $\sigma_t \subset \mathcal{N}_2(\sigma)$, because $|\sigma_t| \leq |\sigma| \leq 1$. In particular, if case a) above occurs, the path σ_{t_0} satisfies the conclusion of Lemma 3.4.10 and we are done. Hence we assume that case b) happens. Let σ_{t_1} be the path coming from the homotopy that intersects Σ in its interior and consider p_1 and q_1 the nearest two distinct points of $\Sigma \cap \sigma_{t_1}$. We obtain in this way two points p_1 and q_1 on Σ joined by a minimizing arc σ_1 such that $\sigma_1 \cap \Sigma = \{p_1, q_1\}$. Moreover, by Claim 3.4.11, $d(\sigma_1, \sigma) \leq 2$ and, by the choice of p_1 and q_1,

$$|\sigma_1| \leq \frac{1}{2}|\sigma_{t_1}| \leq \frac{1}{2}|\sigma| \leq \frac{1}{4}.$$

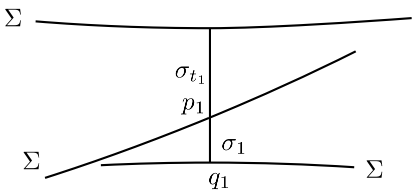

FIGURE 7

Let $\theta_1, \phi_1 \in [0, \pi/2]$ be the angles between σ_1 and Σ at p_1 and q_1 respectively. We assume again that $\max\{\cos\theta_1, \cos\phi_1\} > 2|\sigma|$, otherwise the minimizing path σ_1 would satisfy the conclusion of Lemma 3.4.10 and we would be done.

By iterating this process, we construct two sequences of points p_n and q_n on Σ such that $p_0 = p$, $q_0 = q$, $p_n \neq q_n$ and there is a minimizing path σ_n between p_n and q_n such that $\sigma_n \cap \Sigma = \{p_n, q_n\}$ and $|\sigma_n| \leq \frac{1}{2}|\sigma_{n-1}|$. Moreover, if $\theta_n, \phi_n \in [0, \pi/2]$ are the angles between σ_n and Σ at p_n and q_n, then we make the choice $\cos\theta_n \geq \cos\phi_n$. There are two possibilities:

- either $\cos\theta_n \leq 2|\sigma|$ and the sequences stop at n.
- or $\cos\theta_n > 2|\sigma|$ and the seque

The following claim shows that the sequences stop at uniformly bounded distance.

Claim 3.4.12. — *There is a universal constant $\eta > 0$ such that $d(p, p_n) < \eta$ and $d(p, q_n) < \eta$, whenever p_n and q_n are defined.*

The claim implies that the sequences stop, otherwise $(p_n)_{n\in\mathbb{N}}$ would have a convergent subsequence in the compact ball $B(p, \eta)$, contradicting the fact that the cone-injectivity radius of the limit point is positive. Hence, for the value of n where the sequences stop, the path σ_n satisfies the conclusions of Lemma 3.4.10, because it is at a uniformly bounded distance of σ. Hence Lemma 3.4.10 is proved. □

Proof of Claim 3.4.12. — Let p_n and q_n be two points of the sequences on Σ, and σ_n the minimizing arc between them such that $\sigma_n \cap \Sigma = \{p_n, q_n\}$ and $|\sigma_n| \leq 2^{-n}|\sigma|$. We also assume that $\cos\theta_n > 2|\sigma|$, so that p_{n+1} and q_{n+1} are defined. These points are constructed by considering a homotopy of σ_n as in the beginning of the proof of Lemma 3.4.10. This homotopy gives:

- either a path σ'_n that crosses Σ, and the points p_{n+1} and q_{n+1} are the two nearest different points in $\sigma'_n \cap \Sigma$;
- or a path σ'_n such that the angles θ'_n and ϕ'_n between σ'_n and Σ satisfy
$$\max\{\cos\theta'_n, \cos\phi'_n\} \leq 2|\sigma|.$$
In this case p_{n+1} and q_{n+1} are the end-points of $\sigma_{n+1} = \sigma'_n$ and the sequences stop at $n+1$.

In both cases we have:
$$\max\{d(p_n, p_{n+1}), d(p_n, q_{n+1})\} \leq |\sigma'_n| + d(p_n, \sigma'_n),$$
and $|\sigma'_n| \leq |\sigma_n| \leq |\sigma|/2^n \leq 1/2^{n+1}$. The trigonometric argument of Claim 3.4.11 applies here to give the following inequality,
$$\cos\theta_n \leq \tanh_K |\sigma_n| / \tanh_K(d(p_n, \sigma'_n)).$$
Combining this with the hypothesis $\cos\theta_n > 2|\sigma|$ we get:
$$\tanh_K(d(p_n, \sigma'_n)) \leq \frac{\tanh_K |\sigma_n|}{2|\sigma|} \leq \frac{|\sigma_n|}{2|\sigma|} \leq \frac{1}{2^{n+1}}$$
Since $\tanh_K(x) = x + O(|x|^3)$, it follows from this inequality that there is a universal constant $\eta_0 > 0$ such that $d(p_n, \sigma'_n) \leq \eta_0/2^{n+1}$. Summarizing these inequalities we obtain:
$$\max\{d(p_n, p_{n+1}), d(q_n, q_{n+1})\} \leq (\eta_0 + 1)/2^{n+1}$$
and
$$\max\{d(p, p_{n+1}), d(q, q_{n+1})\} \leq \sum_{i=0}^{n} (\eta_0 + 1)/2^{i+1} < \eta_0 + 1.$$
It suffices to take $\eta = \eta_0 + 1$ to achieve the proof of Claim 3.4.12. □

3.5. Some properties of geometric convergence

In this section we study properties of sequences of pointed cone 3-manifolds in $\mathcal{C}_{[\omega,\pi],a}$ that converge geometrically. During all the section we will assume $\omega \in (0, \pi]$ and $a > 0$.

Proposition 3.5.1. — *Let (C_n, x_n) be a sequence in $\mathcal{C}_{[\omega,\pi],a}$ that converges geometrically to a pointed cone 3-manifold (C_∞, x_∞). Then:*

i) the curvature of C_n *converges to the curvature of* C_∞*;*
ii) $\operatorname{inj}(x_\infty) \le \liminf_{n\to\infty} \operatorname{inj}(x_n)$.

Proof. — Property i) has been proved in Proposition 3.2.6. It follows also from the fact that the sectional curvature may be computed from small geodesic triangles.

To prove Property ii) we distinguish two cases, according to whether the cone-injectivity radius at x_∞ is estimated using singular or non-singular balls. Let $r_\infty = \operatorname{inj}(x_\infty)$. We first assume that for any $0 < \varepsilon < r_\infty$, the ball $B(x_\infty, r_\infty - \varepsilon)$ is standard and non-singular. Geometric convergence implies that for n sufficiently large, $B(x_n, r_\infty - 2\varepsilon)$ is standard in C_n, hence $\operatorname{inj}(x_n) \ge \operatorname{inj}(x_\infty) - 2\varepsilon$. A similar argument applies in the case of singular standard balls. □

Proposition 3.5.2. — *Let* (C_n, x_n) *be a sequence in* $\mathcal{C}_{[\omega,\pi],a}$ *that converges geometrically to a pointed cone 3-manifold* (C_∞, x_∞)*. For any compact subset* $A \subset C_\infty$ *there exists* $n_0 > 0$ *such that for* $n \ge n_0$ *there is an embedding* $f_n : A \to C_n$ *with the following properties:*

i) $f_n(A) \cap \Sigma_n = f_n(A \cap \Sigma_\infty)$*;*
ii) the cone angles at $f_n(A) \cap \Sigma_n$ *approach the cone angles at* $A \cap \Sigma_\infty$ *as* n *goes to infinity.*

Corollary 3.5.3. — *If the limit* C_∞ *of a geometrically convergent sequence* (C_n, x_n) *in* $\mathcal{C}_{[\omega,\pi],a}$ *is compact, then for* n *sufficiently large* C_n *has the same topological type as* C_∞ *(i.e. the pairs* (C_n, Σ_n) *and* $(C_\infty, \Sigma_\infty)$ *are homeomorphic) and the cone angles of* C_n *converge to those of* C_∞*.* □

Proof of Proposition 3.5.2. — Since A is compact, there exists $R > 0$ such that $A \subset B(x_\infty, R)$. The definition of geometric convergence implies that for any $\varepsilon > 0$ and for $n \ge n_0$ (n_0 depending on R, ε and the sequence) there exists a $(1+\varepsilon)$-bilipschitz map $f_n : B(x_\infty, R) \to C_n$ such that

$$f_n(B(x_\infty, R) \cap \Sigma_\infty) = f_n(B(x_\infty, R)) \cap \Sigma_n.$$

This proves Property i) of the proposition. Moreover, by taking $\varepsilon \to 0$ we get Property ii). □

Proposition 3.5.4. — *Let* (C_n, x_n) *be a sequence in* $\mathcal{C}_{[\omega,\pi],a}$ *that converges geometrically to a pointed cone 3-manifold* (C_∞, x_∞)*. Assume that the base point* x_∞ *is non-singular. Let* $A \subset C_\infty$ *be a compact subset containing* x_∞*. Let* ρ_n *(resp.* ρ_∞*) be the holonomy of the cone 3-manifold* C_n *(resp.* C_∞*). Then, we can choose the holonomy representations and the embeddings* f_n *of Proposition 3.5.2 so that:*

i) If the curvature of C_n *does not depend on* n*, then* $\forall \gamma \in \pi_1(A - \Sigma_\infty, x_\infty)$*,*

$$\lim_{n\to\infty} \rho_n(f_{n*}(\gamma)) = \rho_\infty(\gamma).$$

ii) If the curvature $K_n \in [-1,0)$ of C_n converges to 0 as $n \to \infty$, then $\forall \gamma \in \pi_1(A - \Sigma_\infty, x_\infty)$,

$$\lim_{n\to\infty} \rho_n(f_{n*}(\gamma)) = \mathrm{ROT}(\rho_\infty(\gamma)),$$

where $\mathrm{ROT} : \mathrm{Isom}(\mathbb{E}^3) \to O(3)$ *is the surjective morphism whose kernel is the subgroup of translations.*

Proof. — We first prove assertion i). We will assume that A contains a neighborhood of x_∞, by replacing A by a bigger set if necessary. Let $\widetilde{C_\infty - \Sigma_\infty}$, $\widetilde{C_n - \Sigma_n}$ and $\widetilde{A - \Sigma_\infty}$ be the respective universal coverings of $C_\infty - \Sigma_\infty$, $C_n - \Sigma_n$ and $A - \Sigma_\infty$. Let $D_\infty : \widetilde{C_\infty - \Sigma_\infty} \to \mathbb{H}^3_K$ be a developing map of C_∞, $\tilde{\imath} : \widetilde{A - \Sigma_\infty} \to \widetilde{C_\infty - \Sigma_\infty}$ a lift of the inclusion $A - \Sigma_\infty \to C_\infty - \Sigma_\infty$, and $\tilde{f}_n : \widetilde{A - \Sigma_\infty} \to \widetilde{C_n - \Sigma_n}$ a lift of f_n. We claim that we can choose developing maps $D_n : \widetilde{C_n - \Sigma_n} \to \mathbb{H}^3_K$ such that $D_n \circ \tilde{f}_n$ converges to $D_\infty \circ \tilde{\imath}$ uniformly on compact subsets.

To prove this claim, we choose a standard ball $B(x_\infty, \varepsilon) \subset A$ and three points $a, b, c \in B(x_\infty, \varepsilon/2)$ such that x_∞, a, b, c are not contained in a plane. In particular, a point in $B(x_\infty, \varepsilon)$ is determined by its distance to each one of the four points x_∞, a, b, c. We also assume that $B(f_n(x_\infty), \varepsilon)$ is standard. We lift the four points to the universal covering $\tilde{x}_\infty, \tilde{a}, \tilde{b}, \tilde{c} \in \widetilde{A - \Sigma_\infty}$ so that $\tilde{a}, \tilde{b}, \tilde{c} \in B(\tilde{x}_\infty, \varepsilon)$. We choose the developing maps D_n in such a way that $D_n(\tilde{f}_n(x_\infty))$, $D_n(\tilde{f}_n(a))$, $D_n(\tilde{f}_n(b))$ and $D_n(\tilde{f}_n(c))$ converge respectively to $D_\infty(\tilde{\imath}(x_\infty))$, $D_\infty(\tilde{\imath}(a))$, $D_\infty(\tilde{\imath}(b))$ and $D_\infty(\tilde{\imath}(c))$. This choice is possible because f_n is $(1+\varepsilon_n)$-bilipschitz, with $\varepsilon_n \to 0$, and f_n preserves the orientation.

The restriction of $D_n \circ \tilde{f}_n$ to $B(\tilde{x}_\infty, \varepsilon)$ converges uniformly to the restriction of $D_\infty \circ \tilde{\imath}$, because a point of $\mathbb{H}^3_K$ is determined by the distance to four points not contained in a plane. The uniform convergence extends to every compact subset of $\widetilde{A - \Sigma_\infty}$, by covering this subset with standard balls and using the fact that the intersection of two balls, if non-empty, contains four non-coplanar points.

Since $D_n \circ \tilde{f}_n$ converges to $D_\infty \circ \tilde{\imath}$ uniformly on compact subsets, we have that for any $z \in \widetilde{A - \Sigma_\infty}$ and any $\gamma \in \pi_1(A - \Sigma_\infty)$,

$$\rho_n(f_{n*}(\gamma))\big(D_n(f_n(z))\big) \to \rho_\infty(\gamma)\big(D_\infty(\tilde{\imath}(z))\big).$$

Moreover since $D_n(f_n(z)) \to D_\infty(\tilde{\imath}(z))$ and A contains a neighborhood of x_∞, this implies that $\rho_n(f_{n*}(\gamma)) \to \rho_\infty(\gamma)$. This proves assertion i).

Assertion ii) is proved in [**Po1**, Prop. 5.14(i)]. □

3.6. Cone 3-manifolds with totally geodesic boundary

The aim of this last paragraph is to prove a compactness theorem for cone 3-manifolds with totally geodesic boundary. This kind of cone 3-manifolds are only used in Chapter 7. If not specified, a cone 3-manifold is assumed to be without boundary.

Definition 3.6.1. — A *cone 3-manifold with boundary* is a cone 3-manifold with totally geodesic boundary, such that the singular set is orthogonal to the boundary.

To define the cone injectivity radius we use not only standard balls in the model spaces $\mathbb{H}^3_K$ and $\mathbb{H}^3_K(\alpha)$, but also *half standard balls* with totally geodesic boundary. The double of a half standard ball along its boundary is a standard ball. When it is singular, the boundary of the half standard ball is orthogonal to the singular locus.

Let C be a cone 3-manifold with boundary. We define the *cone-injectivity radius* at a point $x \in C$ as

$$\operatorname{inj}(x) = \sup \left\{ \delta > 0 \;\middle|\; \begin{array}{c} B(x,\delta) \text{ is contained in either a standard ball} \\ \text{or a standard half ball in } C \end{array} \right\}.$$

As in the beginning of the chapter, given $a > 0$ and $\omega \in (0, \pi]$, $\mathcal{CB}_{[\omega,\pi],a}$ denotes the set of pointed cone 3-manifolds (C, x), possibly with boundary, with constant curvature in $[-1, 0]$, cone angles in $[\omega, \pi]$, and such that $\operatorname{inj}(x) \geq a$.

Remark 3.6.2. — The set $\mathcal{CB}_{[\omega,\pi],a}$ contains $\mathcal{C}_{[\omega,\pi],a}$, because the limit of cone 3-manifolds with boundary can be a cone 3-manifold without boundary. This happens when, in a converging sequence of pointed cone 3-manifolds, the boundary goes to infinity.

This section is devoted to the proof of the following result:

Compactness theorem for cone 3-manifolds with boundary

For $a > 0$ and $\omega \in (0, \pi]$, the closure of $\mathcal{CB}_{[\omega,\pi],a}$ in $\bigcup_{b>0} \mathcal{CB}_{[\omega,\pi],b}$ is compact for the topology of geometric convergence.

The proof of this theorem follows exactly the same scheme as the proof in the case without boundary. The main difference is the following uniform lower bound of the cone-injectivity radius, that we deduced from the analogous bound in the case without boundary (Proposition 3.2.5 above).

***Proposition 3.6.3* (Uniform lower bound for the cone-injectivity radius in the case with boundary)**

Given $R > 0$, $a > 0$ and $\omega \in (0, \pi]$, there exists a uniform constant $b = b(R, a, \omega) > 0$ such that, for any pointed cone 3-manifold $(C, x) \in \mathcal{CB}_{[\omega,\pi],a}$, the cone-injectivity radius at any point of $B(x, R) \subset C$ is bigger than b.

Proof of the compactness theorem for cone 3-manifolds with boundary. — We follow the proof in the case without boundary.

First, by Gromov compactness criterion, $\mathcal{CB}_{[\omega,\pi],a}$ is relatively compact in the space $\mathcal{L}$ of locally compact metric length spaces equipped with the Hausdorff-Gromov topology. The fact that $\mathcal{CB}_{[\omega,\pi],a}$ satisfies the necessary conditions to apply the compactness criterion follows from Bishop-Gromov inequality, whose proof extends trivially to the case with totally geodesic boundary.

The second step requires the use of Proposition 3.6.3, that we prove later. Using this proposition one shows that, if we have a sequence of pointed cone 3-manifolds with boundary in $\mathcal{CB}_{[\omega,\pi],a}$ that converges in $\mathcal{L}$, then the limit is a pointed cone 3-manifold in $\mathcal{CB}_{[\omega,\pi],b}$ for some $b > 0$. This can be proved by an argument similar to the one given in the proof of Proposition 3.2.6, by using the fact that the Hausdorf-Gromov limit of standard half balls is again a standard half ball.

The final step is to show that Hausdorff-Gromov convergence in $\mathcal{CB}_{[\omega,\pi],a}$ implies geometric convergence. This is proved in Proposition 3.3.1 for cone 3-manifolds without boundary, where the bilipschitz maps are explicitly constructed. These bilipschitz maps are constructed taking account of the singular locus, and the same construction can be made by taking additional account of the boundary. Thus we are left with the proof of Proposition 3.6.3 .

Proof of Proposition 3.6.3. — Given C a cone 3-manifold with totally geodesic boundary, DC denotes its double along the boundary. Since the boundary of C is totally geodesic, DC is still a cone 3-manifold. Given $y \in C$, $\mathrm{inj}_C(y)$ and $\mathrm{inj}_{DC}(y)$ denote the injectivity radius in C and in DC respectively. Note that $\mathrm{inj}_C(y) \leq \mathrm{inj}_{DC}(y)$.

We will also use the notation $B_C(x,R)$ and $B_{DC}(x,R)$ to distinguish the ball in C from the ball in DC. Note that $B_C(x,R) \subseteq B_{DC}(x,R)$.

Given $(C,x) \in \mathcal{CB}_{[\omega,\pi],a}$ and $R > 0$, we want to find a lower bound for the injectivity radius of every point in $B(x,R)$ which depends only on a, R and ω. Since $\mathrm{inj}_C(x) \leq \mathrm{inj}_{DC}(x)$, we have that the pointed cone manifold $(DC,x) \in \mathcal{C}_{[\omega,\pi],a}$. Therefore, by Proposition 3.2.5 there exists a uniform constant $b_0 = b_0(R,\omega,a) > 0$ such that

$$\forall y \in B_{DC}(x, R+4), \qquad \mathrm{inj}_{DC}(y) \geq b_0.$$

We shall deduce from it that, $\forall y \in B_C(x,R)$, $\mathrm{inj}_C(y) \geq b$ for some constant $b = b(b_0,\omega) > 0$ which depends only on b_0 and ω. We do it in several steps (Lemmas 3.6.4, 3.6.7, 3.6.8).

Lemma 3.6.4. — *For every $y \in \partial C \cap B(x, R+2)$, $\mathrm{inj}_{\partial C}(y) > b_0$, where $\mathrm{inj}_{\partial C}(y)$ denotes the injectivity radius in the boundary and b_0 is the constant above.*

Proof. — This is a consequence of the fact that $\mathrm{inj}_{\partial C}(y) \geq \mathrm{inj}_{DC}(y)$, since ∂C is a totally geodesic submanifold of DC. □

Next we define the *normal radius* of a compact region in the boundary.

Definition 3.6.5. — Given $K \subseteq \partial C$ a compact subset of the boundary, we define its *normal radius* as the following supremum:

$$n(K) = \sup \left\{ r > 0 \;\middle|\; \begin{array}{c} \text{two segments of length } r \text{ orthogonal to } \partial C \text{ which} \\ \text{start at different points of } K \text{ do not intersect} \end{array} \right\}$$

Remark 3.6.6. — Since K is compact, its normal radius is well defined.

The normal radius is the supremum of all $r > 0$ such that the normal map defines a collared neighborhood for K. It is the supremum but not the maximum.

Lemma 3.6.7. — *There exists a constant $b_1 = b_1(b_0, \omega) > 0$ depending only on b_0 and ω such that the normal radius is*

$$n(\partial C \cap B_C(x, R+2)) \geq b_1.$$

Proof. — First we bound below the length of an orthogonal segment that starts in a smooth point of $\partial C \cap B_C(x, R+3)$ and hits the singular locus Σ. Thus let $y \in \partial C \cap B_C(x, R+3)$ be a nonsingular point and let σ be a segment between x and Σ which is orthogonal to ∂C. We want to find a lower bound for the length of σ.

Let $z = \sigma \cap \Sigma$. We may assume that $z \in B_C(x, R+4)$. Thus $\mathrm{inj}_{DC}(z) \geq b_0$. Now consider the segment in DC obtained by joining σ to his mirror image along the boundary. Since this segment goes from z to its mirror image, it joins two singular points and its length has to be bigger than the injectivity radius at z. Thus the length of σ is $\geq b_0/2$.

Next we find the lower bound for the normal radius. We choose $\beta > 0$ a constant with the following property: if two points p and q in the singular model space $\mathbb{H}_K(\alpha)$ are joined by two segments of length less than β, then the singular edge of $\mathbb{H}_K(\alpha)$ is at distance at most $b_0/4$ from p and q. This constant exists because $\alpha \geq \omega$, and it depends only on ω and b_0. See Figure 8. We will use the additional fact that the singular locus of $\mathbb{H}_K(\alpha)$ and the segments joining p to q cannot have a common perpendicular plane.

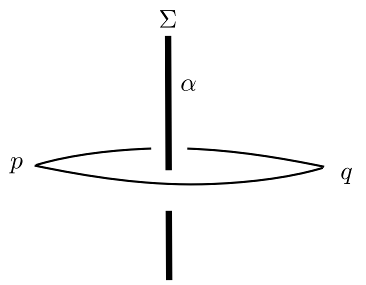

FIGURE 8

Now suppose there are two different segments σ_1, σ_2 with the same length, which are orthogonal to the boundary, and which start at different points of $\partial C \cap B_C(x, R+2)$, but meet at their end-point $\{z\} = \sigma_1 \cap \sigma_2$.

When both σ_1 and σ_2 are nonsingular, we will show by contradiction that the length of σ_1 (and of σ_2) is at least $\inf\{b_0/4, \beta/2\}$. Let us assume that their length is less than $\inf\{b_0/4, \beta/2\}$. We look at the configuration obtained by reflecting $\sigma_1 \cup \sigma_2$ along its boundary. Since $\mathrm{inj}_{DC}(z) > b_0$ this configuration lies in a standard ball of radius $\geq b_0$. By definition of β, we have that the distance between Σ and z is less than $b_0/4$. Moreover, since σ_1 and σ_2 are perpendicular to the boundary, it follows that Σ is not orthogonal to the boundary in the standard ball. Hence we can find a non-singular segment between Σ and $\partial C \cup B_C(x, R+3)$ of length less than $b_0/2$, contradicting the first lower bound of the proof.

If σ_1 and σ_2 are both singular, then their union $\sigma_1 \cup \sigma_2$ is a connected component of Σ. Since the connected components of $D\Sigma \subset DC$ in the ball $B_{DC}(x, R+3)$ have length $\geq b_0$, it follows that the length of σ_1 and σ_2 is $\geq b_0/4$. If σ_1 is singular but σ_2 is not, then we have shown in the beginning of the proof that the length of σ_1 is at least $b_0/2$. □

From Lemmas 3.6.4 and 3.6.7, we have a uniform lower bound for the injectivity radius of points in the collared neighborhood of $\partial C \cap B_C(x, R+1)$ with normal radius $b_1/2$. In particular, such a bound holds for points in $B_C(x, R)$ whose distance to the boundary is $\leq b_1/2$. Hence the following lemma concludes the proof of Proposition 3.6.3.

Lemma 3.6.8. — *There exists a uniform constant $b_2 > 0$ depending only on b_0, ω and b_1 such that for every point $y \in B_C(x, R)$ the following holds: if the distance to ∂C is $\geq b_1/2$, then $\mathrm{inj}_C(y) > b_2$*

Proof of Lemma 3.6.8. — Since $\mathrm{inj}_{DC}(y) \geq b_0$, either the ball $B_{DC}(y, b_0)$ is standard or $B_{DC}(y, b_0)$ is contained in a standard ball.

If $B_{DC}(y, b_0)$ is itself standard, then the ball $B_C(y, \frac{b_1}{2})$ is also standard, because $b_1 \leq b_0$ and $d(y, \partial C) > \frac{b_1}{2}$. Therefore $\mathrm{inj}_C(y) \geq \frac{b_1}{2}$.

When $B_{DC}(y, b_0)$ is contained in a standard ball $B_{DC}(z, r)$ we distinguish two cases according to whether $d(y, z) \leq b_1/8$ or not.

If $d(y, z) \leq b_1/8$, then $d(z, \partial C) \geq b_1/4$. In addition, since $b_1 \leq b_0 \leq r$, the ball $B_C(z, \frac{b_1}{4}) = B_{DC}(z, \frac{b_1}{4})$ is standard and $B_C(y, \frac{b_1}{8}) \subset B_C(z, \frac{b_1}{4})$ by construction. Hence $\mathrm{inj}_C(y) \geq b_1/8$.

If $d(y, z) \geq b_1/8$, then, by trigonometric arguments, we can find a constant $b' > 0$ depending only on b_1 and ω such that the ball $B_{DC}(y, b')$ is standard. By taking $b' < b_1/2$, we have that $B_C(y, b') = B_{DC}(y, b')$ is standard, and therefore $\mathrm{inj}_C(y) \geq b'$. □

This finishes the proof of Proposition 3.6.3 and hence of the compactness theorem in the case with boundary. □

CHAPTER 4

LOCAL SOUL THEOREM FOR CONE 3-MANIFOLDS WITH CONE ANGLES LESS THAN OR EQUAL TO π

The goal of this chapter is to describe the metric structure of a neighborhood of a point with sufficiently small injectivity radius in a hyperbolic cone 3-manifold with cone angles bounded above by π. This description is crucial to study collapsing sequences of cone 3-manifolds in the proofs of Theorems A and B.

We need first the following definition.

Definition 4.0.1. — Let C be a cone 3-manifold and D a cone manifold of dimension less than 3, possibly with silvered boundary ∂D. A surjective map $p\colon C \to D$ is said to be a *cone fiber bundle* if

- on $D - \partial D$, the restriction of p is a locally trivial fiber bundle with fiber a cone manifold. Moreover, if $\dim(D) = 2$ then $p(\Sigma_C) = \Sigma_D$
- on ∂D, the restriction of p is an orbifold fibration. In particular, the fiber over a point of ∂D is an orbifold with some cone angles equal to π.

Local Soul Theorem. — *Given $\omega \in (0,\pi)$, $\varepsilon > 0$ and $D > 1$ there exist $\delta = \delta(\omega,\varepsilon,D) > 0$ and $R = R(\omega,\varepsilon,D) > D > 1$ such that, if C is an oriented hyperbolic cone 3-manifold with cone angles in $[\omega,\pi]$ and if $x \in C$ satisfies* $\mathrm{inj}(x) < \delta$, *then:*

- either C is $(1+\varepsilon)$-bilipchitz homeomorphic to a compact Euclidean cone 3-manifold E of diameter $\mathrm{diam}(E) \leq R\,\mathrm{inj}(x)$*;*

- or there exists $0 < \nu < 1$, depending on x, such that x has an open neighborhood $U_x \subset C$ which is $(1+\varepsilon)$-bilipschitz homeomorphic to the normal cone fiber bundle $\mathcal{N}_\nu(S)$, of radius ν, of the soul S of a non-compact orientable Euclidean cone 3-manifold with cone angles in $[\omega,\pi]$. In addition, according to $\dim(S)$*, the Euclidean non-compact cone 3-manifold belongs to the following list:*

I) (when $\dim(S) = 1$*), $S^1 \ltimes \mathbb{R}^2$, $S^1 \ltimes$ (open cone disk) and the solid pillow (see Figure 1), where $\ltimes$ denotes the metrically twisted product;*

II) (when $\dim(S) = 2$*)*

i) *a product* $T^2 \times \mathbb{R}$*;* $S^2(\alpha,\beta,\gamma)\times\mathbb{R}$*, with* $\alpha+\beta+\gamma=2\pi$ *(the thick turnover);* $S^2(\pi,\pi,\pi,\pi)\times\mathbb{R}$ *(the thick pillow);*

ii) *the orientable twisted line bundle over the Klein bottle* $K^2\tilde{\times}\mathbb{R}$ *or over the projective plane with two silvered points* $\mathbb{P}^2(\pi,\pi)\tilde{\times}\mathbb{R}$*;*

iii) *a quotient by an involution of either* $S^2(\pi,\pi,\pi,\pi)\times\mathbb{R}, T^2\times\mathbb{R}$ *or* $K^2\tilde{\times}\mathbb{R}$*, that gives an orientable bundle respectively over either* $D^2(\pi,\pi)$*, an annulus, or a Möbius strip, with silvered boundary in the three cases (see Figure 2).*

In addition, the $(1+\varepsilon)$*-bilipschitz homeomorphism* $f: U_x \to \mathcal{N}_\nu(S)$ *satisfies the inequality*

$$\max\big(\mathrm{inj}(x), d(f(x),S), \mathrm{diam}(S)\big) \leq \nu/D.$$

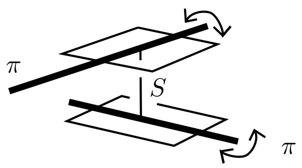

FIGURE 1. The solid pillow. Its soul is the interval $[0,1]$ with silvered boundary.

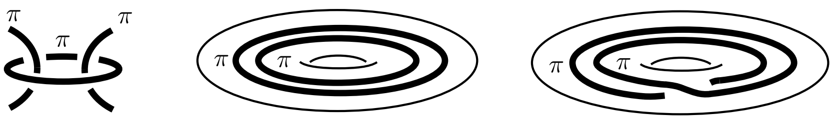

FIGURE 2. From left to right, the non-compact Euclidean cone 3-manifolds with soul $D^2(\pi,\pi)$, an annulus and a Möbius strip, with silvered boundary in every case. They are the respective quotients of $S^2(\pi,\pi,\pi,\pi)\times\mathbb{R}, T^2\times\mathbb{R}$ and $K^2\tilde{\times}\mathbb{R}$ by an involution.

The Euclidean cone 3-manifolds E in the local soul theorem are called the *local models*. We call S the *soul*, because, in each case, S is a totally convex cone submanifold of the local model E, and E is isometric to the normal cone fiber bundle of S.

Remark 4.0.2. — It follows from the proof (cf. [**CGv**]) that $\nu < R\,\mathrm{inj}(x)$. In particular, ν depends on x and this dependence cannot be avoided.

The first step of the proof is Thurston's classification theorem of non-compact orientable Euclidean cone 3-manifolds. We need in fact a simpler classification, because Thurston's classification includes general singular locus, and we consider here only the case where the singular locus is a 1-dimensional submanifold (cf. [**SOK**], [**Ho1**] and [**Zh1**]). After obtaining this classification, we prove the local soul theorem by applying an argument of [**CGv**]. In the last section we extend the local soul theorem to cone 3-manifolds with totally geodesic boundary.

4.1. Thurston's classification theorem of non-compact Euclidean cone 3-manifolds

In this section we prove the following result:

Theorem 4.1.1 **(Thurston).** — *Let E be a non-compact orientable Euclidean cone 3-manifold with cone angles less than or equal to π and a 1-dimensional submanifold as singular locus. Then either $E = \mathbb{R}^3$, $E = \mathbb{R}^3(\alpha) = \mathbb{R} \times$ (open cone disk), or E is one of the local models given in the local soul theorem.*

We prove this theorem by using the soul theorem for Euclidean cone 3-manifolds, which is a generalization of Bieberbach theorem for open flat 3-orbifolds. We need first some definitions. The proof of the soul theorem for Euclidean cone manifolds is analogous to the proof of the soul theorem for Riemannian manifolds of non negative sectional curvature of [**CGl**] (see also [**Sak**]).

Definition 4.1.2. — Let C be a cone 3-manifold with singular locus $\Sigma \subset C$ a 1-dimensional submanifold and curvature $K \in [-1, 0]$.

- The *silvered points* of C are the points of Σ having cone angle π.
- A path $\gamma : [0, l] \to C$ is *geodesic* if it is locally minimizing.
- A path $\gamma : [0, l] \to C$ is *s-geodesic* if it is locally minimizing except for some $t \in (0, l)$ where $\gamma(t)$ is silvered and the following happens: there exist $\varepsilon > 0$ and a neighborhood U of $\gamma(t)$ such that $\gamma : (t - \varepsilon, t + \varepsilon) \to U$ lifts to a minimizing path in the double cover $\tilde{U} \to U$ branched along $\Sigma \cap U$. See Figure 3.
- A subset $S \subset C$ is *totally s-convex* if every s-geodesic path with end-points in S is contained in S.

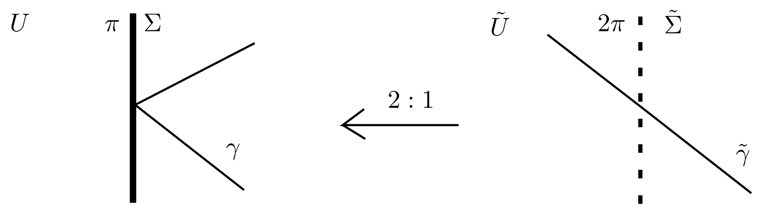

FIGURE 3. Example of s-geodesic.

The notion of s-geodesic generalizes the notion of geodesic, thus total s-convexity is stronger than usual total convexity, as shown in the following example.

Example 4.1.3. — Let A be a totally s-convex subset of a cone 3-manifold C and γ a geodesic path from $p \in A$ to $q \in \Sigma$. If q is a silvered point and γ is orthogonal to Σ, then γ is contained in A, because the path $\gamma * \gamma^{-1}$ is a s-geodesic with end-points in A. In particular, a totally s-convex set intersects all singular components having cone angle π.

***Theorem 4.1.4* (Soul Theorem).** — *Let E be a non-compact Euclidean cone 3-manifold with cone angles $\leq \pi$. Then E contains a compact totally s-convex cone submanifold $S \subset E$ of dimension 0, 1 or 2, with silvered or empty boundary. Moreover E is isometric to the normal cone fiber bundle of S.*

Given a cone submanifold $S \subset E$, there is an $\varepsilon > 0$ such that the tubular neighborhood of radius $\varepsilon > 0$, $\mathcal{N}_\varepsilon(S)$, is a cone bundle over S. When we say that E is isometric to the normal cone fiber bundle of S, we mean that we can choose the radius $\varepsilon = \infty$ and that the metric has the local product structure of the bundle.

The cone submanifold S is called the *soul* of E.

Remark 4.1.5. — When cone angles between π and 2π are allowed, then this theorem does not hold anymore. For instance one can easily construct two dimensional cone manifolds with infinitely many singular points, if we allow the cone angles to be arbitrarily close to 2π. However a weaker version of the soul theorem holds.

We postpone the proof of the soul theorem to Sections 4.2 and 4.3, and now we use it to prove Thurston's Classification Theorem 4.1.1.

Proof of Theorem 4.1.1. — Let E be an orientable non-compact Euclidean cone 3-manifold and Σ its singular locus. Note that every finite covering $\tilde{S} \to S$ (possibly branched at silvered points) induces a covering $\tilde{E} \to E$. Moreover, $\tilde{S}$ is the soul of $\tilde{E}$, because $\tilde{S}$ is totally s-convex and $\tilde{E}$ is the normal cone fiber bundle of $\tilde{S}$. Passing to finite coverings will help us to simplify the proof.

We distinguish three cases, according to the dimension of the soul $S \subset E$.

When $\dim(S) = 0$, then S is a point p. For $\varepsilon > 0$ sufficiently small, the ball $B(p, \varepsilon)$ of radius ε is either a non-singular Euclidean ball or a ball with a singular axis. Hence, since E is isometric to the normal cone fiber bundle of the point p, either E is the Euclidean space $\mathbb{R}^3$, or $E = \mathbb{R}^3(\alpha) = \mathbb{R} \times$ (open cone disk).

When $\dim(S) = 1$, then either S is S^1 or an interval $[0,1]$ with silvered boundary.

If $S = S^1$, then by convexity either $S \subset \Sigma$ or $S \cap \Sigma = \emptyset$. Since E is orientable, $\mathcal{N}_\varepsilon(S)$ is a solid torus, possibly with a singular core. Therefore, E is the (metrically twisted) product of S^1 with an infinite disk, possibly with a singular cone point in the center.

If $S = [0,1]$, then we consider the double covering $S^1 \to [0,1]$ branched along the silvered boundary. The induced double branched covering $\tilde{E} \to E$ is $S^1 \ltimes \mathbb{R}^2$, thus E is the solid pillow ($\mathbb{R}^3$ with two silvered lines; see Figure 1).

When $\dim(S) = 2$, we use the classification of compact Euclidean cone 2-manifolds with geodesic boundary and cone angles $\leq \pi$. This classification is easily deduced from Gauss-Bonnet formula. In particular, either $S = S^2(\alpha, \beta, \gamma)$ with $\alpha + \beta + \gamma = 2\pi$ or S is a Euclidean orbifold having as finite covering $\tilde{S} = T^2$. If $S = S^2(\alpha, \beta, \gamma)$ or $S = T^2$, then S is two sided and $\mathcal{N}_\varepsilon(S) = S \times (-\varepsilon, \varepsilon)$. Therefore $E = S \times \mathbb{R}$. The remaining cases reduce to study finite groups of isometries of T^2 and their orientable isometric extension to $T^2 \times \mathbb{R}$. $\square$

4.2. Totally s-convex subsets in Euclidean cone 3-manifolds.

In this section we give some basic facts about totally s-convex subsets in Euclidean cone 3-manifolds, which are used in the proof of the soul theorem (in Section 4.3). Lemma 4.2.2 shows that totally s-convex subsets appear naturally as level sets of continuous convex functions.

Definition 4.2.1. — A continuous function $f : E \to \mathbb{R}$ on a cone 3-manifold E is *convex* if $f \circ \gamma$ is convex for every geodesic path $\gamma : [0, l] \to E$.

Lemma 4.2.2. — *If $f : E \to \mathbb{R}$ is a continuous convex function then, for every* s-geodesic *path $\gamma : [0, l] \to E$, $f \circ \gamma$ is convex. In particular, the subset $\{x \in E \mid f(x) \leq 0\}$ is totally s-convex.*

Proof. — Every s-geodesic path is locally the limit of geodesic paths, arbitrarily close to the singular set but disjoint from it. It follows by continuity that the inequalities defining convexity are satisfied locally for every s-geodesic path. $\square$

Definition 4.2.3. — Let $A \subset E$ be a smooth submanifold without boundary. We say that A is *totally geodesic* if either $\dim A = 3$ or for every $x \in A$ the following hold:

- if $x \notin \Sigma$, then the second fundamental form of $A \subset E$ at x is trivial;
- if $x \in \Sigma$ and $\dim A = 2$, then A and Σ are orthogonal at x;
- if $x \in \Sigma$ and $\dim A = 1$, then there is a neighborhood $U \subset E$ of x such that $\Sigma \cap U = A \cap U$.

For non-singular points, this definition coincides with the usual definition in Riemannian geometry. We also remark that this is a local notion that does not require A to be complete.

Proposition 4.2.4. — *Let E be a Euclidean cone 3-manifold and $A \subset E$ a non-empty closed totally s-convex subset. Then A is an embedded manifold, possibly with boundary, whose interior $A - \partial A$ is totally geodesic.*

Before proving this proposition we need the following lemma, which describes the local structure of A at the singular points.

Lemma 4.2.5. — *Let $x \in \Sigma$ and let $D^2(x,\varepsilon)$ be the geodesic singular 2-disk transverse to Σ with center x and radius $\varepsilon > 0$. If A is a closed totally s-convex subset such that $A \cap D^2(x,\varepsilon) \neq \emptyset$, then one of the following possibilities happens:*

i) $A \cap D^2(x,\varepsilon) = \{x\}$,

ii) $A \cap D^2(x,\varepsilon)$ *contains a smaller disk* $D^2(x,\delta)$, *with* $0 < \delta < \varepsilon$,

iii) x *is silvered and* $A \cap D^2(x,\varepsilon)$ *is a segment orthogonal to* Σ *at* x.

Proof. — Choose $\varepsilon > 0$ so that $D^2(x,\varepsilon)$ is a disk contained in a standard ball. We prove first that if the cone angle α at x is less than π then the convex hull of a point $y \in A \cap (D^2(x,\varepsilon) - \{x\})$ contains a disk $D^2(x,\delta)$, with $0 < \delta < \varepsilon$. We view $D^2(x,\varepsilon)$ as the quotient of an angular sector S_α whose faces are identified by an isometric rotation, and such that y is obtained by identifying two points $\tilde{y}_1, \tilde{y}_2 \in S_\alpha$, one in each face of S_α. Consider the geodesic path $\tilde{\sigma} : [0,l] \to S_\alpha$ minimizing the distance between $\tilde{y}_1$ and $\tilde{y}_2$ (see Figure 4), it projects to a geodesic loop $\sigma : [0,l] \to D^2(x,\varepsilon)$ based at y. The convex hull of y contains σ and we have

$$d(\sigma, x) = d(y,x)\cos(\alpha/2).$$

By using this formula and the fact that $0 < \cos(\alpha/2) < 1$, we can construct a sequence of concentric geodesic loops converging to x, hence the convex hull of y contains a disk $D^2(x,\delta)$, with $\delta > 0$. This proves the lemma when $\alpha < \pi$.

Assume now that x is a silvered point. Let $y \in A \cap (D^2(x,\varepsilon) - \{x\})$. The set A contains the minimizing path σ from y to x, because $\sigma * \sigma^{-1}$ is a s-geodesic loop based at y. Moreover, if $A \cap D^2(x,\varepsilon)$ contains two such segments, then these segments divide $D^2(x,\varepsilon)$ into two sectors of angles less than π, therefore $D^2(x,\delta) \subset A$ with $\delta > 0$. □

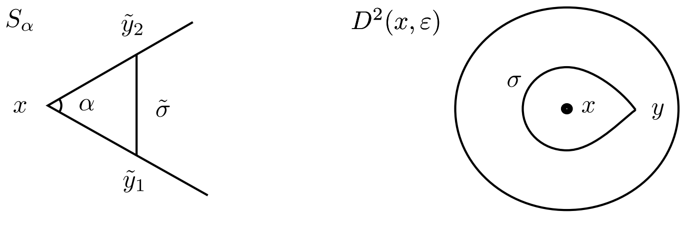

FIGURE 4. The sector S_α and the disk $D^2(x,\varepsilon)$.

Proof of Proposition 4.2.4. — It suffices to prove the result locally: every point $x \in A$ has a neighborhood U such that $\overline{U} \cap A$ is an embedded submanifold, possibly with boundary, whose interior $\overline{U} \cap A - \partial(\overline{U} \cap A)$ is totally geodesic. If x is non-singular then it is just a well known result for locally convex subsets in $\mathbb{R}^3$. Hence we suppose that $x \in \Sigma$.

We choose a neighborhood U of x that has a product structure. More precisely U is isometric to $D^2(0,\delta)\times(-\varepsilon,\varepsilon)$ for some $\varepsilon,\delta>0$, where $D^2(0,\delta)$ is a singular 2-disk of radius δ, with a singularity in its center 0. For this product structure we have that $U\cap\Sigma=\{0\}\times(-\varepsilon,\varepsilon)$ and $x=(0,0)$.

Since the intersection $A\cap U\cap\Sigma$ is a connected subset of $U\cap\Sigma$ containing x, there are the following three possibilities:

a) $A\cap U\cap\Sigma=\{x\}$;
b) $A\cap U\cap\Sigma=\{0\}\times[0,\varepsilon)$, i.e. a subinterval of $\Sigma\cap U$ having x as end-point;
c) $A\cap U\cap\Sigma=\{0\}\times(-\varepsilon,\varepsilon)=\Sigma\cap U$.

By using Lemma 4.2.5 we can describe explicitly all the possibilities for $A\cap\overline{U}$ in each case. In case a), when $A\cap U\cap\Sigma=\{x\}=\{(0,0)\}$, there are 3 subcases:

a1) $A\cap U=\{x\}=\{(0,0)\}$.
a2) $A\cap\overline{U}=\overline{V}\times\{0\}$, where $\overline{V}$ is a convex neighborhood of 0 in $D^2(0,\delta)$. In particular int$(A\cap U)$ is a totally geodesic 2-submanifold transverse to Σ
a3) The point x is silvered and $A\cap U$ is a segment orthogonal to Σ at x.

It follows from this explicit description that the proposition holds in the three subcases a1), a2) and a3).

In case b), when $A\cap U\cap\Sigma=\{0\}\times[0,\varepsilon)$, again there are three subcases:

b1) $A\cap U=A\cap U\cap\Sigma=\{0\}\times[0,\varepsilon)$, (i.e. $A\cap U$ is a subinterval of Σ).
b2) For some $t_0\in[0,\varepsilon)$, $A\cap(D^2(0,\delta)\times\{t_0\})$ contains a singular 2-disk with positive radius.
b3) x is silvered and, for some $t_0\in[0,\varepsilon)$, $A\cap(D^2(0,\delta)\times\{t_0\})$ is a segment perpendicular to Σ at x.

In subcase b1) we have an explicit description of $A\cap U$ and we may conclude that the proposition holds. To give an explicit description in the other cases we need further work.

In subcase b2), we claim that $A\cap U$ is a 3-manifold with boundary and that $x\in\partial(A\cap U)$. First we remark that, for every $t\in(0,\varepsilon)$, the intersection $A\cap(D^2(0,\delta)\times\{t\})$ contains a singular 2-disk with positive radius, because $A\cap U$ contains the convex hull of the union of $A\cap(D^2(0,\delta)\times\{t_0\})$ and $A\cap U\cap\Sigma=\{0\}\times[0,\varepsilon)$.

We parametrize U by cylindrical coordinates $(r,\theta,t)\in[0,\delta)\times[0,\alpha]\times(-\varepsilon,\varepsilon)$, where r is the distance to Σ, $\theta\in[0,\alpha]$ is the angle parameter, α is the singular angle and t is the height parameter. Thus we identify $(r,0,t)$ to (r,α,t), and $(0,\theta,t)$ to $(0,\theta',t)$, for every $\theta,\theta'\in[0,\alpha]$.

By Lemma 4.2.5, if a point belongs to $A\cap U$, then so it does its projection to Σ. Therefore there exists a function $f\colon[0,\alpha]\times[0,\varepsilon]\to[0,\delta]$ such that

$$A\cap\overline{U}=\{(r,\theta,t)\in[0,\delta]\times[0,\alpha]\times[0,\varepsilon]\mid r\leq f(\theta,t)\}.$$

We remark that for every $t\in(0,\varepsilon)$ and every $\theta\in[0,\alpha]$, $f(\theta,t)>0$ because $A\cap(D^2(0,\delta)\times\{t\})$ contains a singular 2-disk with positive radius.

Next we show that f is continuous. The function f is upper semi-continuous because A is closed. Moreover the lower semi-continuity of f at a point (θ, t) can be proved by considering the convex hull of the union of $A \cap (D^2(0,\delta) \times \{t\})$ and $A \cap \Sigma$, because this convex hull has dimension 3.

Since f is continuous, $A \cap \overline{U}$ is a 3-manifold with boundary whose interior is

$$\operatorname{int}(A \cap \overline{U}) = \{(r, \theta, t) \in [0,\delta] \times [0,\alpha] \times (0,\varepsilon) \mid r < f(\theta, t)\}.$$

Hence the proposition holds in subcase b2).

In subcase b3) we claim that $A \cap \overline{U}$ is a 2-manifold that is the union of a family of parallel segments perpendicular to Σ. First we remark that, for every $t \in (0,\varepsilon)$, the intersection $A \cap (D^2(0,\delta) \times \{t\})$ is a segment orthogonal to Σ, because $A \cap U$ contains the convex hull of the union of the segment $A \cap (D^2(0,\delta) \times \{t_0\})$ and $A \cap U \cap \Sigma = \{0\} \times [0,\varepsilon)$. Moreover the segments $A \cap (D^2(0,\delta) \times \{t\})$ are parallel, because if not the convex hull of their union would have dimension 3 and we would be in subcase b2).

Again we parametrize U by cylindrical coordinates (r, θ, t). By the same argument as in subcase b2) we conclude that there exists a continuous function $f\colon(-\varepsilon, \varepsilon) \to [0, \delta]$ such that

$$A \cap \overline{U} = \{(r, \theta, t) \in [0,\delta] \times [0,\alpha] \times [0,\varepsilon] \mid \theta = 0,\ r \leq f(t)\}.$$

Moreover, for $t > 0$, $f(t) > 0$. Hence $A \cap U$ is a 2-dimensional submanifold with boundary and with totally geodesic interior $\{(r,\theta,t) \mid \theta = 0,\ 0 < r < f(t)\}$. Thus the proposition follows from explicit description also in this case.

Finally, in case c), when $A \cap U \cap \Sigma = \{0\} \times (-\varepsilon, \varepsilon)$, there are again three subcases that can be treated with the same method as subcases b1), b2) and b3). These subcases are:

c1) $A \cap U = A \cap U \cap \Sigma = \{0\} \times (-\varepsilon, \varepsilon)$.

c2) $A \cap \overline{U}$ is a 3-submanifold with boundary that contains $U \cap \Sigma = \{0\} \times (-\varepsilon, \varepsilon)$ in its interior.

c3) x is a silvered point and $A \cap \overline{U}$ is a 2-submanifold with boundary, which is the union of parallel segments orthogonal to $U \cap \Sigma$. In particular the interior of $A \cap \overline{U}$ is totally geodesic and $U \cap \Sigma = \{0\} \times (-\varepsilon, \varepsilon)$ is contained in the boundary of $A \cap \overline{U}$.

□

Remark 4.2.6. — In the proof of Proposition 4.2.4, the cases a3), b3) and c3) deal with silvered points. Let $p\colon\tilde{U} \to U$ denote the double cover branched along $U \cap \Sigma$, so that $\tilde{U}$ is non-singular. In the three cases a3), b3) and c3), $\overline{p^{-1}(A \cap U)}$ is a manifold with boundary, of dimension 1 or 2, whose interior is totally geodesic in $\tilde{U}$. Moreover $x \in \partial A$, but in cases a3) and c3) $p^{-1}(x)$ is an interior point of $p^{-1}(A \cap U)$. This motivates the following definition.

Definition 4.2.7. — Let $A \subset E$ be a closed totally s-convex subset. The *silvered boundary* $\partial_S A$ is the set of points $x \in \partial A \cap \Sigma$ that are silvered and the following holds: if $U \subset E$ is a neighborhood of x and $p\colon \tilde{U} \to U$ is the double cover branched along $\Sigma \cap U$, then $p^{-1}(x)$ is an interior point of $p^{-1}(A \cap U)$. We also define the *non-silvered boundary* $\partial_{NS} A$ to be the set of points in ∂A that are not in the silvered boundary: $\partial_{NS} A = \partial A - \partial_S A$. See Figure 5.

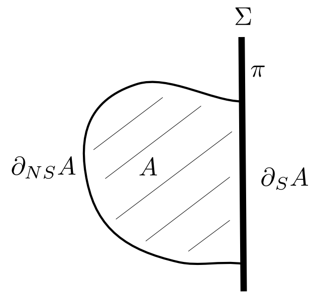

FIGURE 5

Lemma 4.2.8. — *Let $A \subset E$ be a non-empty closed totally s-convex subset in a Euclidean cone 3-manifold. Then the following hold:*

i) The non-silvered boundary $\partial_{NS} A$ is a closed subset of E.
ii) If $\dim A = 0$ or 3, then $\partial_S A = \emptyset$ and $\partial A = \partial_{NS} A$.

Proof. — Let $x \in A \cap \Sigma$ and let $U \subset E$ be a neighborhood of x. By using the explicit description of $U \cap A$ in the proof of Proposition 4.2.4, we have that $x \in \partial_S A$ if and only if x is a silvered point such that, either $U \cap A$ is a segment orthogonal to Σ (case a3)), or $U \cap A$ has dimension 2 and $U \cap \Sigma$ is a piece of $\partial_S A \subseteq \partial A$ (case c3)). This description of points in $\partial_S A$ implies that $\partial_S A$ is open in ∂A, hence assertion i) is proved. We also deduce from the description that if $\partial_S A \neq \emptyset$ then $\dim A = 1$ or 2, which is equivalent to assertion ii). ☐

Proposition 4.2.9. — *Let $A \subset E$ be a non-empty closed totally s-convex subset in a Euclidean cone 3-manifold. If $\dim A < 3$ then every point in $\mathrm{int}(A) \cup \partial_S A = A - \partial_{NS} A$ has a neighborhood $U \subset E$ isometric to the normal cone fibre bundle over $A \cap U$. More precisely:*

- *if $x \in \mathrm{int}(A)$ then U is isometric to the product $(A \cap U) \times B^c(0,\varepsilon)$, where $B^c(0,\varepsilon)$ is a ball of radius $\varepsilon > 0$ and dimension $c = \mathrm{codim}(A)$, maybe with a singularity in its center.*
- *if $x \in \partial_S A$ and $p\colon \tilde{U} \to U$ is the double cover branched along $\Sigma \cap U$, then $\tilde{U}$ is isometric to $p^{-1}(A \cap U) \times B^c(0,\varepsilon)$, where $B^c(0,\varepsilon)$ is a non-singular ball of radius $\varepsilon > 0$ and dimension $c = \mathrm{codim}(A)$.*

Proof. — If $x \in \mathrm{int}(A)$ and $x \notin \Sigma$ then the proposition is clear because $\mathrm{int}(A)$ is totally geodesic.

If $x \in \mathrm{int}(A) \cap \Sigma$ then, by the description given in the proof of Proposition 4.2.4, $A \cap U$ is either in case a2) or in case c1). In case a2), $U \cap A$ is a totally geodesic 2-dimensional disk perpendicular to Σ, and U is isometric to the product of $U \cap A$ with an interval. In case c1), $U \cap A$ is a subinterval of Σ and U is isometric to the product of $U \cap A$ with a singular 2-disk. Hence the proposition holds in both cases.

When $x \in \partial_S A$, $U \cap A$ is either in case a3) or c3). In both cases $p^{-1}(U \cap A)$ is a totally geodesic submanifold of $p^{-1}(A)$ and the proposition follows. □

The following proposition shows that A has a local supporting half-space at every point of $\partial_{NS} A$.

Proposition 4.2.10. — *Let $A \subset E$ be totally s-convex, $x \in A$, $y \in \partial_{NS} A$ and $\gamma : [0, l] \to A$ be a path from x to y that realizes the distance $d(x, \partial_{NS} A)$.*

i) If $y \in \Sigma$, then $\gamma([0, l]) \subset \Sigma$.

ii) Let $B(y, \varepsilon)$ be a standard ball of radius $\varepsilon > 0$ and let $H \subset B(y, \varepsilon)$ be the half-ball bounded by the (maybe singular) totally geodesic disk orthogonal to γ at y. Then $A \cap B(y, \varepsilon) \subseteq H$.

Proof. — Let $y \in \Sigma \cap \partial_{NS}(A)$. We choose a neighborhood $U \subset E$ of y. By using the description in the proof of Proposition 4.2.4, the intersection $U \cap A$ is either in case b1), b2) or b3). It follows from this description that if a path $\gamma : [0, l] \to A$ from x to y realizes $d(x, \partial_{NS} A)$, then it also realizes $d(x, D^2(y, \delta))$, where $D^2(y, \delta)$ is the totally geodesic 2-disk of radius $\delta > 0$ transverse to Σ. In particular, γ is orthogonal to $D^2(y, \delta)$ and assertions i) and ii) hold in the singular case.

If $y \in \partial_{NS} A$ is non-singular, then assertion ii) can be proved by using developping maps to reduce the proof to the case of locally convex subsets in $\mathbb{R}^3$. □

Lemma 4.2.11. — *Let E be a Euclidean cone 3-manifold and let $A \subset E$ be totally s-convex. The function*

$$\begin{array}{rcl} \Phi : A & \to & \mathbb{R} \\ x & \mapsto & d(x, \partial_{NS} A) \end{array}$$

is concave (i.e. $-\Phi$ is convex). Moreover, if for some geodesic path $\gamma : [0, l] \to A$ the function $\Phi \circ \gamma$ is constant, then for every $t \in [0, l]$ there exists a geodesic path orthogonal to γ that realizes the distance from $\gamma(t)$ to $\partial_{NS} A$.

Proof. — Let $\gamma : [0, l] \to A$ be a geodesic path, we want to prove that $\Phi \circ \gamma$ is concave. For $t \in [0, l]$, let $\sigma_t : [0, \lambda] \to A$ be a minimizing path from $\gamma(t)$ to $\partial_{NS} A$ of length $\lambda = d(\gamma(t), \partial_{NS} A)$. Let $\theta \in [0, \pi]$ be the angle between $\gamma'(t)$ and $\sigma_t'(0)$. We claim that there exists some uniform $\varepsilon > 0$ such that, if $|s| < \varepsilon$ and $t + s \in [0, l]$, then

$$\Phi(\gamma(t + s)) \leq \Phi(\gamma(t)) - s \cos(\theta).$$

This inequality shows that $\Phi\circ\gamma$ may be represented locally as the infimum of linear functions, and therefore $\Phi\circ\gamma$ is concave.

To prove this inequality, we consider first the case where $\sigma_t([0,\lambda])\cap\Sigma=\emptyset$. Let $D^2(\sigma_t(\lambda),\delta)$ be a totally geodesic disk with center $\sigma_t(\lambda)$ and radius $\delta>0$ that is orthogonal to σ_t. By Proposition 4.2.10 ii), the disk $D^2(\sigma_t(\lambda),\delta)$ bounds a locally supporting half-ball for A. In particular there exists $\varepsilon>0$ such that, for $|s|<\varepsilon$

$$\Phi(\gamma(t+s)) = d(\gamma(t+s),\partial_{NS}A) \leq d(\gamma(t+s), D^2(\sigma_t(\lambda),\delta)). \tag{2}$$

Moreover, by considering developing maps, we can use elementary trigonometric formulas of Euclidean space to conclude that for $|s|<\varepsilon$

$$d(\gamma(t+s), D^2(\sigma_t(\lambda),\delta)) = d(\gamma(t),\sigma_t(\lambda)) - s\,\cos(\theta) = \lambda - s\cos(\theta),$$

where $\varepsilon>0$ is small enough, so that the tubular neighborhood $\mathcal{N}_\varepsilon(\gamma_t([0,l]))$ embeds in the Euclidean space via developping maps (see Figure 6). Therefore

$$\Phi(\gamma(t+s)) \leq \lambda - s\cos(\theta) = \Phi(\gamma(t)) - s\cos(\theta).$$

Furthermore, the parallel translation of σ_t along γ gives a family of geodesic paths $\{\sigma_{t+s}\mid |s|<\varepsilon\}$, such that σ_{t+s} has length $\lambda - s\,\cos(\theta)$ and minimizes the distance of $\gamma(t+s)$ to $D^2(\sigma_t(\lambda),\delta)$. Therefore, when $\Phi\circ\gamma$ is constant we have equality in (2), $\theta=\pi/2$, and $\{\sigma_{t+s}\mid |s|<\varepsilon\}$ is a family of geodesics of constant length, orthogonal to γ and which minimize the distance to $\partial_{NS}A$.

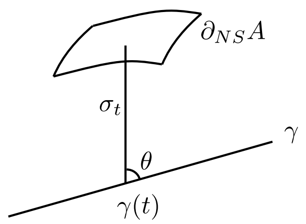

FIGURE 6

When $\sigma_t([0,\lambda])\cap\Sigma\neq\emptyset$, since σ_t is minimizing, either $\sigma_t([0,\lambda])\cap\Sigma=\{\gamma(t)\}$ or $\sigma_t([0,\lambda])\subset\Sigma$ by Proposition 4.2.10 i). In particular, $\gamma(t)\in\Sigma$ and either $\gamma([0,l])\subset\Sigma$ or $\gamma(t)$ is an end-point of γ. Then the argument in the non-singular case goes through in the singular case, by just taking care when we use developing maps close to the singular set.

Finally, note that a compactness argument allows to chose a uniform $\varepsilon>0$. $\square$

4.3. Proof of the soul theorem.

Proof of Theorem 4.1.4 (Soul theorem for Euclidean cone 3-manifolds)

We start by considering Busemann functions. We recall that a *ray emanating from a point* $p \in C$ is a continuous map $\gamma:[0,+\infty) \to E$ such that the restriction on every compact subinterval is minimizing. We assume that the rays are parametrized by arc-length. The *Busemann function associated to* γ is:

$$b_\gamma(x) = \lim_{t \to +\infty} (t - d(x, \gamma(t))).$$

By construction, Busemann functions are Lipschitz, with Lipschitz constant 1. In particular they are continuous.

A Euclidean cone 3-manifold with cone angles less than 2π may be viewed as an Aleksandrov space of curvature ≥ 0. Next lemma is proved in [**Yam**, Prop. 6.2] for those Aleksandrov spaces.

Lemma 4.3.1. — *Busemann functions on Euclidean cone 3-manifolds with cone angles less than 2π are convex.* □

We consider a Euclidean cone 3-manifold E and we fix a point $x_0 \in E$. Following Cheeger and Gromoll [**CGl**] or Sakai [**Sak**, Sect. V.3], for $t \geq 0$, we define

$$A_t = \{x \in E \mid b_\gamma(x) \leq t \text{ for every ray } \gamma \text{ emanating from } x_0\}.$$

Lemma 4.3.2. — *For $t \geq 0$, A_t is a compact totally s-convex subset of E, satisfying:*

1) *If $t_1 \geq t_2 \geq 0$, then $A_{t_2} \subseteq A_{t_1}$ and $A_{t_2} = \{x \in A_{t_1} \mid d(x, \partial_{NS} A_{t_1}) \geq t_1 - t_2\}$. In particular, for $t_2 > 0$, $\partial A_{t_2} = \partial_{NS} A_{t_2} = \{x \in A_{t_1} \mid d(x, \partial_{NS} A_{t_1}) = t_1 - t_2\}$.*
2) *$E = \bigcup_{t \geq 0} A_t$.*
3) *A_t intersects all connected components of Σ.*

Proof. — The set A_t is totally s-convex by Lemmas 4.2.2 and 4.3.1. In order to prove the compactness, we suppose that there is a sequence of points x_n in A_t going to infinity, and we will derive a contradiction. For every $n \in \mathbb{N}$, consider the minimizing path γ_n between x_0 and x_n, which is contained in A_t by convexity. Since the unit tangent bundle at x_0 is compact, the sequence γ_n has a convergent subsequence to a ray γ emanating from x_0 and contained in A_t; that contradicts the definition of A_t.

We recall the following classical inequalities for Busemann functions, which can be proved from triangle inequality. For every ray γ emanating from x_0, every point $x \in E$ and every real $t \geq 0$,

$$d(x, x_0) \geq b_\gamma(x) \geq t - d(x, \gamma(t)).$$

In particular $B(x_0, t) = \{x \in E \mid d(x, x_0) < t\} \subseteq A_t$ for every $t \geq 0$. Thus assertion 2) is clear. Moreover, for $t > 0$, $\dim A_t = 3$. Therefore, by Lemma 4.2.8, $\partial A_t = \partial_{NS} A_t$.

To show assertion 1) we prove first the inclusion $A_{t_2} \subseteq \{x \in A_{t_1} \mid d(x, \partial_{NS} A_{t_1}) \geq t_1 - t_2\}$. Given $x \in A_{t_2}$ and $y \notin \operatorname{int}(A_{t_1})$, we claim that $d(x, y) \geq t_1 - t_2$. By hypothesis, there exists a ray γ emanating from x_0 such that $b_\gamma(x) \leq t_2$ and $b_\gamma(y) \geq t_1$. For every $t > 0$, we have $d(x,y) \geq d(x,\gamma(t)) - t + t - d(y,\gamma(t))$. By taking the limit when $t \to +\infty$, we deduce that $d(x,y) \geq b_\gamma(y) - b_\gamma(x) \geq t_1 - t_2$, as claimed.

To prove the reverse inclusion, we take a point $x \in A_{t_1}$ such that $d(x, \partial_{NS} A_{t_1}) \geq t_1 - t_2$. We claim that for every ray γ emanating from x_0, $b_\gamma(x) \leq t_2$. Note first that, for every $t \geq t_1$, $d(x,\gamma(t)) \geq t - t_1$, because $t_1 \geq b_\gamma(x) \geq t - d(x,\gamma(t))$. Let z be the point in a minimizing path between x and $\gamma(t)$ such that $d(z,\gamma(t)) = t - t_1$. Then $z \notin \operatorname{int} A_{t_1}$, because $b_\gamma(z) \geq t - d(z,\gamma(t)) = t_1$. It follows that $d(x,z) \geq d(x, \partial_{NS} A_{t_1}) = d(x, \partial A_{t_1}) \geq t_1 - t_2$ and

$$t - d(x,\gamma(t)) = t - d(x,z) - d(z,\gamma(t)) \leq t - (t_1 - t_2) - (t - t_1) = t_2.$$

By taking the limit when $t \to +\infty$, $b_\gamma(x) \leq t_2$. This proves assertion 1).

Finally assertion 3) follows from assertion 1) and the following lemma:

Lemma 4.3.3. — *Let $A \subset E$ be totally s-convex and let $A^r = \{x \in A \mid d(x, \partial_{NS} A) \geq r\}$. If $A^r \neq \emptyset$ for some $r > 0$ and Σ_0 is a component of Σ such that $A \cap \Sigma_0 \neq \emptyset$, then $A^r \cap \Sigma_0 \neq \emptyset$.*

Proof. — By Lemma 4.2.11, A^r is totally s-convex. Therefore A^r intersects all components of Σ having cone angle π. In general, let Σ_0 be a component of Σ that intersects A and has cone angle less than π. We distinguish two cases, according to whether Σ_0 is compact or not.

When $\Sigma_0 \cong S^1$, for every $r > 0$ either $A^r \cap \Sigma_0 = \emptyset$ or $\Sigma_0 \subset A^r$, because Σ_0 is a closed geodesic path and A^r is totally s-convex. Therefore, the distance to $\partial_{NS} A$ is constant on Σ_0. Let $r_0 = d(\Sigma_0, \partial_{NS} A)$, then $\Sigma_0 \subseteq A^{r_0}$, because Σ_0 intersects A. We claim that in fact $A^{r_0} = \Sigma_0$. Since A^{r_0} is connected, we prove that $A^{r_0} = \Sigma_0$ by showing that there are no points in $A^{r_0} - \Sigma_0$ in a neighborhood of Σ_0. Seeking a contradiction we suppose that there is a point $y \in A^{r_0} - \Sigma$ in a sufficiently small neighborhood of Σ_0. Then, by Lemma 4.2.5 there is a disk $D^2(x,\delta) \subset A^{r_0}$ of radius $\delta > 0$, centered at a point $x \in \Sigma_0$ and transverse to Σ. The convex hull of $\Sigma_0 \cup D^2(x,\delta)$ gives an open neighborhood of x in A^{r_0}, contradicting the fact that $d(x, \partial_{NS} A) = r_0$. This proves that $A^{r_0} = \Sigma_0$. It follows that $A^r = \emptyset$ for $r > r_0$, and $\Sigma_0 \subseteq A^r$ for $r \leq r_0$.

When $\Sigma_0 \cong \mathbb{R}$, consider $r_0 = \sup\{d(x, \partial_{NS} A) \mid x \in \Sigma_0 \cap A\}$. If $r_0 = \infty$ then there is nothing to prove, hence we can assume $r_0 < \infty$. The intersection $A^{r_0} \cap \Sigma_0$ is either a point or a segment in Σ_0. If it is a segment, the argument above for the closed case shows that $A^{r_0} \cap \Sigma_0 = A^{r_0}$. If the intersection is a point then $\dim(A^{r_0}) = 0$ or 2. In any case $\dim A^{r_0} < \dim A$. Thus $A^r = \emptyset$ for $r > r_0$, and $A^r \cap \Sigma_0 \neq \emptyset$ for $r \leq r_0$.

This finishes the proofs of Lemmas 4.3.2 and 4.3.2. □

We fix a value $t > 0$ and we set $A = A_t$, where A_t is defined as in Lemma 4.3.2. The subset $A \subset E$ is compact totally s-convex and $\dim A = 3$. In particular $\partial_{NS} A = \partial A \neq \emptyset$.

For $r > 0$, we consider

$$A^r = \{x \in A \mid d(x, \partial_{NS}A) \geq r\} \quad \text{and} \quad A^{\max} = \bigcap \{A^r \mid A^r \neq \emptyset\}.$$

If $A^r \neq \emptyset$, then A^r is totally s-convex by Lemma 4.2.11. Let $r_0 = \max\{d(x, \partial_{NS}A) \mid x \in A\}$, then $A^{max} = A^{r_0} = \{x \in A \mid d(x, \partial_{NS}A) = r_0\}$. By Lemma 4.2.11, every geodesic in A^{max} is perpendicular to a geodesic minimizing the distance to $\partial_{NS}A$, hence $\dim A^{max} < \dim A = 3$.

We set $A(1) = A^{max}$. If $\partial_{NS}A(1) = \emptyset$ or $\dim A(1) = 0$, then we take $S = A(1)$. Otherwise, we construct $A(2) = A(1)^{max}$ and so on. Since $\dim A(i+1) < \dim A(i)$, this process stops and we obtain $S = A(i)$ for either $i = 1$, 2 or 3. Thus S is a compact totally s-convex subset with $\partial_{NS}S = \emptyset$ and $\dim S < 3$.

Next we prove that E is isometric to the normal cone fiber bundle of S. The key point of the proof is the following lemma:

Lemma 4.3.4. — *Each point of $E - S$ has a unique minimizing geodesic path to S. Moreover for singular points, this path is contained in Σ.*

Proof. — We consider the following subset of $E - S$

$$X = \left\{ x \in E - S \;\middle|\; \begin{array}{l} x \text{ has more than one minimizing path to } S, \text{ or} \\ x \in \Sigma \text{ has a minimizing path to } S \text{ not contained in } \Sigma \end{array} \right\}$$

Claim 4.3.5. — *X is a closed subset of $E - S$ and $d(X, S) > 0$.*

Proof of the claim. — By Lemma 3.1.4, each point has a finite number of minimizing paths to the totally s-convex submanifold S. Moreover, given a point $x \in E - S$, in a sufficiently small neighborhood of x the minimizing paths to S are obtained by perturbation of those of x. It follows that the property of having a unique minimizing path to S, contained in Σ for singular points, is an open property in $E - S$, and thus $X \subset E - S$ is closed.

Since S is compact, totally s-geodesic and $\partial_{NS}(S) = \emptyset$, by Proposition 4.2.9 S has a metric tubular neighborhood. Thus $d(X, S) > 0$. □

We come back to the proof of Lemma 4.3.4. We want to prove $X = \emptyset$. Seeking a contradiction, we assume $X \neq \emptyset$.

Let $x_0 \in X$ be such that $d(x_0, S) = d(X, S)$. Either x_0 has two minimizing paths to S or $x_0 \in \Sigma$ has a minimizing path to S not in Σ.

We assume first that there are two minimizing paths γ_1 and γ_2 from x_0 to S. If at x_0 the angle $\angle(\gamma_1, \gamma_2) < \pi$, then γ_1 and γ_2 can be deformed to shorter paths, with common origin, that minimize the distance to S, contradicting the definition of x_0. Therefore $\angle(\gamma_1, \gamma_2) = \pi$ and $\gamma_1^{-1} * \gamma_2$ is a geodesic with end-points in S, contradicting the fact that S is totally s-geodesic.

Now we assume that $x_0 \in \Sigma$ has a minimizing path γ to S not contained in Σ. If at x_0 the angle $\angle(\gamma, \Sigma) < \pi/2$, then we can perturb γ to a shorter path, not contained in Σ and minimizing the distance of a point in Σ to S. This contradicts the definition of x_0, hence we can assume that the angle $\angle(\gamma, \Sigma) = \pi/2$. We remark that the point x_0 is not silvered, otherwise $x_0 \in S$, because $\gamma^{-1} * \gamma$ would be a s-geodesic path with end-points in S. By construction there exists a totally s-convex subset $A \subset E$ such that $S \subset \mathrm{int}(A)$ and $x_0 \in \partial_{NS} A$ (this is one of the sets constructed in Lemma 4.3.2 and 4.3.3, i.e. $A = A_t$ for some $t > 0$ or $A = A(i)^r$ for some $r > 0$ and $i = 1$, 2 or 3). Since the angle $\angle(\gamma, \Sigma) = \pi/2$, Proposition 4.2.10 says that γ is tangent to the boundary of a supporting half-space for A at the point x. In particular γ does not go to the interior of A. This contradicts the fact that $S \subset \mathrm{int}(A)$, because $d(S, \partial_{NS} A) > 0$.

This finishes the proof of Lemma 4.3.4. □

We have shown that every point in $E - S$ has a unique minimizing path to S, and that for singular points this path is contained in Σ. By Proposition 4.2.9, since S is compact, totally s-convex and $\partial_{NS}(S) = \emptyset$, for some $\varepsilon > 0$ the tubular neighborhood $\mathcal{N}_\varepsilon(S)$ is isometric to the normal cone fiber bundle of S. The fact that every point in $E - S$ has a unique minimizing geodesic to S implies that the radius ε of the tubular neighborhood can be taken arbitrarily large, and thus E is isometric to the normal cone fiber bundle of S. □

4.4. Proof of the local soul theorem.

Proof. — Seeking a contradiction, we assume that there exist some $\varepsilon_0 > 0$ and some $D_0 > 1$ such that for every $\delta > 0$ and every $R > D_0$ there are a hyperbolic cone 3-manifold $C_{\delta,R}$ with cone angles in $[\omega, \pi]$ and a point $x \in C_{\delta,R}$ with $\mathrm{inj}(x) < \delta$ that do not verify the statement of the local soul theorem with parameters ε_0, D_0 and R. By taking $\delta = \frac{1}{n}$ and $R = n$, we obtain a sequence of pointed hyperbolic cone 3-manifolds $(C_n, x_n)_{n\in\mathbb{N}}$ such that $\mathrm{inj}(x_n) < \frac{1}{n}$ and x_n does not verify the local soul theorem with parameters ε_0, D_0 and $R = n$.

We apply the compactness theorem (Chapter 3) to the sequence of rescaled pointed cone 3-manifolds $(\bar{C}_n, \bar{x}_n)_{n\in\mathbb{N}} = (\frac{1}{\mathrm{inj}(x_n)} C_n, x_n)_{n\in\mathbb{N}}$. Then a subsequence of $(\bar{C}_n, \bar{x}_n)$ converges to a pointed Euclidean cone 3-manifold (C_∞, x_∞).

If the limit C_∞ is compact, then the geometric convergence implies that for some integer n_0 there exists a $(1+\varepsilon_0)$-bilipschitz homeomorphism $f: C_\infty \to \bar{C}_{n_0}$. We can also choose n_0 such that

$$n_0 > \mathrm{diam}(C_\infty).$$

Then the rescaled Euclidean cone 3-manifold $E = \mathrm{inj}(x_{n_0}) C_\infty$ is $(1+\varepsilon_0)$-bilipschitz homeomorphic to C_{n_0} and we have

$$\mathrm{diam}(E) < n_0 \, \mathrm{inj}(x_{n_0}) = R \, \mathrm{inj}(x_{n_0}).$$

Hence x_{n_0} satisfies the statement of the local soul theorem in the compact case and we get a contradiction.

If the limit C_∞ is not compact, then by the soul theorem (Theorem 4.1.4), C_∞ has a soul S_∞ and is the normal cone bundle of S_∞. Since $\operatorname{inj}(x_\infty) \le 1$, the soul S_∞ has dimension 1 or 2. We choose a real number ν_∞ satisfying

$$\nu_\infty > D_0 \max\big(\operatorname{diam}(S_\infty), d(x_\infty, S_\infty) + (1+\varepsilon_0)\varepsilon_0, 1\big).$$

For n_0 sufficiently large, the geometric convergence implies the existence of a $(1+\varepsilon_0)$-bilipschitz embedding $\bar g : \mathcal{N}_{\nu_\infty}(S_\infty) \to \bar C_{n_0}$ such that $d(\bar g(x_\infty), x_{n_0}) < \varepsilon_0$ and $n_0 \ge \nu_\infty$. The image $U = \bar g(\mathcal{N}_{\nu_\infty}(S_\infty))$ is an open neighborhood of x_{n_0}, because

$$d(\partial U, \bar g(x_\infty)) \ge \frac{1}{1+\varepsilon_0} d(\partial \mathcal{N}_{\nu_\infty}(S_\infty), x_\infty) \ge \frac{1}{1+\varepsilon_0}(\nu_\infty - d(x_\infty, S_\infty)) > \varepsilon_0 > d(\bar g(x_\infty), x_{n_0}).$$

As in the compact case, we consider the rescaled 3-manifold $E = \operatorname{inj}(x_{n_0})\, C_\infty$ with soul $S = \operatorname{inj}(x_{n_0})\, S_\infty$. By taking $\nu = \operatorname{inj}(x_{n_0})\, \nu_\infty$, $\bar g^{-1}$ induces a $(1+\varepsilon_0)$-bilipschitz homeomorphism $f : U \to \mathcal{N}_\nu(S)$. Moreover, the constants have been chosen so that $\nu \le n_0 \operatorname{inj}(x_{n_0}) \le 1$ and

$$\max\big(\operatorname{inj}(x_{n_0}), d(f(x_{n_0}), S), \operatorname{diam}(S)\big) \le \nu / D_0.$$

Thus x_{n_0} verifies the statement of the local soul theorem in the non-compact case and we obtain a contradiction again. This finishes the proof of the local soul theorem. □

Corollary 4.4.1. — *Let $(C_n)_{n\in\mathbb{N}}$ be a sequence of hyperbolic cone 3-manifolds such that* $\sup\{\operatorname{inj}(x) \mid x \in C_n\}$ *converges to zero when $n \to \infty$. Then for every $\varepsilon > 0$ and $D > 1$,*

- *either there exists $n_0 = n_0(\varepsilon, D)$ such that for $n \ge n_0$ the local soul theorem with parameters ε, D applies to every point of C_n and the local models are non-compact;*
- *or, after rescaling, a subsequence of $(C_n)_{n\in\mathbb{N}}$ converges to a closed Euclidean cone 3-manifold.*

Proof. — Since the supremum on C_n of the injectivity radius goes to zero, there exists an integer $n_0 > 0$ such that, for $n > n_0$, the local soul theorem applies to every point of C_n with compact or non-compact models. We assume the existence of a sequence of points $(x_k)_{k\in\mathbb{N}}$ such that $x_k \in C_{n_k}$, the local model of x_k is compact, and $n_k \to \infty$ as $k \to \infty$. In particular, for every $k \in \mathbb{N}$ there exists a closed Euclidean cone 3-manifold E_k and a $(1+\varepsilon)$-bilipschitz homeomorphism $f_k : E_k \to C_{n_k}$ such that $\operatorname{diam}(E_k) \le R \operatorname{inj}(x_k)$. Therefore,

$$\operatorname{diam}(C_{n_k}) \le (1+\varepsilon) \operatorname{diam}(E_k) \le (1+\varepsilon) R \operatorname{inj}(x_k).$$

Hence, the diameter of the rescaled cone 3-manifold $\bar{C}_{n_k} = \frac{1}{\mathrm{inj}(x_k)} C_{n_k}$ is uniformly bounded above. By the compactness theorem (Chapter 3) $(\bar{C}_{n_k}, x_{n_k})_{k\in\mathbb{N}}$ has a subsequence converging to a pointed Euclidean cone 3-manifold (E, x_∞). Moreover, the limit E is compact, because the diameter of $\bar{C}_{n_k}$ has a uniform upper bound. □

4.5. Local soul theorem for cone 3-manifolds with boundary

By using the compactness theorem for cone 3-manifolds with boundary, the proof in the previous section leads to the following generalization of the local soul theorem.

Theorem 4.5.1 **(Local Soul Theorem for cone 3-manifolds with boundary)**
Given $\omega \in (0,\pi)$, $\varepsilon > 0$ and $D > 1$ there exist $\delta = \delta(\omega,\varepsilon,D) > 0$ and $R = R(\omega,\varepsilon,D) > D > 1$ such that the following holds. Let C be an oriented hyperbolic cone 3-manifold, possibly with boundary, with cone angles in $[\omega,\pi]$. If $x \in C$ satisfies $\mathrm{inj}(x) < \delta$, then:

- either C is $(1+\varepsilon)$-bilipchitz homeomorphic to a compact Euclidean cone 3-manifold E, possibly with boundary, of diameter $\mathrm{diam}(E) \leq R\,\mathrm{inj}(x)$;

- or there exists $0 < \nu < 1$ (depending on x) such that x has an open neighborhood $U_x \subset C$ which is $(1+\varepsilon)$-bilipschitz homeomorphic to one of the following:

a) The normal cone fiber bundle $\mathcal{N}_\nu(S)$, of radius ν, of the soul S of a non-compact orientable Euclidean cone 3-manifold with cone angles in $[\omega,\pi]$, where E is in the list in the local soul theorem,

b) The quotient of $\mathcal{N}_\nu(S)$ by an isometric involution $\tau : \mathcal{N}_\nu(S) \to \mathcal{N}_\nu(S)$ whose fixed point set is two dimensional and orthogonal to the singular locus.

Moreover the $(1+\varepsilon)$-bilipschitz homeomorphism ($f : U_x \to \mathcal{N}_\nu(S)$ in case a) and $f : U_x \to \mathcal{N}_\nu(S)/\tau$ in case b)) satisfies the inequality

$$\max\big(\mathrm{inj}(x), d(f(x), S), \mathrm{diam}(S)\big) \leq \nu/D.$$

The proof is completely analogous to the case without boundary and we just make some comments. Recall that the proof is by contradiction, and that we find a rescaled pointed sequence

$$(\bar{C}_n, \bar{x}_n)_{n\in\mathbb{N}} = \left(\frac{1}{\mathrm{inj}(x_n)} C_n, x_n\right)_{n\in\mathbb{N}}$$

that we may assume to converge to a pointed Euclidean cone 3-manifold (E, x_∞), possibly with totally geodesic boundary. When E is compact then the discussion is the same as before. When E is non compact, then we distinguish two cases, a) and b), according to whether E has boundary or not. In case a), if E has no boundary, then the same proof as before applies. In case b), if E has nonempty boundary, then the double DE is a Euclidean cone 3-manifold with finite injectivity radius, which is one in the list in case a). Let $\tau : DE \to DE$ be the reflection with respect to ∂E. All we have to prove is that we may choose the soul S to be invariant by τ

(Lemma 4.5.2 below). Once this choice of soul is made, then it is clear that any metric tubular neighborhood of the soul is τ-invariant, and the proof of the local soul theorem applies without further change.

Lemma 4.5.2. — *Let E be a non compact Euclidean cone 3-manifold with totally geodesic boundary, finite injectivity radius and cone angles $\leq \pi$. There is a choice of a soul S of DE so that if $\tau : DE \to DE$ is the reflection with respect to ∂E, then*

$$\tau(S) = S.$$

Proof. — If the soul S of DE is unique, then the lemma is clear because τ is an isometry. If the soul is not unique, then DE is isometric to a product $S \times \mathbb{R}^d$, with $d = 3 - \dim S$. Since τ is an isometry it has to preserve the product structure. We look at the restriction $\tau|_{\mathbb{R}^d}$, which may be trivial or not. When $\tau|_{\mathbb{R}^d}$ is trivial, then any choice of soul is τ invariant. Finally, when $\tau|_{\mathbb{R}^d}$ is not trivial, it has fixed points, because it is an isometric involution of $\mathbb{R}^d$. In this last case it suffices to choose $S \times \{p\}$ for some $p \in \mathbb{R}^d$ fixed by $\tau|_{\mathbb{R}^d}$. $\square$

CHAPTER 5

SEQUENCES OF CLOSED HYPERBOLIC CONE 3-MANIFOLDS WITH CONE ANGLES LESS THAN π

This chapter is devoted to the proof of the following theorem.

Theorem A. — *Let $(C_n)_{n\in\mathbb{N}}$ be a sequence of closed hyperbolic cone 3-manifolds with fixed topological type (C,Σ) such that the cone angles increase and are contained in $[\omega_0,\omega_1]$, with $0<\omega_0<\omega_1<\pi$. Then there exists a subsequence $(C_{n_k})_{k\in\mathbb{N}}$ such that one of the following occurs:*

1) The sequence $(C_{n_k})_{k\in\mathbb{N}}$ converges geometrically to a hyperbolic cone 3-manifold with topological type (C,Σ) whose cone angles are the limit of the cone angles of C_{n_k}.

2) For every k, C_{n_k} contains an embedded 2-sphere $S^2_{n_k}\subset C_{n_k}$ that intersects Σ in three points, and the sum of the three cone angles at $S^2_{n_k}\cap\Sigma$ converges to 2π.

3) There is a sequence of positive reals λ_k approaching 0 such that the subsequence of rescaled cone 3-manifolds $(\lambda_k^{-1}C_{n_k})_{k\in\mathbb{N}}$ converges geometrically to a Euclidean cone 3-manifold of topological type (C,Σ) and whose cone angles are the limit of the cone angles of C_{n_k}.

The proof of Theorem A splits into two cases, according to whether the sequence $(C_n)_{n\in\mathbb{N}}$ collapses or not.

Definition 5.0.1. — We say that a sequence $(C_n)_{n\in\mathbb{N}}$ of cone 3-manifolds *collapses* if the sequence $(\sup\{\mathrm{inj}(x)\mid x\in C_n\})_{n\in\mathbb{N}}$ goes to zero.

Remark 5.0.2. — In the last section §5.5 we strengthen Theorem A by showing that in cases 1) and 2) the sequence $(C_n)_{n\in\mathbb{N}}$ does not collapse. Consequences of this strengthened version of Theorem A are the following Margulis'types results:

Proposition 1. — *Given $0<\omega_0<\omega_1<\pi$, there exists a positive constant $\delta_0=\delta_0(\omega_0,\omega_1)>0$ such that every oriented closed hyperbolic cone 3-manifold with cone angles in $[\omega_0,\omega_1]$ and diameter >1 contains a point x with $\mathrm{inj}(x)\geq\delta_0>0$.*

Stronger thickness results, for general hyperbolic cone 3-manifolds (not assuming any more the singular locus to be a link) can be found in [**BLP1**]).

For cone angles bounded away $2\pi/3$, using Hamilton's theorem (cf. [**Zh2**, Thm. 3.2]) we can get ride of the lower bound on the diameter:

Proposition 2. — *Given $0 < \omega_0 < \omega_1 < 2\pi/3$, there exists a positive constant $\delta_1 = \delta_1(\omega_0, \omega_1) > 0$ such that every oriented closed hyperbolic cone 3-manifold with cone angles in $[\omega_0, \omega_1]$ contains a point x with $\mathrm{inj}(x) \geq \delta_1 > 0$.*

5.1. The non-collapsing case

The following proposition (see [**Zh1**], [**SOK**] and [**Ho1**]) proves Theorem A when the sequence $(C_n)_{n\in\mathbb{N}}$ does not collapse.

Proposition 5.1.1. — *Let $(C_n)_{n\in\mathbb{N}}$ be a sequence of hyperbolic cone 3-manifolds satisfying the hypothesis of Theorem A. If the sequence $(C_n)_{n\in\mathbb{N}}$ does not collapse, then there is a subsequence $(C_{n_k})_{k\in\mathbb{N}}$ that verifies assertion 1) or 2) of Theorem A.*

Proof. — Since the sequence C_n does not collapse, after passing to a subsequence if necessary, there is a positive real number $a > 0$ and, for every $n \in \mathbb{N}$, a point $x_n \in C_n$ such that $\mathrm{inj}(x_n) \geq a$. Thus the sequence $(C_n, x_n)_{n\in\mathbb{N}}$ is contained in $\mathcal{C}_{[\omega_0,\omega_1],a}$, the space of pointed cone 3-manifolds (C, x) with constant curvature in $[-1, 0]$, cone angles in $[w_0, w_1]$, and such that $\mathrm{inj}(x) \geq a > 0$. By the compactness theorem (Chapter 3), the sequence $(C_n, x_n)_{n\in\mathbb{N}}$ has a convergent subsequence, which we denote again by $(C_n, x_n)_{n\in\mathbb{N}}$. Hence, we can assume that the sequence $(C_n, x_n)_{n\in\mathbb{N}}$ converges geometrically to a pointed hyperbolic cone 3-manifold (C_∞, x_∞), which may be compact or not.

If the limit cone 3-manifold C_∞ is compact, then the geometric convergence implies that C_∞ has the same topological type (C, Σ) as the cone 3-manifolds of the sequence C_n. Moreover the cone angles of C_∞ are the limit of the cone angles of C_n. This shows that in this case the assertion 1) of Theorem A holds. If the limit cone 3-manifold is not compact, then the next proposition shows that we get the assertion 2) of Theorem A.

Proposition 5.1.2. — *If the limit cone 3-manifold C_∞ is not compact, then for n sufficiently large, C_n contains an embedded 2-sphere $S_n \subset C_n$ that intersects Σ in three points, and the sum of the three cone angles at $S_n \cap \Sigma$ converges to 2π.*

We start with the following lemma.

Lemma 5.1.3. — *The limit cone 3-manifold C_∞ has finite volume.*

Proof of Lemma 5.1.3. — Since $\mathrm{vol}(C_\infty) = \lim_{R\to\infty} \mathrm{vol}\,\big(B(x_\infty, R)\big)$, it suffices to bound $\mathrm{vol}\,\big(B(x_\infty, R)\big)$ independently of R. From the geometric convergence, for every $R > 0$

there is n_0 so that, for $n > n_0$, there exists a $(1+\varepsilon_n)$-bilipschitz embedding $f_n : B(x_\infty, R) \to (C_n, x_n)$, with $\varepsilon_n \to 0$. Hence, for $R > 0$ and $n > n_0$, $\mathrm{vol}(B(x_\infty, R)) \leq (1+\varepsilon_n)^3 \mathrm{vol}(C_n)$, and we get the bound $\mathrm{vol}(B(x_\infty, R)) \leq 2^3 \mathrm{vol}(C_n)$. According to Schläfli's formula for cone 3-manifolds (cf. [**Ho1**] or [**Po2**]), since the cone angles of the 3-manifolds C_n increase, the sequence $(\mathrm{vol}(C_n))_{n\in\mathbb{N}}$ decreases, hence it is bounded above. This fact follows from [**Po2**, Prop. 4.1], since the sequence (ρ_n) of holonomy representations of the cone 3-manifolds C_n belongs to a piecewise analytical path in the variety of representations (by Remark 2.2.7). ☐

Remark 5.1.4. — The use of Schläfli's formula is justified by the fact that our sequence belongs to a one parameter family of cone 3-manifolds. In the case where we apply Theorem A, it is clear from Chapter 2, §2.2 and §2.3. In general, it is still true because of Hodgson and Kerckhoff's local rigidity theorem [**HK**] and Kojima's deformation theorem [**Koj**].

Proposition 5.1.2 follows from the next one.

Proposition 5.1.5. — *If the limit cone 3-manifold C_∞ is not compact, then its singular set Σ_∞ has a non-compact component.*

Proof of Proposition 5.1.2 from Proposition 5.1.5. — From 5.1.5, there is a connected component Σ^0_∞ of Σ_∞ which is not compact. Since $\mathrm{vol}(C_\infty)$ is finite by Lemma 5.1.3, the cone-injectivity radius along Σ^0_∞ is not bounded away from zero.

By the local soul theorem (Chapter 4), there is a point $y \in \Sigma^0_\infty$ having a neighborhood $(1+\varepsilon)$-bilipschitz homeomorphic to a product $S^2(\alpha,\beta,\gamma) \times (-\nu, \nu)$, where $\nu > 0$ and $S^2(\alpha,\beta,\gamma)$ is a two-dimensional Euclidean cone 3-manifold with underlying space the sphere S^2 and singular set three cone points with singular angles α, β and γ such that $\alpha+\beta+\gamma = 2\pi$.

Since (C_n, x_n) converges geometrically to (C_∞, x_∞), there is an $(1+\varepsilon_n)$-bilipschitz embedding $f_n : S^2(\alpha,\beta,\gamma) \times \{0\} \to C_n$, with $\lim \varepsilon_n = 0$. The image $S^2_n = f_n(S^2(\alpha,\beta,\gamma))$ is a 2-sphere embedded in C_n that intersects the singular set in three points and the sum of the three cone angles $\alpha_n + \beta_n + \gamma_n$ at $S^2_{n_k} \cap \Sigma$ converges to 2π. This proves Proposition 5.1.2. ☐

The remaining of this section is devoted to the proof of Proposition 5.1.5.

Proof of Proposition 5.1.5. — Seeking a contradiction, we suppose that the limit cone 3-manifold C_∞ is not compact, but that all the components of Σ_∞ are compact. We remark first that the number of compact components of Σ_∞ can not be bigger than the number of compact components of $\Sigma_n \cong \Sigma$, because geometric convergence implies that, for every compact subset $A \subset C_\infty$, there is an embedding $f_n : A \to C_n$, with $f_n(A \cap \Sigma_\infty) = f_n(A) \cap \Sigma_n$. In particular Σ_∞ must be compact. Next we use the following lemma of [**Koj**] (see also [**Zh1**]).

Lemma 5.1.6. — *Let C_∞ be a hyperbolic cone 3-manifold of finite volume whose singular set Σ_∞ is compact. Then $C_\infty - \Sigma_\infty$ admits a complete hyperbolic structure of finite volume.*

Proof. — Since Σ_∞ is compact it is a finite collection of disjoint circles. The proof consists in deforming the metric in $\mathcal{N}_\varepsilon(\Sigma_\infty) - \Sigma_\infty$, where $\mathcal{N}_\varepsilon(\Sigma_\infty)$ is a tubular neighborhood of radius $\varepsilon > 0$, so that $C_\infty - \Sigma_\infty$ admits a complete metric of (non-constant) sectional curvature $K \leq -a^2 < 0$. With this complete metric $C_\infty - \Sigma_\infty$ has finite volume, therefore by [**Ebe**, Thm. 3.1] it has only finitely many ends and each end is parabolic. In particular $C_\infty - \Sigma_\infty$ is the interior of a compact manifold with toral boundary. Since strictly negative curvature forbids essential spheres and tori as well as Seifert fibrations, Thurston's hyperbolization theorem (for Haken manifolds) provides a complete hyperbolic structure on $C_\infty - \Sigma_\infty$. See [**Koj**] for the details of the deformation. $\square$

Remark 5.1.7. — If Σ_∞ is compact, then the ends of C_∞ are cusps [**Ebe**, Thm. 3.1]. In particular the ends are topologically $T^2 \times [0, \infty)$. Moreover, if $\rho_\infty : \pi_1(C_\infty - \Sigma_\infty) \to PSL_2(\mathbb{C})$ is the holonomy of C_∞, then the restriction of ρ_∞ to $\pi_1(T^2 \times \{0\})$ is parabolic and faithful. This is a consequence of the fact that, in the proof of Lemma 5.1.6, the metric has not been changed on the ends.

Let $N_\infty \subset C_\infty - \Sigma_\infty$ be a compact core containing the base point x_∞. If Σ_∞ is compact, then the boundary ∂N_∞ is a collection of tori $T_1, \ldots, T_p$ and:

$$C_\infty - \Sigma_\infty = N_\infty \bigcup_{\partial N_\infty} \bigsqcup_{i=1}^{p} T_i \times [0, \infty).$$

We set $X = C - \mathcal{N}(\Sigma) \cong C_n - \mathcal{N}(\Sigma_n)$, where $\mathcal{N}$ denotes an open tubular neighborhood. From the geometric convergence (for n sufficiently large) there is an $(1+\varepsilon_n)$-bilipschitz embedding $f_n : N_\infty \to C_n$, with $\varepsilon_n \to 0$, such that

$$N_n = f_n(N_\infty) \subset C_n - \mathcal{N}(\Sigma_n) \cong X.$$

Claim 5.1.8. — *For n sufficiently large, every connected component of $X - \mathrm{int}(N_n)$ is either a solid torus $S^1 \times D^2$ or a product $T^2 \times [0, 1]$.*

Proof. — First we show that $X - \mathrm{int}(N_n)$ is irreducible for n sufficiently large. Otherwise, after passing to a subsequence, we can assume that $X - \mathrm{int}(N_n)$ is reducible for every n. This implies that there is a ball $B_n \subset X$ such that $N_n \subset B_n$, because X is irreducible. Let $\rho_n : \pi_1(X, x_n) \to PSL_2(\mathbb{C})$ and $\rho_\infty : \pi_1(N_\infty, x_\infty) \to PSL_2(\mathbb{C})$ denote the holonomy representations of C_n and C_∞ respectively. The geometric convergence implies the algebraic convergence of the holonomies (Proposition 3.5.4). This means that for every $\gamma \in \pi_1(N_\infty, x_\infty)$, $\rho_n(f_{n*}(\gamma))$ converges to $\rho_\infty(\gamma)$. Since N_n is contained in a ball, $f_{n*}(\gamma) = 1$, so $\rho_\infty(\gamma) = 1$ for every $\gamma \in \pi_1(N_\infty, x_\infty)$. Since the holonomy representation of C_∞ is non-trivial, we get a contradiction. This proves the irreducibility of $X - \mathrm{int}(N_n)$.

Since $X - \mathrm{int}(N_n)$ is irreducible and ∂N_n is a collection of tori, the claim follows easily from the fact that X is irreducible and atoroidal. $\square$

In order to get a contradiction with the hypothesis that Σ_∞ is compact we need in addition the following claim.

Claim 5.1.9. — *For n sufficiently large, at least one component $X - \mathrm{int}(N_n)$ is a solid torus.*

Proof. — We assume that the claim is false and look for a contradiction. Thus, after passing to a subsequence if necessary, we can assume that all the components of ∂N_n are parallel to the boundary of ∂X; this means that $f_n : N_\infty \to X$ is a homotopy equivalence.

If $T^2 \subset \partial N_\infty$ is a component corresponding to an end of C_∞, then the image $\rho_\infty(\pi_1(T^2, x_\infty))$ is a parabolic subgroup of $PSL_2(\mathbb{C})$ by the remark following Lemma 5.1.6. Furthermore, since C_n converges geometrically to C_∞, for every $\gamma \in \pi_1(T^2, x_\infty)$, $\rho_\infty(\gamma) = \lim_{n\to\infty} \rho_n(f_{n*}(\gamma))$.

Since X has a complete hyperbolic structure, by Mostow's rigidity theorem [**Mos**] and Waldhausen's theorem [**Wa1**] the group $\pi_0(\mathrm{Diff}(X))$ is finite (see also [**Joh**]). Hence, after passing to a subsequence, we can choose $\gamma \in \pi_1(T, x_\infty)$ such that, for every n, $f_{n*}(\gamma)$ is conjugate to a meridian μ_0 of a fixed component Σ_0 of Σ. Since μ_0 is elliptic, $\mathrm{trace}(\rho_n f_{n*}(\gamma)) = \pm 2\cos(\alpha_n/2)$, where $\alpha_n \in [\omega_0, \omega_1]$ is the cone angle of the manifold C_n at the component Σ_0. Since $0 < \omega_0 < \omega_1 \leq \pi$ the sequence $|\,\mathrm{trace}(\rho_n(f_{n*}(\gamma)))|$ is bounded away from 2. As $\rho_\infty(\gamma)$ is parabolic, $|\,\mathrm{trace}(\rho_\infty(\gamma))| = 2$, and we obtain a contradiction with the convergence of $\rho_n(f_{n*}(\gamma))$ to $\rho_\infty(\gamma)$. $\square$

From Claim 5.1.9, there is a collection $T_1, \dots, T_q$ of components of ∂N_∞ such that, for n sufficiently large, $f_n(T_i)$ bounds a solid torus $V_n^i \subset X$, for $i = 1, \dots, q$.

Let $\lambda_n^i \subset f_n(T_i)$ be the boundary of a meridian of the solid torus V_n^i, for $i = 1, \dots, q$. The inverse images $\tilde\lambda_n^1 = f_n^{-1}(\lambda_n^1), \dots, \tilde\lambda_n^q = f_n^{-1}(\lambda_n^q)$ are the meridians of the Dehn fillings of $N_\infty = f_n^{-1}(N_n)$ which give X. More precisely,

$$X = N_\infty \bigcup_{\phi_{i,n}} \bigsqcup_{i=1}^{n} S^1 \times D_i^2,$$

where, for $i = 1, \dots, q$, the gluing maps $\phi_{i,n} : S^1 \times \partial D_i^2 \cong T_i \subset \partial N_\infty$ satisfy $\phi_{i,n}(\{*\} \times \partial D_i^2) = \tilde\lambda_n^i$.

We have now the following claim:

Claim 5.1.10. — *For every $i = 1, 2, \dots, q$, the sequence of simple closed curves $(\tilde\lambda_n^i)_{n\geq n_0}$ represents infinitely many distinct elements in $H_1(T_i)$. Hence, after passing to a subsequence, the length of $\tilde\lambda_n^i$ goes to infinity with n.*

Proof. — If the claim is not true, then, after passing to a subsequence, we can assume that there is an index $i \in \{1, 2, \dots, q\}$ such that the curves $\tilde{\lambda}^i_n$ are all homotopic to a fixed curve $\tilde{\lambda}^i$, for every n.

Since the sequence (C_n, x_n) converges geometrically to (C_∞, x_∞) we have:

$$\rho_\infty(\tilde{\lambda}^i) = \lim_{n\to\infty} \rho_n(f_{n*}(\tilde{\lambda}^i)) = \pm \mathrm{Id}$$

because, for n sufficiently large, $f_n(\tilde{\lambda}^i) = \lambda^i_n$ bounds a meridian disk of a solid torus V^i_n. Since T_i corresponds to a cusp of C_∞, the holonomy $\rho_\infty(\tilde{\lambda}^i)$ is not trivial and we get a contradiction. □

We are now ready to contradict the hypothesis that Σ_∞ is compact. If Σ_∞ is compact, then, by Claims 5.1.8, 5.1.9 and 5.1.10, we have, for $i = 1, \dots, q$, a sequence of curves $(\tilde{\lambda}^i_n)_{n \geq n_0}$ in $T_i \subset \partial N_\infty$ whose lengths go to infinity with n, and so that the 3-manifold obtained by Dehn filling with meridians $\{\tilde{\lambda}^1_n, \dots, \tilde{\lambda}^q_n\}$ is always $X = C - \mathcal{N}(\Sigma)$. According to Thurston's hyperbolic Dehn filling theorem [**Thu1**] (cf. Appendix B), almost all these Dehn fillings are hyperbolic. Furthermore, by Schläfli's formula, almost all of them have different hyperbolic volumes. Thus we get a contradiction, because our Dehn fillings give always the same 3-manifold X. This finishes the proof of Propositions 5.1.5 and 5.1.2. □

5.2. The collapsing case

The next proposition proves Theorem A when the sequence of hyperbolic cone 3-manifolds C_n collapses.

Proposition 5.2.1. — *Let $(C_n)_{n\in\mathbb{N}}$ be a sequence of hyperbolic cone 3-manifolds with the same hypothesis as in Theorem A. If the sequence $(C_n)_{n\in\mathbb{N}}$ collapses, then there is a subsequence $(C_{n_k})_{k\in\mathbb{N}}$ that satisfies assertions* 2) *or* 3) *of Theorem A.*

The proof uses Gromov simplicial volume of a compact oriented 3-manifold M and the dual notion of real bounded cohomology of M, both introduced by M. Gromov [**Gro**] (see also [**Iva**]).

The simplicial volume $\|M\|$ of a compact, orientable, 3-manifold M, with boundary ∂M (possibly empty) is defined as follows

$$\|M\| = \inf \left\{ \sum_{i=1}^n |\lambda_i| \;\middle|\; \begin{array}{l} \sum_{i=1}^n \lambda_i\sigma_i \text{ is a cycle representing a fundamental} \\ \text{class in } H_3(M, \partial M; \mathbb{R}), \text{ where } \sigma_i : \Delta^3 \to M \\ \text{is a singular simplex and } \lambda_i \in \mathbb{R},\ i = 1, \dots, n. \end{array} \right\}$$

In particular, when C is a closed and oriented 3-manifold and $\Sigma \subset C$ is a link, we define the simplicial volume $\|C - \Sigma\| = \|C - \mathrm{int}(\mathcal{N}(\Sigma))\|$, where $\mathcal{N}(\Sigma)$ is a tubular neighborhood of Σ in C.

We are starting now to prove Proposition 5.2.1.

Proof of Proposition 5.2.1. — We are going to show that if assertions 2) and 3) of Theorem A do not hold, then the simplicial volume $\|C-\Sigma\|$ is zero, and this would contradict the hyperbolicity of $C-\Sigma$ (Lemma 5.1.6). To show that the simplicial volume vanishes, we need for a subset of C the notion of abelianity in $C-\Sigma$.

Definition 5.2.2. — We say that a subset $U \subset C$ is *abelian in* $C-\Sigma$ if the image $i_*(\pi_1(U-\Sigma))$ is an abelian subgroup of $\pi_1(C-\Sigma)$, where i_* is the morphism induced by the inclusion $i : (U-\Sigma) \to (C-\Sigma)$.

Definition 5.2.3. — Let C be a hyperbolic cone 3-manifold, $x \in C$, and $\varepsilon, D > 0$. An (ε, D)*-Margulis' neighborhood of abelian type* of x is a neighborhood U_x $(1+\varepsilon)$-bilipschitz homeomorphic to the normal cone fiber bundle $\mathcal{N}_\nu(S)$, of some radius $\nu \leq 1$ depending on x, of the soul S of one of the following non-compact orientable Euclidean cone 3-manifolds:

$$T \times \mathbb{R}, \qquad S^1 \ltimes \mathbb{R}^2, \qquad S^1 \ltimes \text{(cone disk)},$$

where $\ltimes$ denotes the metrically twisted product. Moreover, the $(1+\varepsilon)$-bilipschitz homeomorphism $f : U_x \to N_\nu(S)$ satisfies:

$$\max(\mathrm{inj}(x), d(f(x), S), \mathrm{diam}(S)) \leq \nu/D.$$

Note that a (ε, D)-Margulis' neighborhood of abelian type is abelian in $C-\Sigma$. This definition is motivated by the following lemma, which is the first step in the proof of Proposition 5.2.1.

Lemma 5.2.4. — *Let $(C_n)_{n\in\mathbb{N}}$ be a sequence of hyperbolic cone 3-manifolds which collapses and satisfies the hypothesis of Theorem A. If both assertions 2) and 3) of Theorem A fail to hold, then, for every $\varepsilon, D > 0$, there exists n_0 such that, for $n \geq n_0$, every $x \in C_n$ has a (ε, D)-Margulis' neighborhood of abelian type.*

Proof of Lemma 5.2.4. — Since the sequence collapses, we can apply the local soul theorem (Chapter 4) and we show that the only possible local models are the three ones of abelian type.

More precisely, since the supremum of the cone-injectivity radius converges to zero when n goes to infinity, given $\varepsilon, D > 0$ there exists n_0 such that for $n \geq n_0$ the local soul theorem applies to every point x in C_n. Since we assume that assertion 3) of Theorem A does not hold, by Corollary 4.4.1, the compact models are excluded. Hence we have to consider only the non-compact local models.

From the hypothesis that assertion 2) of theorem A does not hold, we get rid of the product model $S^2(\alpha,\beta,\gamma) \times \mathbb{R}$, where $S^2(\alpha,\beta,\gamma)$ is a Euclidean cone 2-manifold with underlying space the sphere S^2 and singular set three points at which the sum of the three cone angles is $\alpha+\beta+\gamma = 2\pi$.

Since the cone angles belong to $[\omega_0, \omega_1]$, with $0 < \omega_0 < \omega_1 < \pi$, the local models with a cone angle equal to π cannot occur.

Finally, the last model to be eliminated is the one corresponding to the normal bundle of the soul of a twisted fiber bundle over the Klein bottle $K^2\tilde{\times}\mathbb{R}$. A neighborhood U_x $(1+\varepsilon)$-bilipschitz homeomorphic to this model does not intersect the singular set Σ. If this local model occurred, there would be a Klein bottle $K^2\times\{0\}$ embedded in $C-\Sigma$. This would contradict the fact that $C-\Sigma$ admits a complete hyperbolic structure (Lemma 5.1.6). □

Since $C-\Sigma$ admits a complete hyperbolic structure (Lemma 5.1.6), by Gromov [**Gro**] and Thurston [**Thu1**, Ch. 6], $||C-\Sigma|| = \mathrm{vol}(C-\Sigma)/v_3$, where $v_3>0$ is a constant depending only on the dimension. In particular, $||C-\Sigma|| \neq 0$. Then the proof of Proposition 5.2.1 follows from Lemma 5.2.4 and the next proposition:

Proposition 5.2.5. — *There exists a universal constant $D_0>0$ such that, if C is a closed hyperbolic cone 3-manifold where every point has an (ε, D)-Margulis' neighborhood of abelian type, with $\varepsilon<1/2$ and $D>D_0$, then the simplicial volume $||C-\Sigma||$ is zero.*

We prove this proposition in Sections 5.3 and 5.4. In order to show that $||C-\Sigma||$ vanishes, we adapt a construction of Gromov [**Gro**, Sec. 3.4] to the relative case. This construction gives a covering of C by open sets that are abelian in $C-\Sigma$, and the dimension of the covering is 2 in C and 0 in Σ. In fact, Proposition 5.2.5 can be seen as a version of Gromov's isolation theorem [**Gro**, Sec. 3.4] for cone 3-manifolds.

5.3. Coverings à la Gromov

Definition 5.3.1. — For $\eta>0$, a covering $(V_i)_{i\in I}$ of a hyperbolic cone 3-manifold C by open subsets is said to be a *η-covering à la Gromov* if it satisfies:

1) for every $i\in I$, there exists a metric ball $B(x_i,r_i)$ of radius $r_i\leq 1$ that contains V_i;
2) if $B(x_i,r_i)\cap B(x_j,r_j)\neq\emptyset$, then $3/4\leq r_i/r_j\leq 4/3$;
3) for $i\neq j$, $B(x_i,r_i/4)\cap B(x_j,r_j/4)=\emptyset$;
4) every $x\in C$ belongs to an open set V_i such that $d(x,\partial V_i)\geq r_i/3$;
5) for every $i\in I$, $\mathrm{vol}(V_i)\leq \eta\, r_i^3$.

Remark 5.3.2. — Every η-covering à la Gromov of a closed hyperbolic cone 3-manifold is finite, because properties 2) and 3) forbid accumulating sequences.

Our interest in η-coverings à la Gromov comes from the following proposition. Our proof of this proposition follows closely Gromov's proof [**Gro**, Sec. 3.4], for the Riemannian (non-singular) case.

Proposition 5.3.3. — *There exists a universal constant $\eta_0>0$ such that, for any closed hyperbolic cone 3-manifold C admitting a η-covering à la Gromov $(V_i)_{i\in I}$ with*

$\eta \leq \eta_0$, there exists a continuous map from C to a simplicial 2-complex $f : C \to K^{(2)}$ satisfying:

i) for every vertex v of $K^{(2)}$, there is $i(v) \in I$ such that

$$f^{-1}(\mathrm{star}(v)) \subset \bigcup_{V_j \cap V_{i(v)} \neq \emptyset} V_j$$

ii) for every $x \in C$ that belongs to only one open set of the covering, $f(x)$ is a vertex of K^2.

Proof of Proposition 5.3.3. — The proof consist of a sequence of lemmas, as in Gromov's proof [**Gro**, Sec. 3.4]. We recall that a covering has dimension n if every point belongs to at most $n+1$ open sets of the covering.

Lemma 5.3.4. — *There is a universal integer $N > 0$ such that, for every closed hyperbolic cone 3-manifold C and for every $\eta > 0$, the dimension of any η-covering à la Gromov of C is at most N.*

Proof of Lemma 5.3.4. — We shall bound the number N_i of balls $B(x_j, r_j)$ that intersect a given ball $B(x_i, r_i)$. From property 2) of the definition of a η-covering à la Gromov, if $B(x_j, r_j) \cap B(x_i, r_i) \neq \emptyset$, then $3/4 \leq r_j/r_i \leq 4/3$ and we have:

$$\begin{aligned} & B(x_j, r_j) \subset B(x_i, r_i + 2r_j) \subset B(x_i, 4r_i) \\ \text{and} \qquad & B(x_i, 4r_i) \subset B(x_j, 5r_i + r_j) \subset B(x_j, 8r_j). \end{aligned}$$

By using these inclusions and the fact that the balls $(B(x_j, r_j/4))_{j\in I}$ are pairwise disjoint, it follows that the number N_i of balls that intersect a given $B(x_i, r_i)$ is bounded above by:

$$\begin{aligned} N_i &\leq \sup \left\{ \frac{\mathrm{vol}(B(x_i, 4r_i))}{\mathrm{vol}(B(x_j, r_j/4))} \mid B(x_i, r_i) \cap B(x_j, r_j) \neq \emptyset \right\} \\ &\leq \sup_{j\in I} \left\{ \frac{\mathrm{vol}(B(x_j, 8r_j))}{\mathrm{vol}(B(x_j, r_j/4))} \right\}. \end{aligned}$$

Now, the uniform upper bound for N_i comes from Bishop-Gromov inequality (Proposition 3.1.9), which shows that:

$$\frac{\mathrm{vol}(B(x_j, 8r_j))}{\mathrm{vol}(B(x_j, r_j/4))} \leq \frac{\mathrm{v}_{-1}(8r_j)}{\mathrm{v}_{-1}(r_j/4)},$$

where $\mathrm{v}_{-1}(r) = \pi(\sinh(2r) - 2r)$ is the volume of the ball of radius r in the hyperbolic space $\mathbb{H}^3_{-1}$. Since the function $r \mapsto \mathrm{v}_{-1}(8r)/\mathrm{v}_{-1}(r/4)$ is continuous, it is bounded on $[0,1]$. Hence, for any integer N bounding above this function on $[0,1]$, we get $N_i \leq N$ and the lemma is proved. □

Given a η-covering à la Gromov, its nerve K is a simplicial complex and, according to Lemma 5.3.4, the dimension of K is at most N, where N is a uniform constant. Since we work with a compact cone 3-manifold C, every η-covering à la Gromov is finite and its nerve K is compact. We canonically embed K in $\mathbb{R}^p$, where p is the number of vertices of K, which equals the number of open sets of this covering. More precisely, every vertex of K corresponds to a vector of the form $(0, \dots, 1, \dots, 0)$ and the simplices of positive dimension are defined by linear extension.

The proof of Proposition 5.3.3 goes as follows. We start in Lemma 5.3.5 by constructing a Lipschitz map from C to the nerve of the covering $f : C \to K$ which satisfies properties i) and ii) of Proposition 5.3.3. Next, in Lemma 5.3.7, we deform the map f to a Lipschitz map $f_3 : C \to K^{(3)}$ where $K^{(3)}$ is the 3-skeleton of K. Finally, in Lemma 5.3.9, we prove that for $\eta > 0$ sufficiently small we can deform $f_3 : C \to K^{(3)}$ to the 2-skeleton $K^{(2)}$, keeping properties i) and ii) of Proposition 5.3.3. To prove the existence of such a universal constant $\eta_0 > 0$ we need uniform constants in the lemmas, the first example being the upper bound N of the dimension of K.

Lemma 5.3.5. — *Let C be a hyperbolic cone 3-manifold equipped with a η-covering à la Gromov $(V_i)_{1 \leq i \leq p}$. Let $K = K^{(k)}$ be the nerve of this covering, which has dimension $k \leq N$. Then there exists a Lipschitz map $f_k : C \to K$ that verifies properties i) and ii) of Proposition 5.3.3 and in addition:*

iii) *there exists a uniform constant ξ_k, depending only on the dimension k, such that, for $1 \leq i \leq p$,*

$$\forall x, y \in \bigcup_{V_j \cap V_i \neq \emptyset} V_j, \qquad \|f_k(x) - f_k(y)\| \leq \frac{\xi_k}{r_i} d(x, y).$$

In this lemma, d denotes the hyperbolic distance on C and $\|\ \|$ the Euclidean norm on $\mathbb{R}^p$, since we assume that K is canonically embedded in $\mathbb{R}^p$.

Proof of Lemma 5.3.5. — We choose a smooth function $\phi : \mathbb{R} \to [0, 1]$ such that $\phi((-\infty, 0]) = 0$, $\phi([1/3, +\infty)) = 1$, and $|\phi'(t)| \leq 4$ for every $t \in \mathbb{R}$.

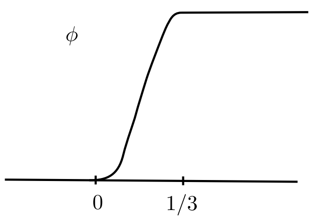

FIGURE 1. The function ϕ

For every $i = 1, \dots, p$, let $\phi_i : \overline{V_i} \to \mathbb{R}$ be the Lipschitz map such that $\forall x \in \overline{V_i}$ $\phi_i(x) = \phi(d(x, \partial V_i)/r_i)$, where ∂V_i is the boundary of V_i. Since ϕ_i vanishes on ∂V_i, we extend it to the whole manifold just by taking zero outside V_i. Then we have the property

$$(3) \qquad \forall x, y \in C, \qquad \|\phi_i(x) - \phi_i(y)\| \leq \frac{4}{r_i} d(x, y)$$

because the map

$$x \mapsto \begin{cases} d(x, \partial V_i) & \text{if } x \in \overline{V_i} \\ 0 & \text{otherwise} \end{cases}$$

has Lipschitz constant 1.

Let $\Delta^{p-1} = \{(u_1, \ldots, u_p) \in \mathbb{R}^p \mid u_1 + \cdots + u_p = 1 \text{ and } u_i \geq 0 \text{ for } i = 1, \ldots, p\}$ be the unit simplex of $\mathbb{R}^p$. We define the map $f_k : C \to \Delta^{p-1}$ to be:

$$\forall x \in C, \qquad f_k(x) = \frac{1}{\sum_{i=1}^{p} \phi_i(x)} (\phi_1(x), \cdots, \phi_p(x)).$$

This map is well defined, since $\sum_{i=1}^{p} \phi_i(x) \geq 1$ by property 4) of a η-covering à la Gromov.

The nerve K of the covering embeds canonically in $\mathbb{R}^p$ as a subcomplex of Δ^{p-1}. Namely, if $V_1, \ldots, V_p$ are the open sets of the covering, then the vertex of K corresponding to V_i is mapped to the i-th vertex $(0, \ldots, 1, \ldots, 0)$ of Δ^{p-1}. By construction the image $f_k(C)$ is contained in $K \subset \Delta^{p-1}$ and satisfies properties i) and ii) of Proposition 5.3.3.

The following claim shows that f_k satisfies property iii).

Claim 5.3.6. — *For $1 \leq i \leq p$ and $\forall x, y \in \bigcup_{V_j \cap V_i \neq \emptyset} V_j$ we have:*

a) $\|(\phi_1(x), \ldots, \phi_p(x)) - (\phi_1(y), \ldots, \phi_p(y))\| \leq \dfrac{8(k+1)}{r_i} d(x, y)$

b) $\|f_k(x) - f_k(y)\| \leq \sqrt{2}(k+1) \|(\phi_1(x), \ldots, \phi_p(x)) - (\phi_1(y), \ldots, \phi_p(y))\|$.

Proof of Claim 5.3.6. — We first prove a).

From inequality (3), for every $i = 1, \ldots, p$, and $\forall x, y \in \bigcup_{V_j \cap V_i \neq \emptyset} V_j$,

$$\|(\phi_1(x), \ldots, \phi_p(x)) - (\phi_1(y), \ldots, \phi_p(y))\|^2 = \sum_{\substack{V_j \cap V_i \neq \emptyset \\ V_l \cap V_j \neq \emptyset}} (\phi_l(x) - \phi_l(y))^2 \leq 4^2 \sum_{\substack{V_j \cap V_i \neq \emptyset \\ V_l \cap V_j \neq \emptyset}} \frac{1}{r_l^2} d(x, y)^2.$$

From Property 2) of a η-covering à la Gromov:

$$\sum_{\substack{V_j \cap V_i \neq \emptyset \\ V_l \cap V_j \neq \emptyset}} \frac{1}{r_l^2} \leq (k+1) \sum_{V_j \cap V_i \neq \emptyset} \left(\frac{4}{3r_j}\right)^2 \leq (k+1)^2 \left(\frac{16}{9r_i}\right)^2 < \left(\frac{2(k+1)}{r_i}\right)^2,$$

where k is the dimension of the covering. Summarizing both inequalities we conclude the proof of assertion a).

Next we prove b). We view f_k as a composition $f_k = g_p \circ (\phi_1, \cdots, \phi_p)$, where $g_p(u_1, \ldots, u_p) = \frac{1}{u_1 + \cdots + u_p}(u_1, \ldots, u_p)$. Since the dimension of the covering is k, $(\phi_1, \cdots, \phi_p)$ maps $\bigcup\limits_{V_j \cap V_i \neq \emptyset} V_j$ into a $(k+1)^2$-dimensional subspace of $\mathbb{R}^p$ (provided that $p \geq (k+1)^2$), which is obtained by putting $p - (k+1)^2$ coordinates equal to zero. In addition, $\sum_j \Phi_j(x) \geq 1$ for every $x \in C$. Hence, by setting $s = \inf\{p, (k+1)^2\}$, it suffices to prove that the restricted map

$$\begin{array}{rcl} g_{s|} : \{(u_1, \ldots, u_s) \in \mathbb{R}^s \mid \sum_j u_j \geq 1, u_i \geq 0\} & \to & \mathbb{R}^s \\ (u_1, \ldots, u_s) & \mapsto & \frac{1}{u_1 + \cdots + u_s}(u_1, \ldots, u_s) \end{array}$$

has Lipschitz constant $\sqrt{2s}$, because $\sqrt{2s} \leq \sqrt{2}(k+1)$. For $u = (u_1, \ldots, u_s) \in \mathbb{R}^s$ satisfying $\sum_j u_j \geq 1$ and $u_i \geq 0$, and for $v \in T_u\mathbb{R}^s$, we claim that:

$$||(D_u g_s)v|| \leq \sqrt{2s}||v||,$$

where $D_u g_s$ is the tangent map of g_s at u. An easy computation shows that $||(D_u g_s)\frac{\partial}{\partial u_i}|| \leq \sqrt{2}$. Therefore, if $v = \sum_i v_i \frac{\partial}{\partial u_i}$, then

$$||(D_u g_s)v|| \leq \sqrt{2}\sum_i |v_i| \leq \sqrt{2s}||v||,$$

by Cauchy-Schwarz inequality. This finishes the proof of Claim 5.3.6 and of Lemma5.3.5

□

Lemma 5.3.7. — *With the hypothesis of Lemma 5.3.5, the Lipschitz map $f_k : C \to K$ can be deformed to a Lipschitz map $f_3 : C \to K^{(3)}$ into the 3-skeleton which satisfies properties i), ii) (from Proposition 5.3.3) and iii) (from Lemma 5.3.5).*

Proof of Lemma 5.3.7. — We start with the map $f_k : C \to K$ obtained in Lemma 5.3.5. If $k = \dim K = 3$, we are done. Hence we assume that $k > 3$ and we prove the lemma by induction: we show that whenever we have a map $f_k : C \to K^{(k)}$ satisfying properties i), ii) and iii) with $k > 3$, then we can deform it to a map into the $(k-1)$-skeleton $K^{(k-1)}$ satisfying the same properties. The key point in the argument is the following technical claim.

Claim 5.3.8. — *Given a Lipschitz map $f_k : C \to K^{(k)}$ satisfying properties i), ii) and iii), there is a uniform constant $\varepsilon_k > 0$ (which depends only on k) such that every k-simplex $\Delta^k \subset K$ contains a point z at distance at least ε_k from the image $f_k(C)$ and the boundary $\partial\Delta^k$.*

Proof Claim 5.3.8. — Let $\varepsilon > 0$ be such that every point in Δ^k is at distance at most $\varepsilon > 0$ from the union $f_k(C) \cup \partial\Delta^k$. We are going to find a uniform constant $\varepsilon_k > 0$ such that $\varepsilon \geq \varepsilon_k$.

Let $\{z_1, \ldots, z_s\} \subset \Delta^k$ be a maximal family of points such that $d(z_i, \partial\Delta^k) \geq 3\varepsilon$ and $\|z_i - z_j\| \geq 3\varepsilon$ for $i \neq j$. There exists a constant $c_1 = c_1(k) > 0$ depending only on the dimension k such that, for ε sufficiently small, we can find at least $c_1\varepsilon^{-k}$ points in this family. So we can assume $s \geq c_1\varepsilon^{-k}$.

By the hypothesis on $\varepsilon > 0$, we can find a family of points $\{y_1, \ldots, y_s\} \subset f_k(C) \cap \Delta^k$ such that $\|z_i - y_i\| \leq \varepsilon$, for $i = 1, \ldots, s$. In particular, $\|y_i - y_j\| > \varepsilon$ (if $i \neq j$) and $d(y_i, \partial\Delta^k) > \varepsilon$. Choose points $\{\bar{y}_1, \ldots, \bar{y}_s\} \subset C$ such that $f_k(\bar{y}_i) = y_i$, $i = 1, \ldots, s$. From property i) of f_k, the points $\{\bar{y}_1, \ldots, \bar{y}_s\}$ belong to $\bigcup_{V_j \cap V_{i(v)} \neq 0} V_j$, where the open set $V_{i(v)}$ corresponds to a vertex v of Δ^k. So, from property iii) we have

$$\forall i \neq j \in \{1, \cdots, s\}, \qquad d(\bar{y}_i, \bar{y}_j) \geq \frac{r_{i(v)}}{\xi_k}\|y_i - y_j\| > \frac{r_{i(v)}}{\xi_k}\varepsilon.$$

This implies that the balls $B(\bar{y}_j, \frac{r_{i(v)}}{2\xi_k}\varepsilon)$ are pairwise disjoint and satisfy

$$B(\bar{y}_j, \frac{r_{i(v)}}{2\xi_k}\varepsilon) \subset \bigcup_{V_j \cap V_{i(v)} \neq 0} V_j \subset B(x_{i(v)}, 4r_{i(v)}),$$

where the last inclusion follows from property 2) of a η-covering à la Gromov. We get the following upper bound for the number s of such balls:

$$s \leq \max_{j=1,\ldots,s} \left(\frac{\mathrm{vol}(B(x_{i(v)}, 4r_{i(v)}))}{\mathrm{vol}(B(\bar{y}_j, \frac{r_{i(v)}}{2\xi_k}\varepsilon))}\right) \leq \max_{j=1,\ldots,s} \left(\frac{\mathrm{vol}(B(\bar{y}_j, 8r_{i(v)}))}{\mathrm{vol}(B(\bar{y}_j, \frac{r_{i(v)}}{2\xi_k}\varepsilon))}\right),$$

because $B(x_{i(v)}, 4r_{i(v)}) \subset B(\bar{y}_j, 8r_{i(v)})$, for $j = 1, \ldots, s$. From Bishop-Gromov inequality (Proposition 3.1.9) we obtain:

$$s \leq \frac{\mathrm{v}_{-1}(8r_{i(v)})}{\mathrm{v}_{-1}(\frac{r_{i(v)}}{2\xi_k}\varepsilon)},$$

where $\mathrm{v}_{-1}(r) = \pi(\sinh(2r) - 2r)$ is the volume of the ball of radius r in the hyperbolic 3-space. There exists a constant $a > 0$ such that $\frac{r^3}{a} \leq \mathrm{v}_{-1}(r) \leq ar^3$ $\forall r \in [0, 8]$. Since $r_{i(v)} \leq 1$, we obtain the upper bound:

$$s \leq \frac{a^2(8r_{i(v)})^3 8\xi_k^3}{r_{i(v)}^3\varepsilon^3} = c_2\varepsilon^{-3},$$

where $c_2 = 2^{12}a^2\xi_k^3 > 0$ depends only on the dimension k. By combining both inequalities $c_1\varepsilon^{-k} \leq s \leq c_2\varepsilon^{-3}$, we conclude that $\varepsilon \geq (c_1/c_2)^{\frac{1}{k-3}}$, with $k > 3$. This finishes the proof of Claim 5.3.8. □

End of the proof of Lemma 5.3.7. — We asume $k > 3$ and we want to construct $f_{k-1} : C \to K^{(k-1)}$. Let $\Delta_1^k, \ldots, \Delta_q^k$ be the k-simplices of K. From Claim 5.3.8, for every k-simplex $\Delta_i^k \subset K$ we can choose a point $z_i \in \Delta_i^k$ so that $d(z_i, f_k(C) \cup \partial\Delta_i^k) > \varepsilon_k$. We consider the map $R_i : K - \{z_i\} \to K$ which is defined by the radial retraction of $\Delta_i^k - \{z_i\}$ onto $\partial\Delta_i^k$ and the identity on $K - \Delta_i^k$. Since the points $\{z_1, \ldots, z_q\}$ do not belong to the image of f_k, the composition

$$f_{k-1} = R_1 \circ \cdots \circ R_q \circ f_k : C \to K$$

is well defined, and the image $f_{k-1}(C)$ lies in the $(k-1)$-skeleton $K^{(k-1)}$. Moreover, it follows from the construction that f_{k-1} satisfies properties i) and ii) of Proposition 5.3.3, because the retractions R_i preserve the vertices and their stars.

For $i = 1, \ldots, q$, the retraction $R_i : K - \{z_i\} \to K$ is piecewise smooth. From the inequality $d(z_i, f_k(C) \cup \partial\Delta_i^k) > \varepsilon_k$, it follows that the local Lipschitz constant of $R_1 \circ \cdots \circ R_q$ is uniformly bounded on the image $f_k(C)$; moreover the bound depends only on the dimension k, because the constant ε_k is uniform, depending only on the dimension k. Thus f_{k-1} satisfies also property iii) of Lemma 5.3.5. $\square$

Next lemma completes the proof of Proposition 5.3.3.

Lemma 5.3.9. — *There exists a universal constant $\eta_0 > 0$ such that, for $\eta < \eta_0$ and for every η-covering à la Gromov of C, the map $f_3 : C \to K^{(3)}$ of Lemma 5.3.7 can be deformed to a continuous map $f_2 : C \to K^{(2)}$ into the 2-skeleton which satisfies properties i) and ii) of Proposition 5.3.3.*

Proof of Lemma 5.3.9. — To deform f_3 to f_2, it suffices to prove that in every 3-simplex $\Delta^3 \subset K$, there is a point $z \in \operatorname{int}(\Delta^3)$ that does not belong to the image $f_3(C)$. Then such a deformation is constructed by composing f_3 with all the radial retractions from $\Delta^3 - \{z\}$ to $\partial\Delta^3$ as in Lemma 5.3.7. The map f_2 will satisfy properties i) and ii) of Proposition 5.3.3 by construction. Next claim shows that $\operatorname{int}(\Delta^3) - f_3(C)$ is non-empty whenever η is less than a universal constant $\eta_0 > 0$. This will conclude the proof of Lemma 5.3.9.

Claim 5.3.10. — *There exists a universal constant $\eta_0 > 0$ such that, if C admits a η-covering à la Gromov with $\eta < \eta_0$, then for every 3-simplex $\Delta^3 \subset K^{(3)}$*

$$\operatorname{vol}(\Delta^3 \cap f_3(C)) < \operatorname{vol}(\Delta^3).$$

Proof of Claim 5.3.10. — Property ii) of the map $f_3 : C \to K^{(3)}$ implies the following inequality for every 3-simplex $\Delta^3 \subset K^{(3)}$:

$$\operatorname{vol}(\Delta^3 \cap f_3(C)) \leq \sum_{V_j \cap V_{i(v)} \neq \emptyset} \operatorname{vol}(f_3(V_j)),$$

where $V_{i(v)}$ is the open set corresponding to a vertex v of Δ^3. The map f_3 is Lipschitz, and from property iii), its restriction to $\bigcup_{V_j \cap V_{i(v)} \neq \emptyset} V_j$ has Lipschitz constant $\xi_3/r_{i(v)}$. Hence, according to the formula giving a bound for the volume of the image of a

Lipschitz map (see [**Fed**, Cor. 2.10.11]), we get:

$$\sum_{V_j \cap V_{i(v)} \neq \emptyset} \operatorname{vol}(f_3(V_j)) \leq \sum_{V_j \cap V_{i(v)} \neq \emptyset} \operatorname{vol}(V_j) \left(\frac{\xi_3}{r_{i(v)}}\right)^3 .$$

Property 5) of a η-covering à la Gromov asserts that $\operatorname{vol}(V_j) \leq \eta\, r_j^3$. Furthermore, from property 2) of these coverings, we have $r_j \leq \frac{4}{3} r_{i(v)}$ whenever $V_j \cap V_{i(v)} \neq \emptyset$. Thus we deduce the following inequalities:

$$\operatorname{vol}(\Delta^3 \cap f_3(C)) \leq \sum_{V_j \cap V_{i(v)} \neq \emptyset} \operatorname{vol}(f_3(V_j)) \leq \eta \left(\frac{4}{3}\xi_3\right)^3 (N+1),$$

where N is the universal upper bound of the dimension of the covering given by Lemma 5.3.4. Hence it suffices to take $\eta_0 < \operatorname{vol}(\Delta^3)/((N+1)(\frac{4}{3}\xi_3)^3)$ to prove the claim. □

5.4. From (ε, D)-Margulis' coverings of abelian type to η-coverings à la Gromov

The aim of this section is to prove Proposition 5.2.5. We recall the statement:

Proposition 5.2.5. — *There exists a universal constant $D_0 > 0$ such that, if C is a closed hyperbolic cone 3-manifold where every point has an (ε, D)-Margulis' neighborhood of abelian type with $\varepsilon < 1/2$ and $D > D_0$, then the simplicial volume $\|C - \Sigma\|$ is zero.*

The proof follows from Proposition 5.3.3 and the following:

Proposition 5.4.1. — *There is a universal constant $b_0 > 0$ such that, if C is a closed hyperbolic cone 3-manifold where each point $x \in C$ has an (ε, D)-Margulis' neighborhood of abelian type with $\varepsilon \leq \frac{1}{2}$ and $D \geq 300$, then C admits a η-covering à la Gromov with $\eta < b_0/D$.*

Moreover, the open sets $(V_i)_{i\in I}$ of the η-covering à la Gromov satisfy the following additional properties:

6) *there is a tubular neighborhood $\mathcal{N}(\Sigma)$ of Σ such that every component of $\mathcal{N}(\Sigma)$ is contained in only one open set of the covering.*
7) $\forall i \in I$, $\displaystyle\bigcup_{V_j \cap V_{i(v)} \neq 0} V_j$ *is abelian in $C - \Sigma$.*

Proof of Proposition 5.2.5. — We choose $D_0 = \max(b_0/\eta_0, 300)$, where $\eta_0 > 0$ is the universal constant of Proposition 5.3.3. From Propositions 5.3.3 and 5.4.1, since every point of C has an (ε, D)-neighborhood of abelian type, we can construct a continuous map from C to a 2-dimensional simplicial complex: $f : C \to K^2$. Moreover properties i) and ii) of Proposition 5.3.3 together with properties 6) and 7) of Proposition 5.4.1 imply that f satisfies:

i′) there is an open tubular neighborhood $\mathcal{N}(\Sigma)$ of Σ such that $f(\overline{\mathcal{N}(\Sigma_i)})$ is a vertex of K^2 for every component Σ_i of Σ.

ii′) for every vertex v of K^2, $f^{-1}(\mathrm{star}(v))$ is abelian in $C - \Sigma$;

Let $C(\lambda_1, \dots, \lambda_q)$ denote the closed 3-manifold obtained by gluing q solid tori to the boundary of the manifold $C - \mathcal{N}(\Sigma)$, so that the boundaries of the meridian disks are identified respectively to the simple closed curves $\lambda_1, \dots, \lambda_q$ in $\partial\overline{\mathcal{N}(\Sigma)}$. More precisely,

$$C(\lambda_1, \dots, \lambda_q) = (C - \mathcal{N}(\Sigma)) \bigcup_{\phi_1, \dots, \phi_q} \bigsqcup_{i=1}^{q} S^1 \times D_i^2,$$

where the gluing maps $\phi_i : \partial\mathcal{N}(\Sigma_i) \to S^1 \times \partial D_i^2$ satisfy $\phi_i(\lambda_i) = (\{*\} \times \partial D_i^2)$, for $i = 1, \dots, q$.

From properties i′) and ii′), the continuous map $f : C \to K^2$ induces a map $\bar{f} : C(\lambda_1, \dots, \lambda_q) \to K^2$. Since abelianity is preserved by quotient, $\bar{f}^{-1}(\mathrm{star}(v))$ is abelian in $C(\lambda_1, \dots, \lambda_q)$, for every vertex v of K^2.

Since $\dim(K^2) = 2$, the closed orientable 3-manifold $C(\lambda_1, \dots, \lambda_q)$ admits an abelian covering of dimension 2 by the pull-back of the stars of the vertices of K^2. The vanishing theorem for simplicial volume (cf. [**Gro**, Sec. 3.1] and [**Iva**]) shows that $||C(\lambda_1, \dots, \lambda_q)|| = 0$. This holds for every choice of simple closed curves on $\partial\mathcal{N}(\Sigma)$. Thus, from Thurston's hyperbolic Dehn filling theorem [**Thu1**] (cf. Appendix B):

$$||C - \Sigma|| = \lim_{\mathrm{Lenght}(\lambda_i) \to \infty} ||C(\lambda_1, \dots, \lambda_q)|| = 0.$$

□

The remaining of this section is devoted to the proof of Proposition 5.4.1.

Proof of Proposition 5.4.1. — Let C be a closed hyperbolic cone 3-manifold so that every point $x \in C$ admits an (ε, D)-Margulis' neighborhood of abelian type, with $\varepsilon < \frac{1}{2}$ and $D > 300$. It means that x has a neihgborhood $U_x \subset C$ that is bilipschitz homeomorphic to the normal cone fiber bundle $\mathcal{N}_\nu(S)$, of some radius $\nu < 1$ depending on x, of the soul S of one of the following non-compact Euclidean cone 3-manifolds: $T^2 \times \mathbb{R}$, $S^1 \ltimes \mathbb{R}^2$, $S^1 \ltimes$ (cone disk). Moreover, the $(1+\varepsilon)$-bilipschitz homeomorphism $f : U_x \to \mathcal{N}_\nu(S)$ satisfies:

a) $\max\big(\mathrm{inj}(x), d(f(x), S), \mathrm{diam}(S)\big) \leq \frac{\nu}{D}$,

(cf. local soul theorem, Chapter 4, and Lemma 5.2.4).

For every point $x \in C$, we define the *abelianity radius* $\mathrm{ab}(x)$ to be:

$$\mathrm{ab}(x) = \sup\{r > 0 \mid B(x, r) \text{ is abelian in } C - \Sigma\}.$$

By using the $(1+\varepsilon)$-bilipschitz homeomorphism $f : U_x \to \mathcal{N}_\nu(S)$ and the upper bound a), we get:

$$\mathrm{ab}(x) \geq \frac{\nu}{1+\varepsilon}(1 - \frac{1}{D}) \geq \frac{\nu}{2} \geq \frac{D}{2}\mathrm{inj}(x).$$

For every $x \in C$ we define $r(x) = \inf(\frac{\mathrm{ab}(x)}{8}, 1)$. Lemmas 5.4.2 and 5.4.3 give the first properties of the balls $B(x, r(x))$.

Lemma 5.4.2. — *Let $x, y \in C$. If $B(x, r(x)) \cap B(y, r(y)) \neq \emptyset$, then*

b) $3/4 \leq r(x)/r(y) \leq 4/3$;

c) $B(x, r(x)) \subset B(y, 4r(y))$.

Proof. — Assume $r(x) \geq r(y)$. Either $r(y) = 1$ or $r(y) = \mathrm{ab}(y)/8$. If $r(y) = 1$, then $r(x) = 1$ and assertion b) is clear. If $r(y) = \mathrm{ab}(y)/8$, by using the inclusion $B(y, 6r(x)) \subset B(x, 8r(x))$ and the fact that $8r(x) \leq \mathrm{ab}(x)$, it follows that $B(y, 6r(x))$ is abelian in $C - \Sigma$. Hence $r(x) \leq \mathrm{ab}(y)/6 \leq 4r(y)/3$ and b) is proved.

Assertion c) follows easily from b) and the inclusion $B(x, r(x)) \subset B(y, 2r(x) + r(y))$. □

Lemma 5.4.3. — *Let $\Sigma = \Sigma_1 \sqcup \cdots \sqcup \Sigma_q$ be the singular set of C. We choose a point x_i in each connected component Σ_i. Then, we have the following properties:*

d) *for $i = 1, \ldots, q$, if $\mu > 0$ is sufficiently small, then $\mathcal{N}_\mu(\Sigma_i) \subset B(x_i, \frac{r(x_i)}{4})$, where $\mathcal{N}_\mu(\Sigma_i)$ is the tubular neighborhood of radius μ around the connected component Σ_i;*

e) $B(x_i, r(x_i)) \cap B(x_j, r(x_j)) = \emptyset$, *for $i \neq j$, $i, j \in \{1, \ldots, q\}$.*

Proof. — Property d) follows from the hypothesis that x_i has an (ε, D)-Margulis' neighborhood of abelian type. Since x_i is singular, the local model is the normal cone fiber bundle $\mathcal{N}_\nu(S)$, of radius $\nu < 1$, of the soul $S = S^1 \times \{\text{cone point}\}$ of the Euclidean cone 3-manifold $S^1 \ltimes$ (cone disk).

Let $f : U_{x_i} \to \mathcal{N}_\nu(S)$ be the $(1+\varepsilon)$-bilipschitz homeomorphism between U_{x_i} and the local model, then $U_{x_i} \cap \Sigma = \Sigma_i = f^{-1}(S) = f^{-1}(S^1 \times \{\text{cone point}\})$. Since $\varepsilon \leq 1/2$, it follows from the upper bound a) that $\mathrm{diam}(\Sigma_i) \leq \mathrm{diam}(S)(1+\varepsilon) \leq 2\frac{\nu}{D}$. Furthermore, since $\nu \leq \inf(1, 2\,\mathrm{ab}(x_i))$, $r(x_i) = \inf(1, \frac{\mathrm{ab}(x_i)}{8})$ and $D > 300$ we get:

$$\mathrm{diam}(\Sigma_i) \leq 2\frac{\nu}{D} \leq \inf(\frac{2}{D}, \frac{4\,\mathrm{ab}(x_i)}{D}) < r(x_i)/9,$$

Hence $\Sigma_i \subset B(x_i, \frac{r(x_i)}{9})$. By taking $\mu \leq \inf\{\frac{r(x_i)}{18} \mid i = 1, \ldots, q\}$ we obtain the inclusion $\mathcal{N}_\mu(\Sigma_i) \subset B(x_i, \frac{r(x_i)}{4})$.

To show property e), we assume that there are $i \neq j$ such that $B(x_i, r(x_i)) \cap B(x_j, r(x_j)) \neq \emptyset$ and we seek a contradiction. From property c) of Lemma 5.4.2, the fact that the balls intersect implies that $B(x_j, r(x_j)) \subset B(x_i, 4r(x_i))$. Hence, by property d), $\Sigma_i \cup \Sigma_j \subset B(x_i, 4r(x_i))$, which is an abelian ball in $C - \Sigma$. This implies that the two peripheral elements of $\pi_1(C - \Sigma)$ represented by the meridians of Σ_i and Σ_j commute. This contradicts the fact that $C - \Sigma$ admits a complete hyperbolic structure (Lemma 5.1.6) □

Construction of the η-covering à la Gromov. — First, we choose a point x_i on each connected component Σ_i of Σ, $i \in \{1, \ldots, q\}$. We fix the points $\{x_1, \ldots, x_q\}$ and we consider all the possible finite sequences of points $\{x_1, \ldots, x_q, x_{q+1}, \ldots, x_p\}$, starting with these fixed q points and having the property that

$$\text{the balls } B\big(x_n, \tfrac{r(x_n)}{4}\big) \text{ are pairwise disjoint.} \tag{4}$$

Note that a sequence satisfying (4) and Lemma 5.4.2 is necessarily finite because C is compact. The following lemma is due to Gromov [**Gro**, Sec. 3.4, Lemma B]:

Lemma 5.4.4. — *Let $x_1, \ldots, x_p$ be a finite sequence in C as above, with the first fixed q points in the singular set. If it is maximal for property (4), then the balls $B\big(x_1, \frac{2}{3}r(x_1)\big), \ldots, B\big(x_p, \frac{2}{3}r(x_p)\big)$ cover C.*

Proof. — Let $x \in C$. By maximality, the ball $B\big(x, \frac{r(x)}{4}\big)$ intersects $B\big(x_i, \frac{r(x_i)}{4}\big)$ for some $i \in \{1, \ldots, p\}$. From property b) of Lemma 5.4.2, $r(x) \leq \frac{4}{3}r(x_i)$ and thus $x \in B\big(x_i, \frac{r(x_i)+r(x)}{4}\big) \subset B\big(x_i, \frac{2}{3}r(x_i)\big)$. $\square$

Let $0 < \mu \leq \inf\{\frac{r(x_i)}{18} \mid i = 1, \ldots, q\}$ so that $\mathcal{N}_\mu(\Sigma_i) \subset B\big(x_i, \frac{r(x_i)}{4}\big)$, as in Lemma 5.4.3 d). Let $x_1, \ldots, x_p$ be a sequence as in Lemma 5.4.4, we consider the covering $(V_i)_{i \in \{1,\ldots,p\}}$ defined by:

$$\begin{cases} V_i = B(x_i, r(x_i)) & \text{for } i = 1, \ldots, q; \\ V_i = B(x_i, r(x_i)) - \mathcal{N}_\mu(\Sigma) & \text{for } i = q+1, \ldots, p. \end{cases} \tag{5}$$

The following Lemma finishes the proof of Proposition 5.4.1.

Lemma 5.4.5. — *There is a universal constant $b_0 > 0$ such that the above covering $(V_i)_{i \in \{1,\ldots,p\}}$ defined by (5) is a η-covering à la Gromov with $\eta < \frac{b_0}{D}$ and satisfies Properties* 6) *and* 7) *of Proposition 5.4.1.*

Proof of Lemma 5.4.5. — We start by checking that the covering satisfies properties 1) to 5) of a η-covering à la Gromov. Property 1) follows from the construction by setting $r_i = r(x_i)$, for $i = 1, \ldots, p$. Property 2) follows immediately from Lemma 5.4.2, and property 3) is the hypothesis 4.

Claim 5.4.6. — *The covering $(V_i)_{i \in \{1,\ldots,p\}}$ satisfies property 4) of a η-covering à la Gromov. That is, $\forall x \in C$ there is an open set V_i such that $x \in V_i$ and $d(x, \partial V_i) > r_i/3$.*

Proof. — Let $x \in C$. From Lemma 5.4.4, $x \in B(x_i, \frac{2}{3}r_i)$ for some $i = 1, \ldots, p$. If $i \in \{1, \ldots, q\}$ (i.e. if $x_i \in \Sigma$) or if $\mathcal{N}_\mu(\Sigma) \cap B(x_i, r_i) = \emptyset$, then by construction (5) $V_i = B(x_i, r_i)$ and we have $d(x, \partial V_i) \geq \frac{r_i}{3}$.

Thus we may assume that $\mathcal{N}_\mu(\Sigma) \cap B(x_i, r_i) \neq \emptyset$. Let $j \in \{1, \ldots, q\}$ be an index so that $\mathcal{N}_\mu(\Sigma_j) \cap B(x_i, r_i) \neq \emptyset$; we can also assume that $d(x, x_j) \geq \frac{2}{3}r_j$. By construction $V_i = B(x_i, r_i) - \mathcal{N}_\mu(\Sigma)$; hence it is enough to show that the distance of x to the component $\mathcal{N}_\mu(\Sigma_j)$ is at least $\frac{1}{3}r_i$ whenever $\mathcal{N}_\mu(\Sigma_j) \cap B(x_i, r_i) \neq \emptyset$.

Since $d(x, \Sigma_j) \geq d(x, x_j) - \operatorname{diam}(\Sigma_j)$, $d(x, x_j) \geq \frac{2}{3} r_j$ and $\operatorname{diam}(\Sigma_j) \leq \frac{r_j}{9}$ (by the proof of Lemma 5.4.3), we get $d(x, \Sigma_j) \geq \frac{5}{9} r_j$. Moreover, from property b) of Lemma 5.4.2, $\frac{4}{3} r_j \geq r_i$, hence:

$$d(x, \Sigma_j) \geq (\tfrac{4}{9} + \tfrac{1}{9}) r_j \geq \tfrac{1}{3} r_i + \tfrac{1}{9} r_j.$$

By the choice of $\mu \leq \inf\{\frac{1}{18} r_j \mid j = 1, \ldots, q\}$ we can conclude that $d(x, \mathcal{N}_\mu(\Sigma_j)) > \frac{1}{3} r_i$. Hence $d(x, \partial V_i) > \frac{1}{3} r_i$ and the claim is proved. □

Property 5) of a η-covering à la Gromov is given by the following claim:

Claim 5.4.7. — *There is a universal constant $b_0 > 0$ such that*

$$\operatorname{vol}(V_i) \leq \operatorname{vol}(B(x_i, r_i)) \leq \frac{b_0}{D} r_i^3, \qquad \textit{for } i = 1, \ldots, p.$$

Proof. — For $i = 1, \ldots, p$, x_i has an (ε, D)-Margulis' neighborhood of abelian type U_x which is $(1+\varepsilon)$-bilipschitz homeomorphic to the normal cone fiber bundle $\mathcal{N}_\nu(S)$, of radius $\nu < 1$, of the soul S of one of the following non-compact Euclidean cone 3-manifolds: $T^2 \times \mathbb{R}$, $S^1 \ltimes \mathbb{R}^2$, $S^1 \ltimes$ (cone disk). The $(1+\varepsilon)$-bilipschitz homeomorphism $f : U_{x_i} \to \mathcal{N}_\nu(S)$ satisfies $\varepsilon < 1/2$ and $\max(\operatorname{inj}(x_i), d(f(x_i), S), \operatorname{diam}(S)) \leq \nu/D$. From these inequalities we deduce that $\operatorname{ab}(x_i) \geq \nu/2$; hence $r_i \geq \nu/16$.

From Bishop-Gromov inequality (Proposition 3.1.9) we get:

$$\operatorname{vol}(B(x_i, r_i)) \leq \operatorname{vol}\left(B(x_i, \tfrac{\nu}{16})\right) \frac{\mathrm{v}_{-1}(r_i)}{\mathrm{v}_{-1}(\frac{\nu}{16})}.$$

Let $a > 0$ be a constant so that $t^3/a \leq \mathrm{v}_{-1}(t) \leq a t^3$ for every $t \in [0, 1]$. Since $\nu \leq 1$ and $r_i \leq 1$, we get:

$$\operatorname{vol}(B(x_i, r_i)) \leq \operatorname{vol}\left(B(x_i, \tfrac{\nu}{16})\right) 2^{12} a^2 \frac{r_i^3}{\nu^3}.$$

Since $d(f(x_i), S) \leq \frac{\nu}{D} \leq \frac{\nu}{300}$, we have the inclusion $f(B(x_i, \frac{\nu}{16})) \subset \mathcal{N}_\nu(S)$. Thus:

$$\operatorname{vol}\left(B(x_i, \tfrac{\nu}{16})\right) \leq (1+\varepsilon)^3 \operatorname{vol}(\mathcal{N}_\nu(S)) \leq 2^3 \operatorname{vol}(\mathcal{N}_\nu(S)),$$

because f is $(1+\varepsilon)$-bilipschitz with $\varepsilon < 1/2$.

By using the upper bound $\operatorname{diam}(S) \leq \nu/D$ and the fact that S is of dimension 1 or 2, a simple computation of Euclidean volumes gives the upper bound:

$$\operatorname{vol}(\mathcal{N}_\nu(S)) \leq 2 \frac{\pi}{D} \nu^3.$$

Thus:

$$\operatorname{vol}(V_i) \leq \operatorname{vol}(B(x_i, r_i)) \leq \frac{b_0}{D} r_i^3, \qquad \text{where } b_0 = 2^{16} \pi a^2.$$

□

Property 6) of Proposition 5.4.1 follows immediately from property d) of Lemma 5.4.3 and the construction of the covering.

Finally, property 7) of Proposition 5.4.1 follows from property c) of Lemma 5.4.2, because

$$\forall i = 1, \ldots, q, \qquad \bigcup_{V_j \cap V_i \neq \emptyset} V_j \subset \bigcup_{V_j \cap V_i \neq \emptyset} B(x_j, r_j) \subset B(x_i, 4r_i)$$

and by construction the ball $B(x_i, 4r_i)$ is abelian in $C - \Sigma$.

This finishes the proof of Proposition 5.4.1, and thus of Theorem A. □

5.5. (ε, D)-Margulis' neighborhood of thick turnover type

The main purpose of this section is to strength the statement of Theorem A by showing that in cases 1) and 2) the sequence $(C_n)_{n\in\mathbb{N}}$ does not collapse. It is a direct consequence of Proposition 5.2.1 and Proposition 5.5.1 below, which shows that case 2) of Theorem A cannot appear if the sequence $(C_n)_{n\in\mathbb{N}}$ collapses. From the proof of Proposition 5.2.1, it suffices to show that in the *non-compact collapse case* (i.e. when case 3 fails to hold), then no (ε, D)-Margulis' neighborhood with local model a thick turnover may appear.

Proposition 5.5.1 holds true for closed connected hyperbolic cone 3-manifolds with cone angles $\leq \pi$.

We use it twice: at the end of this section, to prove two Margulis's type results for closed hyperbolic cone 3-manifolds (Propositions 1 and 2), and also in Chapter 7, in the proof of the uniformization theorem for small 3-orbifolds (Theorem 2).

Proposition 5.5.1. — *Given $\omega > 0$, there is a constant $D_1 > 1$ (depending only on ω), such that if every point of a closed connected hyperbolic cone 3-manifold C with cone angles in $[\omega, \pi]$ admits a (ε, D)-Margulis' neighborhood, with $\varepsilon < 1/2$ and $D > D_1$, then no point of C admits such a (ε, D)-Margulis' neighborhood with local model a thick turnover $S^2(\alpha, \beta, \gamma) \times \mathbb{R}$, with $\alpha + \beta + \gamma = 2\pi$.*

The following is the key lemma for the proof of Proposition 5.5.1.

Lemma 5.5.2. — *Given $\omega > 0$, there is a constant $c = c(\omega)$ such that if a point x of a connected hyperbolic cone 3-manifold C with cone angles in $[\omega, \pi]$ admits a (ε, D)-Margulis' neighborhood with $\varepsilon < 1/2$, $D > 1$ and a local model of type $S^2(\alpha, \beta, \gamma) \times \mathbb{R}$ $(\alpha+\beta+\gamma = 2\pi)$, then, for some $t \in [-\nu/D, \nu/D]$, the preimage $f^{-1}(S^2(\alpha, \beta, \gamma) \times \{t\})$ is contained in the open ball $B(x, c\,\mathrm{inj}(x))$, where $f : U_x \to \mathcal{N}_\nu(S)$ is a $(1+\varepsilon)$-bilipschitz homeomorphism.*

Proof of Lemma 5.5.2. — To simplify the notation, we denote by S the Euclidean cone turnover $S^2(\alpha, \beta, \gamma)$. Since $d(f(x), S) \leq \nu/D$, there exists some $t \in [-\nu/D, \nu/D]$ such that $x \in f^{-1}(S \times \{t\})$. Hence, to prove the inclusion

$$f^{-1}(S \times \{t\}) \subset B(x, c\ \mathrm{inj}(x)),$$

it suffices to prove the inequality $\mathrm{diam}(f^{-1}(S \times \{t\})) \leq c\ \mathrm{inj}(x)$.

Since $\operatorname{diam}(f^{-1}(S \times \{t\})) \leq \frac{3}{2}\operatorname{diam}(S)$, we need only to show that

$$\operatorname{inj}(x) \geq e \operatorname{diam}(S),$$

where the constant $e = e(\omega) > 0$ depends only on ω. Then we can take $c = \frac{3}{2e}$.

In order to prove the inequality $\operatorname{inj}(x) \geq e\operatorname{diam}(S)$, for some constant $e = e(\omega) > 0$, we rescale the hyperbolic cone metric on C by $1/\operatorname{diam}(S)$. Let $\bar{C} = \frac{1}{\operatorname{diam}(S)}C$ be the rescaled cone 3-manifold with constant curvature $\bar{K} = -(\operatorname{diam}(S))^2$. Since $\operatorname{diam}(S) \leq \nu/D < 1$, the (constant) curvature $\bar{K}$ of $\bar{C}$ belongs to $[-1, 0)$. Let $\overline{\operatorname{inj}}(x)$ be the cone injectivity radius of the point $x \in \bar{C}$. Then the proof of Lemma 5.5.2 follows from the following claim:

Claim 5.5.3. — *There is a constant $e = e(\omega) > 0$ such that $\overline{\operatorname{inj}}(x) > e$.*

Proof of Claim 5.5.3. — We denote by $\bar{d}$ the distance on the rescaled cone 3-manifold $\bar{C} = \frac{1}{\operatorname{diam}(S)}C$. Let $y \in f^{-1}(S \times \{t\}) \subset \bar{C}$ be a singular point. Since $\bar{d}(x, y) \leq 2$, by Proposition 3.5.2 (lower bound for the cone injectivity radius) we have only to show that $\overline{\operatorname{inj}}(y) > e'$ for a constant $e' = e'(\omega) > 0$ depending only on ω.

Because of the $(1+\varepsilon)$-bilipschitz homeomorphism $f : U_x \to \mathcal{N}_\nu(S)$ and since y is a singular point, it is sufficient to get such a lower bound for the cone injectivity radius $\overline{\operatorname{inj}}(f(y))$, for $f(y)$ in $\frac{1}{\operatorname{diam}(S)}\mathcal{N}_\nu(S)$. In addition, since the cone 3-manifold $\frac{1}{\operatorname{diam}(S)}\mathcal{N}_\nu(S)$ is isometric to the product

$$\frac{1}{\operatorname{diam}(S)}S \times [-\frac{1}{\operatorname{diam}(S)}\nu, \frac{1}{\operatorname{diam}(S)}\nu],$$

it is sufficient to get a lower bound for $\overline{\operatorname{inj}}(f(y))$ in $\bar{S} = \frac{1}{\operatorname{diam}(S)}S$.

The Euclidean cone turnover $\bar{S}$ is obtained by doubling a Euclidean triangle $\bar{\Delta}$ along its boundary. The longest edge of $\bar{\Delta}$ has length $\operatorname{diam}(\bar{\Delta}) \geq \frac{1}{2}\operatorname{diam}(\bar{S}) = \frac{1}{2}$. Since the angles of $\bar{\Delta}$ belong to $[\omega/2, \pi/2]$, by elementary trigonometric formulas it follows that the two other edges of $\bar{\Delta}$ admit a uniform lower bound e'' depending only on the constant ω.

Since the point $f(y)$ is a vertex of $\bar{\Delta}$, the open ball $B(f(y), e'')$ is a standard singular ball in the Euclidean cone turnover $\bar{S}$. Therefore the cone injectivity radius $\overline{\operatorname{inj}}(f(y))$ satisfies: $\overline{\operatorname{inj}}(f(x)) > e''/2$. This finishes the proof of Claim 5.5.3 and thus of Lemma 5.5.2. □

The proof of Proposition 5.5.1 follows now readily from the following lemma:

Lemma 5.5.4. — *For $\omega > 0$, let $c = c(\omega)$ be the constant given by Lemma 5.5.2. Let C be a closed connected hyperbolic cone 3-manifold with cone angles in $[\omega, \pi]$. If every point of C admits a (ε, D)-Margulis' neighborhood, with $\varepsilon < 1/2$ and $D > D_1 = \max\{1, 2c\}$, then no point of C admits such a (ε, D)-Margulis' neighborhood with local model $S^2(\alpha, \beta, \gamma) \times \mathbb{R}$ $(\alpha + \beta + \gamma = 2\pi)$.*

Proof of Lemma 5.5.4. — We fix the constants $\varepsilon < 1/2$ and $D > D_1 = \max\{1, 2c\}$. Let A be the subset of points $x \in C$ admitting a (ε, D)-Margulis' neighborhood with local model $S^2(\alpha,\beta,\gamma) \times \mathbb{R}$ $(\alpha+\beta+\gamma = 2\pi)$. By hypothesis, A is an open subset of C. To show that A is a closed subset of C, let $(x_n)_{n\in\mathbb{N}} \subset A$ be a sequence of points that converges to a point $x_\infty \in C$. Since the point x_∞ admits by hypothesis a (ε, D)-Margulis'neighborhood U_{x_∞}, for n sufficiently large U_{x_∞} is also a (ε, D)-Margulis'neighborhood for the point $x_n \in A$. Then the following claim shows that U_{x_∞} must have as local model $S^2(\alpha,\beta,\gamma) \times \mathbb{R}$ $(\alpha+\beta+\gamma = 2\pi)$. Hence x_∞ belongs to A and A is a closed subset of C.

Claim 5.5.5. — *Every (ε, D)-Margulis' neighborhood of a point $x \in A$, with $\varepsilon < 1/2$ and $D > D_1 = max\{1, 2c\}$, has local model of type $S^2(\alpha,\beta,\gamma)\times\mathbb{R}$ with $\alpha+\beta+\gamma = 2\pi$.*

Proof of Claim 5.5.5. — Let U_x be a (ε, D)-Margulis' neighborhood of $x \in A$ with $\varepsilon < 1/2$ and $D > D_1 = \max\{1, 2c\}$. Then there is a $(1+\varepsilon)$-bilipschitz homeomorphism $f : U_x \to \mathcal{N}_\nu(S)$, where $\mathcal{N}_\nu(S)$ is the normal cone fibre bundle, with radius $\nu < 1$, of the soul S of a non-compact orientable Euclidean cone 3-manifold. We claim that the soul S is a Euclidean turnover.

By definition of Margulis' neighborhood, we have the following inequality:

$$\max(\mathrm{inj}(x), d(f(x), S), \mathrm{diam}(S)) \leq \nu/D.$$

This inequality together with the inequality $D > 2c$ imply that the open ball $B(x, c\ \mathrm{inj}(x))$ is included in U_x, because $(c\ \mathrm{inj}(x)) < \nu/2$ and $\varepsilon < 1/2$.

Since $x \in A$, it admits also a (ε, D)-Margulis' neighborhood with local model of type $S^2(\alpha,\beta,\gamma) \times \mathbb{R}$ $(\alpha+\beta+\gamma = 2\pi)$. By Lemma 5.5.2 the open ball $B(x, c\ \mathrm{inj}(x))$ contains the preimage $f^{-1}(S^2(\alpha,\beta,\gamma) \times \{t\})$, for some $t \in [-\nu/D, \nu/D]$. Hence U_x contains a cone turnover which is $(1+\varepsilon)$-bilipschitz homeomorphic to a Euclidean cone turnover. A quick inspection of the possible local models given by the local soul theorem in Chapter 4 shows that the local model of U_x has to be of type $S^2(\alpha',\beta',\gamma')\times\mathbb{R}$ with $\alpha'+\beta'+\gamma' = 2\pi$. □

Since the subset A is open and closed in the connected space C, either A is empty or $A = C$. The proof of Lemma 5.5.4 follows from the following claim:

Claim 5.5.6. — *The subset A cannot be equal to C.*

Proof of Claim 5.5.6. — Seeking a contradiction, we assume that $A = C$. Since C is compact, there is a finite covering $\{W_1, ..., W_n\}$ of C:

$$C = \bigcup_{i=1}^{n} W_i,$$

where $W_i = f_i^{-1}(S_i \times [-\nu_i/2D, \nu_i/2D])$, $f_i : U_i \to \mathcal{N}_{\nu_i}(S_i)$ is a $(1+\varepsilon)$-bilipschitz homeomorphism and $\mathcal{N}_{\nu_i}(S_i)$ is the normal cone fibre bundle, with radius $\nu_i < 1$, over a Euclidean cone turnover $S_i = S^2(\alpha_i,\beta_i,\gamma_i)$ with $\alpha_i+\beta_i+\gamma_i = 2\pi$.

By the proof of Claim 5.5.5, if

$$f_i^{-1}(S_i \times [-\nu_i/D, \nu_i/D]) \cap f_j^{-1}(S_j \times [-\nu_j/D, \nu_j/D]) \neq \emptyset,$$

then, for some $t_j \in [-\nu_j/D, \nu_j/D]$, $f_j^{-1}(S_j \times \{t_j\})$ is embedded in U_i. Since U_i is homeomorphic to a product $f_i^{-1}(S_i) \times [a, b]$, it follows that the essential non-separating spheres $f_i^{-1}(S_i)$ and $f_j^{-1}(S_j)$ are homotopic, hence isotopic in C by [**Lau**]. Therefore, by the connectedness of C, all the essential and non-separating spheres $f_i^{-1}(S_i)$, $i \in \{1, ..., n\}$, are parallel in C. Therefore C is homeomorphic to the product $S^2 \times S^1$ and each level 2-sphere meets the singular locus in exactly three points. This shows that the complement $C - \Sigma$ of the singular locus Σ fibers over the circle with fiber a three-punctured 2-sphere. Since the monodromy is, up to isotopy, of finite order, $C - \Sigma$ is Seifert fibered, which is impossible for a hyperbolic cone 3-manifold.

This concludes the proof of Claim 5.5.6, Lemma 5.5.4 and Proposition 5.5.1. □

As a consequence of Proposition 5.5.1 and Theorem A we prove now Propositions 1 and 2.

We prove first Proposition 1.

Proof of Proposition 1. — Seeking a contradiction, we assume that there is a sequence $\{C_n\}_{n \in \mathbb{N}}$ of closed orientable hyperbolic cone 3-manifolds with cone angles in $[\omega_0, \omega_1]$ and diameter ≥ 1, such that $(\sup\{\text{inj}(x) \mid x \in C_n\})_{n \in \mathbb{N}}$ goes to zero. By Theorem A together with Proposition 5.5.1, there is a subsequence that admits a *compact collapse*, corresponding to case 3) of Theorem A. In particular the diameter of the hyperbolic cone 3-manifolds in this subsequence goes to zero, contradicting the hypothesis. □

We prove now Proposition 2.

Proof of Proposition 2. — Seeking a contradiction, we assume that there is a sequence $\{C_n\}_{n \in \mathbb{N}}$ of closed orientable hyperbolic cone 3-manifolds with cone angles in $[\omega_0, \omega_1]$, for some $\omega_1 < 2\pi/3$, and such that $(\sup\{\text{inj}(x) \mid x \in C_n\})_{n \in \mathbb{N}}$ goes to zero. By Theorem A together with Proposition 5.5.1, there is a subsequence that admits a compact collapse, corresponding to case 3) of Theorem A. So the rescaled sequence $\frac{1}{\text{inj}(x_n)} C_n$ converges to a compact Euclidean cone manifold E with cone angles strictly less than $2\pi/3$. In particular, for n large enough, C_n has the same topological type as E.

As in Proposition 2.3.1, by using Hamilton's theorem one can show that there exists a closed spherical 3-orbifold $\mathcal{O}$ with the same topological type as E and with branching indices ≥ 3, because the cone angles of E are $< \frac{2\pi}{3}$. The orbifold $\mathcal{O}$ is Seifert fibered, because it is spherical and of cyclic type [**Dun1, Dun4**]. In addition, since the branching indices are ≥ 3, the singular locus Σ is a union of fibers. Thus the Seifert fibration on $\mathcal{O}$ induces a Seifert fibration of $C_n - \Sigma \cong \mathcal{O} - \Sigma$, which contradicts the hyperbolicity of $C_n - \Sigma$ proved in Lemma 5.1.6 and [**Koj**]. □

CHAPTER 6

VERY GOOD ORBIFOLDS AND SEQUENCES OF HYPERBOLIC CONE 3-MANIFOLDS

This chapter is devoted to the proof of Theorem B.

Theorem B. — *Let $\mathcal{O}$ be a closed orientable connected irreducible very good 3-orbifold with topological type $(|\mathcal{O}|, \Sigma)$ and ramification indices $n_1, \ldots, n_k$. Assume that there exists a sequence of hyperbolic cone 3-manifolds $(C_n)_{n\in\mathbb{N}}$ with the same topological type $(|\mathcal{O}|, \Sigma)$ and such that, for each component of Σ, the cone angles form an increasing sequence that converges to $2\pi/n_i$ when n approaches ∞.*

Then $\mathcal{O}$ contains a non-empty compact essential 3-suborbifold $\mathcal{O}' \subseteq \mathcal{O}$, which is not a product and which is either complete hyperbolic of finite volume, Euclidean, Seifert fibered or Sol.

We recall that a compact orientable 3-suborbifold $\mathcal{O}'$ is essential in a 3-orbifold $\mathcal{O}$ if the 2-suborbifold $\partial\mathcal{O}'$ is either empty or incompressible in $\mathcal{O}$.

The suborbifold $\mathcal{O}'$ of the theorem is not necessarily proper, it can be $\mathcal{O}' = \mathcal{O}$, but it is non-empty. By saying that $\mathcal{O}'$ is complete hyperbolic of finite volume we mean that its interior has a complete hyperbolic structure of finite volume. In particular, $\partial\mathcal{O}'$ is a collection of Euclidean 2-suborbifolds.

The proof of Theorem B splits into two cases, according to whether the sequence of cone 3-manifolds $(C_n)_{n\in\mathbb{N}}$ collapses or not, as in Theorem A. The non-collapsing case does not require the hypothesis very good, and this case will be used in the proof of Theorem B in the general case in next chapter.

6.1. The non-collapsing case

Next proposition proves Theorem B when the sequence of cone 3-manifolds $(C_n)_{n\in\mathbb{N}}$ does not collapse (i.e. $\sup\{\mathrm{inj}(x) \mid x \in C_n\}$ does not converge to zero). The proof of Proposition 6.1.1 does not use the fact that the orbifold $\mathcal{O}$ is very good.

Proposition 6.1.1. — *Let $\mathcal{O}$ and $(C_n)_{n\in\mathbb{N}}$ satisfy the hypothesis of Theorem B. If the sequence $(C_n)_{n\in\mathbb{N}}$ does not collapse, then $\mathcal{O}$ contains a non-empty compact essential 3-suborbifold that is complete hyperbolic of finite volume.*

Proof. — Since the sequence $(C_n)_{n\in\mathbb{N}}$ does not collapse, after passing to a subsequence if necessary, there is a constant $a > 0$ and, for every $n \in \mathbb{N}$, there is a point $x_n \in C_n$ such that $\mathrm{inj}(x_n) \geq a$. Thus, the sequence of pointed cone 3-manifolds (C_n, x_n) is contained in $\mathcal{C}_{[\omega_0,\pi],a}$, for some $\omega_0 > 0$, because the cone angles of C_n converge to angles of the form $2\pi/n_i$.

Since $(C_n, x_n) \in \mathcal{C}_{[\omega_0,\pi],a}$, by the compactness theorem (Chapter 3), after passing to a subsequence, we can assume that $(C_n, x_n)_{n\in\mathbb{N}}$ converges geometrically to a pointed hyperbolic cone 3-manifold (C_∞, x_∞). By hypothesis, the cone angles of C_∞ are of the form $2\pi/m$ with $m \in \mathbb{N}$, hence C_∞ is an orbifold. We distinguish two cases, according to whether the limit 3-orbifold C_∞ is compact or not.

If the limit 3-orbifold C_∞ is compact, then the geometric convergence implies that C_∞ has the same topological type (C, Σ) as the orbifold $\mathcal{O}$. Moreover, the branching indices of C_∞ agree with the ones of $\mathcal{O}$. Therefore as an orbifold $C_\infty = \mathcal{O}$ and $\mathcal{O}$ is a closed hyperbolic orbifold. Thus Proposition 6.1.1 is proved in this case.

If the limit C_∞ is not compact, then we need further work, as in Chapter 5. The first step is the following lemma.

Lemma 6.1.2. — *If the limit 3-orbifold C_∞ is not compact, then*

i) C_∞ has a finite volume,

ii) the ramification locus Σ_∞ of C_∞ has a non-compact component.

Proof. — Assertion i) is Lemma 5.1.3 and assertion ii) is Proposition 5.1.5, both of Chapter 5, whose proofs do not require the cone angles to be strictly less than π but only less than or equal to π. □

Hence, the non-compact orbifold C_∞ is hyperbolic with finite volume. Let $N_\infty \subset C_\infty$ be a compact core corresponding to the thick part of the orbifold. The thin part $C_\infty - N_\infty$ is a union of cusps of the form $F \times (0, +\infty)$, where F is an orientable closed 2-dimensional Euclidean orbifold. Moreover, since Σ_∞ is not compact, at least one of the cusps is singular.

Proposition 6.1.1, in the case where C_∞ is not compact, follows from the following one, because the compact core of C_∞ is not a product.

Proposition 6.1.3. — *Let $N_\infty \subset C_\infty$ be the compact core of the hyperbolic 3-orbifold C_∞. Then N_∞ embeds in $\mathcal{O}$ as an essential 3-suborbifold.*

Proof. — The geometric convergence implies that, for n sufficiently large, there is a $(1+\varepsilon_n)$bilipschitz embedding $f_n : (N_\infty, \Sigma_\infty \cap N_\infty) \to (C_n, \Sigma)$ with $\varepsilon_n \to 0$. Since the 3-orbifold $\mathcal{O}$ and the cone 3-manifolds C_n have the same topological type, we view the image $f_n(N_\infty)$ as a suborbifold of $\mathcal{O}$, which we denote by $N_n \subset \mathcal{O}$. The orbifold

N_n is homeomorphic to N_∞, thus N_n is an orbifold whose interior is hyperbolic of finite volume.

In Lemma 6.1.5 we are going to prove that ∂N_n is incompressible in $\mathcal{O}$, but before we need the following lemma.

Lemma 6.1.4. — *For n sufficiently large, the orbifold $\mathcal{O} - \mathrm{int}(N_n)$ is irreducible.*

Proof. — Seeking a contradiction, we assume that $\mathcal{O} - \mathrm{int}(N_n)$ contains a spherical 2-suborbifold F^2 which is essential. Since $\mathcal{O}$ is irreducible, F^2 bounds a spherical 3-orbifold Δ^3, which is the quotient of a standard 3-ball B^3 by the ortogonal action of a finite subgroup of $SO(3)$, and $N_n \subset \Delta^3$, for n sufficiently large. Since the ramification locus of $\mathcal{O}$ is a link, the only possibility is that Δ^3 is either a non-singular ball B^3 or its quotient by a finite cyclic group. Therefore the topological type of Δ^3 is (B^3, A), where $A = \Delta^3 \cap \Sigma$ is either empty or an unknotted proper arc.

Let $\rho_n : \pi_1(C_n - \Sigma, x_n) \to PSL_2(\mathbb{C})$ be the holonomy representation of C_n and let $f_n : (N_\infty, \Sigma_\infty \cap N_\infty) \to (C_n, \Sigma)$ be the $(1+\varepsilon_n)$-bilipschitz embedding such that $N_n = f_n(N_\infty)$. For n sufficiently large, the representation $\rho_n \circ f_{n*} : \pi_1(N_\infty - \Sigma_\infty, x_\infty) \to PSL_2(\mathbb{C})$ is either cyclic or trivial, since $N_n \subset \Delta^3$. Hence, the holonomy of C_∞ is abelian, because the geometric convergence implies the convergence of the holonomies (Proposition 3.5.4). This contradicts the fact that C_∞ is a complete hyperbolic orbifold of finite volume. □

Lemma 6.1.5. — *For n sufficiently large, the boundary ∂N_n is incompressible in $\mathcal{O}$.*

Proof. — Seeking a contradiction, we suppose that the lemma is not true. So, after passing to a subsequence if necessary, we can assume that ∂N_n is compressible in $\mathcal{O}$ and furthermore that $\mathcal{O} - \mathrm{int}(N_n)$ is irreducible (by Lemma 6.1.4). Let $F_1, \dots, F_p$ be the components of ∂N_∞. By passing again to a subsequence if necessary, we can assume moreover that the embedded components $f_n(F_1), \dots, f_n(F_q)$ are precisely the compressible ones, with $p \geq q$, where $f_n : (N_\infty, \Sigma_\infty \cap N_\infty) \to (C_n, \Sigma)$ is the $(1+\varepsilon_n)$-bilipschitz embedding defining N_n.

For $i = 1, \dots, q$, let λ_n^i be an essential curve on $f_n(F_i)$ which bounds a properly embedded disk in $\mathcal{O} - \mathrm{int}(N_n)$, intersecting Σ in at most one point. Consider the simple closed essential curves $\tilde\lambda_n^i = f_n^{-1}(\lambda_n^i) \subset F_i$, for $i = 1, \dots, q$.

Claim 6.1.6. — *For each $i = 1, \dots, q$, the sequence of simple closed essential curves $(\tilde\lambda_n^i)_{n \geq n_0}$ represents infinitely many different homotopy classes in the fundamental group $\pi_1(F_i)$.*

Proof. — If the claim is false, then, by passing to a subsequence and changing the indices of the F_i, we can suppose that the curves $\tilde\lambda_n^1$ represent a fixed class $\tilde\lambda^1 \in \pi_1(F_1)$ which does not depend on n. Let $\rho_n : \pi_1(C_n - \Sigma, x_n) \to PSL_2(\mathbb{C})$ be the holonomy representation of C_n and $\rho_\infty : \pi_1(C_\infty - \Sigma_\infty, x_\infty) = \pi_1(N_\infty - \Sigma_\infty, x_\infty) \to PSL_2(\mathbb{C})$ be the holonomy of C_∞. From the geometric convergence (Proposition 3.5.4):

$$\rho_\infty(\tilde{\lambda}^1) = \lim_{n\to\infty} \rho_n(f_n(\tilde{\lambda}^1_n)) = \lim_{n\to\infty} \rho_n(\lambda^1_n).$$

Since the curves λ^1_n are compressible in $\mathcal{O}$, their holonomies $\rho_n(\lambda^1_n)$ are either elliptic with bounded order or trivial. Moreover, since $\pi_1(F_1)$ is parabolic and $\tilde{\lambda}^1$ does not bound a discal 2-suborbifold in F_1, the holonomy $\rho_\infty(\tilde{\lambda}^1)$ is non-trivial and parabolic (cf. Appendix B, Lemma B.2.3). Thus we obtain a contradiction comparing $\rho_\infty(\tilde{\lambda}^1)$ with the limit of $\rho_n(\lambda^1_n)$. □

We come back to the proof of Lemma 6.1.5.

Since the orbifold $\mathcal{O} - \mathrm{int}(N_n)$ is irreducible, the toric 2-orbifolds $f_n(F_1), \dots, f_n(F_q)$ bound the quotient of solid tori in $\mathcal{O}$, that we denote by $V_1, \dots, V_q$.

For $i = 1, \dots, q$, the curve λ^i_n bounds a properly embedded discal 2-suborbifold in V_i. Since $\mathrm{int}(N_n) \cong \mathrm{int}(N_\infty)$ admits a complete hyperbolic structure, Claim 6.1.6 and the orbifold version of Thurston's hyperbolic Dehn filling theorem [**DuM**] (cf. Appendix B, §B.1) imply that, for n sufficiently large, the 3-orbifold

$$N_\infty(\tilde{\lambda}^1_n, \dots, \tilde{\lambda}^q_n) = \overline{N}_\infty \cup \bigsqcup_{i=1}^{q} V_i$$

obtained by Dehn filling along the curves $\tilde{\lambda}^1_n, \dots, \tilde{\lambda}^q_n$ is hyperbolic. In particular $N_\infty(\tilde{\lambda}^1_n, \dots, \tilde{\lambda}^q_n)$ has incompressible boundary.

Since the curves $\tilde{\lambda}^i_n = \bar{f}_n^{-1}(\lambda^i_n) \subset F_i$ represent infinitely many different homotopy classes in $\pi_1(F_i)$, it follows from Schläfli's formula for volume that the sequence $(N_\infty(\tilde{\lambda}^1_n, \dots, \tilde{\lambda}^q_n))_{n\in\mathbb{N}}$ contains infinitely many non-homeomorphic 3-orbifolds. We shall obtain a contradiction by showing that in fact all these orbifolds are homeomorphic to finitely many ones. For n large, the boundary $\partial N_\infty(\tilde{\lambda}^1_n, \dots, \tilde{\lambda}^q_n)$ is incompressible in $\mathcal{O}$, because $\mathcal{O} - \mathrm{int}(N_\infty(\tilde{\lambda}^1_n, \dots, \tilde{\lambda}^q_n))$ is irreducible with incompressible boundary and $N_\infty(\tilde{\lambda}^1_n, \dots, \tilde{\lambda}^q_n)$ is hyperbolic. Hence, this 3-suborbifold is a piece of the Bonahon-Siebenmann splitting of the 3-orbifold $\mathcal{O}$ [**BS1**]. Uniqueness of this splitting implies that the 3-orbifolds $N_\infty(\tilde{\lambda}^1_n, \dots, \tilde{\lambda}^q_n)$ are only finitely many. Hence we get the contradiction that proves Lemma 6.1.5 and therefore Proposition 6.1.3. □

6.2. The collapsing case

Next proposition proves Theorem B in the collapsing case.

Proposition 6.2.1. — *Let $\mathcal{O}$ and $(C_n)_{n\in\mathbb{N}}$ satisfy the hypothesis of Theorem B. If the sequence $(C_n)_{n\in\mathbb{N}}$ collapses, then $\mathcal{O}$ contains a non-empty compact essential 3-suborbifold, which is not a product and which is either Euclidean, Seifert or Sol.*

Proof. — Since the sequence $(C_n)_{n\in\mathbb{N}}$ collapses, by Corollary 4.4.1 of the local soul theorem (Chapter 4), either there is a subsequence that, after rescaling, converges to a closed Euclidean cone 3-manifold, or the local soul theorem, with any parameters $\varepsilon > 0$, $D > 0$ and non-compact local models, applies to every point $x \in C_n$. Thus we distinguish again two cases, according to whether we obtain a compact limit or not.

In the first case, for every $n \in \mathbb{N}$ there is $x_n \in C_n$ such that the sequence of rescaled cone 3-manifolds $(\bar{C}_n, x_n) = (\frac{1}{\mathrm{inj}(x_n)} C_n, x_n)$ has a subsequence that converges geometrically to a compact cone 3-manifold $(\bar{C}_\infty, x_\infty)$. The geometric convergence implies that $\bar{C}_\infty$ is a closed Euclidean 3-orbifold with the same topological type and the same branching indices as $\mathcal{O}$. Therefore, as an orbifold $C_\infty = \mathcal{O}$ and so $\mathcal{O}$ is Euclidean. Thus Proposition 6.2.1 holds in this case.

The second case, when we cannot find such a compact limit, is the difficult case to which the remaining of this chapter is devoted. Hence, from now on we suppose that the local soul theorem, with any parameters $\varepsilon > 0$, $D > 0$ and non-compact local models, applies to every point $x \in C_n$, for n sufficiently large.

Lemma 6.2.2. — *Under the hypothesis of the second case, for any $\varepsilon > 0$ and $D > 0$, there exists $n_0 > 0$ such that, for $n \geq n_0$, every $x \in C_n$ has an open neighborhood U_x $(1+\varepsilon)$-bilipschitz homeomorphic to the normal fiber bundle $\mathcal{N}_\nu(S)$, with some radius $\nu < 1$ depending on x, of the soul S of one of the following non-compact orientable Euclidean orbifolds:*

a) $T^2 \times \mathbb{R}$; $S^1 \ltimes \mathbb{E}^2$; $S^1 \ltimes D^2(\frac{2\pi}{p})$;

b) $S^2(\frac{2\pi}{p_1}, \frac{2\pi}{p_2}, \frac{2\pi}{p_3}) \times \mathbb{R}$, with $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$ (thick Euclidean turnover); $S^2(\pi,\pi,\pi,\pi) \times \mathbb{R}$ (thick pillow); the solid pillow;

c) $\mathbb{P}^2(\pi,\pi)\tilde{\times}\mathbb{R}$, which is the twisted orientable line bundle over $\mathbb{P}^2(\pi,\pi)$; and the quotient of $S^2(\pi,\pi,\pi,\pi) \times \mathbb{R}$ by an involution that gives the orientable bundle over $D^2(\pi,\pi)$, with silvered boundary (cf. Figure 1).

Moreover, if $f : U_x \to \mathcal{N}_\nu(S)$ is the $(1+\varepsilon)$-bilipschitz homeomorphism, then

$$\max(\mathrm{inj}(x), d(f(x), S), \mathrm{diam}(S)) \leq \nu/D.$$

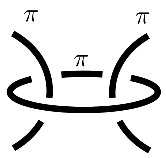

FIGURE 1

We recall that the solid pillow is the orbifold with underlying space $\mathbb{R}^3$ and branching set two straight lines of branching order 2 (cf. Figure 1). It is the quotient of $S^1 \ltimes \mathbb{R}^2$ by an involution.

Proof of Lemma 6.2.2. — From the hypothesis of the second case, for every $\varepsilon > 0$ and $D > 0$, there exists an n_0 such that for $n > n_0$ we can apply the local soul theorem, with parameters $\varepsilon > 0$, $D > 0$ and non-compact local models, to every point $x \in C_n$. Moreover from the hypothesis about the cone angles, the local models are Euclidean non-compact 3-orbifolds. Now it remains to eliminate the Euclidean 3-orbifolds model that are no listed in Lemma 6.2.2 (as in Lemma 5.2.4). Hence, by the local soul theorem, we only have to get rid of the twisted line bundle over the Klein bottle $K^2\tilde{\times}\mathbb{R}^1$ and the two models of Figure 2, which correspond to an orientable bundle over either an annulus or a Möbius strip, with silvered boundary in both cases.

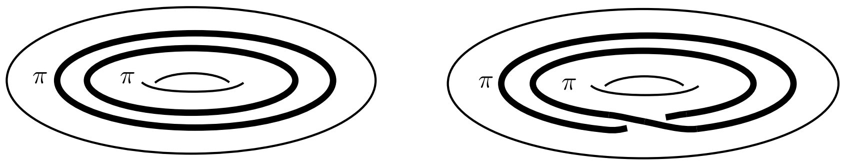

FIGURE 2

Let S be the soul of one of these three Euclidean non-compact orbifolds, and let $\mathcal{N}_r(S)$ denote its normal fiber bundle of radius r. Then $\partial\mathcal{N}_r(S)$ is an incompressible torus in $\mathcal{N}_r(S) - \Sigma$, for every $r > 0$. Therefore the appearance of one of these models would contradict the fact that $C - \Sigma$ is topologically atoroidal and not Seifert fibered. □

As in the previous chapter, the neighborhoods given by Lemma 6.2.2 are called (ε, D)-*Margulis' neighborhoods*. We remark that the (ε, D)-Margulis' neighborhoods of abelian type correspond to the local models listed in a). The local models listed in c) are Seifert fibered and different from a product. If neighborhoods corresponding to these local models appear, then the following lemma proves Proposition 6.2.1.

Lemma 6.2.3. — *If for some $n \geq n_0$ there is a point $x \in C_n$ having a (ε, D)-Margulis' neighborhood of type c) in Lemma 6.2.2, then the orbifold $\mathcal{O}$ contains a non-empty compact essential orientable 3-suborbifold $\mathcal{O}'$ which is Seifert fibered and different from a product.*

Proof of Lemma 6.2.3. — Let S be the soul of one of the Euclidean local models listed in c). This soul is either a projective plane with two cone points $\mathbb{P}^2(\pi,\pi)$ or a disk with two cone points and mirror boundary $\bar{D}^2(\pi,\pi)$. In both cases a regular neighborhood $\mathcal{N}(S)$ of S embeds as a compact suborbifold of $\mathcal{O}$. This suborbifold $\mathcal{O}' = \mathcal{N}(S)$ is Seifert fibered, it is not a product and its boundary $\partial\mathcal{O}' = S^2(\pi,\pi,\pi,\pi)$ is incompressible in $\mathcal{O}'$. It remains to show that either it is also incompressible in $\mathcal{O}$ or $\mathcal{O}$ is Seifert fibered itself.

First note that $\mathcal{O}-\mathrm{int}(\mathcal{O}')$ is irreducible, because the soul $S = \mathbb{P}^2(\pi,\pi)$ or $\bar{D}^2(\pi,\pi)$ cannot be contained in the quotient of a ball by a cyclic action. If $\partial\mathcal{O}'$ is compressible in $\mathcal{O}$, then the orbifold $\mathcal{O}-\mathrm{int}(\mathcal{O}')$ is irreducible with compressible boundary $\partial(\mathcal{O}-\mathrm{int}(\mathcal{O}')) = S^2(\pi,\pi,\pi,\pi)$. Therefore $\mathcal{O}-\mathrm{int}(\mathcal{O}')$ is a pillow and $\mathcal{O}$ is Seifert fibered. □

Lemma 6.2.3 shows that in applying Lemma 6.2.2 we only need to consider local models of types a) and b). The following lemma shows that we must consider local models of type b).

Lemma 6.2.4. — *There is a constant $D_0 > 0$ such that if every point of a closed hyperbolic cone 3-manifold C has a (ε, D)-Margulis' neighborhood of type a) or b), with $\varepsilon < 1/2$ and $D > D_0$, then at least one of the neighborhoods is of type b).*

Proof of Lemma 6.2.4. — By Proposition 5.2.5, there exists a uniform constant $D_0 > 0$ such that, if every point of a closed hyperbolic cone 3-manifold C has a (ε, D)-Margulis' neighborhood of type a) (abelian type), with $\varepsilon < 1/2$ and $D > D_0$, then the simplicial volume $||C-\Sigma|| = 0$. Therefore, there must be a point whose local model is of type b), because the fact that $C-\Sigma$ admits a complete hyperbolic structure (Lemma 5.1.6) implies that $||C-\Sigma|| \neq 0$. □

By using Lemmas 6.2.2, 6.2.3 and 6.2.4, Proposition 6.2.1 follows from Proposition 6.2.6 below:

Definition 6.2.5. — We say that a compact oriented irreducible 3-orbifold $\mathcal{O}$ is a *graph orbifold* if there exists a family of orientable Euclidean closed 2-suborbifolds that decompose $\mathcal{O}$ into Euclidean or Seifert fibered 3-suborbifolds. In particular, Euclidean, Seifert fibered and *Sol* 3-orbifolds are graph orbifolds.

Proposition 6.2.6. — *Let $(C_n)_{n\in\mathbb{N}}$ and $\mathcal{O}$ satisfy the hypothesis of Theorem B. There is a universal constant $D_1 > 0$ such that, if for some n every point of C_n admits a (ε, D)-Margulis' neighborhood of type a) or b), with $\varepsilon < 1/2$ and $D > D_1$, then $\mathcal{O}$ is a graph orbifold.*

Proof of Proposition 6.2.6. — Let $0 < \varepsilon < 1/2$ and $D > D_0$, where D_0 is the constant of Lemma 6.2.4. Assume that, for some n fixed, every point of the hyperbolic cone 3-manifold C_n has a (ε, D)-Margulis' neighborhood of type a) or b).

We choose a point $x_0 \in C_n$ having a (ε, D)-Margulis' neighborhood of type b). It means that x_0 has a neighborhood $U_{x_0} \subset C_n$ with a $(1+\varepsilon)$-bilipschitz homeomorphism $f_0 : U_{x_0} \to \mathcal{N}_{\nu_0}(S)$, where $\mathcal{N}_\nu(S)$ is the normal fiber bundle, with some radius $\nu_0 < 1$ depending onx_0, of the soul S of a non-compact Euclidean 3-orbifold of the family b).

Let $W_0 = f_0^{-1}\big(\overline{\mathcal{N}_{\nu_0/D}(S)}\big) \subset U_{x_0}$ be the inverse image of the closed normal fiber bundle of the soul S, with radius ν_0/D. As a suborbifold of $\mathcal{O}$, W_0 is diffeomorphic to either a thick Euclidean turnover, a thick pillow or a solid pillow.

We need the following proposition, that we shall prove in the next section. We recall that D_0 is the universal constant of Lemma 6.2.4; we can suppose $D_0 > 10^4$.

Proposition 6.2.7. — *Let $\mathcal{O}$ and $(C_n)_{n\in\mathbb{N}}$ satisfy the hypothesis of Theorem B. There is a universal constant $b_1 > 0$ such that, if for some n every point of C_n admits a (ε, D)-Margulis' neighborhood of type a) or b), with $\varepsilon < 1/2$ and $D > D_0 > 10^4$, then C_n admits a η-covering à la Gromov $(V_i)_{i\in I}$ with $\eta < b_1/D$.*

Moreover, there is a choice of x_0 and $W_0 \subset \mathcal{O}$ such that the covering $(V_i)_{i\in I}$ satisfies the additional properties:

6) *W_0 intersects only one open set V_i of the covering;*
7) *for every $i \in I$, $\bigcup_{V_j\cap V_i\neq\emptyset} V_j$ is virtually abelian in $\mathcal{O} - W_0$.*

We say that $U \subset \mathcal{O}$ is *virtually abelian in $\mathcal{O}-W_0$* if, for every connected component U' of $U-p^{-1}(W_0)$, the homomorphism of fundamental groups induced by the inclusion

$$i_* : \pi_1(U') \to \pi_1(\mathcal{O} - \mathrm{int}(W_0))$$

has a virtually abelian image (i.e. the image has a finite index abelian subgroup).

Proof of Proposition 6.2.6 assuming Proposition 6.2.7. — Let $\eta_0 > 0$ be the universal constant of Proposition 5.3.3. We choose $D_1 = \sup(b_1/\eta_0, 10^4)$. Proposition 6.2.7 of this chapter and Proposition 5.3.3 imply the existence of a continuous map $g : C \to K^2$, from C to a simplicial 2-complex K^2, such that:

i) $g(W_0)$ is a vertex of K^2;
ii) for every vertex v of K^2, $g^{-1}(\mathrm{star}(v))$ is virtually abelian in $\mathcal{O} - W_0$.

Since $\mathcal{O}$ is very good, there is a regular finite covering $p : M \to \mathcal{O}$ such that M is a closed 3-manifold. Set $\widetilde{W}_0 = p^{-1}(W_0) \subset M$. By composing g with the projection of the covering map $p : M \to \mathcal{O}$, we have a continuous map $f = g \circ p : M \to K^2$ with the following properties:

i) $f(\widetilde{W}_0)$ is a vertex v_0 of K^2;
ii) for every vertex v of K^2, $f^{-1}(\mathrm{star}(v))$ is virtually abelian in $M - \widetilde{W}_0$.

Now we use the map f to show that all Dehn fillings along the boundary of any connected component of $M - \mathrm{int}(\widetilde{W}_0)$ have simplicial volume zero. Let N be a connected component of $M - \mathrm{int}(\widetilde{W}_0)$. Its boundary ∂N is a union of tori. Let

$$\overline{N} = N \underset{\partial N}{\cup} \bigsqcup_{i=1}^{p} D^2 \times S^1$$

be any closed Dehn filling of N along ∂N. Since $f(\widetilde{W}_0)$ is a vertex v_0 of K^2, the map $f : M \to K^2$ induces a map $\bar{f} : \bar{N} \to K^2$ that coincides with f in N and maps each filling solid torus $D^2 \times S^1$ to the vertex v_0. By property ii), for every vertex $v \in K^2$, $\bar{f}^{-1}(\mathrm{star}(v))$ is virtually abelian in $\overline{N}$. Hence, the closed oriented 3-manifold

$\overline{N}$ admits a virtually abelian covering of dimension 2. By Gromov's vanishing theorem [**Gro**, Sec. 3.1] (cf [**Iva**]) the simplicial volume $||\overline{N}|| = 0$, as claimed.

Next step is the following lemma, whose proof is postponed to the end of the section.

Lemma 6.2.8. — *The 3-orbifold $\mathcal{O} - \mathrm{int}(W_0)$ is irreducible.*

Assuming this lemma, then any connected component N of $M - \mathrm{int}(\widetilde{W}_0)$ is irreducible, by the equivariant sphere theorem ([**DD**], [**MY1, MY2**], [**JR**]). If ∂N is compressible in N, then N is a solid torus because it is irreducible; in particular $||N|| = 0$. If the boundary ∂N is incompressible in N, then the following Lemma shows that the simplicial volume $||N|| = 0$ (cf. [**BDV**]). Therefore $||M - \mathrm{int}(\widetilde{W}_0)|| = 0$.

Lemma 6.2.9. — *Let N be an orientable compact irreducible and ∂-incompressible 3-manifold. Assume that ∂N is a disjoint union of tori and furthermore that any closed orientable 3-manifold $\overline{N}$, obtained from N by Dehn filling, has zero simplicial volume $||\overline{N}|| = 0$. Then N itself has zero simplicial volume $||N|| = 0$.*

Proof of Lemma 6.2.9. — Seeking a contradiction, we assume that $||N|| \neq 0$. According to Jaco-Shalen [**JS**] and Johannson [**Joh**], N splits along incompressible tori into Seifert and simple pieces. By Thurston's hyperbolization theorem the simple pieces are hyperbolic, hence they have non-zero simplicial volume ([**Gro**] and [**Thu1**, Ch. 6]). Since the simplicial volume is additive under gluing along incompressible tori ([**Gro**] and [**Som**]), the assumption that N has non-zero simplicial volume implies that at least one of these geometric pieces N_0 admits a complete hyperbolic structure of finite volume. We distinguish then two cases, according to whether ∂N_0 contains or not some components of ∂N.

In the first case, when $\partial N_0 \cap \partial N \neq \emptyset$, the contradiction is obtained by applying Thurston's hyperbolic surgery theorem (cf. Appendix B) to the components of ∂N_0 which belong to ∂N. This theorem implies that for some closed 3-manifolds $\bar{N}$ obtained by Dehn fillings of N along ∂N, the induced Dehn fillings $\bar{N}_0$ of N_0 along $\partial N_0 \cap \partial N$ give an essential complete hyperbolic submanifold of finite volume in $\bar{N}$. The contradiction then follows from the additivity of the simplicial volume along incompressible tori [**Gro, Som**].

In the second case, when $\partial N_0 \cap \partial N = \emptyset$, the contradiction is obtained by using the fact that ∂N_0 remains incompressible in infinitely many closed Dehn fillings $\bar{N}$ of N along ∂N (by [**CGLS**, Thm. 2.0.1] and [**Gor**, Lemma 7.2]). In particular N_0 would be an essential complete hyperbolic submanifold of finite volume in $\bar{N}$. This would contradict the fact that $||\bar{N}|| = 0$, as in the previous case. □

We prove now that $M - \mathrm{int}(\widetilde{W}_0)$ is a graph manifold. From the proof of Lemma 6.2.9 and the fact that $||M - \mathrm{int}(\widetilde{W}_0)|| = 0$ all the pieces in the Jaco-Shalen and Johannson splitting of $M - \mathrm{int}(\widetilde{W}_0)$ are Seifert fibered; therefore $M - \mathrm{int}(\widetilde{W}_0)$ is a graph manifold.

We deduce that $\mathcal{O} - W_0$ is a graph orbifold, by using the regular covering $p : M - \mathrm{int}(\widetilde{W}_0) \to \mathcal{O} - W_0$ and the results of Meeks and Scott [**MS**], which provide a graph structure on $M - \mathrm{int}(\widetilde{W}_0)$ invariant by the action of the deck transformations group of the covering.

Since the 3-suborbifold $W_0 \subset \mathcal{O}$ is either a thick Euclidean turnover, a thick pillow or a solid pillow, $\mathcal{O}$ admits a graph structure.

This proves Proposition 6.2.6 from Proposition 6.2.7 and Lemma 6.2.8. The proof of Proposition 6.2.7 is given in the next section and the proof of Lemma 6.2.8 comes now.

Proof of Lemma 6.2.8. — Seeking a contradiction, we assume that $\mathcal{O} - W_0$ is reducible. It means that there exists an essential spherical 2-suborbifold $F^2 \subset \mathcal{O}$. Since $\mathcal{O}$ is irreducible, F^2 bounds a discal 3-suborbifold D^3 in $\mathcal{O}$ and D^3 contains W_0. Since the branching locus $\Sigma \subset \mathcal{O}$ is a link, the discal suborbifold D^3 is the quotient of a ball by a finite cyclic orthogonal action. Hence the topological type of D^3 is (B^3, A), where $A = B^3 \cap \Sigma$ is a proper unknotted arc in the ball B^3.

Since $W_0 \subset \mathrm{int}(D^3)$, we already have a contradiction in the case where $W_0 = \mathcal{N}(S^2(\frac{2\pi}{p_1}, \frac{2\pi}{p_2}, \frac{2\pi}{p_3}))$ is a thick Euclidean turnover, because there is no way to embed a 2-sphere in B^3 that intersects A in 3 points.

Hence we assume that W_0 is either a thick pillow or a solid pillow. In both cases, $\Sigma \cap W_0$ is not connected and we find a contradiction using a Dirichlet polyhedron and the fact that $A = B^3 \cap \Sigma$ is connected. More precisely, these local models imply that there is a metric ball $B(x, r) \subset W_0 \subset C_n$ such that $B(x, r) \cap \Sigma$ is not connected. We consider the Dirichlet polyhedron P_x of C_n centered at x. This polyhedron is convex, because the cone angles of C_n are equal to or less than π. By convexity, different connected components of $B(x, r) \cap \Sigma$ give different edges of ∂P_x that belong to different geodesics of $\mathbb{H}^3$. In particular, the holonomy of the meridians of different components of $B(x, r) \cap \Sigma$ are not contained in a cyclic group. This contradicts the inclusion $(W_0, \Sigma \cap W_0) \subset (D^3, \Sigma \cap D^3)$, because $\pi_1(D^3 - \Sigma) \cong \pi_1(B^3 - A)$ is cyclic. Thus we get a contradiction and the lemma is proved. $\square$

6.3. From (ε, D)-Margulis' coverings of type a) and b) to η-coverings à la Gromov

This section is devoted to the proof of Proposition 6.2.7, which constructs the required η-covering à la Gromov.

We recall that we had applied the local soul theorem (Chapter 4) to the hyperbolic cone 3-manifold C_n with parameters (ε, D).

For any point $x_0 \in C_n$ with a (ε, D)-Margulis'neighborhood of type b), there is a neighborhood U_{x_0} and a $(1+\varepsilon)$-bilipschitz homeomorphism $f_0 : U_{x_0} \to \mathcal{N}_{\nu_0}(S)$ where $\mathcal{N}_{\nu_0}(S)$ is a normal fiber bundle, with some radius $\nu_0 \leq 1$ depending on x_0, of the soul S of a non-compact Euclidean cone 3-manifold of type b). We have defined $W_0 = f_0^{-1}\big(\overline{\mathcal{N}_{\nu_0/D}(S)}\big) \subset U_{x_0}$.

We want to prove the following proposition:

Propoisiton 6.2.7. — *Let $\mathcal{O}$ and $(C_n)_{n\in\mathbb{N}}$ satisfy the hypothesis of Theorem B. There is a universal constant $b_1 > 0$ such that, if for some n every point of C_n admits a (ε, D)-Margulis' neighborhood of type a) or b), with $\varepsilon < 1/2$ and $D > D_0 > 10^4$, then C_n admits a η-covering à la Gromov $(V_i)_{i\in I}$ with $\eta < b_1/D$.*

Moreover, there is a choice of x_0 and $W_0 \subset \mathcal{O}$ such that the covering $(V_i)_{i\in I}$ satisfies the additional properties:

6) *W_0 intersects only one open set V_i of the covering;*
7) *for every $i \in I$, $\bigcup_{V_j\cap V_i\neq\emptyset} V_j$ is virtually abelian in $\mathcal{O} - W_0$.*

Proof of Proposition 6.2.7. — In the proof we set $C_n = C$ to simplify notation.

First we describe the choices of $x_0 \in C$ and W_0. Given $\varepsilon > 1/2$ and $D > D_0 > 10^4$, we consider

$$T_{(\varepsilon,D)} = \left\{ x \in C \;\middle|\; \begin{array}{l} x \text{ admits an } (\varepsilon, D)\text{-Margulis'} \\ \text{neighborhood of type b)} \end{array} \right\}$$

Since $D > D_0$, Lemma 6.2.4 implies that $T_{(\varepsilon,D)} \neq \emptyset$. For $x \in T_{(\varepsilon,D)}$, let U_x denote the (ε, D)-Margulis' neighborhood of type b) and let $f : U_x \to \mathcal{N}_{\nu(x)}(S)$ be the $(1+\varepsilon)$-bilipschitz homeomorphism between U_x and the normal fiber bundle, with radius $\nu(x) \le 1$, of the compact soul S of a local model of type b). We choose a point $x_0 \in T_{(\varepsilon,D)}$ such that

$$\nu(x_0) = \nu_0 \ge \frac{1}{1+\varepsilon}\sup\{\nu(x) \mid x \in T_{(\varepsilon,D)}\}.$$

Let $W_0 = f_0^{-1}\big(\overline{\mathcal{N}_{\nu_0/D}(S)}\big) \subset U_{x_0}$ be the inverse image of a closed normal neighborhood of the soul S of radius ν_0/D, where $f_0 : U_{x_0} \to \mathcal{N}_{\nu_0}(S)$ is the $(1+\varepsilon)$-bilipschitz homeomorphism.

For every $x \in C$ we define the virtual abelianity radius (relative to $W_0 \subset \mathcal{O}$):

$$\operatorname{vab}(x) = \sup\{r \in \mathbb{R} \mid B(x,r) \text{ is virtually abelian in } \mathcal{O} - W_0\}.$$

We set $r(x) = \inf\{1, \frac{\operatorname{vab}(x)}{8}\}$.

This definition is analogous to the one given in Section 5.4. For instance, the following lemma has the same proof as Lemma 5.4.2:

Lemma 6.3.1. — *Let $x, y \in C$. If $B(x, r(x)) \cap B(y, r(y)) \neq \emptyset$, then*

a) $3/4 \le r(x)/r(y) \le 4/3$;
b) $B(x, r(x)) \subset B(y, 4r(y))$. □

Lemma 6.3.2. — *For every $x_0 \in W_0$, $W_0 \subset B(x_0, \frac{r(x_0)}{9})$.*

Proof. — This lemma will follow from the inequality

$$\operatorname{diam}(W_0) < r(x_0)/9.$$

Since $W_0 = f_0^{-1}\big(\overline{\mathcal{N}_{\nu_0/D}(S)}\big) \subset U_{x_0}$, where $f_0 : U_{x_0} \to \mathcal{N}_{\nu_0}(S)$ is a $(1+\varepsilon)$-bilipschitz homeomorphism, and $\mathcal{N}_{\nu_0}(S)$ is a normal fiber bundle, with radius ν_0, of the soul S of a non-compact Euclidean cone 3-manifold of type b), we have:

$$\operatorname{diam}(W_0) \le (1+\varepsilon)\operatorname{diam}\big(\mathcal{N}_{\frac{\nu_0}{D}}(S)\big) \le (1+\varepsilon)(\operatorname{diam}(S) + 2\,\frac{\nu_0}{D}) \le 6\,\frac{\nu_0}{D},$$

because $\operatorname{diam}(S) \le \frac{\nu_0}{D}$ and $\varepsilon \le 1/2$. By definition $\operatorname{vab}(x_0) \ge \frac{1}{1+\varepsilon}(\nu_0 - \frac{\nu_0}{D}) \ge \frac{\nu_0}{2}$, moreover $\nu_0 \le 1$ and $D > 10^4$, thus we obtain the following inequalities:

$$\operatorname{diam}(W_0) \le 6\,\frac{\nu_0}{D} \le \inf\{\frac{6}{D}, \frac{12\operatorname{vab}(x_0)}{D}\} < \frac{r(x_0)}{9}.$$

$\square$

Now we give the construction of the η-covering à la Gromov. We fix a point $x_0 \in W_0$, we consider then all the possible finite sequences $\{x_0, x_1, \dots, x_p\}$, starting with x_0, such that:

$$\text{(6)} \qquad \text{the balls } B\Big(x_0, \frac{r(x_0)}{4}\Big), \dots, B\Big(x_p, \frac{r(x_p)}{4}\Big) \text{ are pairwise disjoint.}$$

A sequence satisfying (6) and Lemma 6.3.1 is finite by compactness. Moreover we have the following property, proved in Chapter 5, Lemma 5.4.4.

Lemma 6.3.3. — *If the sequence $\{x_0, x_1, \dots, x_p\}$ is maximal for property (6), then the balls $B(x_0, \frac{2}{3}r(x_0)), \dots, B(x_p, \frac{2}{3}r(x_p))$ cover C.* $\square$

Given a sequence $\{x_0, x_1, \dots, x_p\}$, maximal for property (6) and starting with $x_0 \in W_0$, we consider the covering of C by the following open sets:

$$\begin{cases} V_0 = B(x_0, r(x_0)) \\ V_i = B(x_i, r(x_i)) - W_0, \qquad \text{for } i = 1, \dots, p. \end{cases}$$

Next lemma concludes the proof of Proposition 6.2.7.

Lemma 6.3.4. — *There is a universal constant $b_1 > 0$ such that, for $\varepsilon < 1/2$ and $D > 10^4$, the open sets $V_0, \dots, V_p$ define a η-covering à la Gromov of C, with $\eta < b_1/D$. Moreover this covering satisfies properties 6) and 7) of Proposition 6.2.7.*

Proof. — Lemmas 6.3.2 and 6.3.3 guarantee that the open sets $V_0, \dots, V_p$ cover C. Then by setting $r_i = r(x_i)$ for $i = 1, \dots, p$, properties 1), 2) and 3) of a η-covering à la Gromov follow from the construction and Lemma 6.3.1.

Next claim shows that the covering $(V_i)_{i\in\{0,\dots,p\}}$ satisfies also property 4).

Claim 6.3.5. — *For every $x \in C$ there is an open set V_i, with $i \in \{0, \dots, p\}$, such that $x \in V_i$ and $d(x, \partial V_i) > r_i/3$*

Proof of Claim 6.3.5. — Let $x \in C$, then by Lemma 6.3.3 $x \in B(x_i, \frac{2}{3}r_i)$ for some $i \in \{0, \dots, p\}$; we fix this index i. If $i = 0$ or if $B(x_i, r_i) \cap W_0 = \emptyset$, then $V_i = B(x_i, r_i)$ and the lemma holds. Hence we may assume that $i > 0$ and $B(x_i, r_i) \cap W_0 \neq \emptyset$. Moreover, we can suppose $d(x, x_0) > \frac{2}{3}r_0$. In this case $V_i = B(x_i, r_i) - W_0$ and we claim that $d(x, W_0) > \frac{1}{3}r_i$.

To prove this claim, we use the inequality:

$$d(x, W_0) \geq d(x, x_0) - \operatorname{diam}(W_0) > \tfrac{2}{3}r_0 - \tfrac{2}{9}r_0 = \tfrac{4}{9}r_0, \tag{7}$$

which holds true because $d(x, x_0) > \frac{2}{3}r_0$ by assumption, and $\operatorname{diam}(W_0) < \frac{2}{9}r_0$ by Lemma 6.3.2. Since $B(x_0, r_0) \cap B(x_i, r_i) \neq \emptyset$, Lemma 6.3.1 implies that $r_0 \geq \frac{3}{4}r_i$. Hence inequality (7) becomes $d(x, W_0) > \frac{1}{3}r_i$ and the claim is proved. $\square$

Before proving property 5) of a η-covering à la Gromov, we point out that property 6) of Proposition 6.2.7 is satisfied by construction and Lemma 6.3.2. Moreover property 7) follows from Lemma 6.3.1 and the fact that the balls $B(x_i, 4r_i)$ are virtually abelian in $\mathcal{O} - W_0$.

Next claim proves property 5) of a η-covering à la Gromov and completes the proof of Proposition 6.2.7.

Claim 6.3.6. — *There exists a universal constant $b_1 > 0$ such that*

$$\operatorname{vol}(V_i) \leq \operatorname{vol}\big(B(x_i, r_i)\big) \leq \frac{b_1}{D} r_i^3, \qquad \forall i = 0, \dots, p$$

Proof of Claim 6.3.6. — To estimate the volume of $B(x_i, r_i)$ we use the same method as in Claim 5.4.7 of Chapter 5. To fix notation, for $i = 0, \dots, p$, let $f_i : U_{x_i} \to \mathcal{N}_{\nu_i}(S_i)$ be the $(1+\varepsilon)$-bilipschitz homeomorphism given by the local soul theorem (Chapter 4).

We need the following technical claim, whose proof is postponed to the end of the section.

Claim 6.3.7. — *For $i = 0, \dots, p$, let ν_i denote the radius of the normal fiber bundle of the soul of the Euclidean local model given by the local soul theorem. Then $r_i > \nu_i/2^{11}$.*

Assuming that Claim 6.3.7 holds true, we can compare the volumes of the balls $B(x_i, r_i)$ and $B(x_i, \nu_i/2^{11})$. Since $r_i > \nu_i/2^{11}$, by Bishop-Gromov inequality (Proposition 3.1.9) we get:

$$\operatorname{vol}\big(B(x_i, r_i)\big) \leq \operatorname{vol}\big(B(x_i, \frac{\nu_i}{2^{11}})\big) \frac{\mathrm{v}_{-1}(r_i)}{\mathrm{v}_{-1}(\nu_i/2^{11})},$$

where $\mathrm{v}_{-1}(t) = \pi(\sinh(2t) - 2t)$.

As in Claim 5.4.7 of Chapter 5, let $a > 0$ be a constant such that $t^3/a \leq \mathrm{v}_{-1}(t) \leq a\,t^3$ for every $t \in [0, 1]$. Since $\nu_i \leq 1$ and $r_i \leq 1$, we get:

$$\operatorname{vol}\big(B(x_i, r_i)\big) \leq \operatorname{vol}\big(B(x_i, \frac{\nu_i}{2^{11}})\big) a^2 2^{33} \frac{r_i^3}{\nu_i^3}.$$

Since $d(f_i(x_i), S_i) \leq \nu_i/D < \nu_i\, 10^{-4}$, we have that $f_i\big(B(x_i, \nu_i/2^{11})\big) \subset \mathcal{N}_{\nu_i}(S_i)$. Thus

$$\operatorname{vol}\big(B(x_i, \frac{\nu_i}{2^{11}})\big) \leq (1+\varepsilon)^3 \operatorname{vol}(\mathcal{N}_{\nu_i}(S_i)) \leq 2^3 \operatorname{vol}(\mathcal{N}_{\nu_i}(S_i)),$$

because f_i is $(1+\varepsilon)$-bilipschitz, with $\varepsilon \leq 1/2$.

Using the bound $\operatorname{diam}(S_i) \leq \nu_i/D$ and the fact that the dimension of the soul S_i is 1 or 2, we easily get the upper bound $\operatorname{vol}(\mathcal{N}_{\nu_i}(S_i)) \leq (2\pi/D)\nu_i^3$, as in Claim 5.4.7. Hence

$$\operatorname{vol}\big(B(x_i, r_i)\big) \leq \frac{b_1}{D} r_i^3, \qquad \text{with } b_1 = 2^{37} a^2 \pi.$$

$\square$

Finally the proof of Claim 6.3.7 concludes the proof of Proposition 6.2.7.

Proof of Claim 6.3.7. — For $i = 0, \ldots, p$, let $f_i : U_{x_i} \to \mathcal{N}_{\nu_i}(S_i)$ be the $(1+\varepsilon)$-bilipschitz homeomorphism given by the local soul theorem (Chapter 4). We recall the upper bound

$$\max(\operatorname{inj}(x), d(f_i(x_i), S_i), \operatorname{diam}(S_i)) \leq \frac{\nu_i}{D}.$$

If $i = 0$, then it is clear that $\operatorname{vab}(x_0) \geq \nu_0/2$; thus $r_0 \geq \nu_0/16$, because $\nu_0 \leq 1$.

If $i \geq 1$, then $\operatorname{vab}(x_i) \geq \inf\big(\frac{1}{1+\varepsilon}\nu_i(1 - \frac{1}{D}), d(x_i, W_0)\big)$, because W_0 can intersect the neighborhood U_{x_i}. Since $\varepsilon < 1/2$ and $D > 10^4$, this inequality becomes

$$\operatorname{vab}(x_i) \geq \inf\big(\frac{\nu_i}{2}, d(x_i, W_0)\big).$$

Now we want to find a lower bound for $d(x_i, W_0)$.

Since $d(x_i, x_0) > \frac{r_0}{4}$ by the choice of the sequence $x_0, \ldots, x_p$ (property (6) above) and since $\operatorname{diam}(W_0) \leq \frac{6\nu_0}{D}$ by the proof of Lemma 6.3.2, first we get the following lower bound:

$$d(x_i, W_0) \geq d(x_i, x_0) - \operatorname{diam}(W_0) > \frac{r_0}{4} - \frac{6\nu_0}{D},$$

Moreover, $d(x_i, W_0) > \nu_0(\frac{1}{64} - \frac{6}{D}) > \frac{\nu_0}{128}$, because $r_0 \geq \nu_0/16$. Therefore, since ν_0 and $\nu_i \leq 1$, we obtain:

$$r_i \geq \frac{1}{8}\operatorname{vab}(x_i) \geq \inf\Big(\frac{\nu_i}{2^4}, \frac{\nu_0}{2^{10}}\Big).$$

To compare ν_0 and ν_i we distinguish two cases, according to whether the local model for U_{x_i} is of type a) or b).

If the local model for U_{x_i} is of type b), then by the choice of x_0, we have $\nu_0 \geq \nu_i/2$, hence $r_i \geq \nu_i/2^{11}$.

When the local model for U_{x_i} is of type a), again we distinguish two cases according to whether the intersection $f_i^{-1}(\mathcal{N}_{\nu_i/8}(S_i)) \cap W_0$ is empty or not.

If $f_i^{-1}(\mathcal{N}_{\nu_i/8}(S_i)) \cap W_0 = \emptyset$, then $f_i^{-1}(\mathcal{N}_{\nu_i/8}(S_i))$ is virtually abelian in $\mathcal{O} - W_0$ and we have

$$\operatorname{vab}(x_i) \geq d(x_i, \partial f_i^{-1}(\mathcal{N}_{\nu_i/8}(S_i))) \geq \frac{1}{1+\varepsilon}\left(\frac{\nu_i}{8} - d(f_i(x_i), S_i)\right) \geq \frac{1}{1+\varepsilon}\left(\frac{\nu_i}{8} - \frac{\nu_i}{D}\right) > \frac{\nu_i}{16}$$

and we conclude that $r_i > \nu_i/128$.

If $f_i^{-1}(\mathcal{N}_{\nu_i/8}(S_i)) \cap W_0 \neq \emptyset$, then there exists $y_0 \in W_0$ such that $d(y_0, f_i^{-1}(S_i)) \leq (1+\varepsilon)\nu_i/8 < \nu_i/4$. Hence,

$$\text{for every } x \in W_0, \quad d(x, f_i^{-1}(S_i)) \leq d(y_0, f_i^{-1}(S_i)) + \operatorname{diam}(W_0) \leq \frac{\nu_i}{4} + \frac{6\nu_0}{D}.$$

Since W_0 corresponds to a (ε, D)-Margulis neighborhood of type b), it cannot be contained in a (ε, D)-Margulis neighborhood of type a). In particular, W_0 cannot be contained in $f_i^{-1}(\mathcal{N}_{\nu_i}(S_i))$ and we have:

$$\frac{\nu_i}{4} + \frac{6\nu_0}{D} > \frac{\nu_i}{1+\varepsilon} > \frac{\nu_i}{2}.$$

We deduce that $\nu_0 \geq D\,\nu_i/24 > 32\,\nu_i$, because $D > 10^4$. Thus

$$r_i \geq \inf\left(\frac{\nu_i}{2^4}, \frac{\nu_0}{2^{10}}\right) \geq \nu_i/32$$

and the claim is proved.

This also concludes the proof of Proposition 6.2.7. $\square$

CHAPTER 7

UNIFORMIZATION OF SMALL 3-ORBIFOLDS

An orientable compact 3-orbifold $\mathcal{O}$ is *small* if it is irreducible, its boundary $\partial\mathcal{O}$ is a (perhaps empty) collection of turnovers, and $\mathcal{O}$ does not contain any essential embedded closed orientable 2-suborbifold.

Remark 7.0.1. — If the boundary of a small 3-orbifold $\mathcal{O}$ is not empty, then either $\mathcal{O}$ is a dical 3-orbifold or $\partial\mathcal{O}$ is an union of Euclidean or hyperbolic turnovers.

Example 7.0.2. — In Figure 1 there is an example of a small orbifold with non-empty boundary.

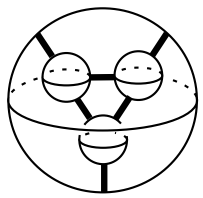

FIGURE 1. This orbifold is small, provided that the ramification indices are sufficiently large [**Dun2**].

This chapter is devoted to the proof of Theorem 2:

Theorem 2. — *Let $\mathcal{O}$ be a compact, orientable, connected, small* 3*-orbifold of cyclic type. Then $\mathcal{O}$ is geometric.*

Remark 7.0.3. — When $\mathcal{O}$ is hyperbolic and $\partial\mathcal{O}$ is not empty, we prove that either $\mathcal{O}$ is a product or $\mathcal{O}$ has finite volume and totally geodesic boundary.

7.1. Desingularization of ramified circle components

Let $\mathcal{O}$ be a compact orientable small 3-orbifold of cyclic type and with topological type (C, Σ). By definition, a small 3-orbifold does not contain any properly embedded essential orientable 2-sided 2-suborbifold. It implies that the underlying space itself does not contain a properly embedded non-separating orientable surface. Let $\Sigma = \Sigma_0 \cup \Sigma_\partial$ be a decomposition of the ramification locus, where Σ_0 corresponds to the circle components of Σ and Σ_∂ to the arcs of Σ.

The purpose of this section is to construct a finite regular covering $\widetilde{\mathcal{O}}$ of $\mathcal{O}$ that desingularizes the circle components $\Sigma_0 \subset \Sigma$.

In particular, if the small orbifold of cyclic type $\mathcal{O}$ is closed, then $\Sigma_0 = \Sigma$ and $\widetilde{\mathcal{O}}$ is a 3-manifold, hence $\mathcal{O}$ is very good.

We start with the following homological lemma (cf. [**Tak1**]).

Lemma 7.1.1. — *Let $\mathcal{O}$ be a compact orientable small 3-orbifold of cyclic type and with topological type (C, Σ). Then :*

i) $H_1(C; \mathbb{Z})$ is finite.

ii) If $\Sigma_0 \subset \Sigma$ is the union of circle components, then the following exact sequence holds:

$$0 \to H_2(\mathcal{N}(\Sigma_0), \partial\mathcal{N}(\Sigma_0); \mathbb{Z}) \to H_1(C - \Sigma_0; \mathbb{Z}) \to H_1(C; \mathbb{Z}) \to 0$$

Proof of Lemma 7.1.1. — Assertion i) is equivalent to $H_1(C; \mathbb{Q}) \cong H_2(C, \partial C; \mathbb{Q}) = 0$. Seeking a contradiction, let us assume that there is a non-separating essential orientable surface $|F|$, properly embedded in C. We can always make it transverse to the ramification locus Σ. Then we choose such a surface $|F|$ with the minimal number of intersection points with Σ. The corresponding orientable 2-suborbifold $F \subset \mathcal{O}$ with underlying space $|F|$ is an essential 2-suborbifold in $\mathcal{O}$, otherwise the incompressibility of $|F|$ implies that one could reduce the number of intersection points with Σ. This contradicts the smallness of $\mathcal{O}$.

We prove assertion ii) with the long exact sequence for the homology of the pair $(C, C - \Sigma_0)$:

$$\begin{aligned} &\cdots \to H_2(C - \Sigma_0; \mathbb{Z}) \to H_2(C; \mathbb{Z}) \to H_2(C, C - \Sigma_0; \mathbb{Z}) \to H_1(C - \Sigma_0; \mathbb{Z}) \to \\ &\to H_1(C; \mathbb{Z}) \to H_1(C, C - \Sigma_0; \mathbb{Z}) \to \dots \end{aligned}$$

The excision property gives an isomorphism

$$H_i(C, C - \Sigma_0; \mathbb{Z}) \cong H_i(\mathcal{N}(\Sigma_0), \partial\mathcal{N}(\Sigma_0); \mathbb{Z}), \qquad \text{for } i \in \{1, 2, 3\}.$$

In particular $H_1(C, C - \Sigma_0; \mathbb{Z}) \cong 0$. Moreover, $H_2(C, C - \Sigma_0; \mathbb{Z})$ is a free abelian group generated by the meridian disks of the tubular neighborhood $\mathcal{N}(\Sigma_0)$ of the circle components of Σ.

To obtain the exact sequence stated in ii), it remains to show that

$$Im\{H_2(C;\mathbb{Z}) \to H_2(C, C-\Sigma_0;\mathbb{Z})\} = \{0\}.$$

In fact, it is sufficient to show that the morphism (with rational coefficients)

$$H_2(C-\Sigma_0;\mathbb{Q}) \to H_2(C;\mathbb{Q})$$

is surjective, since $H_2(C, C-\Sigma_0;\mathbb{Z})$ is a free abelian group. That is a consequence of the two following facts:

1) the morphism $H_2(\partial C;\mathbb{Q}) \to H_2(C;\mathbb{Q})$ is surjective, because $H_2(C, \partial C;\mathbb{Q})$ vanishes by assertion i);
2) this morphism factors through $H_2(C-\Sigma_0;\mathbb{Q})$, because $\Sigma_0 \cap \partial C = \emptyset$.

□

The following proposition gives the construction of the desired regular finite covering $\widetilde{\mathcal{O}}$ of $\mathcal{O}$.

Proposition 7.1.2. — *Let $\mathcal{O}$ be a compact orientable small 3-orbifold of cyclic type and with topological type (C,Σ). There is a finite regular covering $p\colon \widetilde{\mathcal{O}} \to \mathcal{O}$ such that the ramification locus of $\widetilde{\mathcal{O}}$ is $\widetilde{\Sigma} = p^{-1}(\Sigma_\partial)$.*

Proof. — Let $\Phi : \pi_1(C-\Sigma_0) \twoheadrightarrow H_1(C;\mathbb{Z})$ be the surjection obtained by composing the abelianization map $\pi_1(C-\Sigma_0) \to H_1(C-\Sigma_0;\mathbb{Z})$ and the morphism $H_1(C-\Sigma_0;\mathbb{Z}) \to H_1(C;\mathbb{Z})$ induced by inclusion. The kernel $G = \ker\Phi$ is a normal subgroup of finite index in $\pi_1(C-\Sigma_0)$ by Lemma 7.1.1 i).

The exact sequence given by Lemma 7.1.1 ii) shows that Φ induces a surjective morphism

$$\Psi : G \to H_2(\mathcal{N}(\Sigma_0), \partial\mathcal{N}(\Sigma_0);\mathbb{Z}) \to 0.$$

The free abelian group $H_2(\mathcal{N}(\Sigma_0), \partial\mathcal{N}(\Sigma_0);\mathbb{Z}) \cong \oplus_{i=1}^{i=q}\mathbb{Z} < \delta_i >$ is generated by the meridian disks δ_i, $i = 1, \ldots, q$, of the tubular neighborhood $\mathcal{N}(\Sigma_0)$ of the cycle components of Σ. More precisely δ_i is the meridian disk of the tubular neighborhood of the i-th component Σ_0^i of Σ_0.

Let $\{n_1, \ldots, n_q\}$ be the branching indices of the components $\{\Sigma_0^1, \ldots, \Sigma_0^q\}$. Then Ψ induces a surjective morphism $\psi : G \to \oplus_{i=1}^{i=q}\mathbb{Z}_{n_i} < \delta_i >$, where $\mathbb{Z}_{n_i}$ is the ring of integers modulo n_i. So $H = \ker\psi$ is a normal subgroup of finite index in G, hence it is a normal subgroup of finite index in $\pi_1(C-\Sigma_0)$.

We consider now the covering of C branched along Σ_0, associated to the surjective morphism $\beta : \pi_1(C-\Sigma_0) \to \pi_1(C-\Sigma_0)/H$.

For $i \in \{1, \ldots, q\}$ let $\mu_i \subset \pi_1(C-\Sigma_0)$ be a meridian of the i-th component Σ_0^i. Since μ_i corresponds to the boundary of δ_i, the exact sequence of Lemma 7.1.1 ii) shows that μ_i belongs to G. Moreover $\psi(\mu_i)$ is a generator of $\mathbb{Z}_{n_i}$. By construction, it follows that $\beta(\mu_i)$ is a generator of $\mathbb{Z}_{n_i}$, hence has precisely the same order n_i as the branching index of the corresponding component Σ_0^i of Σ_0.

Therefore this branched covering induces a finite regular covering $p : \widetilde{\mathcal{O}} \to \mathcal{O}$ which desingularizes the circle components Σ_0 of Σ. In particular, the ramification locus of $\widetilde{\mathcal{O}}$ is $\widetilde{\Sigma} = p^{-1}(\Sigma_\partial)$. $\square$

In the closed case, Theorem 2 is then a straightforward corollary of Theorem 1 and Proposition 7.1.2 above:

Corollary 7.1.3. — *A closed orientable small 3-orbifold of cyclic type is geometric.* $\square$

The remaining of this chapter is devoted to the proof of Theorem 2 when $\partial\mathcal{O} \neq \emptyset$.

7.2. Uniformization of small 3-orbifolds with non-empty boundary

In this section we always assume that $\mathcal{O}$ is a compact orientable small 3-orbifold of cyclic type with non-empty boundary $\partial\mathcal{O} \neq \emptyset$. In particular $\mathcal{O}$ is either a discal 3-orbifold or it is irreducible and topologically atoroidal, with boundary a collection of hyperbolic and Euclidean turnovers. Hence it is also acylindrical.

By doubling $\mathcal{O}$ along its boundary, the arguments in Section 2.4 of Chapter 2 (Lemmas 2.4.1 to 2.4.7) reduce the proof of Theorem 2 to the proof of Proposition 7.2.1 below. We remark that we do not need Proposition 2.4.9 of Chapter 2 since all boundary components of $\mathcal{O}$ are different from a non-singular torus. Except for that proposition, the hypothesis that the orbifold $\mathcal{O}$ is very good is not used in Section 2.4.

Proposition 7.2.1. — *Let $\mathcal{O}$ be a compact orientable small 3-orbifold of cyclic type and non-empty boundary. If the complement $\mathcal{O} - \Sigma$ of the branching locus admits a complete hyperbolic structure with finite volume and totally geodesic boundary, then $\mathcal{O}$ is geometric.*

Proof of Proposition 7.2.1. — The proof of Proposition 7.2.1 follows the scheme of the proof of Theorem 2 in Sections 2.2 and 2.3 of Chapter 2.

Using the hypothesis that $\mathcal{O}-\Sigma$ admits a complete hyperbolic structure with totally geodesic boundary, we consider the subinterval $J \subseteq [0,1]$ of real numbers $t \in [0,1]$ such that there is a path $\gamma : [0,t] \to R_0$ with the property that, for every $s \in [0,t]$, $\gamma(s)$ is the holonomy of a hyperbolic cone 3-manifold $C(s\alpha)$, with totally geodesic boundary and with the same topological type as $(|\mathcal{O}|, \Sigma)$. The cone angles of $C(s\alpha)$ are $s\alpha = (s\, 2\pi/m_1, \dots, s\, 2\pi/m_q)$, where $\{m_1, \dots, m_q\}$ are the branching indices of $\mathcal{O}$ along Σ. Here R_0 denotes the irreducible component of the representation variety of $\mathcal{O} - \Sigma$ that contains the holonomy of the hyperbolic structure on $\mathcal{O} - \Sigma$, with totally geodesic boundary and whose ends are cusps.

Lemma 7.2.2. — *The subinterval J is non-empty and open.*

Proof. — It is non-empty because $0 \in J$. It is open by Proposition B.3.1 of Appendix B, which is a version of Thurston's hyperbolic Dehn filling theorem. The proof that J is open at points different from 0 is the same as the proof of Proposition 2.2.4, except for the boundary. To control the holonomy at the boundary it suffices to use Lemma B.3.3 of Appendix B, which ensures that, when we deform our holonomy representations, then the representation of the boundary is still the holonomy of a totally geodesic turnover. □

By Lemma 7.2.2, there are three possibilities: either $J = [0,1]$, $J = [0,1)$, or $J = [0,t)$ with $0 < t < 1$. If $J = [0,1]$ then the compact orbifold $\mathcal{O}$ admits a hyperbolic structure with totally geodesic boundary.

Lemma 7.2.3. — *The case $J = [0,t)$, with $0 < t < 1$, does not occur.*

Proof. — We prove it by contradiction: we fix $(t_n)_{n\in\mathbb{N}}$ an increasing sequence in $J = [0,t)$ converging to $t < 1$ and we consider the corresponding sequence of cone manifolds with totally geodesic boundary $\{C_n = C(t_n\alpha)\}_{n\in\mathbb{N}}$. Up to taking a subsequence, either $(C_n)_{n\in\mathbb{N}}$ collapses or not.

If the sequence C_n does not collapse then, by the compactness theorem, we may assume that the sequence of pointed cone 3-manifolds (C_n, x_n) converges to (C_∞, x_∞). Now we apply Theorem A to the sequence obtained by doubling C_n along its boundary, which does not collapse either. Since $\mathcal{O}$ is small, the double along the boundary $D\mathcal{O}$ contains no spherical turnover. This implies that case 2) in Theorem A is excluded because a Euclidean turnover in DC_∞ is spherical in $D\mathcal{O}$, (cf. the proof of Proposition 2.3.1). Therefore Theorem A implies that the double of C_∞ is compact. Thus C_∞ is also compact and has the same topological type as C_n, hence we have a contradiction.

If the sequence C_n collapses, then by applying the strengthened version of Theorem A (Section 5.5) to the sequence of doubles, we deduce that the rescaled sequence $(\frac{1}{\mathrm{inj}(x_n)}C_n, x_n)$ converges to a compact Euclidean cone 3-manifold. Since $t < 1$, it follows that $\mathcal{O}$ has spherical boundary. By irreducibility, the unique possibility is that $\mathcal{O}$ is discal, but this contradicts the hypothesis that $\mathcal{O} - \Sigma$ is hyperbolic.

This finishes the proof of Lemma 7.2.3. □

To complete the proof of Proposition 7.2.1 we are left with the case $J = [0,1)$:

Proposition 7.2.4. — *If $J = [0,1)$ then $\mathcal{O}$ is hyperbolic*

Proof of Proposition 7.2.4. — We fix $(t_n)_{n\in\mathbb{N}}$ an increasing sequence in $J = [0,1)$ converging to 1 and we consider the corresponding sequence of hyperbolic cone 3-manifolds $\{C_n = C(t_n\alpha)\}_{n\in\mathbb{N}}$. In Proposition 7.3.1 below we will show that the sequence $(C_n)_{n\in\mathbb{N}}$ does not collapse, hence we may apply Lemma 7.2.5 below to complete the proof of Proposition 7.2.4.

Lemma 7.2.5. — *If C_n does not collapse, then $\mathcal{O}$ is hyperbolic. In addition the Euclidean boundary turnovers correspond to cusps and the other boundary turnovers are totally geodesic boundary components.*

Proof of Lemma 7.2.5. — We consider the sequence DC_n of doubles of C_n along its totally geodesic boundary. Let $D\mathcal{O}$ denote the double of $\mathcal{O}$. As we indicated in the introduction to Chapter 6, the proof of Theorem B, given in Section 6.1, does not use the existence of a finite regular manifold covering for $D\mathcal{O}$. So it applies to show that $D\mathcal{O}$ contains a non-empty compact essential hyperbolic 3-suborbifold which is not a product. Since $\mathcal{O}$ is small, the arguments at the end of Chapter 2, in the proof of Theorem 1, show that $\mathcal{O}$ itself is hyperbolic, possibly with cusps. □

7.3. The sequence does not collapse

Let $\mathcal{O}$ be an orbifold as in the statement of Propositions 7.2.1 and 7.2.4. As above, let C_n be a sequence of hyperbolic cone 3-manifolds with the same topological type as $\mathcal{O}$ and whose cone angles increase and approach the orbifold angles of $\mathcal{O}$.

Proposition 7.3.1. — *With the hypothesis of Proposition 7.2.4, the sequence C_n does not collapse.*

Proof. — We prove it by contradiction, assuming that C_n collapses. By the local soul theorem in Chapter 4 (the version with boundary and Corollary 4.4.1), we distinguish again two cases, according to whether after rescaling we obtain a compact limit or not:

1) either there is a subsequence that after rescaling converges to a compact Euclidean cone 3-manifold,
2) or the local soul theorem (possibly with boundary), with any parameters $\varepsilon > 0$, $D > 1$ and non-compact local models, applies to every point $x \in C_n$, provided that $n > n_0$ (where n_0 depends on D and ε).

Contradiction in case 1).— In the first case, for every $n \in \mathbb{N}$ there is a point $x_n \in C_n$ such that the sequence of rescaled pointed cone 3-manifolds $(\overline{C}_n, x_n) = (\frac{1}{\mathrm{inj}(x_n)} C_n, x_n)$ has a subsequence that converges geometrically to a compact cone 3-manifold $(\overline{C}_\infty, x_\infty)$. The geometric convergence implies that $\overline{C}_\infty$ is a compact Euclidean 3-orbifold with the same topological type and the same branching indices as $\mathcal{O}$ (In particular $\overline{C}_\infty$ has totally geodesic boundary). Thus $\mathcal{O}$ is Euclidean with totally geodesic boundary. We consider the following non-compact Euclidean orbifold without boundary

$$\mathcal{O} \cup_\partial \partial\mathcal{O} \times [0, \infty).$$

Since $\mathcal{O}$ is Euclidean with totally geodesic boundary, $\partial\mathcal{O}$ has a collar neighborhood which is metrically a product, therefore we can glue $\partial\mathcal{O} \times [0, +\infty)$ so that the metrics match. The ends of $\mathcal{O} \cup_\partial \partial\mathcal{O} \times [0, \infty)$ are $S^2(\alpha, \beta, \gamma) \times [0, +\infty)$, with $\alpha + \beta + \gamma = 2\pi$. If we look at the classification of non-compact Euclidean orbifolds, this fact implies that the only possibility is

$$\mathcal{O} \cup_\partial (\partial\mathcal{O} \times [0, \infty)) \cong S^2(\alpha, \beta, \gamma) \times \mathbb{R}$$

Thus $\mathcal{O} \cong S^2(\alpha, \beta, \gamma) \times [0, 1]$, contradicting the fact the $\mathcal{O} - \Sigma$ is hyperbolic.

Contradiction in case 2).— In the second case, when we cannot find such a compact limit, we apply the local soul theorem to C_n and we consider the induced covering by Margulis' neighborhoods on the double DC_n. Thus we obtain:

***Lemma 7.3.2* (Non-compact collapsing case).** — *For any $\varepsilon > 0$ and $D_1 > 1$, there exists $n_0 > 0$ such that, for $n \geq n_0$, every point x in the double DC_n has an open neighborhood U_x $(1 + \varepsilon)$-bilipschitz homeomorphic to the normal fiber bundle $\mathcal{N}_\nu(S)$, with radius $0 < \nu < 1$ depending on x, of the soul S of one of the following non-compact orientable Euclidean orbifolds E:*

a) $T^2 \times \mathbb{R}$; $S^1 \ltimes \mathbb{E}^2$; $S^1 \ltimes D^2(\frac{2\pi}{p})$;

b) the thick pillow $S^2(\pi, \pi, \pi, \pi) \times \mathbb{R}$; the solid pillow.

Moreover:

i) If $f : U_x \to \mathcal{N}_\nu(S)$ is the $(1 + \varepsilon)$-bilipschitz homeomorphism, then

$$\max(\mathrm{inj}(x), d(f(x), S), \mathrm{diam}(S)) \leq \nu/D_1.$$

ii) Let $\tau_n : DC_n \to DC_n$ denote the involution whose fixed point set is ∂C_n. If $\tau_n(x) = x$ (i.e. $x \in \partial C_n$), then $\tau_n(U_x) = U_x$ and there is an isometric involution $\tau_\infty : \mathcal{N}_\nu(S) \to \mathcal{N}_\nu(S)$ such that:

$$f\tau_n = \tau_\infty f.$$

As in the previous Chapters 5 and 6, the neighborhoods above given by the local soul theorem are called (ε, D)-*Margulis' neighborhoods*, and the corresponding non-compact Euclidean cone manifolds E are called *local models*.

Proof of Lemma 7.3.2. — We observe that the additional property ii) in the statement follows from the fact that we apply the local soul theorem for cone 3-manifolds with totally geodesic boundary as in Section 4.5. The Margulis' neighborhoods on C_n induce Margulis' neighborhoods on the double of C_n with property ii).

To prove the lemma we have to remove five local models from the list of the local soul theorem. These are the models with soul an annulus, a Möbius strip, a disc with two cone points, a projective plane with two cone points, and a Euclidean turnover (when the soul has boundary we assume that it is silvered), cf. Lemma 6.2.2.

If the soul is an annulus or a Möbius strip, then the boundary of the Margulis' neigborhood is a torus. Since $D\mathcal{O} - D\Sigma$ is topologically atoroidal, $D\mathcal{O}$ is the union of the Margulis neighborhood with a solid torus. This contradicts the hyperbolicity of $D\mathcal{O} - D\Sigma$.

If the soul is a disc or a projective plane with two cone points, then the boundary of the Margulis' neigborhood is a pillow. Since $\mathcal{O}$ is small, every pillow in $D\mathcal{O}$ is compressible. Thus $D\mathcal{O}$ is the union of the Margulis neighborhood with a solid pillow, since the Margulis neighborhood is not embedded in a discal 3-orbifold. In particular the underlying space of $D\mathcal{O}$ is S^3, contradicting the fact that $D\mathcal{O}$ is of cyclic type and contains turnovers.

Finally, to remove the case where the soul is a Euclidean turnover (i.e. a thick turnover), we apply Proposition 5.5.1 in Chapter 5. □

The following lemma restricts the type of an (ε, D)-Margulis' neighborhood for a singular point invariant by the reflection τ_n.

Lemma 7.3.3. — *For n sufficiently large, $\varepsilon < 1/2$ and $D > D_1$, every singular point of DC_n, invariant by the reflection τ_n, has a (ε, D)-Margulis' neighborhood of abelian type a) (i.e with local model $S^1 \ltimes D^2(\frac{2\pi}{p})$).*

Proof of Lemma 7.3.3. — Let $\tau_n : C_n \to C_n$ be the reflection through $\partial C_n = \mathrm{Fix}(\tau_n)$. Let $x \in \mathrm{Fix}(\tau_n) \cap \Sigma$ be a singular fixed point of τ_n and let U be a (ε, D)-Margulis' neighborhood of x satisfying properties i) and ii). We choose $\varepsilon < 1/2$ and $D > D_1$.

Seeking a contradiction, let us assume that U is not of the desired type a) (i.e. with local model of type $S^1 \ltimes D^2(\frac{2\pi}{p})$). Then, by Lemma 7.3.2, for n sufficiently large, the local model of U must be of type b): either the thick pillow $S^2(\pi,\pi,\pi,\pi) \times \mathbb{R}$ or the solid pillow.

Let $f : U_x \to \mathcal{N}_\nu(S)$ be the $(1+\varepsilon)$-bilipschitz homeomorphism. Property ii) says that U_x is τ_n-invariant and $\mathcal{N}_\nu(S)$ has an involution τ_∞ such that $f\tau_n = \tau_\infty f$. In particular

$$f(U \cap \partial C_n) = f(U \cap \mathrm{Fix}(\tau_n)) = \mathrm{Fix}(\tau_\infty).$$

Since $\mathcal{N}_\nu(S)$ is either the solid pillow or the thick pillow, and since $\mathrm{Fix}(\tau_\infty)$ is two dimensional and transverse to the singularity, it follows that $\mathrm{Fix}(\tau_\infty)$ is connected and contains at least two singular points with cone angle π. Thus a connected component of $\partial\mathcal{O}$ contains two singular points with ramification 2, which is impossible because the components of $\partial\mathcal{O}$ are non-spherical turnovers □

We now deduce Proposition 7.3.1 from Lemmas 7.3.2, and 7.3.3.

End of the proof of Proposition 7.3.1. — By hypothesis, the union $D\Sigma_\partial$ of components of $D\Sigma$ that meets $\partial\mathcal{O} \subset D\mathcal{O}$ is not empty. Since the 3-orbifold $D\mathcal{O}$ is irreducible, no component of $D\Sigma_\partial$ is contained in a discal 3-suborbifold; hence $D\mathcal{O} - \mathrm{int}(\mathcal{N}(D\Sigma_\partial))$ is an irreducible 3-orbifold.

Let $p : \widetilde{\mathcal{O}} \to \mathcal{O}$ be the finite regular covering given by Proposition 7.1.2, which desingularizes the circle components Σ_0 of the ramification locus Σ of $\mathcal{O}$. Let

$$q : D\widetilde{\mathcal{O}} \to D\mathcal{O}$$

be the induced finite regular covering, then $D\widetilde{\Sigma} = q^{-1}(D\Sigma_\partial)$. Let

$$M = D\widetilde{\mathcal{O}} - \mathrm{int}(\mathcal{N}(D\widetilde{\Sigma}))$$

denote the exterior of the ramification locus of $D\widetilde{\mathcal{O}}$. Then the restriction of q to M is a regular finite manifold covering of the irreducible compact 3-orbifold $D\mathcal{O} - \mathrm{int}(\mathcal{N}(D\Sigma_\partial))$.

Moreover $D\mathcal{O} - \mathrm{int}(\mathcal{N}(D\Sigma_\partial))$ does not contain any essential toric 2-suborbifold. Since $\mathcal{O}$ is small and $D\mathcal{O} - \mathrm{int}(\mathcal{N}(D\Sigma))$ is topologically atoroidal, such an essential toric 2-suborbifold would be homeomorphic to $S^2(\pi,\pi,\pi,\pi)$. Furthermore it would remain essential in $D\mathcal{O}$, because $D\mathcal{O}$ does not contain a spherical turnover, contradicting the fact that $\mathcal{O}$ is small.

Therefore, by the equivariant sphere theorem ([**DD**], [**MY1, MY2**], [**JR**]) the manifold M is irreducible, and it is topologically atoroidal because the characteristic family of essential tori in M is empty by [**MS**]. Thurston's hyperbolization theorem for Haken 3-manifold shows that either M is Seifert fibered or hyperbolic.

If M admits a Seifert fibration, then it is preserved by the deck transformations group of the finite covering $q: M \to D\mathcal{O} - \mathrm{int}(\mathcal{N}(D\Sigma_\partial))$, by Meeks and Scott's results [**MS**]. We can also assume that it is invariant by any lift of the involution of $D\mathcal{O} - \mathrm{int}(\mathcal{N}(D\Sigma_\partial))$ that fixes $\partial\mathcal{O} - \mathcal{N}(\partial\Sigma)$. Thus, the 3-orbifold $D\mathcal{O} - \mathrm{int}(\mathcal{N}(D\Sigma_\partial))$ has a Seifert fibration invariant by this involution. Since the components of $\partial\mathcal{O} - \mathcal{N}(\partial\Sigma)$ are 3 times punctured spheres, by [**Wa1**] they are transverse to the fibration, and $\mathcal{O} - \mathrm{int}(\mathcal{N}(\Sigma_\partial))$ is an I-bundle. It follows that $\mathcal{O}$ is the product of a turnover with an interval, and we have a contradiction with the hyperbolicity of $\mathcal{O} - \Sigma$.

Therefore, from now on we can assume that M admits a complete hyperbolic structure with finite volume. In particular M has a non vanishing simplicial volume $\|M\| > 0$.

We complete the proof of Proposition 7.3.1 by contradiction. By Lemma 7.3.2, for n sufficiently large, $\varepsilon < 1/2$ and $D > D_1$ the local models of the (ε, D)-Margulis' neighborhood of each point in C_n are of type a) or b); moreover by Lemma 7.3.3 for all points of $D\Sigma_\partial$ they are of type a). By a construction of a η-covering à la Gromov on C_n, for n sufficiently large and η sufficiently small, as in Chapters 5 and 6, we will be in position to apply Gromov's vanishing theorem for the simplicial volume of M ([**Gro**, §3.4], [**Iva**]) and thus to get a contradiction.

To achieve that, we reproduce the proof of the "non compact collapsing case" in Sections 5.3 and 5.4, but using the notion of virtually abelian subsets in $D\mathcal{O} - D\Sigma_\partial$ (as in Chapter 6) instead of the notion of abelian subset. This is due to the fact that we have to deal here with local models of type b) which are not abelian but only virtually abelian subsets in $D\mathcal{O} - D\Sigma_\partial$.

More precisely, here is the key lemma (analogous to Proposition 5.4.1) for the construction of the required η-covering à la Gromov on C_n, for n sufficiently large and η sufficiently small.

Lemma 7.3.4. — *With the notation above, there is a universal constant $b_0 > 0$ such that, if for some n every point of C_n has a (ε, D)-Margulis' neighborhood of type a) or b), with $\varepsilon < 1/2$ and $D > \max\{D_1, 300\}$, then C_n admits a η-covering à la Gromov $(V_i)_{i\in I}$ with $\eta < b_0/D$. Moreover, the open sets $(V_i)_{i\in I}$ of the η-covering à la Gromov satisfy the following additional properties:*

6) *$\forall i \in I$, $\bigcup_{V_j\cap V_i\neq\emptyset} V_j$ is virtually abelian in $D\mathcal{O} - D\Sigma_\partial$.*

7) *there is a tubular neighborhood $\mathcal{N}(D\Sigma_\partial)$ of $D\Sigma_\partial$ such that every component of $\mathcal{N}(D\Sigma_\partial)$ is contained in only one open set of the covering.*

Proof of Lemma 7.3.4. — To simplify the notation, we omit the index n of the hyperbolic cone 3-manifold C_n.

The proof is analogous to that of Proposition 5.4.1. The fact that we have to consider a subset $D\Sigma_\partial \subset D\Sigma$ instead of $D\Sigma$ and (ε, D)-Margulis' neighborhood of type *b)* for points on $D\Sigma - D\Sigma_\partial$ does not make any real difference in the proof.

We recall that a subset $U \subset C$ is *virtually abelian in* $D\mathcal{O} - D\Sigma_\partial$, if the image $i_*(\pi_1(U - \Sigma'))$ is a virtually abelian subgroup of $\pi_1(D\mathcal{O} - D\Sigma_\partial)$, where i_* is the morphism induced by the inclusion $i : (U - D\Sigma_\partial) \to (D\mathcal{O} - D\Sigma_\partial)$ and $\pi_1(.)$ denotes the fundamental group of the orbifold.

For every point $x \in C$, we define the *virtual abelianity radius* $\mathrm{vab}(x)$ (relatively to $D\mathcal{O} - D\Sigma_\partial$) to be:

$$\mathrm{vab}(x) = \sup\{r > 0 \mid B(x, r) \text{ is virtually abelian in } D\mathcal{O} - D\Sigma_\partial\}.$$

For every $x \in C$ we define $r(x) = \inf(\frac{\mathrm{vab}(x)}{8}, 1)$.

This definition is analogous to the ones given in Sections 5.4 and 6.3. Since by Lemma 7.3.2 all points of $D\Sigma_\partial$ have abelian (ε, D)-Margulis' neighborhoods, the proofs of Lemmas 5.4.2 to 5.4.5 in Section 5.4 (that give the construction of the η-covering à la Gromov) work without any change, except for the proof of Lemma 5.4.3 e) which is now a consequence of the following claim:

Claim 7.3.5. — *Let $D\Sigma_1$ and $D\Sigma_2$ be two components of $D\Sigma_\partial$. Two peripheral elements of $\pi_1(D\mathcal{O} - D\Sigma_\partial)$ represented respectively by meridians of $D\Sigma_1$ and $D\Sigma_2$ cannot belong to the same virtually abelian subgroup of $\pi_1(D\mathcal{O} - D\Sigma_\partial)$.*

Proof of Claim 7.3.5. — Let μ_1 and μ_2 be two peripheral elements in $\pi_1(D\mathcal{O} - D\Sigma_\partial)$ represented respectively by a meridian m_1 of $\mathcal{N}(D\Sigma_1)$ and m_2 of $\mathcal{N}(D\Sigma_2)$. Let $\bar{\mu}_1$ and $\bar{\mu}_2$ the two peripheral elements of $\pi_1(M)$ corresponding to some lifts of m_1 and m_2 on ∂M. Since M admits a complete hyperbolic structure on its interior, they correspond to two parabolic elements in $\pi_1(M)$ with different fixed points on the sphere at infinity. Hence, $\bar{\mu}_1$ and $\bar{\mu}_2$ always generate a non-elementary group in $\pi_1(M) \subset PSL_2(\mathbb{C})$. In particular, μ_1 and μ_2 cannot belong to the same virtually abelian subgroup of $\pi_1(D\mathcal{O} - D\Sigma_\partial)$.

This finishes the proofs of Claim 7.3.5 and thus of Lemma 7.3.4. □

To finish the proof of Proposition 7.3.1 we choose a constant

$$D \geq \max\{D_0, D_1, b_0/\eta_0, 300\},$$

where D_0 and η_0 are the constants given in Chapter 5, b_0 and D_1 are the constant given in Lemmas 7.3.4 and 7.3.2. Then Lemma 7.3.4, together with Proposition 5.3.3, implies the existence of a continuous map $g : D\mathcal{O} \to K^2$, from $D\mathcal{O}$ to a 2-dimensional simplicial complex K^2 such that:

i′) for every component $D\Sigma_i$ of $D\Sigma_\partial$ there is an open tubular neighborhood $\mathcal{N}(D\Sigma_i)$ of $D\Sigma_i$ such that $g(\overline{\mathcal{N}(D\Sigma_i)})$ is a vertex of K^2;

ii′) for every vertex v of K^2, $g^{-1}(\mathrm{star}(v))$ is virtually abelian in $D\mathcal{O} - D\Sigma_\partial$.

By composing the restriction of g to the 3-orbifold $D\mathcal{O} - \mathrm{int}(\mathcal{N}(D\Sigma_\partial))$ with the projection of the covering map $q : M \to D\mathcal{O} - \mathrm{int}(\mathcal{N}(D\Sigma_\partial))$, we get a continuous map $f = g \circ q : M \to K^2$ with the following properties:

i′) the components of ∂M are mapped by f to distinct vertices of K^2;

ii′) for every vertex v of K^2, $f^{-1}(\mathrm{star}(v))$ is virtually abelian in M .

On every closed Dehn filling $\overline{M}$ of M, we use the map f to construct a covering of dimension 2 of $\overline{M}$ by virtually abelian open subsets, as in the proof of Proposition 5.2.5 in Section 5.4. Then, by Gromov's vanishing theorem ([**Gro**, §3.1], cf [**Iva**]) the simplicial volume of every closed Dehn fillings $\overline{M}$ of M along ∂M vanishes. Since M has a complete hyperbolic structure with finite volume, Thurston's hyperbolic Dehn filling theorem [**Thu1**] (cf. Appendix B) gives a contradiction. This finishes the proof of Proposition 7.3.1, and hence of Theorem 2. ☐

CHAPTER 8

HAKEN 3-ORBIFOLDS

The purpose of this chapter is to give a detailed overview of the proof of Theorem 3 (Thurston's hyperbolization theorem for Haken 3-orbifolds) that we need to complete the proof of Thurston's orbifold theorem without assuming the orbifold to be very good.

Thusrton's Orbifold Theorem. — *Let $\mathcal{O}$ be a compact, connected, orientable, irreducible, 3-orbifold of cyclic type. If $\mathcal{O}$ is topologically atoroidal, then $\mathcal{O}$ is geometric.*

Definition 8.0.1. — A compact orientable 3-orbifold is *Haken* if it is irreducible and it does not contain an essential turnover, but it contains an incompressible 2-suborbifold different from a turnover.

Theorem 3 **(Thurston's hyperbolization theorem for Haken 3-orbifolds)**
Let $\mathcal{O}$ be a compact orientable connected Haken 3-orbifold. If $\mathcal{O}$ is topologically atoroidal and not Seifert fibered, nor Euclidean, then $\mathcal{O}$ is hyperbolic.

Remark 8.0.2. — The word Haken may lead to confusion: it is not true that a compact orientable irreducible 3-orbifold containing an orientable incompressible properly embedded 2-suborbifold is Haken in our meaning.

Example 8.0.3. — In Figure 1 of Chapter 7 there is an example of a small (hence non-Haken) 3-orbifold with a non-empty boundary. By doubling it along its boundary, one gets a closed irreducible 3-orbifold which is not Haken, but contains an essential embedded 2-sided 2-suborbifold.

We show in Section 8.1 how to deduce Thurston's orbifold theorem from Theorem 2 and Theorem 3. Next, in §8.2, we give some basic properties of Haken 3-orbifolds. Finally, in the remaining of the chapter, we discuss the proof of Thurston's hyperbolization theorem for Haken 3-orbifolds. We do not intend here to give a full detailed proof, but only to point out the main modifications with respect to the proof in the manifold case. Since we follow closely the exposition of Thurston's hyperbolization theorem for Haken 3-manifolds, given in [**McM1**] and [**Ot2**] (cf. also [**OP**]), with a bit of courage the details can be worked out using these two references.

8.1. Proof of Thurston's Orbifold theorem

Let $\mathcal{O}$ be a compact orientable connected irreducible topologically atoroidal 3-orbifold. By [**Dun2**, Thm. 12] there exists in $\mathcal{O}$ a (possibly empty) maximal collection $\mathcal{T}$ of disjoint embedded pairwise non parallel essential turnovers. Since $\mathcal{O}$ is irreducible and topologically atoroidal, any turnover in $\mathcal{T}$ is hyperbolic (i.e. has negative Euler characteristic).

When $\mathcal{T}$ is empty, Thurston's orbifold theorem reduces to Theorem 2 or 3 according to whether $\mathcal{O}$ is small or Haken.

When $\mathcal{T}$ is not empty, we first cut open the orbifold $\mathcal{O}$ along the turnovers of the family $\mathcal{T}$. By maximality of the family $\mathcal{T}$, the closure of each component of $\mathcal{O} - \mathcal{T}$ is a compact orientable irreducible topologically atoroidal 3-orbifold that does not contain any essential embedded turnover.

Let $\mathcal{O}'$ be one of these connected components. By the previous case, $\mathcal{O}'$ is either hyperbolic, Euclidean or Seifert fibered. Since, by construction, $\partial\mathcal{O}'$ contains at least one hyperbolic turnover T, $\mathcal{O}'$ must be hyperbolic. Moreover any such hyperbolic turnover T in $\partial\mathcal{O}'$ is a Fuchsian 2-suborbifold, because there is a unique conjugacy class of faithful representations of the fundamental group $\pi_1(T)$ into $PSL_2(\mathbb{C})$.

If the convex core of $\mathcal{O}'$ is 2-dimensional, then $\mathcal{O}'$ is a product $T \times [0,1]$, where T is a hyperbolic turnover. In this case the 3-orbifold $\mathcal{O}$ is Seifert fibered.

Therefore we can assume that all the connected components of $\mathcal{O} - \mathcal{T}$ have 3-dimensional convex cores. In this case the hyperbolic turnovers are totally geodesic boundary components of the convex cores. Hence the hyperbolic structures of the components of $\mathcal{O} - \mathcal{T}$ can be glued together along the hyperbolic turnovers of the family $\mathcal{T}$ to give a hyperbolic structure on the 3-orbifod $\mathcal{O}$. □

8.2. Fundamental results on Haken 3-orbifolds

For the rest of this chapter we are not assuming anymore the 3-orbifold $\mathcal{O}$ to be of cyclic type. In particular the ramification locus Σ may be a trivalent graph.

We have for Haken 3-orbifolds fundamental results which are analogous to the results known for Haken 3-manifolds. This follows mainly from Dunbar's theorem [**Dun2**] which shows that a Haken 3-orbifold $\mathcal{O}$ admits a *strong hierarchy*:

Dunbar's Theorem [**Dun2**]. — *Let $\mathcal{O}$ be a compact orientable Haken 3-orbifold. There is a finite sequence of pairs*

$$(\mathcal{O}_1, F_1) \to (\mathcal{O}_2, F_2) \to \ldots \to (\mathcal{O}_n, \emptyset),$$

such that:

i) $\mathcal{O}_1 = \mathcal{O}$

ii) F_i *is a 2-sided essential (connected orientable) 2-dimensional suborbifold in* $\mathcal{O}_i$ *which is not a turnover*

iii) *If* $\partial\mathcal{O}_i$ *is neither empty nor an union of turnovers, then* $\partial F_i \neq \emptyset$.

iv) $\mathcal{O}_{i+1} = \mathcal{O}_i \setminus int(\mathcal{N}(F_i))$ *(cut* $\mathcal{O}_i$ *along* F_i*).*

v) $\mathcal{O}_n$ *is a disjoint union of discal 3-orbifolds and Euclidean or hyperbolic thick turnovers (i.e* $\{turnover\} \times [0,1]$*).*

Remark 8.2.1. — Property iii) is not explicitly stated in Dunbar's paper. To find essential subordifolds, the key point in Dunbar's paper is to use Culler and Shalen's technique [**CS**] about curves of representations. This technique allows to construct suborbifolds satisfying property iii), because any nontrivial curve of representations of a hyperbolic manifold induces a nontrivial curve of representations of its boundary. This follows for instance from the results of [**Kap**, Ch. 9] .

The following proposition is a straightforward generalization of Waldhausen's theorem [**Wa3**]:

Proposition 8.2.2. — *Let* $\mathcal{O}$ *be a compact orientable Haken* 3*-orbifold. Then its universal cover* $\tilde{\mathcal{O}}$ *is homeomorphic to* $B^3 \setminus \Gamma$ *where* Γ *is a closed subset of* ∂B^3.

Proof. — To prove Proposition 8.2.2 it is sufficient to show that $\mathcal{O}$ is a good orbifold, because then the proof reduces to Waldhausen's proof (cf. [**Wa3**]).

The fact that a Haken 3-orbifold is good follows from [**Tak2**, Theorem A] by induction on the length of a strong hierarchy for $\mathcal{O}$. □

When $\mathcal{O}$ is a good 3-orbifold, we can use the following extensions of the Loop theorem and Dehn's Lemma, required in cut and paste methods for 3-orbifolds (cf. [**Tak2, TaY1**]). They are derived from the equivariant Dehn's Lemma (cf. [**JR**], [**MY1, MY2**]):

Dehn's Lemma. — *Let* $\mathcal{O}$ *be an orientable good 3-orbifold with boundary. Let* $\gamma \subset \partial\mathcal{O} - \Sigma$ *be a simple closed non singular curve. Assume that* γ *represents an element of finite order* n *in* $\pi_1(\mathcal{O})$. *Then there is a discal 2-suborbifold* Δ *properly embedded in* $\mathcal{O}$ *such that* $\partial\Delta = \gamma$. □

Loop Theorem. — *Let* $\mathcal{O}$ *be an orientable good 3-orbifold with boundary. Let* $F \subset \mathcal{O}$ *be a connected component. If* $\ker(\pi_1(F) \to \pi_1(\mathcal{O})) \neq \{1\}$, *then there is a discal 2-suborbifold* Δ *properly embedded in* $\mathcal{O}$ *such that* $\Delta \cap F = \partial\Delta$ *and* $\partial\Delta$ *does not bound a discal 2-suborbifold in* F. □

Corollary 8.2.3. — *Let* $\mathcal{O}$ *be an orientable good 3-orbifold. Let* $F \subset \mathcal{O}$ *be a properly embedded 2-sided incompressible 2-suborbifold. Then:*

i) $\ker(\pi_1(F) \to \pi_1(\mathcal{O})) = \{1\}$.

ii) If $\mathcal{O}'$ is any connected component of the 3-orbifold obtained by cutting open $\mathcal{O}$ along F, $\ker(\pi_1(\mathcal{O}') \to \pi_1(\mathcal{O})) = \{1\}$. □

Remark 8.2.4. — Since a Haken 3-orbifold is good and irreducible, by Corollary 8.2.3 either it is a discal 3-orbifold or it has an infinite fundamental group.

A direct consequence of the equivariant sphere theorem for 3-manifolds (cf. [**DD**], [**JR**], [**MY1, MY2**]) is:

Corollary 8.2.5. — *Let $\mathcal{O}$ be an orientable good 3-orbifold. If $\mathcal{O}$ is irreducible, then any manifold covering of $\mathcal{O}$ is irreducible.*

Remark 8.2.6. — To get the irreducibility of any orbifold covering in Corollary 8.2.5, one has to use the fact that every smooth action of a finite group on a 3-ball is conjugate to linear action. In the cyclic case that follows from the solution of the Smith Conjecture (cf. [**MB**]) and in the general case from the works of Meeks and Yau [**MY3**] and of Kwasik and Schultz [**KS**].

Definition 8.2.7. — Let $\mathcal{O}$ be an orientable 3-orbfifold. A subgroup in $\pi_1(\mathcal{O})$ is *peripheral subgroup* if it is conjugate to a subgroup of the fundamental group of a boundary component.

The following result is a generalization of Waldhausen'classical result [**Wa3**] due to Y. Takeuchi and M. Yokoyama [**TaY2**, Thm. 5.6] (cf. also [**Tak1**]):

Theorem 8.2.8. — *Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two orientable Haken 3-orbifolds with incompressible boundaries. Let $h\colon \pi_1(\mathcal{O}_1) \to \pi_1(\mathcal{O}_2)$ be an isomorphism which sends peripheral subgroups into peripheral subgroups. Then there is a homeomorphism between $\mathcal{O}_1$ and $\mathcal{O}_2$ that induces h.* □

Another fundamental result we need is the following:

Theorem 8.2.9 **(Torus Theorem).** — *Let $\mathcal{O}$ be a compact orientable good 3-orbifold. If $\pi_1(\mathcal{O})$ contains a subgroup $\mathbb{Z} \oplus \mathbb{Z}$ which is not peripheral, then either $\mathcal{O}$ contains an orientable essential toric 2-suborbifold or $\mathcal{O}$ is Euclidean or Seifert fibered.*

Proof of Theorem 8.2.9. — Like in Scott's proof [**Sc1**] of the Torus theorem for 3-manifolds, the first step is given by the following claim:

Claim 8.2.10. — *Let $\mathcal{O}$ be a compact orientable good irreducible 3-orbifold. Let $A = \mathbb{Z} \oplus \mathbb{Z} \subset \pi_1(\mathcal{O})$ be a non-peripheral subgroup. Then the covering $\hat{\mathcal{O}}$ of $\mathcal{O}$ associated to this $\mathbb{Z} \oplus \mathbb{Z}$ subgroup is a non-compact irreducible 3-manifold with at least two ends.* □

The proof of Claim 8.2.10 is the same as in the case of 3-manifold (cf. [**Sc1**]; see also [**Ja**, Thm.VII.2.2]), once one observes that $\hat{\mathcal{O}}$ must be a 3-manifold, because $\hat{\mathcal{O}}$ is a good irreducible 3-orbifold whose fundamental group $\mathbb{Z} \oplus \mathbb{Z}$ has no torsion. In particular $\hat{\mathcal{O}}$ has a compact core homeomorphic to $T^2 \times [0,1]$.

In fact by the wok of J. Hass, H. Rubinstein and P. Scott [**HRS**], one can even show that $\hat{\mathcal{O}}$ has a manifold compactification to $T^2 \times [0,1]$. In the Haken case, that follows already from J. Simon's theorem (cf. [**Ja**, Thm.VII.4]).

Therefore any infinite cyclic covering of $\hat{\mathcal{O}}$ has only one end, so we can apply Dunwoody and Swenson algebraic torus theorem [**DS**] to show that:

i) the group $\pi_1(\mathcal{O})$ either splits non-trivially over a virtual $\mathbb{Z} \oplus \mathbb{Z}$ subgroup, or
ii) $\pi_1(\mathcal{O})$ is an extension of a virtual $\mathbb{Z}$ by a virtual surface group, or
iii) $\pi_1(\mathcal{O})$ is virtually $\mathbb{Z}^3$.

One may remark that the virtual $\mathbb{Z} \oplus \mathbb{Z}$ subgroup obtained in case i) is not always commensurable to the original subgroup A.

Anyway, since $\mathcal{O}$ is a good orbifold, it follows from [**TaY1**] that the algebraic splitting given in case i) may be realized by a geometric one. Hence it corresponds to a splitting along an orientable essential toric 2-suborbifold, since its fundamental group is virtually $\mathbb{Z} \oplus \mathbb{Z}$.

In case ii) $\pi_1(\mathcal{O})$ is virtually the fundamental group of a compact orientable Seifert 3-manifold. Then it follows from Scott's result [**Sc2**] that it is finitely covered by a Seifert 3-manifold. Therefore, by [**MS**], it is a Euclidean or a Seifert fibered orbifold.

In case iii) the orbifold $\mathcal{O}$ is finitely covered by a closed irreducible 3-manifold, that is homeomorphic to the 3-torus T^3 by Waldhausen's theorem [**Wa3**]. Then it follows from [**MS**] that $\mathcal{O}$ is an Euclidean orbifold. □

The following homotopic characterization of compact orientable good 3-orbifolds, which are topologically atoroidal and not Euclidean nor Seifert fibered, is used in the proof of Theorem 3:

Proposition 8.2.11. — *Let $\mathcal{O}$ be a compact orientable irreducible good 3-orbifold with infinite fundamental group. If $\mathcal{O}$ is topologically atoroidal then either $\mathcal{O}$ is Euclidean, Seifert fibered, or $\pi_1(\mathcal{O})$ is not virtually abelian and every $\mathbb{Z} \oplus \mathbb{Z}$ subgroup is peripheral.*

Proof of Proposition 8.2.11. — If $\pi_1(\mathcal{O})$ contains a non peripheral $\mathbb{Z} \oplus \mathbb{Z}$ subgroup, then by Theorem 8.2.9 either $\mathcal{O}$ contains a 2-sided embedded essential toric 2-sub-orbifold, or it is Euclidean or Seifert fibered. Since $\mathcal{O}$ is topologically atoroidal, it must be Euclidean or Seifert fibered.

If $\pi_1(\mathcal{O})$ is virtually abelian, since $\mathcal{O}$ is good, irreducible and compact, it is finitely covered by a compact orientable irreducible 3-manifold M such that $\pi_1(M)$ is isomorphic to $\mathbb{Z}$, $\mathbb{Z} \oplus \mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ (cf. [**Ja**, Chap. V]). Since M is irreducible, it follows that M is homeomorphic to either $S^1 \times D^2$, $T^2 \times [0,1]$ or the 3-torus T^3. Hence $\mathcal{O}$ is Euclidean or Seifert fibered by [**MS**]. □

Remark 8.2.12. — In [**Mai**], S. Maillot has proved that a small orientable closed 3-orbifold, whose fundamental group contains an infinite cyclic normal subgroup, is Seifert fibered, hence geometric.

Definition 8.2.13. — We say that a compact orientable 3-orbifold $\mathcal{O}$ is *homotopically atoroidal* if $\pi_1(\mathcal{O})$ is not virtually abelian and every $\mathbb{Z}\oplus\mathbb{Z}$ subgroup is peripheral.

Now we can restate Thurston's hyperbolization theorem as:

Theorem 8.2.14 **(Thurston's hyperbolization for Haken 3-orbifolds)**
Let $\mathcal{O}$ be a compact orientable connected Haken 3-orbifold. If $\mathcal{O}$ is homotopically atoroidal, then $\mathcal{O}$ is hyperbolic.

The proof of this theorem follows exactly the scheme of the proof for Haken 3-manifolds (cf. [**Thu2, Thu3, Thu5**], [**McM1**], [**Kap**], [**Ot1, Ot2**]; see also [**Boi**] for an overview)

In the non-fibered case, the inductive gluing step, based on Thurston's fixed point theorem [**McM1**] (cf. [**Ot2**]) and Maskit's combination theorem [**Mas1, Mas2**], is carried in an analogous way. The initial step requires an equivariant and stronger version of Andreev's theorem (cf. [**An1, An2**], [**Thu1**, Ch. 13]).

In the fibered case over a 1-orbifold, either it is fibered over the circle or it is fibered over a compact interval with silvered ends. When it is fibered over the circle, the orbifold is of cyclic type and very good, so Theorem 8.2.14 follows already from our Theorem 1.

When it is fibered over a compact interval, there is a 2-fold covering that fibers over the circle, hence that is hyperbolic by the first case. Then using the argument given by Bonahon and Siebenmann in [**BS3**], as in Lemma 2.4.7, one shows that this covering involution is conjugate to an isometry. Another proof follows from Takeuchi and Yokoyama's generalization of Waldhausen's theorem (cf. Theorem 8.2.8).

8.3. Kleinian groups

We call *Kleinian group* a discrete subgroup of $PSL_2(\mathbb{C})$.

In the following we will always assume that Γ is non-elementary (i.e not virtually abelian) and is finitely generated.

The action of Γ on the hyperbolic space $\mathbb{H}^3$ extends to a conformal action on the sphere at infinity S^2_∞, which we identify with the Riemann sphere $\mathbb{C}\cup\{\infty\}$. The sphere at infinity S^2_∞ is partitioned into the *domain of discontinuity* Ω and the *limit set* Λ.

The domain of discontinuity Ω is the maximal open Γ-invariant subset of S^2_∞, on which Γ acts properly discontinuously. The limit set Λ is the closure of the set of fixed points of non-elliptic elements of Γ. It is the smallest non-empty closed Γ-invariant subset of S^2_∞.

The *Kleinian orbifold* $\overline{\mathcal{O}} := (\mathbb{H}^3 \cup \Omega)/\Gamma$ is an orientable 3-orbifold with a complete hyperbolic structure on its interior $\mathcal{O} = \mathbb{H}^3/\Gamma$ and a conformal structure on its boundary $\partial\overline{\mathcal{O}} = \Omega/\Gamma$.

Remark 8.3.1. — A Kleinian orbifold $\overline{\mathcal{O}}$ is always very good, by Selberg's Lemma [**Sel**] (cf. [**Rat**]).

Since Γ is not elementary, if the domain of discontinuity Ω is not empty, then it admits a unique complete metric of curvature -1 (called the Poincaré metric), which is conformal to the underlying Euclidean metric and such that Γ acts as a group of isometries of this metric. Thus $\partial\overline{\mathcal{O}} = \Omega/\Gamma$ is a hyperbolic 2-orbifold. Moreover, Ahlfors' finiteness theorem asserts that $\partial\overline{\mathcal{O}} = \Omega/\Gamma$ has finite area when Γ is finitely generated.

Proposition 8.3.2 **(Ahlfors Finiteness Theorem [Ah])**
Let Γ be a finitely generated Kleinian group which is not elementary. If $\Omega(\Gamma) \neq \emptyset$, then $\Omega(\Gamma)/\Gamma$ is a finite-area hyperbolic orientable 2-orbifold in the Poincaré metric. $\square$

The *convex core* of a hyperbolic 3-orbifold $\mathcal{O} = \mathbb{H}^3/\Gamma$ is the quotient $C(\mathcal{O}) = \widetilde{C}(\Lambda)/\Gamma$ of the convex hull $\widetilde{C}(\Lambda) \subset \mathbb{H}^3$ of the limit set Λ of Γ.

By construction, the convex core $C(\mathcal{O})$ is the smallest closed convex subset of $\mathcal{O}$ such that the inclusion map $i : C(\mathcal{O}) \to \mathcal{O}$ induces an orbifold-homotopy equivalence. Because of the convexity, any closed geodesic 1-suborbifold is contained in $C(\mathcal{O})$. In general $C(\mathcal{O})$ is not a differentiable suborbifold of $\mathcal{O}$ because $\partial C(\mathcal{O})$ is not smooth, but it is "bent" along some geodesic 1-suborbifolds. A way to avoid this difficulty is to consider a closed δ-neighborhood of $C(\mathcal{O})$, for $\delta > 0$:

$$C_\delta(\mathcal{O}) = \{x \in \mathcal{O} | d(x, C(\mathcal{O})) \leq \delta\}.$$

That is now a 3-suborbifold of $\mathcal{O}$ of class C^1, with a smooth strictly convex boundary. Moreover, $C_\delta(\mathcal{O})$ does not depend (up to diffeomorphism) of $\delta > 0$ and, for all $\delta > 0$, $C_\delta(\mathcal{O})$ is diffeomorphic to the Kleinian 3-orbifold $\overline{\mathcal{O}} = (\mathbb{H}^3 \cup \Omega(\Gamma))/\Gamma$. We call it the *thickened convex core*.

Remark 8.3.3. — Associated to a non-elementary Kleinian group we have:

1) The complete hyperbolic 3-orbifold $\mathcal{O} = \mathbb{H}^3/\Gamma$.
2) The Klenian orbifold (with boundary) $\overline{\mathcal{O}} = (\mathbb{H}^3 \cup \Omega(\Gamma))/\Gamma$.
3) The thickened convex core $C_\delta(\mathcal{O}) \subset \mathcal{O}$, where $C_\delta(\mathcal{O}) \hookrightarrow \mathcal{O}$ is an orbifold- homotopy equivalence.

Definition 8.3.4. — A complete hyperbolic 3-orbifold $\mathcal{O}$ is *geometrically finite* if for some (hence for every) $\delta > 0$ the volume of a thickened convex core is finite:

$$vol(C_\delta(\mathcal{O})) < \infty.$$

Remark 8.3.5. — This notion of geometrically finite hyperbolic 3-orbifold is the natural generalization of *uniform lattices* (i.e. compact finite volume hyperbolic 3-orbifolds). For a complete hyperbolic 3-orbifold $\mathcal{O} = \mathbb{H}^3/\Gamma$, an equivalent definition of geometrically finiteness is that some (hence every) Dirichlet fundamental domain of the Kleinian group Γ has a finite number of sides (cf. [**MT**, Chap. 3]).

Let Γ be a non-elementary Kleinian group. Given a point $x \in \mathbb{H}^3$ and a real number $\mu > 0$ one defines:

$$\Gamma_\mu(x) = \langle \gamma \in \Gamma \mid d(x, \gamma x) \leq \mu \rangle$$

$\Gamma_\mu(x)$ is the group generated by those elements of Γ which move the point x a distance at most μ.

Let $T_\mu(\Gamma) = \{x \in \mathbb{H}^3 \mid \Gamma_\mu(x) \text{ is infinite}\}$; it is a closed Γ-invariant subset of $\mathbb{H}^3$.

According to B. Bowditch [**Bow**], for a complete hyperbolic orientable 3-orbifold $\mathcal{O}$ with fundamental group Γ one defines the *μ-thin part* of $\mathcal{O}$ to be:

$$thin_\mu(\mathcal{O}) = T_\mu(\Gamma)/\Gamma$$

It is a closed subset of $\mathcal{O}$. The closure of $\mathcal{O} - thin_\mu(\mathcal{O})$ is the *μ-thick* part $thick_\mu(\mathcal{O})$.

There exists a universal constant *(the Margulis' constant)* $\mu_0 > 0$ such that for $\mu \leq \mu_0$ each connected component of the μ-thin part is either a *cuspidal end* of rank 1 or 2 (i.e corresponds to the Γ-conjugacy class of a maximal infinite virtually rank 1 or 2 parabolic subgroup of Γ) or is a tubular neighborhood of a closed geodesic 1-suborbifold of $\mathcal{O}$ (called a Margulis tube).

The union of the cuspidal ends of the μ-thin part (for $\mu \leq \mu_0$) is called the *cuspidal part* of $\mathcal{O}$ and denoted by $cusp(\mathcal{O})$.

If $\mathcal{O}$ is geometrically finite, then $vol(C_\delta(\mathcal{O})) < \infty$ and all the closed geodesic 1-suborbifolds are in $C_\delta(\mathcal{O})$. For this reason, there is no arbitrarily small closed geodesic. Hence there is a real number $\mu(\mathcal{O}) > 0$ such that $thin_\mu(\mathcal{O}) = cusp(\mathcal{O})$ for all $\mu < \mu(\mathcal{O})$.

The geometrically finiteness of $\mathcal{O}$ is equivalent to the fact that the μ-thick part $thick_\mu(C_\delta(\mathcal{O})) = C_\delta(\mathcal{O}) \cap thick_\mu(\mathcal{O})$ of a convex core is compact for $\mu < \mu(\mathcal{O})$. In this case the μ-thin part $thin_\mu(C_\delta(\mathcal{O})) = C_\delta(\mathcal{O}) \cap thin_\mu(\mathcal{O})$ is a disjoint union of finitely many cuspidal ends of finite volume.

Then each connected component of the μ-thin part $thin_\mu(C_\delta(\mathcal{O}))$ is isometric to the quotient of either a cylinder cusp (rank 1 cusp) or a torus cusp (rank 2 cusp) by a finite group of isometries. Thus it is isometric to $F \times [0, \infty)$ with the metric $e^{-2t}ds^2 + dt^2$, where (F, ds^2) is a compact orientable annular or toric flat 2-suborbifold. The possibilities for F are: an annulus $S^1 \times [0,1]$, a disk $D^2(\pi, \pi)$ with two branching points of order 2; a torus T^2; a pillow $S^2(\pi,\pi,\pi,\pi)$; a turnover $S^2(\frac{2\pi}{p_1},\frac{2\pi}{p_2},\frac{2\pi}{p_3})$, with $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}=1$.

For a geometrically finite hyperbolic 3-orbifold, the compact 3-orbifold pair

$$(M,P) := (thick_\mu(C_\delta(\mathcal{O})), \partial\, thick_\mu(C_\delta(\mathcal{O})) \cap thin_\mu(C_\delta(\mathcal{O})))$$

is independent of $\delta > 0$ and of $\mu < \mu(\mathcal{O})$. The suborbifold $P \subset \partial M$ is an union of toric and annular orientable 2-suborbifolds, coresponding to the track of the truncated cuspidal ends

The following proposition is a straightforward consequence of properties of geometrically finite Kleinian groups (cf. [**Mor1**]).

Proposition 8.3.6. — *Let $\mathcal{O}$ be an orientable geometrically finite hyperbolic 3-orbifold. The compact 3-orbifold pair*

$$(M,P) := (thick_\mu(C_\delta(\mathcal{O})), \partial\, thick_\mu(C_\delta(\mathcal{O})) \cap thin_\mu(C_\delta(\mathcal{O})))$$

has the following topological properties:

1) M is a compact orientable, irreducible and very good 3-orbifold.

2) $P \subset \partial M$ is a disjoint union of incompressible toric and annular 2-suborbifolds.

3) Every $\mathbb{Z}\oplus\mathbb{Z}$ subgroup in $\pi_1 M$ is conjugate to a subgroup of some $\pi_1(P_i)$, where $P_i \subset P$ is a connected component (such a $\mathbb{Z}\oplus\mathbb{Z}$ must be parabolic).

4) Any properly embedded annular 2-suborbifold $(A,\partial A) \hookrightarrow (M,P)$ whose boundary rests on essential curves in P is parallel to P. □

Definition 8.3.7. — An orbifold pair (M,P) which satisfies properties 1) to 4) of Proposition 8.3.6 is called a *pared 3-orbifold*. The 2-suborbifold P is called the *parabolic locus* of the pared orbifold, and the compact 2-suborbifold $\partial_0 M = \overline{\partial M \setminus P}$, the boundary of the pared orbifold.

This boundary $\partial_0 M$ is said to be *super-incompressible*, if it is incompressible and there is no embedded essential annular 2-suborbifold $(A,\partial A) \hookrightarrow (M,\partial M)$ with one boundary component on $\partial_0 M$ and the other on P.

The pared 3-orbifold (M,P) is said *Haken* when the 3-orbifold M is Haken.

For the rest of this chapter we use the following terminology:

Definition 8.3.8. — A pared 3-orbifold is *hyperbolic* if there exists an orientable geometrically finite hyperbolic 3-orbifold $\mathcal{O}$ such that for $\delta > 0$ and $0 < \mu \leq \mu(\mathcal{O})$:

$$(M,P) := (thick_\mu(C_\delta(\mathcal{O})), \partial\, thick_\mu(C_\delta(\mathcal{O})) \cap thin_\mu(C_\delta(\mathcal{O})))$$

Here is the precise version of Thurston's hyperbolization theorem, needed for the proof of Theorem 8.2.14.

Theorem 8.3.9 **(Thurston's hyperbolization for pared 3-orbifolds)**

Every compact orientable Haken pared 3-orbifold with a non-virtually abelian fundamental group $\pi_1(M)$ is hyperbolic.

***Remark 8.3.10* (cf. [Mor1]).** — Compact pared 3-orbifolds with virtually abelian fundamental groups correspond to the pairs: $(\Delta^3, \emptyset)$, with Δ^3 a discal 3-orbifold; $(F \times [0, \infty), F \times \{0\})$, with F a closed orientable toric 2-orbifold; (V, P), where V is a solid torus or pillow with possibly a ramified soul, and P is empty or an annular 2-suborbifold. They admit hyperbolic metric, but with infinite volume.

A compact orientable Haken 3-orbifold is built up inductively from discal 3-orbifolds and/or Euclidean or hyperbolic thick turnovers by gluing along incompressible suborbifolds of the boundary. So this version of Thurston's hyperbolization theorem for pared 3-orbifolds is needed at the inductive step, where pared 3-orbifolds appear naturally.

Thurston's *mirror trick* (see §8.6, [**Kap**], [**Ot2**], [**Pau**]) allows to reduce the gluing inductive step to the final gluing step where the whole boundary $\partial_0 M$ is involved. Hence at the inductive step one has a hyperbolic (perhaps not connected) pared 3-orbifold (M, P) with super-incompressible boundary and gluing instructions encoded by an orientation-reversing involution $\tau : \partial_0 M \to \partial_0 M$. In particular the manifold M/τ obtained after gluing has a boundary which is a (possibly empty) union of closed orientable toric 2-orbifolds.

We devote the next sections §8.4 and §8.5 to explain the proof of this gluing step. Then in §8.6, we explain the mirror trick.

8.4. Thurston's gluing theorem

The main step of Thurston's hyperbolization theorem for Haken 3-orbifolds is the following theorem

***Theorem 8.4.1* (Thurston's gluing theorem).** — *Let (M, P) be a pared 3-orbifold with super-incompressible non-empty boundary $\partial_0 M$. Let $\tau : \partial_0 M \to \partial_0 M$ be an orientation-reversing smooth involution which permutes the boundary components by pairs, i.e. $\tau = (f, f^{-1}) : \partial_0^+ M \amalg \partial_0^- M \to \partial_0^- M \amalg \partial_0^+ M$. Assume that each connected component of the pared orbifold (M, P) is hyperbolic. Then the quotient orbifold M/τ is hyperbolic if and only if M/τ is homotopically atoroidal.*

The proof of Thurston's gluing theorem splits into two totally different cases according whether or not the pared 3-orbifold (M, P) is a I-bundle over a compact 2-orbifold.

In the first case the quotient 3-orbifold M/τ is fibred over a closed 1-orbifold. As remarked before, this case can be handled using our Theorem 1.

So, from now on, we assume that the pared 3-orbifold (M, P) is not a I-bundle over a compact 2-orbifold. By Stallings's 3-dimensional h−cobordism theorem [**Sta**] (cf. [**Hem**, Chap. 10]) and the fact that finite group actions on a product are standard (cf. [**MS**]), this hypothesis is equivalent to the condition that $\pi_1(M)$ does not contain a finite index subgroup isomorphic to the fundamental group of an orientable 2-orbifold (cf. [**Tak1**], §8.6).

Using Maskit's combination theorem ([**Mas1, Mas2**]; cf. also [**Kap**], [**Ot2**]), we show now how Thurston reduces the proof of the gluing theorem to a fixed point theorem in the non-virtually fibered case.

We recall first the definition of the *Teichmüller space* $\mathcal{T}(F)$ of a connected hyperbolic orientable 2-orbifold F of topological type $(|F|, \Sigma_F)$, where Σ_F is a finite number of points. Two hyperbolic structures (F, s_1) and (F, s_2) are equivalent if there is an isometry $h : (F, s_1) \to (F, s_2)$, which is properly isotopic to the identity by an isotopy of the pair $(|F|, \Sigma_F)$. The *Teichmüller space* $\mathcal{T}(F)$ is the set of equivalence classes of hyperbolic structures with finite area on $\text{int}(F)$.

There is a natural distance on $\mathcal{T}(F)$. The *Teichmüller distance* between two equivalence classes represented by (F, s_1) and (F, s_2) is defined by:

$$\Delta(s_1, s_2) = \frac{1}{2} inf\{log(K(h))\},$$

where the infimum is taken among all the quasiconformal homeomorphims $h : (F, s_1) \to (F, s_2)$ that are properly isotopic to the identity by an isotopy of the pair $(|F|, \Sigma_F)$, and where $K(h)$ is the eccentricity of h. This distance turns $\mathcal{T}(F)$ into a complete metric space.

In general the Teichmüller space $\mathcal{T}(F)$ of a possibly non connected orientable hyperbolic 2-orbifold F is the product of the Teichmüller spaces of the connected components. The Teichmüller distance is then defined as the maximum of the distances on the Teichmüller spaces of the components.

We introduce now the *space of hyperbolic structures* of a hyperbolic pared 3-orbifold.

Definition 8.4.2. — Let (M, P) be a connected orientable pared 3-orbifold. Let $\Sigma \subset M$ be the ramification locus. A hyperbolic structure on (M, P) is a pair $(\mathcal{O}, [f])$, where $\mathcal{O}$ is a geometrically finite hyperbolic 3-orbifold and $[f]$ is a proper homotopy class of orientation preserving homeomorphisms

$$f : (M, P) \to (thick_\mu(C_\delta(\mathcal{O})), \partial\, thick_\mu(C_\delta(\mathcal{O})) \cap thin_\mu(C_\delta(\mathcal{O}))),$$

for $\delta > 0$ and $0 < \mu \leq \mu(\mathcal{O})$. Two hyperbolic structures $(\mathcal{O}_1, [f_1])$ and $(\mathcal{O}_2, [f_2])$ are equivalent if $(f_2)^{-1} \circ f_1$ is properly homotopic to an isometry.

The set of equivalence classes of hyperbolic structures on the hyperbolic pared 3-orbifold (M, P) is denoted $\mathcal{GF}(M, P)$.

In the non-connected case, $\mathcal{GF}(M, P)$ is the product $\prod \mathcal{GF}(M_i, P_i)$ of the spaces $\mathcal{GF}(M_i, P_i)$ of the connected components M_i of M.

Remark 8.4.3. — 1) If M is connected and if $(\mathcal{O}_1, [f_1])$ and $(\mathcal{O}_2, [f_2])$ are two hyperbolic structures on M, then by A. Marden [**Mar**] the proper homotopy class of $(f_2)^{-1} \circ f_1$ can be realized by a homeomorphism that extends to a quasi-conformal homeomorphism between the boundaries of the associated Kleinian orbifolds $\overline{\mathcal{O}_1}$ and $\overline{\mathcal{O}_2}$. We call such a homeomorphism a *quasiconformal extension* in the proper homotopy class $[(f_2)^{-1} \circ f_1]$. Thus the set $\mathcal{GF}(M,P)$ is also called the *space of quasi-conformal deformations* of the hyperbolic pared 3-orbifold (M,P)

This allows to define the Teichmüller distance Δ on $\mathcal{GF}(M,P)$ by:

$$\Delta((\mathcal{O}_1,[f_1]),(\mathcal{O}_2,[f_2])) = \frac{1}{2} inf\{log(K(\phi))\},$$

where the infimum is taken over all the quasi-conformal extensions $\phi : \overline{\mathcal{O}_1} \to \overline{\mathcal{O}_2}$ in the proper homotopy class $[(f_2)^{-1} \circ f_1]$, and $K(\phi)$ is the eccentricity of the restriction $\phi : \partial\overline{\mathcal{O}_1} \to \partial\overline{\mathcal{O}_2}$.

Moreover, Teichmüller's theory shows that this infimum is realized by a unique quasi-conformal extension in the proper homotopy class of $[(f_2)^{-1} \circ f_1]$.

2) In the non-connected case, the distance on the product space $\mathcal{GF}(M,P)$ is defined as the maximum of the Teichmüller distances on the factors. This distance turns $\mathcal{GF}(M,P)$ into a complete metric space.

Let (M,P) be an orientable hyperbolic pared 3-orbifold. The Ahlfors-Bers map $\partial : \mathcal{GF}(M,P) \to \mathcal{T}(\partial_0 M)$ assigns to an equivalence class of hyperbolic structures $(\mathcal{O},[f]) \in \mathcal{GF}(M,P)$ the equivalence class in $\mathcal{T}(\partial_0 M)$ of finite area hyperbolic structures induced on the boundary $\partial\overline{\mathcal{O}}$ of the corresponding Kleinian 3-orbifold $\overline{\mathcal{O}}$ (by Ahlfors finiteness theorem).

Ahlfors-Bers theorem gives a parametrization of the quasi-conformal deformation space $\mathcal{GF}(M)$ by the Teichmüller space $\mathcal{T}(\partial_0 M)$.

***Proposition 8.4.4* (Ahlfors-Bers Theorem).** — *Let (M,P) be an orientable hyperbolic pared 3-orbifold, with non-empty boundary $\partial_0 M$. The map $\partial : \mathcal{GF}(M,P) \to \mathcal{T}(\partial_0 M)$ is a homeomorphism.*

Remark 8.4.5. — If $\partial_0 M$ is empty then $\mathcal{GF}(M,P)$ is reduced to a point by Mostow's rigidity theorem [**Mos**].

Proof of Proposition 8.4.4. — The proof of Proposition 8.4.4 follows from the classical case of manifolds [**AB**], by using the following equivariant definitions of $\mathcal{GF}(M,P)$ and $\mathcal{T}(\partial_0 M)$.

Since M is a hyperbolic 3-orbifold, it is very good. Let $q : N \to M$ be a finite regular manifold covering of M. Let $Q = q^{-1}(P)$, then (N,Q) is a hyperbolic pared 3-manifold.

Let $G \subset Diff^+(N,Q)$ be the covering group of transformations. It has natural isometric actions on the deformation spaces $\mathcal{GF}(N,Q)$ and $\mathcal{T}(\partial_0 N)$, depending only on the image of G in the mapping class group $\pi_0 Diff(N,Q)$. Moreover the spaces $\mathcal{GF}(M,P)$ and $\mathcal{T}(\partial_0 M)$ can be identified respectively with the fixed points set $\mathcal{GF}(N,Q)^G$ and $\mathcal{T}(\partial_0 N)^G$ of these actions. From now on, we will always make these identifications, that allow to consider $\mathcal{GF}(M,P)$ and $\mathcal{T}(\partial_0 M))$ as metric subspaces of $\mathcal{GF}(N,Q)$ and $\mathcal{T}(\partial_0 N)$.

By the Bers embedding, the Teichmüller space $\mathcal{T}(\partial_0 N)$ carries a complex structure. Hence the identification of $\mathcal{T}(\partial_0 M)$ with $\mathcal{T}(\partial_0 N)^G$ allows to consider $\mathcal{T}(\partial_0 M)$ as a complex submanifold of $\mathcal{T}(\partial_0 N)$, since the action of G is by holomorphic transformations on $\mathcal{T}(\partial_0 N)$, cf. [**Gard**].

By definition of the Ahlfors-Bers map $\overline{\partial} : \mathcal{GF}(N,Q) \to \mathcal{T}(\partial_0 N)$, one has the inclusion $\overline{\partial}(\mathcal{GF}(N,Q)^G) \subset \mathcal{T}(\partial_0 N)^G$. Moreover, its restriction is the Ahlfors-Bers map $\partial : \mathcal{GF}(M,P) \to \mathcal{T}(\partial_0 M)$. Then the bijectivity of ∂ follows from the proof of the bijectivity of $\overline{\partial}$. $\square$

From now on we assume that (M,P) is a compact orientable pared 3-orbifold with *super-incompressible boundary* $\partial_0 M$.

We define Thurston's skinning map, $\sigma : \mathcal{T}(\partial_0 M) \to \mathcal{T}(\partial_0 \widetilde{M})$, where $\partial_0 \widetilde{M}$ is the boundary of the pared 3-orbifold with the reversed orientation. To define this map we use the following notation: for a connected component $F \subset \partial_0 M$, the index F specifies the projection on the factor $\mathcal{T}(F)$ of $\mathcal{T}(\partial_0 M)$.

By the Ahlfors-Bers theorem, to a point $s \in \mathcal{T}(\partial_0 M)$ corresponds a geometrically finite hyperbolic structure $(\mathcal{O}_s,[f_s]) \in \mathcal{GF}(M,P)$. The covering $\mathcal{O}_s^F$ of $\mathcal{O}_s$, associated to the component $F \subset \partial_0 M$ determines a quasi-Fuchsian structure on the product pared 3-orbifold $(F,\partial F) \times [0,1]$, because the Kleinian group $\pi_1(\mathcal{O}_s)$ has no accidental parabolic. This is a consequence of the fact that $\partial_0 M$ is super-incompressible.

Let $(s_F, s'_F) \in \mathcal{T}(F) \times \mathcal{T}(\widetilde{F})$ be the Ahlfors-Bers parameters of this quasi- Fuchsian structure, where $s_F \in \mathcal{T}(F)$ is the coordinate on the factor $\mathcal{T}(F)$ of $s \in \mathcal{T}(\partial_0 M)$.

Definition 8.4.6. — The *skinning map* $\sigma : \mathcal{T}(\partial_0 M) \to \mathcal{T}(\partial_0 \widetilde{M})$ is defined by $\sigma(s)_F = s'_F$ for every point $s \in \mathcal{T}(\partial_0 M)$.

In the following we denote by $\tau^\star : \mathcal{T}(\partial_0 M) \to \mathcal{T}(\partial_0 \widetilde{M})$ the involution induced by the involution τ.

Thurston has reformulated Maskit's combination theorem [**Mas1, Mas2**] into a fixed point criterium for the map $\tau^\star \circ \sigma : \mathcal{T}(\partial_0 M) \to \mathcal{T}(\partial_0 M)$.

***Proposition 8.4.7* (Gluing criterium).** — *Let (M,P) be a hyperbolic pared 3-orbifold, with super-incompressible non-empty boundary $\partial_0 M$ and which is not an interval bundle. Let $\tau : \partial_0 M \to \partial_0 M$ be an orientation-reversing smooth involution which permutes the boundary components by pairs. If the map $\tau^\star \circ \sigma : \mathcal{T}(\partial_0 M) \to \mathcal{T}(\partial_0 M)$ has a fixed point, then the quotient orbifold M/τ is hyperbolic* $\square$

Remark 8.4.8. — The proof follows from the definition of the skinning map and Maskit's combination theorem. The algebraic part of Maskit's combination theorem remains valid for Kleinian group with torsion. For the topological part one can invoke Takeuchi's generalization of Waldhausen's theorem (cf. Theorem 8.2.8). In fact following [**Ot2**, §2.1], only the trivial I-bundle case is needed ([**Tak1**, §6], [**TaY2**]) together with the Baer-Nielsen's theorem (cf. [**Zie**]).

This gluing criterium shows that the gluing theorem 8.4.1 is equivalent to a fixed point theorem, that we discuss in the next section.

8.5. Thurston's Fixed Point Theorem

The goal of this section is to explain the proof the following theorem, that implies the gluing theorem 8.4.1.

Theorem 8.5.1 **(Thurston's fixed point theorem).** — *Let (M,P) be a hyperbolic pared 3-orbifold, with super-incompressible non-empty boundary $\partial_0 M$ and which is not an interval bundle. Let $\tau : \partial_0 M \to \partial_0 M$ be an orientation-reversing smooth involution which permutes the boundary components by pairs. Then the map $\tau^\star \circ \sigma : \mathcal{T}(\partial_0 M) \to \mathcal{T}(\partial_0 M)$ has a fixed point if and only if the quotient 3-orbifold M/τ is homotopically atoroidal.*

Since $\mathcal{T}(\partial_0 M)$ is complete with respect to the Teichmüller distance, to prove Thurston's fixed point Theorem 8.5.1 one has to study the contraction properties of the map $\tau^\star \circ \sigma$. It is a holomorphic map on the Teichmüller space $\mathcal{T}(\partial_0 M)$ such that $\|d(\tau^\star \circ \sigma)\| \leq 1$, since the Teichmüller metric coincides with the Kobayashi metric (cf. [**Gard**]).

Moreover, since M is not an interval bundle, $\tau^\star \circ \sigma$ strictly (but not uniformly) decreases the Teichmüller distance. This follows from the facts that $\tau^\star$ is an isometry and that $\|d\sigma\| < 1$ pointwise.

Coming back to the notation of §8.4, let $q : N \to M$ be a finite regular manifold covering of M. Let $Q = q^{-1}(P)$, then (N, Q) is a hyperbolic pared 3-manifold, which is not an interval bundle.

Let $G \subset Diff^+(N, Q)$ be the covering group of transformations. Then the Teichmüller space $\mathcal{T}(\partial_0 M)$ can be identified (as a metric subspace) with the fixed points set $\mathcal{T}(\partial_0 N)^G$ of the natural isometric action of G on $\mathcal{T}(\partial_0 N)$. We recall that the Teichmüller space $\mathcal{T}(\partial_0 N)$ carries a complex structure, on which G acts holomorphically. Whence we can consider $\mathcal{T}(\partial_0 M)$ as a complex submanifold of $\mathcal{T}(\partial_0 N)$. Moreover it is a convex subset for the Teichmüller metric: the Teichmüller geodesic between two points of $\mathcal{T}(\partial_0 N)^G$ is contained in $\mathcal{T}(\partial_0 N)^G$.

We consider the skinning map of the manifold covering $\overline{\sigma} : \mathcal{T}(\partial_0 N) \to \mathcal{T}(\partial_0 \widetilde{N})$. Then $\overline{\sigma}(\mathcal{T}(\partial_0 N)^G) \subset \mathcal{T}(\partial_0 \widetilde{N})^G$, since by the Ahlfors-Bers map $\overline{\partial}(\mathcal{GF}(N, Q)^G) = \mathcal{T}(\partial_0 N)^G$. By construction, the restriction of $\overline{\sigma}$ to $\mathcal{T}(\partial_0 N)^G$ coincides with σ.

Moreover, the involution $\tau : \partial_0 M \to \partial_0 M$ induces an isometry $\tau^\star : \mathcal{T}(\partial_0 N)^G \to \mathcal{T}(\partial_0 \widetilde{N})^G$, that may not extend to the whole Teichmüller space $\mathcal{T}(\partial_0 N)$.

Since $\tau^\star$ is an isometry, to prove Thurston's fixed point Theorem 8.5.1, one has only to study the contraction properties (with respect to the Teichmüller distance) of the restriction of $\overline{\sigma}$ to $\mathcal{T}(\partial_0 N)^G$.

We will use McMullen's detailed analysis of the derivative and coderivative of the skinning map of the manifold covering $\overline{\sigma} : \mathcal{T}(\partial_0 N) \to \mathcal{T}(\partial_0 \widetilde{N})$, to study the contraction properties of the map $\tau^\star \circ \sigma$. (cf. [**McM1**]; see also [**Ot2**], [**OP**]).

For example, if the pared 3-orbifold (M, P) is acylindrical, the quotient 3-orbifold M/τ is always homotopically atoroidal. The pared 3-orbifold (M, P) is acylindrical iff the pared 3-manifold (N, Q) is acylindrical. McMullen [**McM1**] shows that in this case the skinning map $\overline{\sigma}$ contracts strictly uniformly the Teichmüller distance on $\mathcal{T}(\partial_0 N)$. Therefore $\tau^\star \circ \overline{\sigma}$ contracts strictly uniformly the Teichmüller distance on the closed subspace $\mathcal{T}(\partial_0 N)_G$, hence has a fixed point since Teichmüller space is complete.

When the pared 3-orbifold (M, P) contains an essential annulus, some gluing involutions τ may produce homotopically non-atoroidal 3-orbifolds M/τ. Therefore one must take account of τ in the proof of the fixed point theorem. This is why in this case one proves only that some fixed iterate $(\tau^\star \circ \sigma)^K$ is strictly uniformly contracting on some $\tau^\star \circ \sigma$-invariant closed subset of $\mathcal{T}(\partial_0 M)$, as in the classical manifold case.

In the following, we fix $K = C + S$, where C is the number of components of $\partial_0 M$ and S is the maximal number of homotopy classes of disjoint simple closed curves or arcs with silvered end points in $\partial_0 M$, not parallel to ∂P.

We choose an arbitrary point $s_0 \in \mathcal{T}(\partial_0 M)$ and we denote by L the Teichmüller distance $\Delta(s_0, \tau^\star \circ \sigma(s_0))$. We define $\mathcal{T}(\partial_0 M)_L \subset \mathcal{T}(\partial_0 M)$ to be the set of points which are moved a Teichmüller distance less than L by the map $\tau^\star \circ \sigma$. Then $\mathcal{T}(\partial_0 M)_L$ is a closed $\tau^\star \circ \sigma$-invariant subset since $\tau^\star \circ \sigma$ decreases the distance. Moreover for every point $s \in \mathcal{T}(\partial_0 M)_L$ the Teichmüller geodesic between s and $\tau^\star \circ \sigma(s)$ lies in $\mathcal{T}(\partial_0 M)_L$ because of the triangle inequality.

The following theorem is a consequence of McMullen's work [**McM1**] (cf. [**Ot2**], [**OP**]).

Theorem 8.5.2. — *Let (M, P) be a hyperbolic pared 3-orbifold, satisfying the hypothesis of the fixed point theorem 8.5.1 and such that the quotient 3-orbifold M/τ is homotopically atoroidal. Let $\mathcal{T}(\partial_0 M)_L \subset \mathcal{T}(\partial_0 M)$ and the integer $K > 0$ be defined as above. Then there is a uniform constant $c < 1$ such that the norm of the derivative of $(\tau^\star \circ \sigma)^K$ at every point $s \in \mathcal{T}(\partial_0 M)_L$ verifies $\|d(\tau^\star \circ \sigma)^K_s\| \le c < 1$.*

Theorem 8.5.2 shows that the map $(\tau^\star \circ \sigma)^K$ is uniformly strictly contracting on the path formed by the union of all the positive iterates by $(\tau^\star \circ \sigma)^K$ of the Teichmüller geodesic joining s_0 to $\tau^\star \circ \sigma(s_0)$. Since $\mathcal{T}(\partial_0 M)$ is complete, that implies the existence of a fixed point for $(\tau^\star \circ \sigma)^K$, and hence for $(\tau^\star \circ \sigma)$.

We give now a sketch (without details) of the proof of Theorem 8.5.2.

Sketch of the proof of Theorem 8.5.2. — We recall that the *moduli space* $\mathcal{M}(\partial_0 N)$ is the quotient of the Teichmüller space by the natural action of the mapping class group. By Baer's theorem (cf. [**Zie**]) the moduli space $\mathcal{M}(\partial_0 M)$ can be identified with the image of $\mathcal{T}(\partial_0 N)^G$ in $\mathcal{M}(\partial_0 N)$. In particular it is a closed subspace of $\mathcal{M}(\partial_0 N)$.

One of the key results in McMullen's article [**McM1**, Thm 5.3] is:

***Theorem 8.5.3* (McMullen's contraction Theorem)).** — *If the pared 3-manifold (N, Q) is not an interval bundle, then for any point $s \in \mathcal{T}(\partial_0 N)$*

$$||d\overline{\sigma}_s|| < c([s]) < 1,$$

where c is a continuous function of the modular class of s, $[s] \in \mathcal{M}(\partial_0 M)$. □

Since σ is a contraction and τ is an isometry, it follows that if $||d(\tau^\star \circ \sigma)^K_s||$ is near 1, then $||d(\sigma)_{(\tau^\star\circ\sigma)^k(s)}||$ is also near 1 for all $0 \leq k \leq K-1$. Then by the discussion above and by using Mumford's compactness theorem [**Mum1**], a straightforward corollary of Theorem 8.5.3 is:

Corollary 8.5.4. — *If the pared 3-orbifold (M, P) is not an interval bundle, then either Theorem 8.5.2 is true, or there are points $s \in \mathcal{T}(\partial_0 M)_L$ such that the hyperbolic 2-orbifolds $(\partial_0 M, (\tau^\star \circ \sigma)^k(s))$ develop closed short geodesics, for $0 \leq k \leq K-1$.* □

Remark 8.5.5. — In our context, a closed geodesic has to be understood in the orbifold sense: it is either a closed curve or an arc with silvered end-points.

One proves now Theorem 8.5.2 by contradiction. Let $s \in \mathcal{T}(\partial_0 M)_L$ be such that $||d(\tau^\star \circ \sigma)^K_s|| \geq 1 - \eta$ for some $\eta > 0$ sufficiently small (depending only on the pared 3-orbifold (M, P)). Choose $\varepsilon > 0$ such that $\log(\varepsilon) + 2KL \leq \log(\mu_0(2))$, where $\mu_0(2)$ is the 2-dimensional Margulis constant.

Using Corollary 8.5.4 and following [**McM1**, §7.3], (cf. also [**Ot2**, Facts 6.13 to 6.15]), one shows the existence of an integer $0 \leq k \leq C-1$ such that the hyperbolic 2-orbifold $(\partial_0 M, s_0)$, with $s_0 = (\tau^\star \circ \sigma)^k(s)$ has the following properties:

i) it contains a closed geodesic α_0 shorter than $\varepsilon/2$;
ii) there is an essential immersed annular 2-orbifold A_0 in the pared 3-orbifold (M, P) with two boundaries components $\partial A_0 = \alpha_0 \cup \gamma_0 \subset \partial_0 M$, where γ_0 is homotopic to a closed geodesic shorter than $\varepsilon/2$ for the hyperbolic metric $\sigma(s_0)$ on $\partial_0 M$.

The length of the geodesic α_1 homotopic to $\tau(\gamma_0)$ for the hyperbolic metric $(\tau^\star \circ \sigma)(s_0)$ on $\partial_0 M$ is equal to the length of the geodesic homotopic to γ_0 for the hyperbolic metric $\sigma(s_0)$, which by construction of γ_0 has length shorter than $\varepsilon/2$ (cf. [**McM1**, §7.3], [**Ot2**, Fact 6.14]). Then one shows that the closed geodesic α_1 on the hyperbolic surface $(\partial_0 M, \tau^\star \circ \sigma)(s_0))$ verifies property ii) (cf. [**McM1**, §7.3], [**Ot2**, Facts 6.13 and 6.14]).

By iterating this construction, one produces a sequence of $S+1$ immersed essential annular 2-orbifolds A_i, $i = 0, \dots, S$, in the pared 3-orbifold (M, P) joining two closed 1-orbifolds α_i and γ_i in $\partial_0 M$ such that:

1) α_i is a closed geodesic, shorter than $\varepsilon/2$, for the hyperbolic metric $(\tau^\star \circ \sigma)^i(s_0)$ on $\partial_0 M$.
2) $\tau \circ (\gamma_i)$ is homotopic to (α_{i+1}).

Since the closed geodesic α_i is shorter than $\varepsilon/2$ for the hyperbolic metric $(\tau^\star \circ \sigma)^i(s_0)$ on $\partial_0 M$, it has length at most $\mu_0(2)$ for the hyperbolic metric s_0. That follows from the choice of ε and the fact that the Teichmüller distance $\Delta(s_0, \tau^\star \circ \sigma)^i(s_0))$ is at most $L\,K$ by the triangle inequality.

By Margulis lemma, the closed geodesics of length less than $\mu_0/2$ on the hyperbolic 2-orbifold $(\partial_0 M, s_0)$ is a collection of disjoint closed simple curves or simple arcs with silvered end points. Hence, by the definition of the integer S, the closed 1-orbifolds α_i, $i = 0, \dots, S$, belongs to at most S homotopy classes on $\partial_0 M$. Therefore at least two among the closed 1-orbifolds α_i, say α_m and α_n, must be homotopic on $\partial_0 M$. It follows that the result of gluing with τ the boundaries of the annular 2-orbifolds A_i, for $m \leq i \leq n$, can be closed up by an homotopy between α_m and α_n. In contradiction with the hypothesis, this would produce an essential map of a toric 2-orbifold into the 3-orbifold M/τ, because by construction each annular 2-orbifold A_i is essential into the pared 3-orbifold (M, P). □

8.6. Thurston's Mirror Trick

In this section we describe the topological part of the inductive step, which allows to reduce its proof to Thurston's gluing Theorem 8.4.1. It is in this part of the proof that the existence of a finite strong hierarchy for a Haken 3-orbifold is used. It allows to associate to any orientable compact Haken 3-orbifold a *complexity*, and thus to argue by induction on this complexity.

To define strong hierarchies adapted to the notion of pared 3-orbifolds, we need the following definitions:

Definition 8.6.1. — A compact properly embedded 2-suborbifold $(F, \partial F) \hookrightarrow (M, \partial M)$ is *super-essential* in a compact orientable pared 3-orbifold (M, P) if it satisfies the following properties:

- F is essential in M.
- There is no essential embedded annulus $(S^1 \times [0,1], S^1 \times \{0\}, S^1 \times \{1\}) \hookrightarrow (M, \partial M, F)$, such that the boundary component $S^1 \times \{1\} \subset F$ is not parallel to ∂F.
- Any connected component of ∂F isotopic to a loop in ∂P already lies in P.
- ∂F meets transversely ∂P in the minimum number of points in its isotopy class.

Definition 8.6.2. — An *orbifoldbody* is a compact orientable Haken 3-orbifold that can be cut along a (possibly empty) collection of disjoint two-sided properly embedded discal 2-suborbifolds into a disjoint union of discal 3-orbifolds and/or thick turnovers.

Such a minimal collection of disjoint two-sided properly embedded discal 2-suborbifolds is called a *complete system of meridian disks* for the orbifoldbody.

Remark 8.6.3. — It follows from [**Dun2**] that a Haken 3-orbifold is an orbifoldbody iff any properly embedded orientable essential 2-suborbifold in it is a discal 2-suborbifold.

The following proposition introduces a notion of hierarchy for Haken pared 3-orbifolds, which follows from Dunbar's construction of a strong hierarchy for Haken 3-orbifolds [**Dun2**, Thms. 10, 11, 12] (see also [**Mor1**, §4], [**Ot2**, §7] and [**Pau**] for the case of 3-manifolds). When the relative homology group with rational coefficients $H_1(|M|, \partial|M|; \mathbb{Q})$ vanishes, to construct the desired super-essential splitting 2-suborbifold one uses Culler and Shalen's method via ideal points of the character variety of the hyperbolic pared 3-orbifold (M, P) [**CS**].

Proposition 8.6.4. — *A compact orientable Haken pared 3-orbifold has a partial hierarchy of the following type: there is a finite sequence of compact orientable pared 3-orbifolds (M_0, P_0),...,(M_n, P_n) such that:*

i) $(M_0, P_0) = (M, P)$;

ii) for $k \leq n-1$, there is a connected super-essential 2-dimensional suborbifold $F_k \subset M_k$ which is not discal nor a turnover; moreover if $\partial_0 M_k$ is neither empty nor an union of turnovers, $\partial F_k \neq \emptyset$;

iii) M_{k+1} is the 3-orbifold obtained by splitting M_k along F_k. Moreover, P_{k+1} is the union of toric and annular 2-orbifolds, obtained by cutting P_k along ∂F_k and by forgetting the components that are discal;

iv) M_n is an orbifoldbody. □

The integer n is called the *length* of the hierarchy. It has an upperbound depending only on the pared 3-orbifold, because of the orbifold version of Haken-Kneser finiteness theorem (cf. [**Dun2**, Thm. 12]). We associate to a Haken pared 3-orbifold (M, P) the integer $\ell(M, P)$ that is the greatest possible length for a strong hierarchy given by Proposition 8.6.4. We call it the *length* of (M, P).

A compact connected orientable pared 3-orbifold (M, P) of minimal length is an orbifoldbody with parabolic locus a (possibly empty) collection of incompressible toric and annular 2-orbifolds. Moreover the length of a pared 3-orbifold decreases strictly by cutting along any connected properly embedded super-essential 2-suborbifold which is not discal, nor a turnover.

Thurston's mirror trick consists in associating to a compact orientable pared 3-orbifold (M, P) a compact orientable irreducible atoroidal *mirrored* 3-orbifold with boundary a (possibly empty) union of toric 2-orbifolds.

Definition 8.6.5. — A *mirrored 3-orbifold* is a pair $(\hat{M}, H)$ where $\hat{M}$ is a compact orientable irreducible 3-orbifold with boundary a (possibly empty) union of toric 2-orbifolds, and $H = (\mathbb{Z}_2)^k \subset Diff(\hat{M})$ is a non-trivial finite abelian group that is generated by orientation reversing symmetries through properly embedded orientable essential 2-suborbifolds in $\hat{M}$.

We call the non-orientable quotient 3-orbifold $\mathcal{K} = \hat{M}/H$ the *kaleidoscope* associated to the mirrored 3-orbifold $(\hat{M}, H)$.

Remarks on kaleidoscopes. — A kaleidoscope $\mathcal{K} = \hat{M}/H$ is a non-orientable 3-orbifold that is locally modeled on the quotient of $\mathbb{R}^3$ by one or the following groups:

i) a finite cyclic rotation group,
ii) a finite abelian group generated by a rotation and a reflection through a coordinate plane orthogonal to the axis of the rotation,
iii) a subgroup of the group of eight elements generated by the reflections through the three coordinate planes.

The ramification locus $\Theta \subset \mathcal{K}$ is the union of a 1-suborbifold Σ, that is the quotient $\hat{\Sigma}/H$ of the ramification locus $\hat{\Sigma}$ of $\hat{M}$, with a 2-suborbifold whose underlying space is $\partial|\mathcal{K}| - int(|\partial\mathcal{K}|)$.

The interior points with an isotropy group generated by reflections through two or three coordinate planes, together with the boundary points with an isotropy group generated by a reflection through a coordinate plane form a 1-suborbifold $\mathcal{G} \subset \Theta$. Its underlying space is a trivalent graph in $\partial|\mathcal{K}| - int(|\partial\mathcal{K}|)$ whose vertices correspond to interior points with isotropy group $(\mathbb{Z}/2\mathbb{Z})^3$ or boundary points with isotropy group $(\mathbb{Z}/2\mathbb{Z})^2$. We call the closure of a connected component of $\Theta - \{\mathcal{G} \cup \Sigma\}$ a *mirror* of $\mathcal{K}$.

One can see the 3-orbifold $\mathcal{K} = \hat{M}/H$ as a right angled kaleidoscope obtained by *silvering* the connected components of $\partial_0 M - \mathcal{G}$ that are neither a *square*, nor a *bigon* with a branching point. It is F. Bonahon's observation that for the proof of Thurston's hyperbolization theorem for Haken 3-orbifolds we need to consider only this kind of kaleidoscopes.

Conversely, according to [**Mor1**, Lemma 14.1] or [**Ot2**, §7.2] a Kaleidoscope can always be mirrored along its silvered faces to get a compact orientable mirrored 3-orbifold.

This dictionary between mirrored 3-orbifolds and kaleidoscopes will be very useful to prove some equivariant properties of mirrored 3-orbifolds.

Definition 8.6.6. — The *underlying type* of a mirrored 3-orbifold $(\hat{M}, H)$ or of the associated kaleidoscope $\mathcal{K} = \hat{M}/H$ is the pair (M, P), where M is the compact orientable 3-orbifold obtained from the kaleidoscope $\mathcal{K}$ by *erasing* the mirrors, and P is obtained from $\partial\mathcal{K}$ by *discarding* the components which are either a square or a bigon with a branching point. Since $H \neq \{1\}$, $\partial_0 M = \partial|\mathcal{K}| - P$ is not empty. The ramification locus of the 3-orbifold M is obtained from $\Sigma = \hat{\Sigma}/H$ by forgetting the silvering at its end points. In particular the topological type of the 3-orbifold M is given by the pair $(|\mathcal{K}|, \Sigma)$.

Definition 8.6.7. — Given a compact orientable pared 3-orbifold (M, P), we denote by $\mathcal{AM}(M, P)$ the set of homotopically atoroidal mirrored 3-orbifolds with underlying type (M, P).

Proposition 8.6.8. — *Let (M, P) be a compact orientable pared 3-orbifold, then there is a homotopically atoroidal mirrored 3-orbifold $(\hat{M}, H) \in \mathcal{AM}(M, P)$ such that $\partial\hat{M}/H = P$.*

Proof of Proposition 8.6.8. — Since $H \neq \{1\}$, $\partial_0 M \neq \emptyset$. Then, the proof of Proposition 8.6.8 reduces to the construction of a trivalent graph $\mathcal{G} \subset \partial_0 M$ with the following properties:

1) $\mathcal{G} \cap \Sigma = \emptyset$, where Σ is the ramification locus of the 3-orbifold M;
2) the closure of every connected component of $\partial_0 M - \mathcal{G}$ is either:
   i) a n-gon with $n \geq 5$ vertices of $\mathcal{G}$ or
   ii) an annulus with one boundary component lying in ∂P, and at least one vertex of $\mathcal{G}$ in the other boundary component;
3) if $\gamma \subset \partial_0 M$ is an embedded closed curve which intersects $\mathcal{G}$ transversely in at most four points, then γ bounds a discal 2-suborbifold $\Delta \subset \partial_0 M$ whose intersection $\mathcal{G} \cap \Delta$ is homeomorphic to one of the following forms (see Figures 1 and 2):
   a) the cone over $\mathcal{G} \cap \gamma$ when $\mathcal{G} \cap \gamma$ contains at most 3 points;
   b) two disjoint arcs with end points $\mathcal{G} \cap \gamma$, possibly connected in Δ by an edge of $\mathcal{G}$.

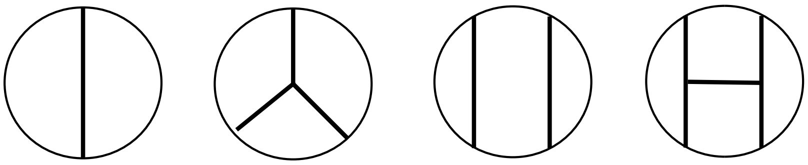

FIGURE 1. Property 3) when Δ is a nonsingular disk.

Then the compact Kaleidoscope $\mathcal{K}$ with underlying type (M, P) and obtained by silvering all the connected components of $\partial_0 M - \mathcal{G}$ is irreducible and topologically atoroidal. The irreducibility follows from the irreducibility of the 3-orbifold M, the super-incompressibility of $\partial_0 M$ and properties 1), 2) and 3 a) of the graph $\mathcal{G}$. The topological atoroidality follows from the topological properties of the pared 3-orbifold (M, P) and properties 1), 2) and 3 b) of $\mathcal{G}$.

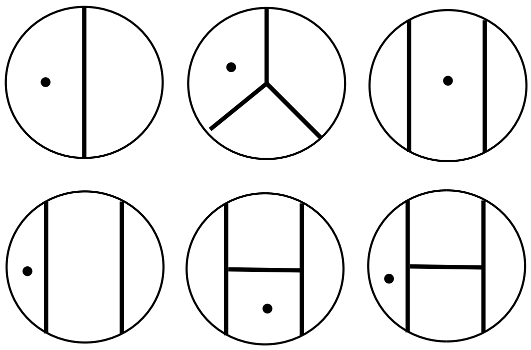

FIGURE 2. Property 3) when Δ is a disk with one cone point.

By [**BS1**] and [**MS**] $\hat{M}$ is either homotopically atoroidal, Euclidean or Seifert fibered. Since the mirrors are n-gons with $n \geq 5$ vertices, $\hat{M}$ contains essential hyperbolic 2-suborbifolds. Hence it cannot be Euclidean and it has infinite fundamental group. Moreover, by [**MS**] $\hat{M}$ is not Seifert fibered, since the fibration cannot be preserved by the group H. Otherwise some of the reflecting essential 2-suborbifold in $\hat{M}$ would be saturated, because $\mathcal{G}$ has at least one trivalent vertex.

Construction of the trivalent graph $\mathcal{G}$.— To construct the trivalent graph $\mathcal{G}$ with the required properties, we fix a closed collar neighborhood $\mathcal{C}(P)$ of the parabolic locus P in ∂M such that the neighborhoods are pairwise disjoint. We consider a triangulation $\mathcal{T}$ of $\partial M - int(\mathcal{C}(P))$ such that the ramification locus $\partial_0 M \cap \Sigma$ belongs to the 0-skeleton of $\mathcal{T}$, any 2-simplex in $\mathcal{T}$ intersects Σ in at most one vertex and $\partial\mathcal{C}(P)$ belongs to the 1-skeleton of $\mathcal{T}$.

We first get a refined triangulation $\mathcal{T}'$ by modifying $\mathcal{T}$ inside each triangle as follows: we subdivide each edge of the triangle in its middle point, we add the edges (parallel to the edges of the triangle) which join them, then we reproduce this modification in the created triangle and we join each vertex of the new homothetic triangle obtained by an edge to the corresponding vertex of the initial triangle (see Figure 3). This construction is due to E. Giroux (cf. [**Ot2**, §7], [**Pau**]).

Let ${}^t\mathcal{T}'$ be the cellulation dual to the triangulation $\mathcal{T}'$, obtained by putting a 0-cell in the interior of each 2-simplex of $\mathcal{T}'$ and of each 1-simplex of $\mathcal{T}' \cap \partial\mathcal{C}(P)$, a 1-cell for each 1-simplex of $\mathcal{T}'$ and each 0-simplex of $\mathcal{T}' \cap \partial\mathcal{C}(P)$, and a 2-cell for each interior 0-simplex of $\mathcal{T}'$.

Let $\mathcal{G}$ be the 1-skeleton of the cellulation ${}^t\mathcal{T}'$. It is a trivalent graph that verifies property 1) because $\Sigma \cap \partial_0 M$ belongs to the 0-skeleton of $\mathcal{T}'$. It verifies also property 2) since by construction each interior 0-simplex of $\mathcal{T}'$ is at least pentavalent and each 0-simplex of $\mathcal{T}' \cap \partial\mathcal{C}(P)$ is at least tetravalent.

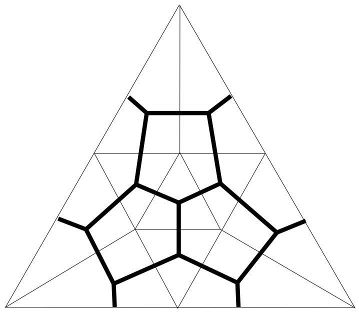

FIGURE 3. A refined triangle and its dual cellulation.

Property 3) is a consequence of the fact that a closed curve γ intersecting $\mathcal{G}$ in at most 4 points gives rise to a closed path $\overline{\gamma}$ contained in the 1-skeleton of $\mathcal{T}'$ and following at most 4 edges. By the construction of the triangulation $\mathcal{T}'$ such closed path $\overline{\gamma}$ is contained in the union of at most 2 triangles of the initial triangulation $\mathcal{T}$ with a common edge or a common vertex. Since at most one vertex of each triangle of the triangulation $\mathcal{T}$ belongs to the ramification locus Σ of M, the disk bounded by γ contains at most one point of ramification and verifies property 3.

Moreover, since no connected component of $\partial_0 M - \mathcal{G}$ is a rectangle, nor a bigon with a branching point, $\partial \hat{M}/H = P$. □

Definition 8.6.9. — A compact orientable mirrored 3-orbifold $(\hat{M}, H)$ is said hyperbolic if it admits a H-invariant complete hyperbolic structure with finite volume.

Given a compact orientable pared 3-orbifold (M, P), let $\mathcal{HM}(M, P)$ denote the set of hyperbolic mirrored 3-orbifolds with underlying type (M, P). Clearly $\mathcal{HM}(M, P) \subset \mathcal{AM}(M, P)$.

Remark 8.6.10. — The existence of a complete H-invariant hyperbolic structure with finite volume on the mirrored 3-orbifold $(\hat{M}, H)$ is equivalent to the existence of a compete hyperbolic structure with finite volume on the associated kaleidoscope $\mathcal{K} = \hat{M}/H$. In particular all the mirrors are totally geodesic and the dihedral angle at a common edge between two mirrors is $\pi/2$.

Such a kaleidoscope is said *hyperbolic*.

When the orbifold M is not fibered over a 1-orbifold, the following hyperbolization theorem is proved by induction on the length $\ell(M, P)$ of the pared 3-orbifold (M, P) by using the gluing theorem 8.4.1.

Theorem 8.6.11 **(Hyperbolization theorem for mirrored 3-orbifolds)**
Let (M, P) be a compact orientable Haken pared 3-orbifold, then

$$\mathcal{HM}(M, P) = \mathcal{AM}(M, P).$$

First we show how to deduce the hyperbolization theorem for Haken pared 3-orbifolds (Theorem 8.3.9, and thus Theorem 3) from the hyperbolization theorem for mirrored 3-orbifolds

Proof of Theorem 8.3.9 from Theorem 8.6.11. — Let (M, P) be a compact orientable Haken pared 3-orbifold. By Proposition 8.6.8 there is a mirrored 3-orbifold $(\hat{M}, H) \in \mathcal{AM}(M, P)$ such that $\partial\hat{M}/H = P$. Since Theorem 8.6.11 shows that the mirrored 3-orbifold is hyperbolic, Theorem 8.3.9 follows from the following:

Claim 8.6.12. — *Let $(\hat{M}, H)$ be a compact orientable hyperbolic mirrored 3-orbifold with underlying type (M, P) and such that $\partial\hat{M}/H = P$. If $\pi_1(M)$ is not virtually abelian, then (M, P) is a hyperbolic pared 3-orbifold.*

Proof of Claim 8.6.12. — The kaleidoscope $\mathcal{K} = \hat{M}/H$ has a complete hyperbolic structure on its interior with totally geodesic mirrors. Moreover its boundary $\partial K = P$ corresponds to cuspidal ends of this structure. Hence any thickened convex core $C_\delta(\mathcal{K})$ of the hyperbolic 3-orbifold $\mathcal{K}$ is of finite volume and the 3-orbifold pair $(thick_\mu(C_\delta(\mathcal{K})), \partial\, thick_\mu(C_\delta(\mathcal{K})) \cap thin_\mu(C_\delta(\mathcal{K})))$ (for $\mu < \mu(\mathcal{K})$) is diffeomorphic to (M, P). When $\pi_1(M)$ is not virtually abelian, this gives a geometrically finite hyperbolic structure on the pared 3-orbifold (M, P). ☐

We explain now the proof of the hyperbolization theorem for mirrored 3-orbifold (Theorem 8.6.11) when M is not fibred over a 1-orbifold.

We first need some extra results about *good splitting 2-suborbifolds* in a mirrored 3-orbifold.

Let (M, P) be a compact orientable pared 3-orbifold and let $(F, \partial F) \subset (M, \partial M)$ be a properly embedded orientable 2-suborbifold. In the following we denote by (M_F, P_F) the pair obtained by cutting M and P along F and by forgetting, after cutting P, the components that are discal. We will use analogous notations when cutting a mirrored 3-orbifold or a kaleidoscope along some properly embedded two-sided 2-suborbifold.

Definition 8.6.13. — Let $(\hat{M}, H)$ be a mirrored 3-orbifold with underlying type (M, P). A *good splitting 2-suborbifold* is a H-equivariant properly embedded super-essential orientable 2-suborbifold $(\hat{F}, \partial\hat{F}) \hookrightarrow (\hat{M}, \partial\hat{M})$ such that the non-orientable 2-suborbifold $\mathcal{F} = \hat{F}/H \subset \mathcal{K} = \hat{M}/H$ corresponds, after erasing the mirrors of $\mathcal{K}$, to a properly embedded, orientable 2-suborbifold $(F, \partial F) \subset (M, \partial_0 M)$ with non-empty boundary and which is either:

i) a complete system of meridian disks if M is an orbifoldbody, or
ii) a connected super-essential 2-suborbifold that is not discal, nor a turnover.

We call $F \subset M$ the *underlying type* of $\hat{F}$ and $\mathcal{F}$.

The following two lemmas are crucial to reduce the proof of Theorem 8.6.11 to the gluing theorem 8.4.1.

Lemma 8.6.14. — *Let $(\hat{M}, H)$ be a compact orientable mirrored 3-orbifold. If its underlying type (M, P) is a Haken pared 3-orbifold, then there is a good splitting 2-suborbifold in $(\hat{M}, H)$.*

Proof of Lemma 8.6.14. — Let $\mathcal{K} = \hat{M}/H$ be the kaleidoscope associated to the mirrored 3-orbifold $(\hat{M}, H)$. Using the fact that the underlying space M of $\mathcal{K}$ is a Haken 3-orbifold, the proof consists in finding a properly embedded non-orientable super-essential 2-suborbifold $(\mathcal{F}, \partial\mathcal{F}) \subset (\mathcal{K}, \partial\mathcal{K})$ with the following properties:

i) the underlying type F of $\mathcal{F}$ is either a complete system of meridian disks if M is an orbifoldbody, or a connected super-essential 2-dimensional suborbifold which is not discal, nor a turnover and with a non-empty boundary if $\partial_0 M_k \neq \emptyset$.
ii) $|\mathcal{F}|$ intersects transversely the graph $|\mathcal{G}|$ in the minimal number of points among all the properly embedded non-orientable super-essential 2-suborbifold satisfying property i).

Then the lift $\hat{F}$ of $\mathcal{F}$ to $\hat{M}$ is a good splitting 2-suborbifold. We refer to [**Ot2**, Lemma 7.3] and leave the details to the reader. □

Lemma 8.6.15. — *Let $(\hat{M}, H) \in \mathcal{AM}(M, P)$ be a compact orientable homotopically atoroidal mirrored 3-orbifold. Let $\hat{F} \subset \hat{M}$ be a good splitting 2-suborbifold, with underlying type F. Then there is a compact orientable homotopically atoroidal mirrored 3-orbifold $(\hat{M}', H') \in \mathcal{AM}(M_F, P_F)$ and a subgroup $H'' \subset H'$ such that $H'/H'' \cong H$ and the mirrored 3-orbifold $(\hat{M}', H'')$ has for underlying type the pared 3-orbifold $(\hat{M}_{\hat{F}}, (\partial\hat{M})_{\hat{F}})$.*

Proof of Lemma 8.6.15. — Let $\hat{F} \subset \hat{M}$ be a good splitting 2-suborbifold and let $\mathcal{F} = \hat{F}/H$ be the properly embedded non-orientable super-essential 2-suborbifold of the kaleidoscope $\mathcal{K} = \hat{M}/H$. We denote by $F \subset M$ the properly embedded orientable super-essential 2-suborbifold obtained from $\mathcal{F}$ after erasing the mirrors of $\mathcal{K}$.

By cutting the kaleidoscope $\mathcal{K}$ along $\mathcal{F}$ one gets a kaleidoscope $\mathcal{K}_{\mathcal{F}}$ whose boundary contains two copies $\mathcal{F}^+$ and $\mathcal{F}^-$ of $\mathcal{F}$.

Let (M_F, P_F) be the compact orientable pared 3-orbifold obtained by cutting (M, P) along the super-essential 2-suborbifold F, then the proof follows from the following claim:

Claim 8.6.16. — *There is a homotopically atoroidal kaleidoscope $\mathcal{K}'$ with underlying type (M_F, P_F) such that:*

1) $\mathcal{F}^+ \cup \mathcal{F}^-$ is an union of mirrors of $\mathcal{K}'$;
2) the kaleidoscope $\mathcal{K}_{\mathcal{F}}$ can be obtained from the kaleidoscope $\mathcal{K}'$ by erasing the mirrors contained in $\mathcal{F}^+ \cup \mathcal{F}^-$.

We first deduce Lemma 8.6.15 from Claim 8.6.16.

By the equivariant characteristic toric decomposition [**BS1**], the mirrored 3-orbifold $(\hat{M}', H')$ such that $\hat{M}'/H' = \mathcal{K}'$ is homotopically atoroidal and has the same underlying type (M_F, P_F) as $\mathcal{K}$; thus $(\hat{M}', H') \in \mathcal{AM}(M_F, P_F)$.

Let $H'' \subset H'$ be the subgroup generated by the reflections in the mirror group H' corresponding to the mirrors contained in $\mathcal{F}^+ \cup \mathcal{F}^-$. Then the mirrored 3-orbifold $(\hat{M}', H'')$ has for underlying type the pared 3-orbifold $(\hat{M}_{\hat{F}}, (\partial\hat{M})_{\hat{F}})$. □

Proof of Claim 8.6.16. — Let F^+ and F^- be the two copies of F in ∂M_F and $\pi : M_F \to M$ be the quotient map given by identification of F^+ with F^-. We denote by $\mathcal{G}_F \subset \partial_0 M_F$ the trivalent graph $\pi^{-1}(\mathcal{G} \cup \partial F)$, where $\mathcal{G} \subset \partial_0 M$ is the trivalent graph whose edges belongs in the kaleidoscope $\mathcal{K}$ to the intersection of two mirrors or of a mirror and a connected component of $\partial\mathcal{K}$. The proof consists in adding to $\mathcal{G}_F$ edges which are contained in $F^+ \cup F^-$ to get a trivalent graph $\mathcal{G}' \subset \partial_0 M_F$ so that:

a) no component of $(F^+ \cup F^-) - \mathcal{G}'$ is a rectangle, nor a bigon with a branching point;
b) the kaleidoscope, obtained by silvering the component of $\partial_0 M_F - \mathcal{G}'$ which are not rectangle, nor a bigon with a branching point, is homotopically atoroidal.

This can be thought as a relative version of the proof of Proposition 8.6.8. We refer to [**Ot2**, Lemma 7.4] or [**Pau**, Lemma 2.32], and leave the details to the reader. □

Now we start explaining the proof of the hyperbolization theorem for mirrored 3-orbifolds. We consider only the case where the underlying type is not a bundle over a 1-orbifold.

Proof of Theorem 8.6.11 in the non-fibered case. — The first step of the induction is given by Andreev's theorem and some generalizations of it due to Thurston. These results show that homotopically atoroidal kaleidoscopes with underlying types either $(\Delta^3, \emptyset)$, $(T_e \times [0,1], T_e \times \{1\})$ or $(T_h \times [0,1], \emptyset)$ are hyperbolic, where Δ^3 is a discal 3-orbifold, T_e an Euclidean turnover and T_h a hyperbolic turnover.

For a proof of these theorems we refer to the original articles by E.M. Andreev [**An1, An2**] (cf. [**Riv**]), and to Thurston's notes [**Thu1**, Thm. 13.6.1, 13.6.5 and 13.7.1], where the approach via pattern of circles gives the desired generalizations to handle the cases with underlying types $(T_e \times [0,1], T_e \times \{1\})$ or $(T_h \times [0,1], \emptyset)$ (cf. also [**Bro**], [**Kap**]).

Theorem 8.6.17 **(Andreev-Thurston's Theorem).** — *Let $\mathcal{K}$ be a compact kaleidoscope with underlying type either $(\Delta^3, \emptyset)$, $(T_e \times [0,1], T_e \times \{1\})$ or $(T_h \times [0,1], \emptyset)$. Then $\mathcal{K}$ is hyperbolic iff $\mathcal{K}$ is topologically atoroidal and acylindrical.* □

For topologically atoroidal and acylindrical kaleidoscopes with underlying type a pared 3-manifold $(B^3, \emptyset)$, or $(T^2 \times [0,1], T^2 \times \{1\})$, or $(F \times [0,1], \emptyset)$, where T^2 is the 2-torus and F is a hyperbolic surface, the existence of the desired hyperbolic structure follows from an argument of circle pattern (cf. [**Thu1**, Thm. 13.6.2 and 13.7.1], [**Mor1**, §15]). Moreover, this hyperbolic structure is uniquely determined, up to isometry, by the combinatorial structure of the Kaleidoscope (i.e the cellulation of $\partial_0 M$ given by the mirrors and whose 1-skeleton is the trivalent graph $\mathcal{G}$).

Hence any symmetry of the kaleidoscope is reflected in it. That gives the desired hyperbolic structures on the topologically atoroidal and acylindrical kaleidoscopes with underlying types $(\Delta^3, \emptyset)$, $(T_e \times [0,1], T_e \times \{1\})$ or $(T_h \times [0,1], \emptyset)$, since they are finitely covered by Kaleidoscopes with underlying type one of the above pared 3-manifolds.

The proof in the thick hyperbolic turnover case uses the assumption that $T_h \times \{1\}$ is a single mirror. The statement of Theorem 8.6.17 can be deduced from this case by cutting the kaleidoscope $\mathcal{K}$ with underlying type $(T_h \times [0,1], \emptyset)$ into two pieces along the hyperbolic turnover $T_h \times \{1/2\}$. One obtains two topologically atoroidal and aspherical kaleidoscopes $\mathcal{K}_1$ and $\mathcal{K}_2$ by silvering each copy of the hyperbolic turnover $T_h \times \{1/2\}$. Then one gets the hyperbolic structure on $\mathcal{K}$ by gluing the hyperbolic structures on $\mathcal{K}_1$ and $\mathcal{K}_2$, given by the proof above, along the two totally geodesic hyperbolic turnovers corresponding to a copy of $T_h \times \{1/2\}$ in $\mathcal{K}_1$ and $\mathcal{K}_2$.

To handle the inductive step we need the following equivariant version of the gluing theorem 8.4.1.

Proposition 8.6.18. — *Let $(\hat{M}, H) \in \mathcal{AM}(M, P)$, where M is not a bundle over a closed 1-orbifold. Let $\hat{F} \subset \hat{M}$ be a good splitting 2-suborbifold, with underlying type $F \subset M$. Let (M_F, P_F) be the pared 3-orbifold obtained by cutting open (M, P) along F. If $\mathcal{AM}(M_F, P_F) = \mathcal{HM}(M_F, P_F)$, then the mirrored 3-orbifold $(\hat{M}, H)$ is hyperbolic.*

Proof of Proposition 8.6.18. — Let $(\hat{M}_{\hat{F}}, (\partial \hat{M})_{\hat{F}})$ be the pared 3-orbifold obtained by cutting open the pared 3-orbifold $(\hat{M}, \partial \hat{M})$ along the super-essential 2-suborbifold $\hat{F}$. Then Lemma 8.6.15 implies the following claim:

Claim 8.6.19. — *With the hypothesis of Proposition 8.6.18, the pared 3-orbifold $(\hat{M}_{\hat{F}}, (\partial \hat{M})_{\hat{F}})$ admits a H-invariant hyperbolic structure.*

Proof of Claim 8.6.19. — Let $(\hat{M}', H') \in \mathcal{AM}(M_F, P_F)$ be the compact orientable mirrored 3-orbifold given by Lemma 8.6.15. Since by hypothesis $\mathcal{AM}(M_F, P_F) = \mathcal{HM}(M_F, P_F)$, the mirrored 3-orbifold $(\hat{M}', H')$ is hyperbolic. Hence the 3-orbifold $\hat{M}'$ admits a complete H'-invariant hyperbolic structure with finite volume. This hyperbolic structure is also invariant by the subgroup $H'' \subset H'$, given by Lemma 8.6.15. In particular, the kaleidoscope $\hat{M}'/H''$ has a H-invariant hyperbolic structure. Since $\partial \hat{M}'/H'' = (\partial \hat{M})_{\hat{F}}$, Claim 8.6.12 shows that the pared 3-orbifold $(\hat{M}_{\hat{F}}, (\partial \hat{M})_{\hat{F}})$ has a H-invariant hyperbolic structure. □

By definition of a good splitting 2-suborbifold and the fact that the 3-orbifold M is not a bundle over a closed 1-orbifold, the pared 3-orbifold $(\hat{M}_{\hat{F}}, (\partial \hat{M})_{\hat{F}})$ is not a bundle over a 1-orbifold (cf. [**Ot2**, Prop. 7.6]). Moreover its boundary $\partial_0 \hat{M}_{\hat{F}}$ is super-incompressible.

Since the pared 3-orbifold $(\hat{M}_{\hat{F}}, (\partial \hat{M})_{\hat{F}})$ is hyperbolic, the gluing theorem 8.4.1 shows that the interior of the compact 3-orbifold $\hat{M}$ admits a complete hyperbolic structure with finite volume. There are several ways to get a H-invariant hyperbolic structure.

One way, as in [**Ot2**, §8], is to establish a H-equivariant version of the gluing theorem 8.4.1. One can define a natural action of H on the Teichmüller space $\mathcal{T}(\partial_0 \hat{M}_{\hat{F}})$ and consider the the fixed point set $\mathcal{T}(\partial_0 \hat{M}_{\hat{F}})^H$ of this action. This set is not empty by Claim 8.6.19. Hence, it is a closed non-empty subset of $\mathcal{T}(\partial_0 \hat{M}_{\hat{F}})$, invariant by the map $\tau^\star \circ \sigma$. Since $\tau^\star \circ \sigma$ is contracting, the unique fixed point in $\mathcal{T}(\partial_0 \hat{M}_{\hat{F}})$ given by the fixed point Theorem 8.5.1 must belong to $\mathcal{T}(\partial_0 \hat{M}_{\hat{F}})^H$. Now the existence of a H-invariant complete hyperbolic structure on $\hat{M}$ follows from a H-equivariant version of Maskit's combination theorem (cf. [**Ot2**, §8]).

Another way is to verify that Kapovich's proof of his homeomorphism theorem for homotopically atoroidal kaleidoscopes, with underlying type an orientable Haken 3-manifold [**Kap**, Thm. 7.30], extends to the case where the underlying type is an orientable Haken 3-orbifold. One follows Kapovich's proof to show that an isomorphism of the fundamental groups of the two homotopically atoroidal Kaleidoscopes induces an isomorphism of the fundamental group of the underlying types that preserves the peripheral structures. Then by Takeuchi and Yokoyama's result (cf. Theorem 8.2.8) this isomorphism can be realized by a homeomorphism between the two underlying types. The last task is to construct such a homeomorphism which preserves the two mirror structures on the boundaries of the underlying types (cf. [**Kap**, §7.5]). $\square$

We now finish the proof of Theorem 8.6.11.

If the length of the pared 3-orbifold $\ell(M, P) = 0$, then M is an orbifoldbody.

Let $(\hat{M}, H) \in \mathcal{AM}(M, P)$. By Lemma 8.6.14 there is in $\hat{M}$ a good splitting 2-suborbifold $\hat{F}$ with underlying type a complete system of meridian disks $F \subset M$. Since each connected component of the pared 3-orbifold (M_F, P_F), obtained by cutting open (M, P) along F, is either a discal 3-orbifold or a thick Euclidean or hyperbolic turnover, $\mathcal{AM}(M_F, P_F) = \mathcal{HM}(M_F, P_F)$ by Theorem 8.6.17. Hence $(\hat{M}, H)$ is hyperbolic by Proposition 8.6.18. Therefore, we have shown that $\mathcal{AM}(M, P) = \mathcal{HM}(M, P)$ when $\ell(M, P) = 0$.

Let (M, P) be a Haken pared 3-orbifold with length $\ell(M, P) = \ell > 0$ and assume Theorem 8.6.11 to hold for all Haken pared 3-orbifolds with length $< \ell$. Let $(\hat{M}, H) \in \mathcal{AM}(M, P)$, by Lemma 8.6.14 there is a good splitting 2-suborbifold $\hat{F} \subset \hat{M}$ with underlying type a properly embedded orientable super-essential 2-suborbifold $F \subset M$. Hence $\ell(M_F, P_F) < \ell(M, P)$ by definition of the length. Since by the induction hypothesis $\mathcal{AM}(M_F, P_F) = \mathcal{HM}(M_F, P_F)$, Proposition 8.6.18 shows that the mirrored 3-orbifold $(\hat{M}, H)$ is hyperbolic. Therefore $\mathcal{AM}(M, P) = \mathcal{HM}(M, P)$ and

Theorem 8.6.11 is proved for all Haken pared 3-orbifolds of length $\ell > 0$. This finishes the induction step and the proof of Theorem 8.6.11. $\square$

CHAPTER 9

EXAMPLES

This chapter is devoted to examples of the different phenomena that occur in the proof of the orbifold theorem when we increase the cone angles of a hyperbolic cone 3-manifold, until some angle $\leq \pi$. There are two kinds of phenomena: collapses and the appearance of an essential Euclidean cone 2-manifold, namely a Euclidean turnover $S^2(\alpha, \beta, \gamma)$ (with $\alpha + \beta + \gamma = 2\pi$), a pillow $S^2(\pi, \pi, \pi, \pi)$, or $\mathbb{RP}^2(\pi, \pi)$.

The chapter is organized in eight sections, one for each example. The first examples are Euclidean collapses at angle $< \pi$, i.e. hyperbolic cone 3-manifolds that collapse to a point and that, after rescaling, converge to a Euclidean cone 3-manifold with the same topological type. In Examples 9.1 and 9.2, the collapsing angle lies in $[\frac{2\pi}{3}, \pi)$. By the use of Hamilton's theorem, the corresponding orbifold with cone angle π is spherical, but in these examples one can check explicitly that we have a continuous family of spherical cone structures with cone angles between the angle of Euclidean collapse and π. Example 9.3 exhibits a Euclidean collapse at angle π.

Example 9.4 is devoted to several collapses at angle π where the corresponding orbifolds at angle π are Seifert fibered and have one of the following geometries: Nil, $SL_2(\mathbb{R})$ or $\mathbb{H}^2 \times \mathbb{R}$. In Example 9.5 we show another collapse at angle π, where the corresponding orbifold at angle π has geometry Sol. Geometries $\mathbb{S}^3$ and $\mathbb{S}^2 \times \mathbb{R}$ do not occur as direct collapses: the $\mathbb{S}^3$ case is eliminated in Appendix A and the case $\mathbb{S}^2 \times \mathbb{R}$ is eliminated in Lemma 9.5.2.

The last three examples are devoted to Euclidean cone 2-submanifolds that appear when we increase the cone angles. There are three kinds of essential Euclidean cone 2-manifolds that can appear: a turnover $S^2(\alpha, \beta, \gamma)$ with $\alpha + \beta + \gamma = 2\pi$, a pillow $S^2(\pi, \pi, \pi, \pi)$ or its non-orientable quotient $\mathbb{RP}^2(\pi, \pi)$.

When a Euclidean turnover $S^2(\alpha, \beta, \gamma)$ appears, a cusp must open, because by Proposition 5.5.1 we cannot have a collapse (cf. remark 5.0.2). But when a pillow

$S^2(\pi,\pi,\pi,\pi)$ or $\mathbb{RP}^2(\pi,\pi)$ appear, we may have a cusp opening, a collapse, or a combination of both phenomena. Example 9.6 illustrates the turnover case, Example 9.7, the pillow case, and Example 9.8, the $\mathbb{RP}^2(\pi,\pi)$ case.

9.1. A Euclidean collapse at angle $\frac{2\pi}{3}$

Consider $C(\alpha)$ the cone 3-manifold with underlying space S^3, singular set the figure-eight knot and cone angle α (see Figure 1). It has been shown in [**HLM1**] that $C(\alpha)$ is

i) hyperbolic for $\alpha \in (0, \frac{2\pi}{3})$,
ii) Euclidean for $\alpha = \frac{2\pi}{3}$, and
iii) spherical for $\alpha \in (\frac{2\pi}{3}, \pi]$.

It follows from Appendix A that the same phenomenon occurs for any hyperbolic two-bridge knot or link in S^3: if $C(\alpha)$ is a cone 3-manifold with underlying space space S^3, singular set any hyperbolic two-bridge knot and cone angle α, then it has a Euclidean collapse at some angle $\alpha \in [\frac{2\pi}{3}, \pi)$.

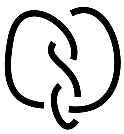

FIGURE 1. The cone 3-manifold $C(\alpha)$.

For the Whitehead link, Suárez [**Sua**] has constructed an explicit family of hyperbolic, Euclidean and spherical cone 3-manifolds between angles 0 and π.

9.2. Another Euclidean collapse before π

Let $C(\alpha)$ be the cone 3-manifold with underlying space $\mathbb{RP}^3$, cone angle α and singular set the knot described as follows: we view $\mathbb{RP}^3$ as the result of an integer surgery with coefficient 2 on one component of the Whitehead link, and the singular set is the remaining component (see Figure 2).

The orbifold $C(\pi)$ is spherical, because its double cover is the Seifert fibered manifold with description: $(Oo0 \mid 0; (2,1), (2,1), (2,1))$ (here we follow the notation of [**Mon**]).

Lemma 9.2.1. — *For* $0 < \alpha < \arccos\left(\frac{1}{2} - \sqrt{2}\right) < \pi$, *the cone 3-manifold* $C(\alpha)$ *is hyperbolic. In addition it is Euclidean for angle* $\alpha = \arccos\left(\frac{1}{2} - \sqrt{2}\right)$.

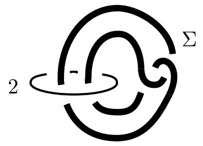

FIGURE 2. The cone 3-manifold $C(\alpha)$.

Proof. — The open manifold $M = C(\alpha) - \Sigma$ is fibered over S^1 with fiber a punctured torus and homological monodromy $\left(\begin{smallmatrix}1&2\\1&3\end{smallmatrix}\right)$ (see [**HMW**, Prop. 3]). By Thurston's hyperbolization theorem [**Thu4, Ot1**], M is hyperbolic. Since the underlying space of $C(\alpha)$ is $\mathbb{RP}^3$, it does not contain any turnover. Thus, by Theorem A and Appendix A, there exists a Euclidean collapse at some angle $< \pi$.

Next we compute the angle where the Euclidean collapse occurs, by using the results of [**Po1**]. The variety of characters is computed in [**Po1**, Ch. 4, Ex. 2] . Following the notation in [**Po1**], the component of the variety of characters in which we are interested is:

$$(x_1^2 - 2)y_1^2 = 4(x_1^2 - 1)$$

where x_1 and y_1 are the traces of some elements of $\pi_1(M)$. In addition, the trace of the meridian is $y_0 = \frac{1}{2}x_1 y_1$ and the Reidemeister torsion with respect to the meridian is

$$\mathbb{T}_{(M,\mu)} = \pm\big(\frac{1}{2}y_1^2 - x_1^2\big).$$

By [**Po1**, Thm. 5.13] if χ is the character of the Euclidean collapse then $\mathbb{T}_{(M,\mu)}(\chi) = 0$, (cf. also [**HLM2**]). Thus, at the Euclidean collapse we obtain $x_1^2 = 2 \pm \sqrt{2}$, $y_1^2 = 2x_1^2$ and $y_0^2 = 3 \pm 2\sqrt{2}$. By writing $y_0 = \pm 2\cos\frac{\alpha}{2}$, we have $\cos\alpha = \frac{1}{2} \pm \sqrt{2}$. Hence the Euclidean collapse occurs at angle

$$\alpha = \arccos\big(\frac{1}{2} - \sqrt{2}\big) \approx 2.72 \in \big(\frac{2\pi}{3}, \pi\big).$$

□

9.3. A Euclidean collapse at angle π

This example is picked up from [**HLMW**], where the authors attribute it to Thurston, (see also the applications of Andreev's theorem in Thurston's Notes [**Thu1**]). Let $C(\alpha)$ denote the cone 3-manifold with underlying space S^3, singular set the Borromean rings and cone angle $\alpha \in (0, \pi]$ (Figure 3).

It is shown explicitly in [**HLMW**] that $C(\alpha)$ is hyperbolic for $\alpha < \pi$ and that $C(\pi)$ is Euclidean. More precisely, they construct a continuous family of polyhedra $P(\alpha)$ in the space $\mathbb{H}_{K(\alpha)}$ of constant curvature $K(\alpha)$, for $\alpha \in (0, \pi]$. The cone 3-manifold

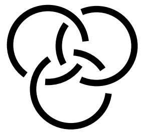

FIGURE 3. The cone 3-manifold $C(\alpha)$.

$C(\alpha)$ is obtained by gluing the faces of $P(\alpha)$ by a isometries. The curvature, which is continuous, satisfies

$$\begin{cases} K(\alpha) < 0 & \text{for } \alpha \in (0,\pi) \\ K(\alpha) = 0 & \text{for } \alpha = \pi \end{cases}$$

Since $K(\alpha)$ is a continuous function on α, it follows that $C(\alpha)$ has a Euclidean collapse at angle π. We remark that the orbifold $C(\pi)$ is Seifert fibered.

Remark 9.3.1. — J.P. Otal pointed out that for the cone angle $\alpha \in (\pi, 2\pi)$, one can explicitly show that $C(\alpha)$ is spherical.

9.4. Collapses at angle π for Seifert fibered geometries

We give a family of examples of cone 3-manifolds that collapse at angle π and the orbifolds at angle π have geometry Nil, $SL_2(\mathbb{R})$ or $\mathbb{H}^2 \times \mathbb{R}$.

The other Seifert fibered geometries $\mathbb{S}^3$ and $\mathbb{S}^2 \times \mathbb{R}$ do not occur as such collapses, by Appendix A for $\mathbb{S}^3$ and Lemma 9.5.2 for $\mathbb{S}^2 \times \mathbb{R}$.

Let $C_{n,h}(\alpha)$ be the cone 3-manifold with underlying manifold the Lens space $L(|n|,1)$, cone angle α and singular set described in Figure 4, where $L(|n|,1)$ is represented by integer surgery of coefficient n on the trivial knot, with $n \in \mathbb{Z} - \{0\}$.

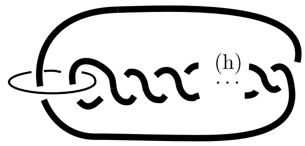

FIGURE 4. The cone 3-manifold $C_{n,h}(\alpha)$, where h is the number of half-twists of Σ. The underlying space is $L(|n|,1)$.

We assume that the number of half-twists of Σ is $h \geq 2$ and that $n \geq 4$ or $n \leq -5$.

Lemma 9.4.1. — *For $h \geq 2$ and $n \geq 4$ or $n \leq -5$, the orbifold $C_{n,h}(\pi)$ has geometry:*

- *Nil when $n = 4$ and $h = 2$;*
- *$\mathbb{H}^2 \times \mathbb{R}$ when $n = -2h$;*

– $\widetilde{SL_2}(\mathbb{R})$ *otherwise.*

Proof. — To prove that the orbifold $C_{n,h}(\pi)$ is geometric, we use the fact that it is Seifert fibered. This Seifert fibration follows from viewing the link with two components in Figure 4 as a Montesinos link. To decide the geometry of the orbifold we use two invariants:

– the sign of the Euler characteristic of the basis.
– the vanishing or not of the Euler class.

The basis of the Seifert fibration is a 2-orbifold B^2 with underlying space a disk. It has silvered boundary, and contains a cone point of order $|n| > 0$ and a dihedral corner of order $h \geq 2$ (see Figure 5).

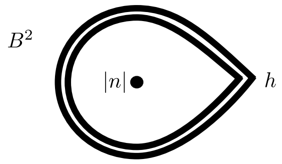

FIGURE 5. The basis B^2 of the Seifert fibration of the orbifold $C_{n,h}(\pi)$.

In particular the Euler characteristic of the basis is

$$\chi(B^2) = \frac{1}{|n|} + \frac{1}{2h} - \frac{1}{2}.$$

Thus $\chi(B^2) = 0$ iff $n = 4$ and $h = 2$. Otherwise $\chi(B^2) < 0$. Therefore B^2 is Euclidean when $n = 4$ and $h = 2$, and B^2 is hyperbolic otherwise.

In addition, we also use the vanishing or not of the rational Euler number [**BS2**]. To compute this number, we use a double cover of the orbifold, which is the Seifert fibered manifold $(Oo0 \mid 0; (n,1), (n,1), (h,1))$. The rational Euler number of the orbifold is

$$e_0 = \frac{1}{2}\left(-\frac{2}{n} - \frac{1}{h}\right) = -\frac{1}{n} - \frac{1}{2h}.$$

Thus $e_0 = 0$ if and only if $n = -2h$. It follows that $C_{n,h}(\pi)$ has a product geometry if and only if $n = -2h$. $\square$

Proposition 9.4.2. — *If $h \geq 2$ and $n \geq 4$ or $n \leq -5$, then $C_{n,h}(\alpha) - \Sigma$ is hyperbolic.*

Corollary 9.4.3. — *The cone 3-manifold $C_{n,h}(\alpha)$ is hyperbolic for $\alpha \in [0, \pi)$ and it collapses at angle π.*

Proof of Proposition 9.4.2. — By Thurston's hyperbolization theorem, it suffices to check that $C_{n,h}(\alpha) - \Sigma$ is irreducible and atoroidal and that it is not Seifert fibered.

To prove that $C_{n,h}(\alpha)-\Sigma$ is irreducible and atoroidal, we look at the double cover of the orbifold

$$M_{n,h} \to C_{n,h}(\pi),$$

which is a small Seifert fibered manifold. Since $M_{n,h}$ has geometry Nil, $\mathbb{H}^2 \times \mathbb{R}$ or $SL_2(\mathbb{R})$ and it is small, it is irreducible and atoroidal. It follows that the orbifold $C_{n,h}(\pi)$ is also irreducible and atoroidal, and, by a standard argument, the manifold $C_{n,h}(\pi) - \Sigma$ is also irreducible and atoroidal.

We prove that $C_{n,h}(\alpha) - \Sigma$ is not Seifert fibered by contradiction. Assuming that $C_{n,h}(\alpha) - \Sigma$ admits a Seifert fibration, then it extends to a Seifert fibration of the underlying manifold $L(|n|, 1)$, so that Σ is one of the fibers, because $L(|n|, 1)$ is irreducible. In particular we have two Seifert fibrations on the orbifold $C_{n,h}(\pi)$: one of them contains Σ as a fiber and the other one was used in Lemma 9.4.1. We lift both fibrations to the manifold double cover $M_{n,h}$. Both lifted fibrations are homotopic, by [**OVZ**], and the generic fiber generates the center of $\pi_1(M_{n,h})$, which is isomorphic to $\mathbb{Z}$, because:

- $M_{n,h}$ has geometry Nil, $\mathbb{H}^2 \times \mathbb{R}$ or $SL_2(\mathbb{R})$,
- and at least one of the fibrations (hence both) has an orientable basis.

Let $\tau : M_{n,h} \to M_{n,h}$ denote the involution associated to the covering so that $M_{n,h}/\tau \cong C_{n,h}(\pi)$. We obtain the contradiction by looking at the action of τ on the center $\mathbb{Z}$ of $\pi_1(M_{n,h})$, because for one of the fibrations the action is trivial, but for the other one the action is non trivial. □

Proof of Corollary 9.4.3. — Since $L(|n|, 1)$ is irreducible, $C_{n,h}(\alpha)$ contains no turnover. In addition, since $C_{n,h}(\pi)$ is not spherical, Theorem A implies that $C_{n,h}(\alpha)$ is hyperbolic for $\alpha \in (0, \pi)$. By the proof of Theorem B, since $C_{n,h}(\pi)$ is atoroidal and Seifert fibered, it follows that $C_{n,h}(\alpha)$ collapses at $\alpha = \pi$ □

9.5. A collapse at π for *Sol* geometry

This is an example of collapse at angle π into *Sol* geometry. Let $\mathcal{O}$ be the orbifold fibered over S^1 with fiber a pillow $S^2(2, 2, 2, 2)$. We assume that the monodromy

$$\phi : S^2(2, 2, 2, 2) \to S^2(2, 2, 2, 2)$$

fixes a singular point of the pillow $S^2(2, 2, 2, 2)$ and therefore we view it as an element of the mapping class group of a disk with three points, i.e. the braid group with three strings. We take $\phi = \sigma_1\sigma_2^{-1}$, where $\langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$ is the Artin presentation of the braid group with three strings.

The underlying space of this orbifold is $S^2 \times S^1$ and it has two singular components. A surgery description of this orbifold is given in Figure 6 below.

The orbifold $\mathcal{O}$ has a double cover which fibers over S^1 with fiber a torus T^2 and monodromy a composition of Dehn twists along two curves that generate $\pi_1(T^2)$.

FIGURE 6. Surgery description of $\mathcal{O}$. All branching indices are 2.

This double cover has geometry *Sol*, and one can easily check that the involution of the covering is geometric. Therefore $\mathcal{O}$ has geometry *Sol*.

Let $C(\alpha)$ denote the cone 3-manifold with the same topological type as $\mathcal{O}$ and cone angle α.

Lemma 9.5.1. — *For $\alpha \in (0, \pi)$, the cone 3-manifold $C(\alpha)$ is hyperbolic and it collapses at angle π.*

Proof. — By Thuston's hyperbolization theorem [**Thu4, Ot1**], the manifold $\mathcal{O} - \Sigma$ is hyperbolic because it is fibered over S^1 and its monodromy is pseudo-Anosov (because viewed as an element of the braid group of a three times punctured disk, it is just $\sigma_1\sigma_2^{-1}$).

Since Σ represents a trivial cycle in $H_1(S^2 \times S^1, \mathbb{Z}/2\mathbb{Z})$, Σ cannot intersect a sphere in three points. Hence $C(\alpha)$ does not contain any turnover, even if the underlying manifold is reducible. In addition, since $\mathcal{O}$ has *Sol* geometry, it cannot be spherical, and by Theorem A we conclude that $C(\alpha)$ is hyperbolic for $\alpha \in (0, \pi)$.

By the proof of Theorem B, either it collapses or $\mathcal{O}$ contains an essential hyperbolic suborbifold with toric boundary different from a product. Since $\mathcal{O}$ is *Sol*, the only possibility is that the family $C(\alpha)$ collapses at angle π. $\square$

A similar example at angle 2π is developed in detail in [**Sua**], where a family of cone manifolds that collapses at angle 2π is shown to converge to a circle, and the corresponding manifold has geometry *Sol*.

To finish the sections of collapses, we prove that orbifolds with geometry $\mathbb{S}^2 \times \mathbb{R}$ cannot do not have the topological type of a hyperbolic cone manifold.

Lemma 9.5.2. — *If $\mathcal{O}$ is an orbifold of cyclic type, with geometry $\mathbb{S}^2 \times \mathbb{R}$ and with branching locus Σ, then $\mathcal{O} - \Sigma$ is not hyperbolic.*

Proof. — If the orbifold $\mathcal{O}$ has geometry $\mathbb{S}^2 \times \mathbb{R}$, then we write $\mathcal{O} = S^2 \times S^1/G$, where G acts on $S^2 \times S^1$ isometrically (in particular it preserves the product structure). In addition, the stabilizers of the action of G are at most cyclic, by hypothesis.

We shall prove that $S^2 \times S^1 - \Sigma_G$ is not hyperbolic, where

$$\Sigma_G = \{x \in S^2 \times S^1 \mid \text{ the stabilizer } G_x \text{ of } x \text{ is nontrivial}\}.$$

Since G preserves the product structure, the components of Σ_G are either vertical (equal to $\{p\} \times S^1$) or horizontal (contained in $S^2 \times \{q\}$).

If all components of Σ_G are vertical, then $S^2 \times S^1 - \Sigma_G$ is Seifert fibered, and if all components of Σ_G are horizontal, then $S^2 \times S^1 - \Sigma_G$ is not irreducible. Hence we assume that Σ_G contains at least a vertical component σ_v and an horizontal one σ_h. Let $S^2 \times \{q\}$ be the horizontal sphere that contains σ_h. If $g \in G$ is an element that fixes the vertical fibre σ_v, then it preserves each horizontal sphere; in particular $g(\sigma_h) \subset S^2 \times \{q\}$. Since σ_h and $g(\sigma_h)$ are geodesics in $S^2 \times \{q\}$, they intersect, and since Σ is a link, the only possibility is that $\sigma_h = g(\sigma_h)$. Thus the restriction $g|_{S^2 \times \{q\}}$ is a rotation that fixes the north and south poles of $S^2 \times \{q\}$, and σ_h is the equator of $S^2 \times \{q\}$. It follows that $S^2 \times S^1 - \Sigma_G$ is homeomorphic to the exterior of the link in S^3 of Figure 7, the key ring, which is Seifert fibered. $\square$

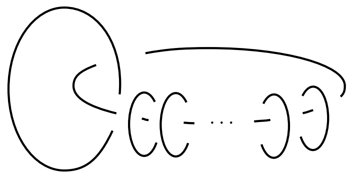

FIGURE 7. $S^2 \times S^1 - \Sigma_G$ is homeomorphic to the exterior of the key ring in S^3. The two biggest circles correspond to the vertical components of Σ_G (one with 0 and the other with ∞-surgery), and the other circles correspond to the horizontal components of Σ_G.

9.6. An essential Euclidean turnover: opening of a cusp

We show an example of Euclidean turnover at angle $\frac{2\pi}{3}$, that corresponds to the opening of a cusp. This example is used by Dunfield in [**Dunf**] to find an ideal point of the character variety with a non-trivial root of unity. He finds a sixth root of unity, because it corresponds to the opening of a cusp, with transverse section a Euclidean turnover $S^2(\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3})$. There is a similar example in [**Ho1**].

We now describe the example. Let $C(\alpha)$ be the cone 3-manifold with underlying space $S^2 \times S^1$, cone angle α and singular set described as follows. We view $S^2 \times S^1$ as the result of 0-surgery on one component of the 2-bridge link 7^2_1 in Rolfsen's table, and the singular set is the other component of the link (see Figure 8).

This cone 3-manifold contains a turnover $S^2(\alpha, \alpha, \alpha)$, whose intersection with the exterior of the link is a three times punctured disk, represented in Figure 8. When $\alpha = 2\pi/3$, this turnover is Euclidean and when $\alpha < 2\pi/3$, it is hyperbolic.

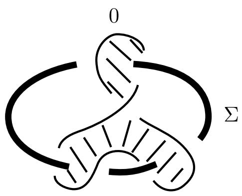

FIGURE 8. The surgery description of $C(\alpha)$. The disk that bounds the component with 0 surgery extends to a turnover in $C(\alpha)$, because it meets Σ in three points.

We will assume that $\alpha \leq \frac{2\pi}{3}$. Let $C_0(\alpha) = C(\alpha) - \mathcal{N}(S^2(\alpha,\alpha,\alpha))$ be the cone 3-manifold (with boundary) obtained by cutting open $C(\alpha)$ along this turnover. The cone 3-manifold $C_0(\alpha)$ has boundary two hyperbolic or Euclidean turnovers $S^2(\alpha,\alpha,\alpha)$ and its underlying space is $S^2 \times I$. The singular set consists of three arcs, as described in Figure 9.

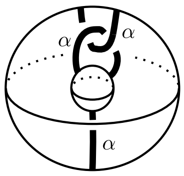

FIGURE 9. The cone 3-manifold $C_0(\alpha)$, with cone angles α.

Proposition 9.6.1. — *For $\alpha \in (0, \frac{2\pi}{3})$, the cone 3-manifold $C_0(\alpha)$ is hyperbolic with totally geodesic boundary. In addition, the interior of $C_0(\frac{2\pi}{3})$ is complete hyperbolic with cusps and, for a good choice of the base point x_α, we have*

$$\lim_{\alpha \to \frac{2\pi}{3}} (C_0(\alpha), x_\alpha) = (\mathrm{int}(C_0(\tfrac{2\pi}{3})), x_{\frac{2\pi}{3}}).$$

The following corollary says that a cusp appears at angle $\frac{2\pi}{3}$.

Corollary 9.6.2. — *For $\alpha \in (0, \frac{2\pi}{3})$, the cone 3-manifold $C(\alpha)$ is hyperbolic. In addition, for a good choice of the base point x_α, we have*

$$\lim_{\alpha \to \frac{2\pi}{3}} (C(\alpha), x_\alpha) = (\mathrm{int}(C_0(\tfrac{2\pi}{3})), x_{\frac{2\pi}{3}}).$$

Proof of Corollary 9.6.2. — For $\alpha < \frac{2\pi}{3}$, since $\partial C_0(\alpha)$ has two totally geodesic components which are $S^2(\alpha,\alpha,\alpha)$, and since a turnover is rigid, we can glue the components of $\partial C_0(\alpha)$ to obtain the hyperbolic structure on $C(\alpha)$. The assertion about the limits follows from the proposition, because the boundary of $C_0(\alpha)$ goes to infinity when $\alpha \to \frac{2\pi}{3}$. □

Proof of Propostion 9.6.1. — We draw $C_0(\alpha)$ in a more convenient way in Figure 10 (a), in order to view it as a truncated tetrahedron $P(\alpha)$ with faces identified.

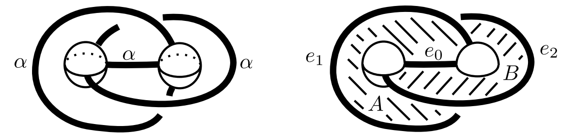

FIGURE 10. (a) The cone 3-manifold $C_0(\alpha)$. (b) The ideal triangles.

We label the singular edges e_0, e_1 and e_2, and we consider the two truncated triangles A and B in Figure 10 (b), which are the glued faces of the truncated tetrahedron $P(\alpha)$. This truncated tetrahedron $P(\alpha)$ is represented in Figure 11, where the angle of the edges with label e_0 is $\frac{\alpha}{4}$ and the angle of the edges e_1 and e_2 is α. When $\alpha = \frac{2\pi}{3}$, $P(\frac{2\pi}{3})$ is an ideal tetrahedron, with vertices in $\partial\mathbb{H}^3 = S^2_\infty$, but when $\alpha < \frac{2\pi}{3}$ the vertices of $P(\alpha)$ lie outside $\partial\mathbb{H}^3 = S^2_\infty$, therefore its vertices are truncated by totally geodesic triangles orthogonal to the faces.

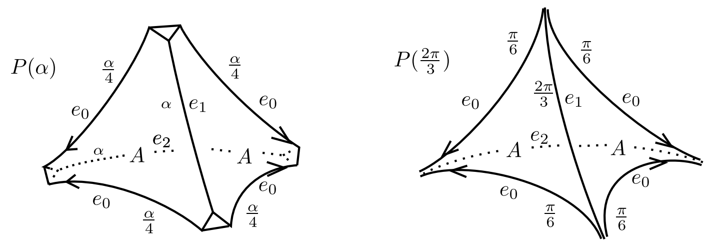

FIGURE 11. The truncated tetrahedron $P(\alpha)$ and the ideal one $P(\frac{2\pi}{3})$.

The face identifications are obtained by rotations around the edges e_1 and e_2. We observe that the edges of the truncated tetrahedra correspond precisely to the singular edges of $C_0(\alpha)$. It follows from this construction that $C_0(\alpha)$ and $\operatorname{int} C_0(\frac{2\pi}{3})$ have the hyperbolic structures stated in the proposition. Since when $\alpha \to \frac{2\pi}{3}$ the polygon $P(\alpha)$ converges to $P(\frac{2\pi}{3})$, we also have the assertion about the limits. □

Remark 9.6.3. — We could keep deforming by increasing α after $\frac{2\pi}{3}$, but then the cusps would be filled to create singular vertices, because the vertices of $P(\alpha)$ lie in $\mathbb{H}^3$ for $\alpha > \frac{2\pi}{3}$. One can even show that $P(\alpha)$ collapses to a Euclidean polyhedron and it becomes spherical at $\alpha = \pi$ (some egdes have angle π and $P(\pi)$ is a lens in $\mathbb{S}^3$).

A similar example is quoted in [**Ho1**] by doing surgery 0 and ∞ on the Whitehead link. In Figure 12 we give a surgery description of Hogson's example, which has two singular components. If we take Hodgson's example with cone angles α everywhere and we cut open along the turnover, then we obtain again the cone 3-manifold $C_0(\alpha)$ above.

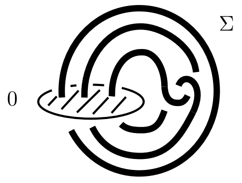

FIGURE 12. Hodgson's example, with the turnover represented by a disk bounding the component with surgery coefficient 0.

9.7. A pillow

In this example we shall combine the two tangles of Figure 13.

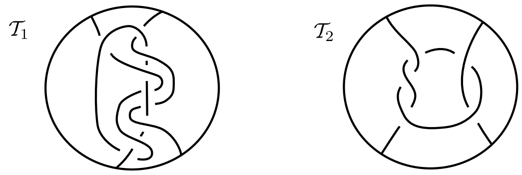

FIGURE 13. The tangles $\mathcal{T}_1$ and $\mathcal{T}_2$.

We view each $\mathcal{T}_i$ as an orbifold with underlying space the 3-ball, ramification locus the strings of the tangle, and ramification indices 2. The boundary of $\mathcal{T}_i$ is a pillow $S^2(2,2,2,2)$. The orbifold $\mathcal{T}_1$ is the quotient of exterior of the figure eight knot by the involution of Figure 14 (a), and $\mathcal{T}_2$ is the quotient of the trefoil knot by the involution of Figure 14 (b).

Hence, by Theorem 1, $\mathcal{T}_1$ is hyperbolic with a cusp. The orbifold $\mathcal{T}_2$ is Seifert fibered with basis B^2, where B^2 is a 2-orbifold with underlying space a disk, three boundary arcs of this disk are mirrors connected by two dihedral corners of order 2 and 3, and ∂B^2 is an interval with silvered end-points (see Figure 15).

Consider the orbifolds

$$\begin{aligned} \mathcal{O}_{11} &= \mathcal{T}_1 \cup_\partial \mathcal{T}_1, \\ \mathcal{O}_{12} &= \mathcal{T}_1 \cup_\partial \mathcal{T}_2, \\ \mathcal{O}_{22} &= \mathcal{T}_2 \cup_\partial \mathcal{T}_2, \end{aligned}$$

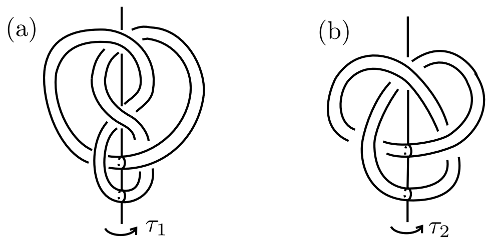

FIGURE 14. (a) $\mathcal{T}_1$ is the quotient of the figure eight knot exterior by the involution τ_1. (b) $\mathcal{T}_2$ is the quotient of the trefoil knot exterior by the involution τ_2.

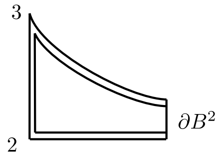

FIGURE 15. The basis B^2 of the Seifert fibration of $\mathcal{T}_2$.

glued along the boundary. We do not specify the gluing map, except for $\mathcal{O}_{22}$, where we require the fibrations not to be compatible by the gluing map.

Lemma 9.7.1. — *The manifolds $\mathcal{O}_{11} - \Sigma$, $\mathcal{O}_{12} - \Sigma$ and $\mathcal{O}_{22} - \Sigma$ are hyperbolic.*

Proof. — By Thurston's hyperbolization theorem, it suffices to check that $\mathcal{O}_{ij} - \Sigma$ is irreducible, atoroidal and not Seifert fibered.

We recall that $\mathcal{T}_1$ and $\mathcal{T}_2$ have a double cover which is irreducible and topologically atoroidal. Thus $\mathcal{O}_{ij}$ is finitely covered by an irreducible manifold which contains a unique essential torus, up to isotopy (here the choice of the gluing map for $\mathcal{O}_{22}$ is relevant). Therefore the orbifold $\mathcal{O}_{ij}$ itself is irreducible and has a unique essential toric suborbifold, which is the pillow $\partial\mathcal{T}_i \cong S^2(2,2,2,2)$, by [**BS1**]. Then it follows easily that $\mathcal{O}_{ij} - \Sigma$ is irreducible and atoroidal.

The manifold $\mathcal{O}_{ij} - \Sigma$ is not Seifert fibered, because it contains a properly embedded essential separating sphere with four punctures. □

Let $C_{ij}(\alpha)$ denote the cone 3-manifold with the same topological type as $\mathcal{O}_{ij}$ and cone angle α.

Proposition 9.7.2. — *For $0 < \alpha < \pi$, $C_{ij}(\alpha)$ is hyperbolic. When $\alpha = \pi$, an essential pillow appears and the following occurs:*

i) For the family $C_{11}(\alpha)$, there is an opening of a cusp that splits $\mathcal{O}_{ij}$ into two copies of $\mathcal{T}_1$.

ii) The family $C_{22}(\alpha)$ collapses.

iii) For the family $C_{12}(\alpha)$, depending on the choice of the base point, there is either an opening of a cusp (corresponding to a base point in $\mathcal{T}_1$) or a partial collapse (corresponding to a base point in $\mathcal{T}_2$).

Proof. — First we remark that $\mathcal{O}_{ij}$ contains no turnover because the underlying space is S^3. In addition, the fundamental group of $\mathcal{O}_{ij}$ is infinite. Thus, by Theorem A and Hamilton's theorem, if $C_{ij}(\alpha)$ is the corresponding cone 3-manifold, then it is hyperbolic for $\alpha \in (0, \pi)$.

At angle π, the 3-orbifold $\mathcal{T}_1$ is hyperbolic and the 3-orbifold $\mathcal{T}_2$ is Seifert fibered (with geometry $\mathbb{H}^2 \times \mathbb{R}$). The different cases are discussed by appealing to the proof of Theorem B. The cone manifold $C_{11}(\alpha)$ does not collapse at angle π, but a cusp opens that splits the cone 3-manifold into two copies of $\mathcal{T}_1$, which is a hyperbolic orbifold. The family $C_{12}(\alpha)$ has a partial collapse: if we chose a base point in the thick part of $\mathcal{T}_1$ then we have again an opening of a cusp, but if we choose a base point in $\mathcal{T}_2$ we have a collapse. Finally, $C_{22}(\alpha)$ collapses at angle π, for any choice of base point. □

Remark 9.7.3. — The example of the orbifold $\mathcal{O}_{22}$ is illustrated in Figure 16, which has been communicated to us by M. Lozano and has inspired the whole Section 9.7. In [**HLM3**] they compute the character variety of $\mathcal{O}_{22} - \Sigma$, which can be used to show that there is a collapse at angle π.

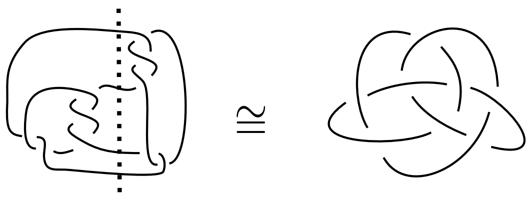

FIGURE 16. The singular set of $\mathcal{O}_{22}$ is the knot 8_{16} in Rolfsen's table. The pillow (corresponding to a Conway sphere) is represented by a dotted line.

9.8. An incompressible $\mathbb{RP}^2(\pi, \pi)$

Let $\overline{\mathcal{N}_1(\mathbb{RP}^2(\pi, \pi))}$ denote the closed tubular neighborhood (of radius 1) of $\mathbb{RP}^2(\pi, \pi) \times \{0\}$ in the orientable line bundle on $\mathbb{RP}^2(\pi, \pi)$. Alternatively, $\overline{\mathcal{N}_1(\mathbb{RP}^2(\pi, \pi))}$ is the quotient of $S^2(\pi, \pi, \pi, \pi) \times [-1, 1]$ by the involution $\tau_0 \times \tau_1$ where τ_0 is the antipodal map on $S^2(\pi, \pi, \pi, \pi)$ and τ_1 changes the sign on $[-1, 1]$. Note that

$$\partial\overline{\mathcal{N}_1(\mathbb{RP}^2(\pi, \pi))} \cong S^2(\pi, \pi, \pi, \pi).$$

We consider the closed orbifold

$$\mathcal{O} = \mathcal{T}_1 \cup_\partial \mathcal{N}_1(\mathbb{RP}^2(\pi, \pi)).$$

In the notation of Example 9.7, the orbifold $\mathcal{O}$ is doubly covered by $\mathcal{O}_{11} = \mathcal{T}_1 \cup_\partial \mathcal{T}_1$, because $\partial \mathcal{N}_1(\mathbb{RP}^2(\pi, \pi))$ is doubly covered by $S^2(\pi, \pi, \pi, \pi) \times [-1, 1]$.

It follows easily from the discussion in Example 9.7 that this provides an example of cusp opening at angle π, because an essential $\mathbb{RP}^2(\pi, \pi)$ appears. The cusp section is a pillow $S^2(\pi, \pi, \pi, \pi)$, which is the boundary of the tubular neighborhood of $\mathbb{RP}^2(\pi, \pi)$.

APPENDIX A

LIMIT OF HYPERBOLICITY FOR SPHERICAL 3-ORBIFOLDS

BY MICHAEL HEUSENER[1] AND JOAN PORTI

Let $\mathcal{O}$ be a spherical 3-orbifold of cyclic type. We denote the ramification locus by $\Sigma \subset \mathcal{O}$; it is a k-component link $\Sigma := \Sigma_1 \sqcup \cdots \sqcup \Sigma_k$. During this appendix we assume that the complement $\mathcal{O} - \Sigma$ of the branching locus admits a complete hyperbolic structure of finite volume. For $t > 0$ small enough, let $C(t\alpha)$ be the hyperbolic cone manifold with topological type $(|\mathcal{O}|, \Sigma)$ and cone angles $t\alpha = (t2\pi/m_1, \ldots, t2\pi/m_k)$, where m_i is the ramification index along the component Σ_i (see Chapter 2, Proposition 2.2.4). Let t_∞ be the limit of hyperbolicity, i.e. $C(t\alpha)$ is a hyperbolic cone manifold for all $t \in J := [0, t_\infty)$.

The aim of this appendix is to prove that the hyperbolic cone manifolds $C(t\alpha)$ cannot degenerate directly to the spherical orbifold $\mathcal{O}$, i.e. we shall prove:

Main Proposition. — *Let $\mathcal{O}$ be a spherical 3-orbifold of cyclic type. If the complement $\mathcal{O} - \Sigma$ of the branching locus admits a complete hyperbolic structure of finite volume then the limit of hyperbolicity t_∞ is contained in the open interval $(0, 1)$.*

We obtain the following corollary from this proposition:

Corollary. — *The cone manifold $C(t_\infty\alpha)$ is Euclidean.*

Proof of the corollary. — By the main proposition we have $0 < t_\infty < 1$. Proceeding as in the proof of Proposition 2.3.1 of Chapter 2, we see that $C(t_\infty\alpha)$ is a Euclidean cone manifold with the same topological type as $\mathcal{O}$ and with cone angles $(t_\infty\alpha_1, \ldots, t_\infty\alpha_k)$. □

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Remark A.0.1. — The main proposition does not follow from Proposition 5.2.1 of Chapter 5. The proof of the "Collapsing Case" requires the use of the simplicial volume and does not give information about the collapse itself. If we had a method which described the collapse at the angle π in a geometric way we could probably see directly that a hyperbolic cone manifold cannot degenerate to a spherical orbifold.

Example A.0.2. — Let $\mathcal{O}(\alpha,\beta;n)$ be the 3-orbifold whose ramification locus is the 2-bridge knot or link $b(\alpha,\beta)\subset S^3$ and with branching index n. The orbifold $\mathcal{O}(\alpha,\beta;2)$ is spherical, and the 2-fold branched covering of $(S^3,b(\alpha,\beta))$ is the lens space $L(\alpha,\beta)$ which itself is a spherical space form. The complement of the branching locus supports a complete hyperbolic metric of finite volume iff $|\beta|>1$. The orbifold $\mathcal{O}(5,3;3)$ is Euclidean, and the orbifolds $\mathcal{O}(5,3;n)$, $n>3$, are hyperbolic. Note that $\mathcal{O}(\alpha,\beta;n)$, $n\geq 3$, is hyperbolic if $\alpha>5$ and $|\beta|>1$. These orbifolds and their limits of hyperbolicity were studied in [**HLM2**].

The *strategy of the proof* of the main proposition is the following. We assume that $t_\infty=1$ and we seek a contradiction. We consider a sequence $t_n\to 1$ in $J=[0,1)$ and the corresponding sequence of holonomy representations (ρ_n). By the construction in Chapter 2, we may assume that the sequence (ρ_n) belongs to an algebraic curve $\mathcal{C}$. This curve $\mathcal{C}$ has a natural compactification $\bar{\mathcal{C}}$ that consists in adding some ideal points. Up to a subsequence, ρ_n converges to a point in the compactification $\rho_\infty\in\bar{\mathcal{C}}$.

In Lemma A.1.1 we show that ρ_∞ is not an ideal point (i.e. ρ_∞ is a representation), by using Culler-Shalen theory about essential surfaces associated to ideal points and Lemma A.0.3. In fact, we prove that ρ_∞ is an irreducible representation into $SU(2)$ (Lemma A.1.2). Then we prove that ρ_∞ is μ-regular (see Definition A.0.4), which implies that, for n large, ρ_n is conjugate to a representation into $SU(2)$. We have obtained a contradiction, because the holonomy representation of a hyperbolic cone manifold of finite volume can not be contained in $SU(2)$.

Spherical 3-orbifolds. — Let $\mathcal{O}$ be a spherical 3-orbifold $\mathcal{O}=S^3/G$, where $G\subset SO(4)$ is finite. The orbifold $\mathcal{O}$ is very good, its universal covering is S^3, and its fundamental group $\pi_1(\mathcal{O})$ is the group of covering transformations, i.e. $\pi_1(\mathcal{O})=G$ is a finite group. There is a surjection $\pi_1(\mathcal{O})\to\pi_1(|\mathcal{O}|)$ where $|\mathcal{O}|$ is the underlying manifold (see [**DaM**]). The 3-manifold $|\mathcal{O}|$ is hence a rational homology sphere which contains the link Σ. Note that $\Sigma\subset|\mathcal{O}|$ is a prime link.

We denote respectively by $\mu_1,\dots,\mu_k$ and $m_1,\dots,m_k$ the meridians and ramification indices of the components $\Sigma_1,\dots,\Sigma_k$ of Σ. We assume that each meridian μ_i is represented by a simple closed curve in $\partial\mathcal{N}(\Sigma)$ which bounds a properly embedded orbifold disk in $\mathcal{N}(\Sigma)$. Here $\mathcal{N}(\Sigma)$ denotes a tubular neighborhood of $\Sigma\subset\mathcal{O}$.

In what follows, we shall make use of the following lemma:

Lemma A.0.3. — *Let $F \subset \mathcal{O} - \mathcal{N}(\Sigma)$ be a properly embedded orientable surface (so ∂F may be empty). If F is incompressible and non boundary-parallel then there is a meridian μ_i such that $\partial F \cap \mu_i \neq \emptyset$.*

Proof. — Let F be a properly embedded, orientable, incompressible, non boundary-parallel surface in $\mathcal{O} - \mathcal{N}(\Sigma)$. If ∂F has empty intersection with the meridians of Σ then we obtain a closed surface $\bar{F} \subset |\mathcal{O}|$, and hence the link $\Sigma \subset |\mathcal{O}|$ is sufficiently large (following the definition in [**CS**, §5.1]). This is impossible because S^3 is a regular branched covering of $(|\mathcal{O}|, \Sigma)$ and, according to [**CS**, Thm. 5.1.1], such a covering contains either an incompressible surface of higher genus or a non-separating sphere (see also [**GL**, Theorem 1]). □

Varieties of representation and characters. — Let Γ be a finitely generated group. The *variety of characters* $X(\Gamma)$ is the quotient in the algebraic category of the action of $SL_2(\mathbb{C})$ by conjugation on the variety of representations $R(\Gamma) = \mathrm{Hom}(\Gamma, SL_2(\mathbb{C}))$. Following [**CS**], $X(\Gamma)$ is an affine complex variety, but it is not necessarily irreducible. For a representation $\rho \in R(\Gamma)$, its projection onto $X(\Gamma)$, denoted by χ_ρ, is called the character of ρ. The character χ_ρ may be interpreted as a map:

$$\begin{aligned} \chi_\rho : \Gamma &\to \mathbb{C} \\ \gamma &\mapsto \mathrm{tr}(\rho(\gamma)). \end{aligned}$$

For any $\gamma \in \Gamma$, the trace function $\tau_\gamma \colon R(\Gamma) \to \mathbb{C}$, $\tau_\gamma(\rho) = \mathrm{tr}(\rho(\gamma))$, is invariant under conjugation. Therefore, it factors through $R(\Gamma) \to X(\Gamma)$ to the rational function

$$\begin{aligned} I_\gamma : X(\Gamma) &\to \mathbb{C} \\ \chi_\rho &\mapsto \chi_\rho(\gamma) = \mathrm{tr}(\rho(\gamma)). \end{aligned}$$

We use the notation $X(\mathcal{O} - \Sigma) = X(\pi_1(\mathcal{O} - \Sigma))$.

Definition A.0.4. — Let $\rho : \pi_1(\mathcal{O} - \Sigma) \to SL_2(\mathbb{C})$ be an irreducible representation such that $\rho(\mu_1) \neq \pm\mathrm{Id}, \dots, \rho(\mu_k) \neq \pm\mathrm{Id}$. We say that ρ is *μ-regular* if the following conditions are satisfied:

i) $H_1(\mathcal{O} - \Sigma; Ad\rho) \cong \mathbb{C}^k$, where k is the number of components of Σ.
ii) The function $I_\mu = (I_{\mu_1}, \dots, I_{\mu_k}) : X(\mathcal{O} - \Sigma) \to \mathbb{C}^k$ is locally biholomorphic at χ_ρ.

The following lemma is proved in [**Po1**, Prop. 5.24] and is going to be used at the end of the proof of the main proposition.

Lemma A.0.5. — *Let $\rho : \pi_1(\mathcal{O} - \Sigma) \to SU(2)$ be an irreducible representation such that $\mathrm{tr}(\rho(\mu_i)) \neq \pm 2$ for all $i = 1, \dots, k$.*

If ρ is μ-regular then there exists an open neighborhood $U \subset R(\mathcal{O}-\Sigma)$ of ρ such that for every representation $\rho' \in U$, ρ' is conjugate to a representation into $SU(2)$ if and only if $\mathrm{tr}(\rho'(\mu_i)) \in \mathbb{R}$ *for all* $i = 1, \ldots, k$. □

A.1. Proof of the main proposition

Beginning of the proof. — Let $t_n \in [0,1)$ be a sequence that converges to t_∞. We choose a lift $\rho_n \in R(\mathcal{O}-\Sigma)$ of the holonomy representation of the hyperbolic cone manifold $C(t_n\alpha)$. We may assume that the sequence (ρ_n) is contained in a complex curve $\mathcal{C} \subset R(\mathcal{O}-\Sigma) \subset \mathbb{C}^N$ (see the proof of Lemma 2.3.2). Now let $\bar{\mathcal{C}} \subset \mathbb{P}^N$ be the projective closure of $\mathcal{C}$ and let $\tilde{\mathcal{C}}$ be the non-singular projective curve whose function field F is isomorphic to that of $\mathcal{C}$ (see [**CS**] for details). Following [**CS**], we call the points of $\tilde{\mathcal{C}}$ which correspond to points of $\bar{\mathcal{C}} - \mathcal{C}$ *ideal points*. We might assume (by passing to a subsequence) that (ρ_n) is contained in the non-singular part of $\mathcal{C}$ and hence we have that $(\rho_n) \subset \tilde{\mathcal{C}}$. The sequence (ρ_n) converges since $\tilde{\mathcal{C}}$ is compact.

Each point $\tilde{x} \in \tilde{\mathcal{C}}$ gives us a discrete valuation $\nu_{\tilde{x}}\colon F^* \to \mathbb{Z}$ with valuation ring A. The ring A consists exactly of those functions which do not have a pole at $\tilde{x}$.

The curve $\mathcal{C} \subset R(\mathcal{O}-\Sigma)$ gives us a *tautological* representation $P\colon \pi_1(\mathcal{O}-\Sigma) \to SL_2(F)$ (see [**CS**]). For a fixed point $\tilde{x} \in \tilde{\mathcal{C}}$ we obtain therefore a representation $P\colon \pi_1(\mathcal{O}-\Sigma) \to SL_2(F)$ where F is a field with a discrete valuation. The group $\pi_1(\mathcal{O}-\Sigma)$ acts hence on the associated Bass–Serre–Tits tree which will be denoted by T. An element $\gamma \in \pi_1(\mathcal{O}-\Sigma)$ fixes a point of T if and only if $\tilde{\tau}_\gamma$ does not have a pole at $\tilde{x}$ where $\tilde{\tau}_\gamma\colon \tilde{\mathcal{C}} \to \mathbb{P}^1$ denotes the rational function determined by τ_γ.

Lemma A.1.1. — *The sequence (ρ_n) does not converge to an ideal point if $t_\infty = 1$.*

Proof. — Assume that $t_\infty = 1$ and that (ρ_n) converges to an ideal point $\tilde{x} \in \mathcal{C}$.

Let $\mu_1, \ldots, \mu_k$ be the meridians of Σ. Since $\mathrm{tr}(\rho_n(\mu_i)) = \pm 2\cos(t_n\pi/m_i)$ converges to $\pm 2\cos(\pi/m_i)$ it follows that $\tilde{\tau}_{\mu_i}$ does not have a pole at $\tilde{x}$. The image $P(\mu_i)$ is therefore contained in a vertex stabilizer of T. We obtain hence an incompressible non boundary-parallel surface $F \subset \mathcal{O} - \mathcal{N}(\Sigma)$ such that $F \cap \mu_i = \emptyset$ for $i = 1, \ldots, k$ (see [**CS**, Prop. 2.3.1]). This surface can not exist by Lemma A.0.3. □

Lemma A.1.2. — *If $t_\infty = 1$ then the sequence (ρ_n) converges to a representation $\rho_\infty \in R(\mathcal{O}-\Sigma)$ which has the following properties:*

i) ρ_∞ factors through a representation of $\pi_1(\mathcal{O})$ into $PSL_2(\mathbb{C})$;
ii) ρ_∞ is conjugate to a representation into $SU(2)$;
iii) ρ_∞ is irreducible.

Proof. — The sequence (ρ_n) converges to a representation $\rho_\infty \in R(\mathcal{O}-\Sigma)$ by Lemma A.1.1 and we have

$$\mathrm{tr}(\rho_\infty(\mu_i)) = \pm 2\cos(\pi/m_i), \text{ for } i = 1, \ldots, k.$$

In particular $\rho_\infty(\mu_i^{m_i}) = \pm\,\mathrm{Id}$ and therefore ρ_∞ factors trough $\pi_1(\mathcal{O}-\Sigma) \to \pi_1(\mathcal{O})$ to a representation of $\pi_1(\mathcal{O})$ into $PSL_2(\mathbb{C})$. Note that $\pi_1(\mathcal{O})$ is the quotient of $\pi_1(\mathcal{O}-\Sigma)$ by the normal subgroup generated by $\{\mu_1^{m_1}, \dots, \mu_k^{m_k}\}$. This proves i).

Assertion ii) follows from i): $\pi_1(\mathcal{O})$ is finite and up to conjugation $SU(2)$ is the only maximal compact subgroup of $SL_2(\mathbb{C})$.

Assume that ρ_∞ is reducible. It follows from ii) that ρ_∞ is abelian because every reducible representation into $SU(2)$ is conjugate to a diagonal representation. The representations ρ_n are all irreducible (see [**Po1**, Prop. 5.4]). The abelian representation ρ_∞ is therefore the limit of irreducible representations which implies the existence of a reducible metabelian (but not abelian) representation $\rho'_\infty \in R(\mathcal{O}-\Sigma)$ such that $\mathrm{tr}(\rho_\infty(g)) = \mathrm{tr}(\rho'_\infty(g))$ for all $g \in G$ (see [**HLM2**]). Since $\rho'_\infty(\mu_i) = \pm 2\cos(\pi/m_i)$ it follows that the image of ρ'_∞ is finite. We obtain that ρ'_∞ is conjugate to a representation into $SU(2)$. This contradicts the fact that ρ'_∞ is metabelian and non-abelian. Hence the lemma is proved. □

With the help of the next lemma we are able to prove the main proposition.

Lemma A.1.3. — *If $t_\infty = 1$ then the limit representation ρ_∞ is μ-regular.*

End of the proof of the main proposition. — Assume that $t_\infty = 1$. The sequence (ρ_n) converges to an irreducible representation $\rho_\infty\colon \pi_1(\mathcal{O}-\Sigma) \to SU(2)$ such that $\mathrm{tr}(\rho_\infty(\mu_i)) \neq \pm 2$. The representation ρ_∞ is μ-regular by Lemma A.1.3 and hence we can apply Lemma A.0.5.

The image of ρ_n is contained in $SU(2)$ up to conjugation if n is sufficiently large by Lemma A.1.3; note that $\mathrm{tr}(\rho_n(\mu_i)) \in \mathbb{R}$. This contradicts the fact that ρ_n is the holonomy of a compact hyperbolic cone manifold (see [**Po1**, Prop. 5.4]). □

It remains to prove Lemma A.1.3. Before we start with the proof we briefly recall how to define the homology of an orbifold $\mathcal{O}$ with twisted coefficients $Ad\rho$. Let ρ be a representation of $\pi_1(\mathcal{O})$ into $PSL_2(\mathbb{C})$ and let K be a CW-complex whose underlying space is the orbifold $\mathcal{O}$ such that the ramification locus Σ is contained in the 1-skeleton. The CW-complex K lifts to a $\pi_1(\mathcal{O})$-equivariant CW-complex $\tilde{K}$ over K the universal covering of $\mathcal{O}$. Set:

$$C_*(K; Ad\rho) = sl_2(\mathbb{C}) \otimes_{\pi_1(\mathcal{O})} C_*(\tilde{K}; \mathbb{Z}),$$

where $\gamma \in \pi_1(\mathcal{O})$ acts on the right on the Lie algebra $sl_2(\mathbb{C})$ via the adjoint of $\rho(\gamma^{-1})$. Note that $C_*(\tilde{K}; \mathbb{Z})$ is not a free $\pi_1^o(\mathcal{O})$–module (see [**Po1**, Section 1.2] for the details). There is a natural boundary map $\partial_i : C_i(K; Ad\rho) \to C_{i-1}(K; Ad\rho)$ (induced by the boundary operator on $C_*(\tilde{K}; \mathbb{Z})$) and the homology $H_*(\mathcal{O}; Ad\rho)$ is defined. This homology does not depend on the CW-complex and on the conjugacy class of ρ. When $\Sigma = \emptyset$ (i.e. $\mathcal{O}$ is a manifold), this is the usual homology with twisted coefficients.

Proof of Lemma A.1.3. — In order to compute $H_1(\mathcal{O}-\Sigma; Ad\rho_\infty)$ we consider the homology of the orbifold. Note that, since ρ_∞ induces a representation of $\pi_1(\mathcal{O})$ into $PSL_2(\mathbb{C})$, the adjoint representation of $\pi_1(\mathcal{O})$ into the endomorphism group of the Lie algebra $sl_2(\mathbb{C})$ is well defined.

Step 1. — $H_*(\mathcal{O}, Ad\rho_\infty) \cong 0$.

The universal covering of $\mathcal{O}$ is S^3. The projection $\pi\colon S^3 \to \mathcal{O}$ induces a map

$$\pi_* : H_*(S^3, sl_2(\mathbb{C})) \to H_*(\mathcal{O}, Ad\rho_\infty)$$

where $H_*(S^3, sl_2(\mathbb{C})) \cong H_*(S^3, \mathbb{C}) \otimes_{\mathbb{C}} sl_2(\mathbb{C})$ is the homology of S^3 with non-twisted coefficients $sl_2(\mathbb{C}) \cong \mathbb{C}^3$. Since we work over $\mathbb{C}$, we can construct a right inverse to π_* by using the transfer map (see [**Bre**, Chapter III])

$$s_*\colon H_*(\mathcal{O}, Ad\rho_\infty) \to H_*(S^3, sl_2(\mathbb{C})),$$

i.e. $\pi_* \circ s_* = \mathrm{Id}$. In particular s_* is injective and its image is invariant by the action of $\pi_1(\mathcal{O})$. The homology $H_*(S^3, sl_2(\mathbb{C}))$ is only non-trivial in dimensions 0 and 3. Since ρ_∞ is irreducible, the subspace of $sl_2(\mathbb{C})$ invariant by $\pi_1(\mathcal{O})$ is trivial, hence $H_*(\mathcal{O}, Ad\rho_\infty) \cong 0$.

Step 2. — $H_1(\mathcal{O}-\Sigma; Ad\rho_\infty) \cong \mathbb{C}^k$.

We apply a Mayer-Vietoris argument (adapted to the orbifold situation) to the pair $(\mathcal{N}(\Sigma), \mathcal{O}-\mathcal{N}(\Sigma))$, where $\mathcal{N}(\Sigma)$ is a tubular neighborhood of Σ. Since $H_*(\mathcal{O}, Ad\rho) \cong 0$, we have a natural isomorphism induced by the inclusion maps:

$$H_1(\mathcal{N}(\Sigma); Ad\rho_\infty) \oplus H_1(\mathcal{O}-\mathcal{N}(\Sigma); Ad\rho_\infty) \cong H_1(\partial\mathcal{N}(\Sigma); Ad\rho_\infty) \tag{8}$$

The homology groups $H_1(\mathcal{N}(\Sigma), Ad\rho_\infty)$ and $H_1(\partial\mathcal{N}(\Sigma), Ad\rho_\infty)$ are easily computed, and they have dimension k and $2k$ over $\mathbb{C}$ respectively (see [**Po1**, Lemma 2.8 and Prop. 3.18] for instance). Therefore $H_1(\mathcal{O}-\Sigma; Ad\rho_\infty) \cong \mathbb{C}^k$.

Step 3. — χ_{ρ_∞} is a smooth point of $X(\mathcal{O}-\Sigma)$ with local dimension k.

By an estimate of Thurston [**Thu1**, Thm. 5.6], see also [**CS**, Thm. 3.2.1], the dimension of $X(\mathcal{O}-\Sigma)$ at χ_{ρ_∞} is $\geq k$. In addition, since $H^1(\mathcal{O}-\Sigma; Ad\rho_\infty)$ contains the Zariski tangent space $T_{\chi_\rho}X(\mathcal{O}-\Sigma)$, and $H^1(\mathcal{O}-\Sigma; Ad\rho_\infty)$ is dual to $H_1(\mathcal{O}-\Sigma; Ad\rho_\infty)$, $\dim(T_{\chi_\rho}X(\mathcal{O}-\Sigma)) \leq k$. Thus $\dim(T_{\chi_\rho}X(\mathcal{O}-\Sigma)) = k$ and χ_{ρ_∞} is a smooth point.

Step 4. — $I_\mu = (I_{\mu_1}, \dots, I_{\mu_k}) : X(\mathcal{O}-\Sigma) \to \mathbb{C}^k$ is locally biholomorphic at χ_{ρ_∞}.

Viewing $H_1(\mathcal{O}-\Sigma; Ad\rho_\infty)$ as the Zariski cotangent space $T_{\chi_\rho}X(\mathcal{O}-\Sigma)$, the proof consists in finding a basis for $H_1(\mathcal{O}-\Sigma; Ad\rho_\infty)$ that can be interpreted as the set of differential forms $\{dI_{\mu_1}, \dots, dI_{\mu_k}\}$.

Let $\Sigma_1 \sqcup \cdots \sqcup \Sigma_k = \Sigma$ be the decomposition of Σ in connected components. Choose $\lambda_1, \dots, \lambda_k \in \pi_1(\mathcal{O}-\Sigma)$ such that λ_i, μ_i generate $\pi_1(\partial(\mathcal{N}(\Sigma_i))) \cong \mathbb{Z} \oplus \mathbb{Z}$, for $i = 1, \dots, k$. Since $\mathrm{tr}(\rho(\mu_i)) \neq \pm 2$, we may assume that $\mathrm{tr}(\rho(\lambda_i)) \neq \pm 2$, up to replacing λ_i by $\lambda_i\mu_i$ if necessary. If we identify homology groups with cotangent spaces, then the differential form dI_{λ_i} generates $H_1(\mathcal{N}(\Sigma_i); Ad\rho_\infty) \cong \mathbb{C}$ and $\{dI_{\lambda_i}, dI_{\mu_i}\}$

is a basis for $H_1(\partial\mathcal{N}(\Sigma_i); Ad\rho_\infty) \cong \mathbb{C}^2$ (see for instance [**Po1**, Lemma 3.20] or [**Ho2**] for these computations). It follows from the Mayer-Vietoris isomorphism (8) that $\{dI_{\mu_1}, \ldots, dI_{\mu_k}\}$ is a basis for $H_1(\mathcal{O} - \Sigma; Ad\rho_\infty) \cong T_{\chi_\rho} X(\mathcal{O} - \Sigma)$. Therefore $I_\mu = (I_{\mu_1}, \ldots, I_{\mu_k})$ is locally biholomorphic at χ_{ρ_∞} and ρ_∞ is μ-regular. $\square$

APPENDIX B

THURSTON'S HYPERBOLIC DEHN FILLING THEOREM

We give a proof of Thurston's hyperbolic Dehn filling theorem for completeness. In the manifold case, the proof is given in Thurston's notes [**Thu1**], and it has been generalized to orbifolds by Dunbar and Meyerhoff [**DuM**].

We follow Thurston's proof [**Thu1**], taking care of the smoothness of the variety of representations. For the smoothness, we use an argument from [**Zh1, Zh2**]. There is another approach in [**PP**] without using these results in the manifold case.

We prove the theorem for manifolds in Section B.1, and for orbifolds in Section B.2. In Section B.3 we prove it for a special case of manifolds with totally geodesic boundary.

B.1. The manifold case

Let M be a compact 3-manifold with boundary $\partial M = T_1^2 \cup \cdots \cup T_k^2$ a non-empty union of tori, whose interior is complete hyperbolic with finite volume. Thurston's hyperbolic Dehn filling theorem provides a parametrization of a space of hyperbolic deformations of this structure on $\operatorname{int}(M)$. The parameter for these deformations is the generalized Dehn filling coefficients, which describe the metric completion of the ends of $\operatorname{int}(M)$.

For each boundary component T_j^2 we fix two oriented simple closed curves μ_j and λ_j that generate $\pi_1(T_j^2)$. The completion of the structure on the j-th end of $\operatorname{int}(M)$ is described by the generalized Dehn filling coefficients $(p_j, q_j) \in \mathbb{R}^2 \cup \{\infty\} = S^2$, so that the structure at the j-th end is complete iff $(p_j, q_j) = \infty$. The interpretation of the coefficients $(p_j, q_j) \in \mathbb{R}^2$ is the following:

- If $p_j, q_j \in \mathbb{Z}$ are coprime, then the completion at the j-th torus is a non-singular hyperbolic 3-manifold, which topologically is the Dehn filling with surgery meridian $p_j\mu_j + q_j\lambda_j$.

- When $p_j/q_j \in \mathbb{Q} \cup \{\infty\}$, let $m_j, n_j \in \mathbb{Z}$ be coprime integers such that $p_j/q_j = m_j/n_j$. The completion is a cone 3-manifold obtained by gluing a torus with singular core. The surgery meridian is $m_j\mu_j + n_j\lambda_j$ and the cone angle of the singular component is $2\pi|m_j/p_j|$.
- When $p_j/q_j \in \mathbb{R} - \mathbb{Q}$, then the completion (by equivalence classes of Cauchy sequences) is not topologically a manifold. These singularities are called of Dehn type, cf. [**Ho2**].

Theorem B.1.1 **(Thurston's Hyperbolic Dehn Filling [Thu1])**
There exists a neighborhood of $\{\infty, \dots, \infty\}$ *in* $S^2 \times \cdots \times S^2$ *such that the complete hyperbolic structure on* $\mathrm{int}(M)$ *has a space of hyperbolic deformations parametrized by the generalized Dehn filling coefficients in this neighborhood.*

Proof. — The proof has three main steps. The first one is the construction of the algebraic deformation of the holonomy of the complete structure on $\mathrm{int}(M)$. The second step is to associate generalized Dehn filling coefficients to this deformation and the third one is the construction of the developing maps with the given holonomies. These steps are treated in paragraphs B.1.1, B.1.2 and B.1.3 respectively.

B.1.1. Algebraic deformation of the holonomies. — We recall some notation. Let $R(M) = \mathrm{Hom}(\pi_1(M), SL_2(\mathbb{C}))$ be the variety of representations of $\pi_1(M)$ into $SL_2(\mathbb{C})$, and $X(M) = R(M)//SL_2(\mathbb{C})$ its variety of characters. Both are affine algebraic complex varieties (not necessarily irreducible). For a representation $\rho \in R(M)$, its character χ_ρ is its projection to $X(M)$ and can be viewed as the map $\chi_\rho : \pi_1(M) \to \mathbb{C}$ defined by $\chi_\rho(\gamma) = \mathrm{trace}(\rho(\gamma))$, for every $\gamma \in \Gamma$. Given an element $\gamma \in \pi_1(M)$ we will also consider the rational function

$$\begin{aligned} I_\gamma : X(M) &\to \mathbb{C} \\ \chi_\rho &\mapsto \chi_\rho(\gamma) = \mathrm{trace}(\rho(\gamma)). \end{aligned}$$

Recall that $\mu_1, \dots, \mu_k$ is a family of simple closed curves, one for each boundary component of M. We will consider the map

$$I_\mu = (I_{\mu_1}, \dots, I_{\mu_k}) : X(M) \to \mathbb{C}^k.$$

Let $\rho_0 \in R(M)$ be a lift of the holonomy representation of $\mathrm{int}(M)$ and let $\chi_0 \in X(M)$ denote its character. The main result we need about deformations is the following:

Theorem B.1.2. — *The map* $I_\mu = (I_{\mu_1}, \dots, I_{\mu_k}) : X(M) \to \mathbb{C}^k$ *is locally bianalytic at* χ_0.

Proof. — We follow the proofs of [**Thu1**] and [**Zh1, Zh2**]. We prove first that I_μ is open at χ_0. Let $X_0(M)$ be any irreducible component of $X(M)$ that contains χ_0. In order to prove that I_μ is open we use the following two facts:

- By an estimate of Thurston [**Thu1**, Thm. 5.6], see also [**CS**, Thm. 3.2.1], $\dim X_0(M) \geq k$.
- The character χ_0 is an isolated point of $I_\mu^{-1}(I_\mu(\chi_0))$, by Mostow Rigidity.

By the openness principle [**Mum2**], it follows that I_μ is open at χ_0.

Moreover, I_μ is either locally bianalytic or a branched cover. Let $V_0 \subset X(M)$ and $V_1 \subset \mathbb{C}^k$ be respective neighborhoods of χ_0 and $I_\mu(\chi_0)$ such that the restriction $I_\mu|_{V_0} : V_0 \to V_1$ is either bianalytic or a branched cover. If $I_\mu|_{V_0}$ was a branched cover, then the ramification set would be a proper subvariety $W \subset V_0$ such that $\overline{I_\mu(W)}$ would be also a proper subvariety of V_1. In addition, if $I_\mu|_{V_0}$ was a branched cover, then the restriction of $I_\mu|_{V_0 - W}$ would be a cover of $V_1 - \overline{I_\mu(W)}$ of degree $d > 1$. Hence it suffices to show that $I_\mu|_{V_0}$ has only one preimage in a Zariski dense subset of $V_1 \subset \mathbb{C}^k$. This set is

$$S = \left\{ \left(\epsilon_1 2 \cos \frac{\pi}{n_1}, \dots, \epsilon_q 2 \cos \frac{\pi}{n_q} \right) \mid n_i > N_0 \right\}$$

for some N_0 sufficiently large, where the coefficients $\epsilon_i = \pm 1$ are chosen so that $I_\mu(\chi_0) = (\epsilon_1 2, \dots, \epsilon_q 2)$. We have that for $\chi \in X(M)$ in a neighborhood of χ_0, if $I_\mu(\chi) \in S$ then χ is the character of the holonomy of a hyperbolic orbifold, and therefore it is unique by Mostow rigidity.

Along this proof we have used twice that deformations of the holonomy imply deformations of the structure (every time we used Mostow rigidity). The techniques in Paragraph B.1.3 below apply to construct such deformations of structures. □

Remark B.1.3. — A stronger version of this theorem can be found in Kapovich's book, [**Kap**, Thm. 9.34 and Remark 9.41], where the dimension of certain cohomology groups with twisted coefficients are computed. These computations are an infinitesimal rigidity result, similar to rigidity results of Calabi-Weil [**Wei**], Raghunathan [**Rag**] and Garland [**Garl**], and they imply Theorem B.1.2 above.

B.1.2. Dehn filling coefficients. — In order to define the Dehn filling coefficients (p_j, q_j), we must introduce first the holomorphic parameters u_j and v_j. Following [**Thu1**], if we view the holonomy of μ_j and λ_j as affine transformations of $\mathbb{C} = \partial \mathbb{H}^3 - \{\infty\}$ (∞ being a point fixed by μ_j and λ_j), u_j and v_j are branches of the logarithm of the linear part of the holonomy of μ_j and λ_j respectively. For the definition we use branched coverings.

Definition B.1.4. — Let $U \subset \mathbb{C}^k$ be a neighborhood of the origin and $W \subset X(M)$ a neighborhood of χ_0. We define $\pi : U \to W$ to be the branched covering such that

$$I_{\mu_j} \pi(u) = \epsilon_j 2 \cosh(u_j/2) \qquad \text{for every } u = (u_1, \dots, u_k) \in U,$$

and $\epsilon_j \in \{\pm 1\}$ is chosen so that $I_{\mu_j}(\pi(0)) = \chi_0(\mu_j) = \operatorname{trace}(\rho_0(\mu_j)) = \epsilon_j 2$.

Remark B.1.5. — From this definition, $u = (u_1, \dots, u_k)$ is just the parameter of a neighborhood of the origin $U \subset \mathbb{C}^k$ and its geometric interpretation comes from the branched covering $\pi : U \to V \subset X(M)$. We also remark that $\pi(u) = \chi_0$ iff $u = 0$.

Now we shall associate a representation to each $u \in U$ by considering an analytic section
$$s : V \subset X(M) \to R(M)$$
such that $s(\chi_0) = \rho_0$. This section may be constructed easily by using [**CS**, Prop. 3.1.2] or [**Po1**, Prop. 3.2]. We use the notation:
$$\rho_u = s(\pi(u)) \in R(M) \qquad \text{for every } u \in U.$$

Lemma B.1.6. — *For $j = 1, \dots, k$, there is an analytic map $A_j : U \to SL_2(\mathbb{C})$ such that for every $u \in U$:*
$$\rho_u(\mu_j) = \epsilon_j A_j(u) \begin{pmatrix} e^{u_j/2} & 1 \\ 0 & e^{-u_j/2} \end{pmatrix} A_j(u)^{-1} \qquad \textit{with } \epsilon_j = \pm 1.$$

Proof. — Let $\epsilon_j \in \{\pm 1\}$ be such that $I_{\mu_j}(\chi_0) = \chi_0(\mu_j) = \epsilon_j 2$. We fix a vector $w_2 = (w_2^1, w_2^2) \in \mathbb{C}^2$ that is not an eigenvector for ρ_0 and we set
$$w_1(u) = \left(w_1^1(u), w_1^2(u)\right) = \left(\epsilon_j \rho_u(\mu_j) - e^{-u_j/2}\right) w_2.$$
Since $\epsilon_j e^{\pm u_j/2}$ are the eigenvalues for $\rho_u(\mu_j)$, the following is the matrix of a change of basis that has the required properties for the lemma:
$$A_j(u) = \frac{1}{\sqrt{w_1^1(u) w_2^2 - w_1^2(u) w_2^1}} \begin{pmatrix} w_1^1(u) & w_2^1 \\ w_1^2(u) & w_2^2 \end{pmatrix}.$$
□

Lemma B.1.7. — *There exist unique analytic functions $v_j, \tau_j : U \to \mathbb{C}$ such that $v_j(0) = 0$ and for every $u \in U$:*
$$\rho_u(\lambda_j) = \pm A_j(u) \begin{pmatrix} e^{v_j(u)/2} & \tau_j(u) \\ 0 & e^{-v_j(u)/2} \end{pmatrix} A_j(u)^{-1}.$$

In addition:

i) $\tau_j(0) \in \mathbb{C} - \mathbb{R}$;
ii) $\sinh(v_j/2) = \tau_j \sinh(u_j/2)$;
iii) v_j *is odd in* u_j *and even in* u_l, *for* $l \neq j$;
iv) $v_j(u) = u_j(\tau_j(u) + O(|u|^2))$.

Proof. — The existence and uniqueness of v_j and τ_j, as well as point ii), follow straightforward from the commutativity between λ_j and μ_j. We remark that the uniqueness of v_j uses the hypothesis $v_j(0) = 0$, because this fixes the branch of the logarithm. To prove i) we recall that $\rho_0(\mu_j)$ and $\rho_0(\lambda_j)$ generate a rank two parabolic group, because ρ_0 is the holonomy of a complete structure. In particular 1 and $\tau_j(0)$ generate a lattice in $\mathbb{C}$ and therefore $\tau_j(0) \notin \mathbb{R}$.

To prove iii) we remark that the points $(\pm u_1, \pm u_2, \dots, \pm u_k)$ project to the same character in $X(M)$ independently of the signs $\pm$, hence

$$v_j(\pm u_1, \pm u_2, \dots, \pm u_k) = \pm v_j(u_1, u_2, \dots, u_k).$$

This equality, combined with points i) and ii) imply that v_j is odd in u_j and even in u_l for $l \neq j$. Finally, iv) follows easily from the previous points. $\square$

Definition B.1.8. — [**Thu1**] For $u \in U$ we define the *generalized Dehn filling coefficients* of the j-th cusp $(p_j, q_j) \in \mathbb{R}^2 \cup \{\infty\} \cong S^2$ by the formula:

$$\begin{cases} (p_j, q_j) = \infty & \text{if } u_j = 0 \\ p_j u_j + q_j v_j = 2\pi\sqrt{-1} & \text{if } u_j \neq 0 \end{cases}$$

The equality $v_j = u_j(\tau_j(u) + O(|u|^2))$, with $\tau_j(0) \in \mathbb{C} - \mathbb{R}$, implies:

Proposition B.1.9. — *The generalized Dehn filling coefficients are well defined and*

$$\begin{aligned} U &\to S^2 \times \cdots \times S^2 \\ u &\mapsto (p_1, q_1), \dots, (p_k, q_k) \end{aligned}$$

defines a homeomorphism between U and a neighborhood of $\{\infty, \dots, \infty\}$. $\square$

B.1.3. Deforming developing maps. — Let $D_0 : \widetilde{\operatorname{int} M} \to \mathbb{H}^3$ be the developing map for the complete structure on $\operatorname{int}(M)$, with holonomy ρ_0. The following proposition completes the proof of Theorem B.1.1.

Proposition B.1.10. — *For each $u \in U$ there is a developing map $D_u : \widetilde{\operatorname{int} M} \to \mathbb{H}^3$ with holonomy ρ_u, such that the completion of* $\operatorname{int} M$ *is given by the generalized Dehn filling coefficients of u.*

Proof. — We write $\operatorname{int}(M) = N \cup C_1 \cup \cdots \cup C_k$, where $N \cong M$ is a compact core of $\operatorname{int}(M)$, $C_j \cong T^2 \times [0, +\infty)$, $C_j \cap N \cong T^2 \times [0, 1]$ and $C_j \cap C_l = \emptyset$ for $j \neq l$. We shall construct D_u separately for $\tilde{N}$ and for $\tilde{C}_j$, and then glue the pieces. We shall construct a family of maps $\{D_u\}_{u \in U}$ that will be continuous on u for the compact $\mathcal{C}^1$-topology. This means that if $\{u_n\}_{n \in \mathbb{N}}$ is a sequence in U converging to $u_\infty \in U$, then D_{u_n} converges to D_{u_∞} uniformly on compact subsets, and the tangent map of D_{u_n} also converges to D_{u_∞} uniformly on compact subsets.

Lemma B.1.11. — *There exists a family of local diffeomorphism $D_u^0 : \tilde{N} \to \mathbb{H}^3$, which depends on $u \in U$ continuously for the compact $\mathcal{C}^1$-topology, such that D_u^0 is ρ_u-equivariant and $D_0^0 = D_{0|\tilde{N}}$.*

Proof. — This is a particular case of [**CEG**, Lemma 1.7.2], but we repeat their proof here because we will use the gluing technique. We fix $u \in U$ and we construct D_u^0 a family continuous on u for the compact $\mathcal{C}^1$-topology.

We start with a finite covering $\{U_1, \dots, U_n\}$ of a neighborhood of N. Let $p : \widetilde{\operatorname{int}(M)} \to M$ denote the universal covering projection and let V_1 be a connected component of $p^{-1}U_1 = \bigsqcup_{\gamma\in\pi_1} \gamma V_1$. We define $\Delta_1 : V_1 \to \mathbb{H}^3$ to be the restriction of D_0^0 and we extend it ρ_t-equivariantly to $p^{-1}U_1 = \bigsqcup_{\gamma\in\pi_1} \gamma V_1$.

We would like to define $\Delta_i : p^{-1}U_i \to \mathbb{H}^3$ in the same way and to construct D_u^0 by gluing $\Delta_1, \dots \Delta_n$, but we must be careful with the equivariance and the continuity on u for the compact $\mathcal{C}^1$-topology. The next step will be to try to extend Δ_1 to a map on $p^{-1}U_1 \cap p^{-1}U_2$. To be precise, we take $\{U_1', \dots, U_n'\}$ a shrinking of $\{U_1, \dots, U_n\}$ that covers N and we will define $\Delta_2 : p^{-1}U_1' \cap p^{-1}U_2' \to \mathbb{H}^3$ as an extension of $\Delta_1|_{p^{-1}U_1'}$. Thus we define $\Delta_2|_{p^{-1}U_1'} = \Delta_1|_{p^{-1}U_1'}$ and we extend it to $p^{-1}U_2'$ as follows.

We choose V_2 a connected component of $p^{-1}U_2$ and V_2' a connected component of the smaller neighborhood $p^{-1}U_2'$ which is contained in V_2. In particular

$$\overline{V_2'} \subset \operatorname{int}(V_2).$$

Let $\phi : \overline{V_2} \to [0,1]$ a $\mathcal{C}^\infty$-bump function such that:

- ϕ restricted to $V_2' \cap p^{-1}U_1'$ is constant equal to 1,
- the closure of the support of ϕ is contained in $V_2 \cap p^{-1}U_1$.

By using ϕ, we define $f : V_2 \to \mathbb{H}$ as:

$$f = \phi\Delta_1 + (1-\phi)D_0^0.$$

This is, f equals Δ_1 on the intersection $V_2' \cap p^{-1}U_1'$ and equals D_0^0 on $V_2 - p^{-1}U_1$. In addition f depends continuously on u for the compact $\mathcal{C}^1$-topology (we remark that ϕ is independent of u). We define $\Delta_2|_{V_2'} = f|_{V_2'}$ and we extend it ρ_u-equivariantly to $p^{-1}U_2' = \bigsqcup_{\gamma\in\pi_1} \gamma V_2'$. In this way Δ_2 is ρ_u-equivariant and the construction depends continuously on u for the compact $\mathcal{C}^1$-topology.

Now we can continue by an induction process and making successive shrinking to get D_u^0 defined on a neighborhood of $\tilde{N}$. Finally, since D_0^0 is a local diffeomorphism and N is compact, the compact $\mathcal{C}^1$-topology implies that D_u^0 is a local diffeomorphism for u close to 0. □

Lemma B.1.12. — *There exists a family of local embeddings $D_u^j : \tilde{C}_j \to \mathbb{H}^3$ which is continuous on $u \in U$ for the compact $\mathcal{C}^1$-topology, such that D_u^j is ρ_u-equivariant, $D_0^j = D_{0|\tilde{C}_j}$ and the structure on C_j can be completed as described by the generalized Dehn filling parameters.*

Before proving this lemma, we prove the following one, that concludes the proof of Proposition B.1.10.

Lemma B.1.13. — *There exists a family of local embeddings $D_u : \widetilde{\operatorname{int}}(M) \to \mathbb{H}^3$ which depends continuously on $u \in U$ for the compact $\mathcal{C}^1$-topology, such that D_u is ρ_u-equivariant and D_0 is the developing map of the complete structure on* int(M). *In addition, away from a compact set it coincides with the maps $D_u^1, \dots D_u^k$ of Lemma B.1.12.*

Proof of Lemma B.1.13. — We already have D_u^0 and D_u^j, defined on the respective universal coverings of N and C_j. We want to glue these maps by using bump functions again. Recall that $C_j \cong T^2 \times [0,+\infty)$ and $N \cap C_j \cong T^2 \times [0,1]$. Thus it suffices to work with a partition of the unit subordinate to the covering $\{[0,3/4),(1/4,1]\}$ of the interval $[0,1]$ to glue the maps on the universal coverings. □

Proof of Lemma B.1.12. — The universal covering $\tilde{C}_j$ is homeomorphic to $\mathbb{R}^2 \times [0,+\infty)$. We suppose that the action of the fundamental group is given by:

$$\begin{array}{rcl} \mu_j : \mathbb{R}^2 \times [0,+\infty) & \to & \mathbb{R}^2 \times [0,+\infty) \\ (x,y,t) & \mapsto & (x+1,y,t) \end{array} \quad \text{and} \quad \begin{array}{rcl} \lambda_j : \mathbb{R}^2 \times [0,+\infty) & \to & \mathbb{R}^2 \times [0,+\infty) \\ (x,y,t) & \mapsto & (x,y+1,t) \end{array}.$$

By Lemmas B.1.6 and B.1.6, we may assume that

$$\rho_u(\mu_j) = \pm \begin{pmatrix} e^{u_j/2} & 1 \\ 0 & e^{-u_j/2} \end{pmatrix} \qquad \rho_u(\lambda_j) = \pm \begin{pmatrix} e^{v_j(u)/2} & \tau_j(u) \\ 0 & e^{-v_j(u)/2} \end{pmatrix}.$$

Since the cusp is complete for the initial hyperbolic structure when $u = 0$, we also assume that the restriction of the developing map $D_0^j = D_{0|\tilde{C}_j}$ is:

$$\begin{array}{rcl} D_0^j : \mathbb{R}^2 \times [0,+\infty) & \to & \mathbb{H}^3 \cong \mathbb{C} \times (0,+\infty) \\ (x,y,t) & \mapsto & (x+\tau_j(0)y, e^t) \end{array}$$

Here we use the half space model $\mathbb{H}^3 \cong \mathbb{C} \times (0,+\infty)$ for the hyperbolic space.

We consider the family of maps $D_u^j : \mathbb{R}^2 \times [0,+\infty) \to \mathbb{H}^3$ defined by

$$D_u^j(x,y,t) = \begin{cases} \left(\dfrac{e^{u_j x + v_j(u) y} - 1}{e^{u_j/2} - e^{-u_j/2}}, e^{t + \operatorname{Re}(u_j x + v_j(u) y)} \right) & \text{if } u_j \neq 0; \\ (x + \tau_j(u) y, e^t) & \text{if } u_j = 0. \end{cases}$$

The map D_u^j is ρ_u-equivariant and it is also a local diffeomorphism. Since $v_j(u) = u_j(\tau_j(u) + O(|u|^2))$, D_u^j varies continuously on $u \in U$ for the compact $\mathcal{C}^1$-topology.

The following claim finishes the proof of the lemma.

Claim B.1.14. — *The hyperbolic structure on C_j induced by D_u^j is complete iff $u_j = 0$. If $u_j \neq 0$ then the metric completion of C_j is the completion described by the Dehn filling parameters.*

Proof. — When $u_j = 0$, the structure is complete since it is the quotient of a horoball by a rank two parabolic group (cf. [**BP**] or [**Rat**]).

We assume that $u_j \neq 0$. For every $t \in [0,+\infty)$, $D_u^j(\mathbb{R}^2 \times \{t\})$ is the set of points that are at distance $d(t)$ from the geodesic γ having end-point

$$\bar{\gamma} \cap \partial \mathbb{H}^3 = \left\{ \frac{-2}{\sinh(u_j/2)}, +\infty \right\}.$$

The distance $d(t)$ satisfies

$$\sinh(d(t))\, 2\, |\sinh(u_j/2)| e^t = 1.$$

In particular, for fixed $u \in U$, $d(t) \to 0$ when $t \to +\infty$, and

$$D_u^j\left(\mathbb{R}^2 \times [t,+\infty)\right) = \mathcal{N}_{d(t)}(\gamma) - \gamma,$$

where $\mathcal{N}_{d(t)}(\gamma)$ is the tubular neighborhood of γ of radius $d(t)$.

We deal first with the case where (p_j, q_j) is a pair of coprime integers. Let $r_j, s_j \in \mathbb{Z}$ be such that $p_j s_j - q_j r_j = 1$. Consider the linear isomorphism

$$\begin{aligned} \Phi : \mathbb{R}^2 &\to \mathbb{R}^2 \\ (a,b) &\mapsto (p_j a + r_j b, q_j a + s_j b) \end{aligned}$$

The equality $p_j u_j + v_j q_j = 2\pi\sqrt{-1}$ implies that

$$D_u^j(\Phi(a,b),t) = \left(\frac{e^{a2\pi\sqrt{-1}+bl} - 1}{e^{u_j/2} - e^{-u_j/2}}, e^{t+b\ \mathrm{Re}(l_j)} \right)$$

where $l_j = r_j u_j + s_j v_j$. An easy computation shows that

$$\mathrm{Re}(l_j) = \mathrm{Im}(u_j \bar{v}_j)/(2\pi),$$

which is non-zero, because $v_j(u) = u_j(\tau_j(u) + O(|u|^2)$ and $\mathrm{Im}(\tau_j(0)) \neq 0$.

It follows that for every $t > 0$, $D_u^j : \mathbb{R}^2 \times [t,+\infty) \to \mathbb{H}^3$ factorizes to a homeomorphism

$$(\mathbb{R}^2 \times [t,+\infty))\Big/\langle p_j\mu_j + q_j\lambda_j \rangle \cong \mathcal{N}_{d(t)}(\gamma) - \gamma,$$

where $\langle p_j\mu_j + q_j\lambda_j \rangle$ denotes the cyclic group generated by $p_j\mu_j + q_j\lambda_j$. In addition, the holonomy of $r_j\mu_j + s_j\lambda_j$ preserves γ and acts on γ as a translation of length $\mathrm{Re}(l_j) \neq 0$. It follows that the completion of C_j is obtained by adding the quotient of γ by this translation, and topologically this is the Dehn filling with meridian $p_j\mu_j + q_j\lambda_j$.

Next we study the case where $p_j/q_j \in \mathbb{Q} \cup \{\infty\}$. Let (m_j, n_j) be a pair of coprime integers such that $m_j/n_j = p_j/q_j$ and set $\alpha_j = 2\pi m_j/p_j$. Consider the singular space $\mathbb{H}^3_{\alpha_j}$ defined in the first chapter. The developing map $D_u^j : \tilde{C}_j \to \mathbb{H}^3 - \gamma$ induces a developing map $D'_u : \tilde{C}_j \to \mathbb{H}^3_{\alpha_j} - \Sigma$, because the universal coverings of $\mathbb{H}^3 - \gamma$ and of $\mathbb{H}^3_{\alpha_j} - \Sigma$ are isometric. Now the discussion in the precedent case applies, and we conclude that the completion of C_j consist of adding a singular geodesic, with cone angle α_j, and the topological filling meridian is $m_j\mu_j + n_j\lambda_j$.

In the last case, where $p_j/q_j \in \mathbb{R} - \mathbb{Q}$, the holonomy of C_j acts faithfully on the geodesic γ. Since $\pi_1(C_j) \cong \mathbb{Z} \oplus \mathbb{Z}$, this action is non discrete. It follows easily that the completion can not be Hausdorff.

This finishes the proof of the claim, of Proposition B.1.10 and of Theorem B.1.1. □

The proof yields not only the existence of a one parameter family of cone 3-manifold structures but also gives a path of corresponding holonomies in the variety $R(M)$ of representations of $\pi_1(M)$ into $SL_2(\mathbb{C})$. A corollary of the proof of Thurston's hyperbolic Dehn filling theorem is the following:

Corollary B.1.15. — *For any real numbers $\alpha_1, \ldots, \alpha_k \geq 0$ there exist $\varepsilon > 0$ and a path $\gamma : [0, \varepsilon) \to R(M)$, such that, for every $t \in [0, \varepsilon)$, $\gamma(t)$ is a lift of the holonomy of a hyperbolic structure on M corresponding to the generalized Dehn filling coefficients*

$$\big((p_1, q_1), \ldots, (p_k, q_k)\big) = \big((2\pi/(\alpha_1 t), 0), \ldots, (2\pi/(\alpha_k t), 0)\big).$$

When $\alpha_j t = 0$, the structure at the j-th cusp is complete; otherwise its completion is a cone 3-manifold obtained by adding to T_j^2 a solid torus with meridian curve μ_j and singular core with cone angle $\alpha_j t$.

B.2. The orbifold case

B.2.1. Dehn filling on orbifolds. — Let $\mathcal{O}$ be a compact 3-orbifold whose boundary components are Euclidean 2-orbifolds. Each component of $\partial\mathcal{O}$ is one of the following:

- a 2-torus $T^2 \cong S^1 \times S^1$;
- a pillow $P = S^2(2,2,2,2) \cong T^2/(\mathbb{Z}/2\mathbb{Z})$;
- a turnover $S^2(n_1, n_2, n_3)$ with $\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} = 1$.

A turnover cannot bound the quotient of a solid torus, hence we can not do any Dehn filling on it. This is coherent with the fact that turnovers are rigid, and do not allow to define Dehn filling parameters. For 2-torus T^2, we define the Dehn filling coefficients exactly in the same way as for manifolds. Next we give the details of the definition for a pillow.

Definition B.2.1. — A *solid pillow* is a 3-ball with two unknoted singular arcs with ramification indices 2. In a solid pillow, a *meridian disk* is a proper non-singular disc of the solid pillow that splits it into two balls with a singular arc each one (see Figure 1).

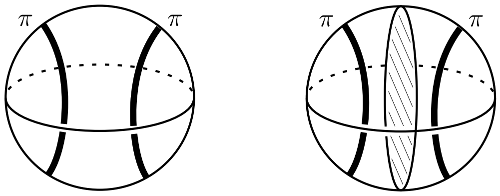

FIGURE 1. The solid pillow. Its boundary is the pillow. The figure on the right represents a meridian disk in the solid pillow.

A meridian disk in a solid pillow is unique up to orbifold isotopy. The solid pillow is the quotient of the solid torus by $\mathbb{Z}/2\mathbb{Z}$, and the meridian disk of the solid pillow lifts to two parallel meridian disks of the solid torus.

The boundary of the solid pillow is the pillow $S^2(2,2,2,2)$, hence we have the following definition.

Definition B.2.2. — Let $\mathcal{O}$ be a 3-orbifold, let $P \subset \partial\mathcal{O}$ be a boundary component with $P \cong S^2(2,2,2,2)$, and let $\mu \subset P$ be a simple closed curve that splits P into two discs with two cone points each one. The *Dehn filling* of $\mathcal{O}$ with *surgery meridian* μ is the orbifold $\mathcal{O} \cup_\phi S$, where S is a solid pillow and $\phi : P \to \partial S$ is an orbifold homeomorphism that identifies μ with the boundary of a meridian disc.

As for manifolds, the Dehn filling only depends on the orbifold isotopy class of the surgery meridian. To describe the orbifold isotopy classes of these curves, we need to recall some elementary facts about the fundamental group of the pillow. Since $S^2(2,2,2,2) \cong T^2/(\mathbb{Z}/2\mathbb{Z})$, we have an exact sequence:

$$1 \to \mathbb{Z} \oplus \mathbb{Z} \to \pi_1(S^2(2,2,2,2)) \to \mathbb{Z}/2\mathbb{Z} \to 1.$$

The sequence splits, and the generator of $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{Z}\oplus\mathbb{Z}$ by mapping each element to its inverse.

We list some elementary properties of the fundamental grup of an orbifold in the following lemma, whose proof is an easy exercise.

Lemma B.2.3. — *Given $\gamma \in \pi_1(S^2(2,2,2,2))$ with $\gamma \neq 1$, then:*

i) Either γ is torsion free or has order two.
ii) The element γ is torsion free iff $\gamma \in \ker\big(\pi_1(S^2(2,2,2,2)) \to \mathbb{Z}/2\mathbb{Z}\big) \cong \mathbb{Z}\oplus\mathbb{Z}$.
iii) If γ is torsion free, then γ is represented by n times a simple closed loop that splits $S^2(2,2,2,2)$ into two discs with two singular points each one. □

Definition B.2.4. — We call $\ker\big(\pi_1(S^2(2,2,2,2)) \to \mathbb{Z}/2\mathbb{Z}\big) \cong \mathbb{Z}\oplus\mathbb{Z}$ the *torsion free* subgroup of $\pi_1(S^2(2,2,2,2))$.

Remark B.2.5. — For a Dehn filling on a pillow, the surgery meridian gives, up to sign, a primitive element of the torsion free subgroup of $\pi_1(S^2(2,2,2,2))$. Thus, to describe a Dehn filling it suffices to give a primitive element of the torsion free subgroup, up to sign.

B.2.2. The hyperbolic Dehn filling theorem. — Let $\mathcal{O}$ be a compact 3-orbifold with boundary such that $\mathrm{int}(\mathcal{O})$ is hyperbolic with finite volume. Each boundary component of $\mathcal{O}$ is a Euclidean 2-orbifold. As for manifolds, the completion of the deformed hyperbolic structures on $\mathrm{int}(\mathcal{O})$ is described by generalized Dehn filling parameters. Assume that $\partial\mathcal{O}$ has

$$\begin{cases} \text{- } k \text{ non-singular tori,} \\ \text{- } l \text{ pillows, and} \\ \text{- } m \text{ turnovers,} \end{cases}$$

with $k+l>0$. For each torus T_j^2 in $\partial\mathcal{O}$, we fix μ_j and λ_j two generators of $\pi_1(T_j^2)$, that are represented by two simple loops in T_j^2. For each pillow P_j^2 in $\partial\mathcal{O}$, we also fix μ_j and λ_j two generators of the torsion free subgroup of $\pi_1(P_j^2)$, that represent two simple closed curves in P_j^2 (each curve bounds a disc with two cone points).

For a torus T_j^2 ($j \leq k$), the interpretation of the the generalized Dehn filling coefficients is the same as in the manifold case. For a pillow P_j^2 ($k+1 \leq j \leq k+j$), we also associate generalized Dehn filling coefficients $(p_j, q_j) \in \mathbb{R}^2 \cup \{\infty\}$ such that $(p_j, q_j) = \infty$ iff the structure at the j-th cusp is complete. When $(p_j, q_j) \in \mathbb{R}^2$ the interpretation is very similar to the manifold case:

- If $p_j, q_j \in \mathbb{Z}$ are coprime, then the completion at the j-th end is a non-singular hyperbolic 3-orbifold, which topologically is the Dehn filling with surgery meridian $p_j\mu_j + q_j\lambda_j$.
- When $p_j/q_j \in \mathbb{Q} \cup \{\infty\}$, let $m_j, n_j \in \mathbb{Z}$ be coprime integers such that $p_j/q_j = m_j/n_j$. The completion is a cone 3-manifold obtained by gluing a solid pillow with singular core. This core is a segment with silvered boundary (see Figure 2), and therefore there are singularities which are not of cyclic type. The surgery meridian is $m_j\mu_j + n_j\lambda_j$ and the cone angle of the singular component is $2\pi|m_j/p_j|$.
- When $p_j/q_j \in \mathbb{R} - \mathbb{Q}$, the completion (by equivalence classes of Cauchy sequences) is not topologically a manifold. These singularities are called of Dehn type, cf. [**Ho2**].

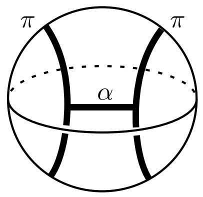

FIGURE 2. The solid pillow with a singular soul with cone angle α.

Theorem B.2.6 **(Thurston's Hyperbolic Dehn Filling for orbifolds [DuM])**

There exists a neighborhood of $\{\infty, \overset{(k+j)}{\dots}, \infty\}$ *in* $S^2 \times \overset{(k+j)}{\cdots} \times S^2$ *such that the complete hyperbolic structure on* $\mathrm{int}(\mathcal{O})$ *has a space of hyperbolic deformations parametrized by the generalized Dehn filling coefficients in this neighborhood.*

Proof. — The proof has the same steps as in the manifold case, but it is more involved. We give the three main steps in the next three paragraphs.

B.2.3. Algebraic deformation of holonomies. — The holonomy representation of $\pi_1(\mathcal{O})$ into $PSL_2(\mathbb{C})$ may not lift to a representation into $SL_2(\mathbb{C})$, because $\pi_1(\mathcal{O})$ has elements of finite order which are rotations. One could work with the variety of representations into $PSL_2(\mathbb{C})$, but in order to use some results of Section B.1, we will work with representations of $\mathcal{O}-\Sigma$ into $SL_2(\mathbb{C})$, where $\Sigma=\Sigma_{\mathcal{O}}$ is the ramification set of $\mathcal{O}$.

Assume that Σ consists of n_0 circles and n_1 arcs (thus it has n_0+n_1 components). Let $\gamma_1,\dots,\gamma_{n_0+n_1}\in\pi_1(\mathcal{O}-\Sigma)$ represent meridians of the components of Σ. Let $\rho_0:\pi_1(\mathcal{O}-\Sigma)\to SL_2(\mathbb{C})$ denote a lift of the restriction of the holonomy of $\operatorname{int}(\mathcal{O})$. Its character is denoted by χ_0. We define:

$$\begin{aligned}\mathcal{R}(\mathcal{O}) &= \{\rho\in R(\mathcal{O}-\Sigma)\mid \operatorname{trace}(\rho(\gamma_j))=\operatorname{trace}(\rho_0(\gamma_j)),\ \text{for } j=1,\dots,n_0+n_1\},\\ \mathcal{X}(\mathcal{O}) &= \{\chi\in X(\mathcal{O}-\Sigma)\mid \chi(\gamma_j)=\chi_0(\gamma_j),\ \text{for } j=1,\dots,n_0+n_1\}.\end{aligned}$$

A representation in $\mathcal{R}(\mathcal{O})$ composed with the natural projection $SL_2(\mathbb{C})\to PSL_2(\mathbb{C})$ factors to a representation $\pi_1(\mathcal{O})\to PSL_2(\mathbb{C})$ because the restriction $\operatorname{trace}(\rho(\gamma_j))=\operatorname{trace}(\rho_0(\gamma_j))$ implies that $\rho(\gamma_j)$ is a rotation of the same angle as $\rho_0(\gamma_j)$. For the same reason, if $\chi_\rho\in\mathcal{X}(\mathcal{O})$, then ρ factors to a representation $\pi_1(\mathcal{O})\to PSL_2(\mathbb{C})$.

The elements $\mu_1,\dots,\mu_{k+l}\in\pi_1(\mathcal{O})$ represent a family of simple closed curves, one for each boundary component of $\mathcal{O}$ different from a turnover. As in the manifold case, we consider the map

$$I_\mu=(I_{\mu_1},\dots,I_{\mu_{k+l}}):\mathcal{X}(\mathcal{O})\to\mathbb{C}^{k+l}.$$

Let χ_0 be the character of the holonomy ρ_0 of the complete structure on $\operatorname{int}\mathcal{O}$.

Theorem B.2.7. — *The map $I_\mu=(I_{\mu_1},\dots,I_{\mu_{k+l}}):\mathcal{X}(\mathcal{O})\to\mathbb{C}^{k+l}$ is locally bianalytic at χ_0.*

Proof. — The proof or the theorem follows the same scheme as the proof of Theorem B.1.2 in the manifold case. The only different thing is the lower bound of the dimension of $\mathcal{X}_0(\mathcal{O})$, where $\mathcal{X}_0(\mathcal{O})$ is a component of $\mathcal{X}(\mathcal{O})$ that contains χ_0. This is done in the following lemma.

Lemma B.2.8. — $\dim(\mathcal{X}_0(\mathcal{O}))\geq k+l$.

Proof of Lemma B.2.8. — Let $X_0(\mathcal{O}-\Sigma)$ be a component of $X(\mathcal{O}-\Sigma)$ that contains $\mathcal{X}_0(\mathcal{O})$. Let $\mathcal{N}(\Sigma)$ denote a tubular neighborhood of Σ. By Thurston's estimate [**Thu1**, Thm 5.6], see also [**CS**, Thm. 3.2.1], we have that

$$\dim(X_0(\mathcal{O}-\Sigma))\geq k+n_0-\frac{3}{2}\chi(\partial(\mathcal{O}-\mathcal{N}(\Sigma))),$$

because $k+n_0$ is the number of torus components of $\partial(\mathcal{O}-\mathcal{N}(\Sigma))$. Since each pillow meets 4 singular arcs, each turnover meets 3 singular arcs, and each singular arc meets the boundary twice, we have that

$$4l + 3m = 2n_1.$$

In addition the Euler characteristic of the boundary is

$$\chi(\partial(\mathcal{O} - \mathcal{N}(\Sigma))) = -m - 2l.$$

Combining these equalities, Thurston's estimate can be reformulated as:

$$\dim(X_0(\mathcal{O} - \Sigma)) \geq k + n_0 + n_1 + l.$$

Since $\mathcal{X}_0(\mathcal{O})$ is a subset of $X_0(\mathcal{O} - \Sigma)$ defined by $n_0 + n_1$ equations, lemma B.2.8 follows. This also finishes the proof of Theorem B.2.7 $\square$

B.2.4. Generalized Dehn filling coefficients. — As in the manifold case, by using Theorem B.2.7, we choose a neighborhood $V \subset \mathcal{X}(\mathcal{O})$ of χ_0, a neighborhood $U \subset \mathbb{C}^{k+l}$ of the origin, and a branched covering $\pi : U \to V \subset \mathcal{X}(\mathcal{O})$ of order 2^{k+l} defined by

$$I_{\mu_j}(\pi(u)) = I_{\mu_j}(\pi(u_1, \ldots, u_{k+l})) = \epsilon_j 2\cosh(u_j/2) \qquad \text{for } j = 1, \ldots, k+l,$$

where the coefficients $\epsilon_j \in \{\pm 1\}$ are chosen so that $I_{\mu_j}(\chi_0) = \operatorname{trace}(\rho_0(\mu_j)) = \epsilon_j 2$.

Following the manifold case, we choose an analytic section $s : V \to \mathcal{R}(\mathcal{O})$ and we use the notation $\rho_u = s(\pi(u)) \in \mathcal{R}(\mathcal{O})$.

Recall that for $j = 1, \ldots, k$, the j-th boundary component of $\mathcal{O}$ is a torus T_j and μ_j and λ_j generate $\pi_1(T_j^2)$. For $j = k+1, \ldots, k+l$, the j-th boundary component of $\mathcal{O}$ is a pillow P_j and μ_j and λ_j generate the torsion free subgroup of $\pi_1(P_j)$. We choose $\theta_j \in \pi_1(P_j)$ any element of order two, so that the following is a presentation of the fundamental group

$$\pi_1(P_j) = \langle \mu_j, \lambda_j, \theta_j \mid \mu_j\lambda_j = \lambda_j\mu_j,\ \theta_j^2 = 1,\ \theta_j\mu_j\theta_j = \mu_j^{-1},\ \theta_j\lambda_j\theta_j = \lambda_j^{-1} \rangle.$$

We recall that $\pi_1(P_j)$ is a quotient of $\pi_1(P_j - \Sigma)$.

Lemma B.2.9. — *Let $\tilde{\mu}_j, \tilde{\theta}_j \in \pi_1(P_j - \Sigma)$ be two elements that project to $\mu_j, \theta \in \pi_1(P_j)$.*

i) For $j = 1, \ldots, k+l$, there is an analytic map $A_j : U \to PSL_2(\mathbb{C})$ such that for every $u \in U$:

$$\rho_u(\tilde{\mu}_j) = \varepsilon_j A_j(u) \begin{pmatrix} e^{u_j/2} & 1 \\ 0 & e^{-u_j/2} \end{pmatrix} A_j(u)^{-1}.$$

ii) In addition, for $j = k+1, \ldots, k+l$, $A_j : U \to PSL_2(\mathbb{C})$ satisfies:

$$\rho_u(\tilde{\theta}_j) = \pm A_j(u) \begin{pmatrix} i & 0 \\ -i(e^{u_j/2} - e^{-u_j/2}) & -i \end{pmatrix} A_j(u)^{-1}$$

where $i = \sqrt{-1}$.

Proof. — We give only the proof for pillows, the proof for tori being the proof of Lemma B.1.6. We fix $w_3 \in \mathbb{C}^2$ such that w_3 is not an eigenvector for $\rho_u(\tilde{\theta}_j)$ and set $w_2(u) = (\rho_u(\tilde{\theta}_j) - i)w_3 \neq 0$, so that $(\rho_u(\tilde{\theta}_j) + i)w_2(u) = 0$, because $\pm i$ are the eigenvalues for $\rho_u(\tilde{\theta}_j)$.

The matrix $\rho_0(\tilde{\mu}_j)$ is parabolic, hence it does not diagonalize. This means that $\rho_0(\tilde{\mu}_j)$ has only a one dimensional eigenspace with eigenvalue ϵ_j. Therefore, up to replacing i by $-i$, we have:

$$\ker(\rho_0(\tilde{\mu}_j) - \epsilon_j \operatorname{Id}) \cap \ker(\rho_0(\tilde{\theta}_j) + i \operatorname{Id}) = \{0\}.$$

In particular $w_2(u) = (w_2^1(u), w_2^2(u))$ is not an eigenvector for $\rho_0(\tilde{\mu}_j)$. As in Lemma B.1.6, we take $w_1(u) = (w_1^1(u), w_1^2(u)) = (\epsilon_j \rho_u(\mu_j) - e^{-u_j/2})w_2$, where $\epsilon_j = \pm 1$ is the eigenvalue for $\rho_0(\tilde{\mu}_j)$. We define:

$$A_j(u) = \frac{1}{\sqrt{w_1^1(u)w_2^2(u) - w_1^2(u)w_2^1(u)}} \begin{pmatrix} w_1^1(u) & w_2^1(u) \\ w_1^2(u) & w_2^2(u) \end{pmatrix},$$

and it is clear from this construction that i) holds.

To prove ii), since $(\rho_u(\tilde{\theta}_j) + i)w_2(u) = 0$, we have that $\rho_u(\tilde{\theta}_j)$ is of the form:

$$\rho_u(\tilde{\theta}_j) = \pm A_j(u) \begin{pmatrix} * & 0 \\ * & -i \end{pmatrix} A_j(u)^{-1}.$$

Therefore point ii) follows from the fact that $\rho_u(\tilde{\theta}_j) \in SL_2(\mathbb{C})$ and the relation $\rho_u(\theta_j)\rho_u(\mu_j)\rho_u(\theta_j^{-1}) = \pm\rho_u(\mu_j^{-1})$, because ρ_u factors to a representation of $\pi_1(P_j)$ into $PSL_2(\mathbb{C})$. □

The following lemma has exactly the same proof as Lemma B.1.7, again because ρ_u factors to a representation of $\pi_1(P_j)$ into $PSL_2(\mathbb{C})$.

Lemma B.2.10. — *Let $\tilde{\lambda}_j \in \pi_1(P_j - \Sigma)$ be an element that projects to $\lambda_j \in \pi_1(P_j)$. For $j = 1, \ldots, k + l$, there exist unique analytic functions $v_j, \tau_j : U \to \mathbb{C}$ such that $v_j(0) = 0$ and for every $u \in U$:*

$$\rho_u(\tilde{\lambda}_j) = \pm A_j(u) \begin{pmatrix} e^{v_j(u)/2} & \tau_j(u) \\ 0 & e^{-v_j(u)/2} \end{pmatrix} A_j(u)^{-1}.$$

In addition:

i) $\tau_j(0) \in \mathbb{C} - \mathbb{R}$*;*
ii) $\sinh(v_j/2) = \tau_j \sinh(u_j/2)$*;*
iii) v_j *is odd in* u_j *and even in* u_r*, for* $r \neq j$*;*
iv) $v_j = u_j(\tau_j(u) + O(|u|^2))$. □

Following the manifold case, we define:

Definition B.2.11 (**[Thu1]**). — For $u \in U$ and $j = 1, \ldots, k+l$, we define the *generalized Dehn filling coefficients* of the j-th cusp $(p_j, q_j) \in \mathbb{R}^2 \cup \{\infty\} \cong S^2$ by the formula:

$$\begin{cases} (p_j, q_j) = \infty & \text{if } u_j = 0; \\ p_j u_j + q_j v_j = 2\pi\sqrt{-1} & \text{if } u_j \neq 0. \end{cases}$$

The following proposition follows also from Lemma B.2.10 i) and iv).

Proposition B.2.12. — *The generalized Dehn filling coefficients are well defined and*

$$U \to S^2 \times \overset{(k+l)}{\cdots} \times S^2$$
$$u \mapsto (p_1, q_1), \ldots, (p_{k+l}, q_{k+l})$$

defines a homeomorphism between U and a neighborhood of $\{\infty, \ldots, \infty\}$. □

B.2.5. Deformation of developing maps. — Let $D_0 : \widetilde{\operatorname{int}\mathcal{O}} \to \mathbb{H}^3$ be the developing map for the complete structure on $\operatorname{int}\mathcal{O}$, with holonomy ρ_0. The following is the orbifold version of Proposition B.1.10 and completes the proof of Theorem B.2.6.

Proposition B.2.13. — *For each $u \in U$ there is a developing map $D_u : \widetilde{\operatorname{int}\mathcal{O}} \to \mathbb{H}^3$ with holonomy ρ_u, such that the completion of $\operatorname{int}\mathcal{O}$ is given by the generalized Dehn filling coefficients of u.*

Proof. — The proof is analogous to the proof of Proposition B.1.10, but it needs to be adapted to orbifolds.

First we need an orbifold version of Lemma B.1.11. In the proof of Lemma B.1.11 we use a finite covering $\{U_1, \ldots, U_n\}$ of a neighborhood of a compact core of the manifold $N \subset \operatorname{int}(M)$, such that each U_i is simply connected. In the orbifold case, we have to use simply connected subsets U_i such that if $U_i \cap \Sigma_{\mathcal{O}} \neq \emptyset$ then U_i is the quotient of a ball by an orthogonal rotation. With this choice of U_i, one can generalize the argument in Lemma B.1.11 by using the fact that, for every torsion element $\gamma \in \pi_1(\mathcal{O})$, the fixed point set of $\rho_u(\gamma)$ depends analytically on $u \in U$. By using these remarks, Lemma B.1.11 can easily be generalized to orbifolds, as well as Lemma B.1.13.

It only remains to prove a version of Lemma B.1.12 for orbifolds. This lemma gives the precise developing maps for the ends. In the orbifold case, we have to distinguish the kind of end of $\operatorname{int}(\mathcal{O})$, according to the associated component of $\partial\mathcal{O}$. For tori, Lemma B.1.12 applies. We do not have to worry about turnovers because they are rigid. Thus we only need an orbifold version of Lemma B.1.12 for pillows. It is Lemma B.2.14 below, that concludes the proof of Proposition B.2.13. Let C_j denote the j-th end of $\mathcal{O}$. If $k+1 \leq j \leq k+l$ then $C_j \cong P_j \times [0, +\infty)$, where P_j is a pillow.

Lemma B.2.14. — *For $k+1 \leq j \leq k+l$, there exists a family of local embeddings $D_u^j : \tilde{C}_j \to \mathbb{H}^3$ which is continuous on $u \in U$ for the compact $\mathcal{C}^1$-topology, such that:*

i) D^j_u *is* ρ_u*-equivariant,*
ii) $D^j_0 = D_{0|\tilde{C}_j}$ *and*
iii) the structure on C_j *can be completed as described by the generalized Dehn filling parameters.*

Proof. — The universal covering $\tilde{C}_j$ is homeomorphic to $\mathbb{R}^2 \times [0, +\infty)$. With the notation above, the group $\pi_1(C_j) \cong \pi_1(P_j)$ is generated by μ_j, λ_j and θ_j. We may assume that their action on $\tilde{C}_j$ by deck transformations is the following:

$$\begin{cases} \mu_j(x,y,t) = (x+1, y, t) \\ \lambda_j(x,y,t) = (x, y+1, t) \\ \theta_j(x,y,t) = (-x,-y,t) \end{cases} \qquad \text{for every } (x,y,t) \in \mathbb{R}^2 \times [0,+\infty).$$

By Lemmas B.2.9 and B.2.10, we may assume:

$$\rho_u(\mu_j) = \pm \begin{pmatrix} e^{u_j/2} & 1 \\ 0 & e^{-u_j/2} \end{pmatrix}, \qquad \rho_u(\lambda_j) = \pm \begin{pmatrix} e^{v_j(u)/2} & \tau_j(u) \\ 0 & e^{-v_j(u)/2} \end{pmatrix}$$

$$\text{and } \rho_u(\theta_j) = \pm \begin{pmatrix} i & 0 \\ -i(e^{u_j/2} - e^{-u_j/2}) & -i \end{pmatrix}.$$

When $u = 0$, the cusp is complete and therefore the developing map $D^j_0 = D_{0|\tilde{C}^j}$ is:

$$\begin{aligned} D^j_0 : \mathbb{R}^2 \times [0,+\infty) &\to \mathbb{H}^3 \cong \mathbb{C} \times (0,+\infty) \\ (x,y,t) &\mapsto (x + \tau_j(0)y, e^t) \end{aligned}$$

The family of maps $D^j_u : \mathbb{R}^2 \times [0,+\infty) \to \mathbb{H}^3$ that proves the lemma is the following:

$$D^j_u(x,y,t) = \begin{cases} \left(\dfrac{a(u,t)\, e^{u_j x + v_j(u) y} - 1}{e^{u_j/2} - e^{-u_j/2}}, a(u,t)\, e^{t + \mathrm{Re}(u_j x + v_j(u) y)} \right) & \text{if } u_j \neq 0; \\ (x + \tau_j(u) y, e^t) & \text{if } u_j = 0. \end{cases}$$

where $a(u,t) = \left(1 + e^t |e^{u_j/2} - e^{-u_j/2}|\right)^{-1/2}$. We remark that in Lemma B.1.13 we used the same family but with $a(u,t) \equiv 1$, since we did not require the equivariance by θ_j.

The family D^j_u is a family of ρ_u-equivariant local diffeomorphisms that depends continuously on $u \in U$ for the compact $\mathcal{C}^1$-topology. The completion of C_j for the structure induced by D^j_u is the one described by the Dehn coefficients and it can be proved in the same way as Claim B.1.14.

This concludes the proof of Lemma B.2.14 and of Theorem B.2.6. □

B.3. Dehn filling with totally geodesic turnovers on the boundary

The aim of this last section is to prove Proposition B.3.1, which is a version with boundary of hyperbolic Dehn filling theorem used in Chapter 7.

Let N^3 be a three manifold with boundary and $\Sigma \subset N^3$ a 1-dimensional properly embedded submanifold. This is the case for instance when N^3 is the underlying space of an orbifold and Σ its branching locus.

We will assume that every component of ∂N^3 is a 2-sphere that intersects Σ in three points. We define the non-compact 3-manifold with boundary

$$M^3 = N^3 - \Sigma.$$

Each component of ∂M^3 is a disjoint union of 3 times punctured spheres. Each end of M^3 is the product of $[0, +\infty)$ with a torus or an annulus, according to whether the corresponding component of Σ is a circle or an arc.

We also assume that M^3 admits a hyperbolic structure with totally geodesic boundary, whose ends are cusps (of rang one or two, according to whether the corresponding component of Σ is an arc or a circle). As a metric space M^3 is complete of finite volume, and the boundary components are three times punctured spheres. The double of M^3 along the boundary is a complete hyperbolic manifold with finite volume and without boundary. Let k denote the number of connected components of Σ.

Proposition B.3.1. — *For any real numbers $\alpha_1, \dots, \alpha_k \geq 0$ there exist $\varepsilon > 0$ and a path $\gamma : [0, \varepsilon) \to R(M^3)$, such that, for every $t \in [0, \varepsilon)$, $\gamma(t)$ is a lift of the holonomy of a hyperbolic structure on M^3 whose metric completion is a cone manifold structure with totally geodesic boundary, topological type (N^3, Σ), and cone angles $(\alpha_1 t, \dots, \alpha_k t)$.*

Proof. — We follow the same argument as in the proof of Theorem B.1.1. For the algebraic part, we choose $\{\mu_1, \dots, \mu_k\} \subset \pi_1(M^3)$ a system of meridians for Σ. As in Theorem B.1.2 we have:

Proposition B.3.2. — *The map $I_\mu = (I_{\mu_1}, \dots, I_{\mu_k}) : X(M^3) \to \mathbb{C}^k$ is locally bianalytic at χ_0, where χ_0 is the character of the lift of holonomy of the complete structure on M^3.* □

The proof of Proposition B.3.2 follows precisely the same argument as Theorem B.1.2: Thurston's estimate gives $\dim(X_0(M^3)) \geq k$, and one can also apply the argument about Mostow rigidity to the double of M^3. Moreover we use the following lemma:

Lemma B.3.3. — *Let $\rho_0 : \pi_1(S^2 - \{*, *, *\}) \to SL_2(\mathbb{R})$ be the holonomy of a hyperbolic turnover or of a hyperbolic 3 times punctured sphere. Let $\{\rho_t\}_{t \in [0,\varepsilon)}$ be a deformation of ρ_0 in $R(S^2 - \{*, *, *\}, SL_2(\mathbb{C}))$ such that, for each meridian $\mu \in \pi_1(S^2 - \{*, *, *\})$ and for each $t \in (0, \varepsilon)$, $\rho_t(\mu)$ is a rotation. Then ρ_t is conjugate to the holonomy of a hyperbolic turnover (i.e. it is Fuchsian).*

Proof. — If ρ_0 is a holonomy representation, then ρ_0 is irreducible. Since irreducibility is an open property, we may assume that ρ_t is irreducible. The group

$\pi_1(S^2 - \{*, *, *\})$ is free of rank 2, generated by two meridians a and b such that the product ab is also a meridian. To prove the claim we use the fact that the conjugacy class of an irreducible representation is determined by the traces of a, b and ab. Thus if $\rho_t(a)$, $\rho_t(b)$ and $\rho_t(ab)$ are rotations, then ρ_t is conjugate to the holonomy of the hyperbolic turnover that has cone angles given by $\rho_t(a)$, $\rho_t(b)$ and $\rho_t(ab)$. This finishes the proof. □

By using the fact that deformations of holonomy imply deformations of the structure and Lemma B.3.3 we obtain the following remark:

Remark B.3.4. — When we deform the holonomy of a hyperbolic cone structure on M^3 with totally geodesic boundary so that the meridians are mapped to rotations, then the deformed representations are still the holonomy of a hyperbolic structure with totally geodesic boundary.

All the explicit deformations constructed in Section B.1 can be used here, combined with Lemma B.3.3, to prove Proposition B.3.1. □

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