Hyperbolic isometries versus symmetries of links

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Abstract

We prove that every finite group is the orientation-preserving isometry group of the complement of a hyperbolic link in the 3-sphere.

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1 Introduction

There are several well-known results in the study of finite group actions on special 3-manifolds, starting with the work [Mi] of Milnor, who gave a list of all finite groups which are susceptible to act freely and orientation-preserving on spheres. In the 3-dimensional case, by elliptization of three-manifolds (see [P1], [P2], [P3], [CM] and [B]) and Thurston's orbifold theorem (see [T] and [BLP], [BP] and [CHK]), the only finite groups which can act on $\mathbf{S}^3$ preserving the orientation are the finite subgroups of $SO(4)$. It happens also that the list of finite groups acting on integral and, more generally, $\mathbf{Z}/2$-homology spheres is quite restricted [MZ].

Definition 1. *A* group of symmetries *of a (non-oriented) link L in $\mathbf{S}^3$ is a finite group G acting on $\mathbf{S}^3$ preserving the orientation and leaving L invariant.*

In particular, a group of symmetries is a finite subgroup of $SO(4)$. Indeed, if L is a non-trivial knot, it follows from Smith's conjecture [MB] that the only possible groups of symmetries for L are either cyclic or dihedral. Note also that, in general, a link does not have a unique (maximal) group of symmetries [S] (up to conjugation), but this is indeed the case if the link is hyperbolic. Observe in fact, that each symmetry of the link L induces an orientation-preserving diffeomorphism of the complement of the link, which on its turn, if L is hyperbolic, gives rise to an isometry of the hyperbolic structure. (For basic facts and definitions in hyperbolic geometry the reader is referred to [R].)

One can then ask: *What is the relation between symmetries of a hyperbolic link and isometries of its complement?* As knots are determined by their complements [GL], isometries of the complement of a knot are also symmetries.

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However, when the link has several components, the group of symmetries of a hyperbolic link is only a subgroup of the group of isometries of its complement. In general this subgroup can be proper, for a generic isometry does not need to preserve a peripheral structure on the cusps.

Examples of isometries of exteriors of hyperbolic links which are not induced by symmetries can be found in [HW], where both groups are computed using Jeff Week's program SnapPea for links with at most nine crossings. For most links in their list both groups coincide, and when they do not, the index of the symmetry group in the isometry one is rather small (≤ 4). Also, the symmetry groups which appear are of very special types (either abelian, or dihedral).

This behavior is certainly due to the limited number of crossings and components considered. Indeed, in this paper we prove:

Theorem 1. *Every finite group G is the group of orientation preserving isometries of the complement of some hyperbolic link in $\mathbf{S}^3$. Moreover, G acts freely.*

It was already known that every finite group can be realized as an isometry group of some closed hyperbolic 3-manifold: Kojima proved that every finite group can be realized as the full group of isometries of a closed hyperbolic manifold [K], with a not necessarily free action. Besides, Cooper and Long showed that every finite group acts freely on some hyperbolic rational homology sphere [CL]; in this case the full isometry group might a priori be larger.

Although the aforementioned results are related to the one presented in this paper, it deserves to be stressed that there are some peculiarities which come from the fact that we want to control several things at the same time: the structure of the manifold (i.e. the complement of a link in $\mathbf{S}^3$), the type of action (i.e. free), and the fact that no extra isometries are introduced.

Given a finite group G, it is not difficult to construct a link in $\mathbf{S}^3$ (and even a hyperbolic one, thanks to a result of Myers [My1]) whose complement admits an effective free G-action (see Proposition 2), but it is rather delicate to ensure that G coincides with the whole isometry group of the complement. Notice that the naive idea to drill out some "asymmetric" link in a G-equivariant way does not work, for the fact of removing simple closed curves has the effect of possibly increasing the group of isometries.

The idea is thus to choose the link in such a way that its complement contains some very rigid structure (in our case, totally geodesic pants) and use it to control the isometry group.

As a by-product of our result we are able to prove that Cooper and Long's result can be rigidified, that is, that the hyperbolic rational homology sphere on which the group G acts freely can be chosen so that G coincides with the full orientation preserving isometry group.

Corollary 1. *Any finite group G is the full orientation-preserving isometry group of a hyperbolic rational homology sphere. Moreover G acts freely.*

Before passing to the proof of the result, which will be the content of Sections 2 and 3, we state the following:

Proposition 1. *For any finite group H acting on $\mathbf{S}^3$, there exists a hyperbolic link whose group of symmetries is precisely H.*

Proof. We start by noticing that Myer's theorem works in an H-invariant setting (it suffices to choose an H-invariant triangulation) if one does not require the resulting hyperbolic link to be connected (see the proof of Lemma 1 for a similar reasoning). In this case one can indeed exploit the naive idea to add some "asymmetric" but H-invariant link to ensure that H coincides with the whole group of symmetries. More precisely, given any finite group G acting on $\mathbf{S}^3$, take $|H|$ copies of a knot K which admits no symmetry, in such a way that each copy is contained in a ball which misses all the fixed-point sets of elements of H, and such that the pairs (ball,K) are freely permuted by the action of H. It is now possible, using a result of Myers [My2], to find an H-invariant link Λ so that no component of Λ is the knot K and the complement of $L = \Lambda \cup G \cdot K$ is hyperbolic. It is now straightforward to see that the group of symmetries of the link L is precisely H. □

Note finally that in both constructions, Theorem 1 and Proposition 1, the number of components of the links constructed is pretty large. It would be interesting to understand whether, for a given group G, there is some bound on the number of components of a link on which G acts as group of symmetries/isometries and under which conditions the lower bound can be realized.

2 Construction of the link

The following result is easy and is the starting point to prove our main result.

Proposition 2. *Every finite group acts effectively and freely by orientation-preserving isometries on the complement of some hyperbolic link in the* 3*-sphere.*

Proof. Fix a finite group G. It is not difficult to find a closed 3-manifold M on which G acts effectively and freely; moreover, Cooper and Long [CL] showed that M can be chosen to be hyperbolic and a rational homology sphere. The Dehn surgery theorem of Lickorish [L] and Wallace [W] assures the existence of a link $\mathcal{L}$ contained in M whose exterior is contained in the 3-sphere, i.e. $M \setminus \mathcal{L} = \mathbf{S}^3 \setminus L$, where L is a link. We want to show that $\mathcal{L}$ can be chosen G-invariant, thanks to a general position argument. Take the quotient $(M, \mathcal{L})/G$. By perturbing slightly the image of $\mathcal{L}$ inside the quotient, we can assume that it has no self-intersection. Note that the complement of the G-invariant link of M obtained this way is again contained in $\mathbf{S}^3$, for we can assume that the perturbation performed did not affect the isotopy class of the components corresponding to the original link. Now, using a result of Myers [My1], we can find in $(M, \mathcal{L})/G$ a knot $\mathcal{K}$ whose exterior is hyperbolic. The preimage in M of $\mathcal{L}/G \cup \mathcal{K}$ is a link with the desired properties. □

Proof of Theorem 1. We start with G acting on $\mathbf{S}^3 \setminus L$ for some link $L \subset \mathbf{S}^3$, as in Proposition 2. A priori the group $\mathrm{Isom}^+(\mathbf{S}^3 \setminus L)$ can be larger than G, and we shall modify the link so that both groups are precisely the same. Denote by N the quotient $(\mathbf{S}^3 \setminus L)/G$. Choose a genus-2 unknotted handlebody contained in a ball of N. Using [My1], we can find a hyperbolic knot $\mathcal{K}$ in $N \setminus \mathring{H}$, whose exterior is, moreover, anannular. Fix three meridional curves C_i, $i = 1, 2, 3$, on the boundary of H as shown in Figure 1. Observe that each C_i

is non-separating, but the three of them cut ∂H in two pairs of pants. The manifold $(N \setminus \mathring{H}) \setminus (\mathcal{K} \cup \bigcup_{i=1}^{3} C_i)$ has a non-compact boundary, consisting of two cusped pairs of pants, and admits a hyperbolic structure with totally geodesic boundary: this last fact follows from Thurston's hyperbolization theorem, since the manifold is atoroidal and its compact core is anannular. Notice that the curves C_i correspond to rank one cusps.

We consider the handlebody H.

Lemma 1. *There exists a link Λ inside H such that $H \setminus (\Lambda \cup \bigcup_{i=1}^{3} C_i)$ satisfies the following properties:*

1. *It is hyperbolic with totally geodesic boundary.*
2. *It contains a unique geodesic η of minimal length.*
3. *Its orientation preserving isometry group is trivial.*

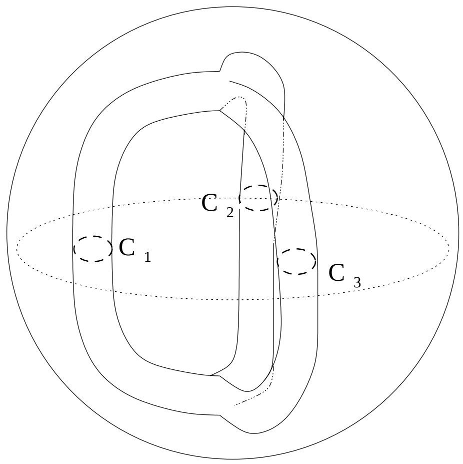

Figure 1: The handlebody H contained in a ball and the three curves C_i.

Proof. Consider the group $\mathbf{Z}/2 \times \mathbf{D}_3$, where $\mathbf{D}_3$ denotes the dihedral group of order 6, which coincides with the symmetric group on three elements. This group acts on the pair $(H, \bigcup_{i=1}^{3} C_i)$ as illustrated in Figure 2a: the central element of order two is the hyperelliptic involution fixing all three curves C_i while exchanging the two pairs of pants (its "axis" is a circle in the figure), the elements of $\mathbf{D}_3$ leave invariant each pair of pants and permute the curves C_i. Consider now the quotient orbifold (see Figure 2b)

$$\mathcal{O} = (H \setminus \bigcup_{i=1}^{3} C_i) \Big/ \mathbf{Z}/2 \times \mathbf{D}_3.$$

Remove from $\mathcal{O}$ a simple closed curve γ contained in a ball which does not meet the singular locus. Choose a triangulation of the compact core of $\mathcal{O} \setminus \gamma$ which contains the singular locus: such triangulation lifts to a $\mathbf{Z}/2 \times \mathbf{D}_3$-equivariant triangulation of the compact core of $H \setminus (\tilde{\gamma} \cup \bigcup_{i=1}^{3} C_i)$, where $\tilde{\gamma}$ denotes the

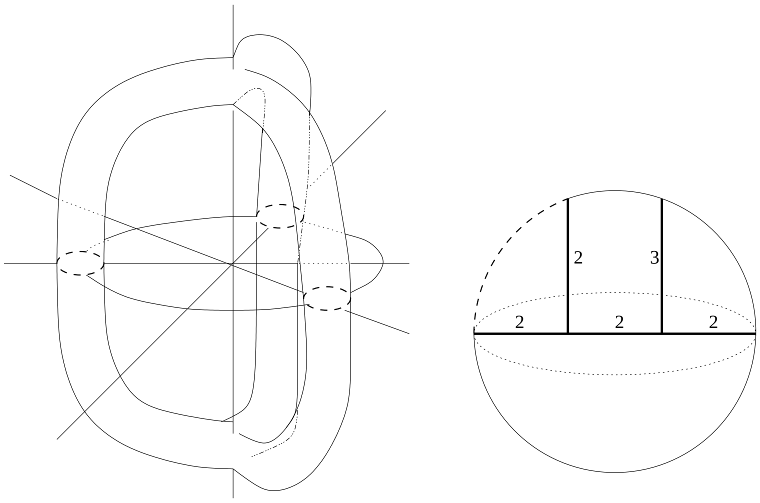

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Figure 2: The group $\mathbf{Z}/2 \times \mathbf{D}_3$ acting on H and the quotient orbifold $\mathcal{O}$.

lift of γ. Taking the second baricentric subdivision of such triangulation, one can get a "special handle decomposition" and the very same proof as in [My1, Theorem 6.1] shows that one can find a hyperbolic link $\Lambda_0 \subset H \setminus (\tilde{\gamma} \cup \bigcup_{i=1}^3 C_i)$ which is $\mathbf{Z}/2 \times \mathbf{D}_3$-invariant and whose exterior

$$W = H \setminus (\tilde{\gamma} \cup \Lambda_0 \cup \bigcup_{i=1}^{3} C_i)$$

has anannular compact core. Consider now $\tilde{\gamma}$: it consists of twelve connected components, for each of which we choose a meridian-longitude system (μ, λ). One can perform hyperbolic Dehn surgery with meridian curves $\mu + n\,\lambda$, $n \gg 1$ so that the lengths of the surgered geodesics are pairwise distinct and of shortest length inside the resulting manifold $H \setminus (\Lambda \cup \bigcup_{i=1}^3 C_i)$, Λ being the image of Λ_0 after surgery.

Consider now an orientation-preserving isometry of $H \setminus (\Lambda \cup \bigcup_{i=1}^3 C_i)$. Since they have minimal length, the twelve geodesics obtained by hyperbolic surgery must be left invariant by the isometry, which thus induces an isometry of the exterior of the geodesics. This means that the isometry must act as one of the elements of $\mathbf{Z}/2 \times \mathbf{D}_3$, since it is determined by its action on the boundary and since $\mathbf{Z}/2 \times \mathbf{D}_3$ is the complete group of positive isometries of the boundary. But only the identity element of $\mathbf{Z}/2 \times \mathbf{D}_3$ extends to $H \setminus (\Lambda \cup \bigcup_{i=1}^3 C_i)$, for all twelve geodesics must be left setwise fixed, because their lengths are pairwise distinct. $\square$

The boundary components of $H \setminus (\Lambda \cup \bigcup_{i=1}^3 C_i)$ are two totally geodesic pairs of pants with cusps (topologically, they are $\partial H \setminus \bigcup_{i=1}^3 C_i$). By abuse of notation, C_i denotes the curve and also the corresponding cusp.

Lemma 2. *The only embedded geodesic pair of pants in $H \setminus (\Lambda \cup \bigcup_{i=1}^3 C_i)$ having cusps C_1, C_2 and C_3 are its boundary components.*

Proof. Let P be an embedded pair of pants with cusps C_1, C_2 and C_3. By the geometry of Margulis tubes, P must be disjoint from the cores of the filling tori, thus $P \subset W$ (recall that W was defined in the proof of Lemma 1, and that it is the result of removing the twelve shortest geodesics). The pants P can be rendered totally geodesic also in W, because W minus an open tubular neighborhood of P

$$W \setminus \mathcal{N}(P)$$

has still an irreducible, atoroidal, and anannular compact core. (This follows from the fact that every simple closed curve in P is either compressible or boundary parallel.) By Thurston's hyperbolization, $W \setminus \mathcal{N}(P)$ is hyperbolic with totally geodesic boundary, consisting of two copies of P that we can glue back by an isometry.

Next we claim that P is equivariant by the action of $\mathbf{Z}/2 \times \mathbf{D}_3$. To prove it, we look carefully at the geometry of the cusps C_i. A horospherical section of the cups C_i is a Euclidean annulus A_i, and P intersects the annulus A_i in a circle metrically parallel to the boundary components ∂A_i. Thus the intersection of P with those annuli must be equivariant. Since the action of $\mathbf{Z}/2 \times \mathbf{D}_3$ on P is determined by its restriction to the cusps, it follows that P is equivariant.

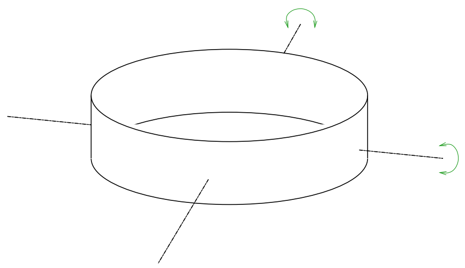

Figure 3: The action of $\mathbf{Z}/2 \oplus \mathbf{Z}/2$ on the annulus A_i.

Now we consider the possible quotients of P by its stabilizer, and look at how they project on $W/(\mathbf{Z}/2 \times \mathbf{D}_3)$. We remark that, on the circle of the intersection of P with each horospherical annulus A_i, the projection to the quotient restricts to a map of the circle $P \cap A_i$ to its quotient which is either 2 to 1, or 4 to 1 (Figure 3). In the former case, the quotient is a circle, in the latter, an interval with mirror boundary. This implies that the stabilizer of P must contain at least three orientation preserving involutions, one for each cusp, and corresponding to the product of the two involutions whose axes are pictured in Figure 3 (i.e. a rotation on the S^1 factor). As the orientation preserving isometry group of the pants is $\mathbf{D}_3$, it follows that the only possible quotients for P are the ones described in Figure 4.

In fact case b in Figure 4 is not possible, as the mirror points have to agree with singular points of order 2 in the orbifold $\mathcal{O}$ in Figure 2b.

We claim that the quotient of P is parallel to the boundary in $\mathcal{O}$. To see that, notice that $W/(\mathbf{Z}/2 \times \mathbf{D}_3)$ is the exterior in $\mathcal{O}$ of an anannular hyperbolic link with two components, γ and the quotient of Λ_0. Those components cannot be separated by the quotient of P, because otherwise the disc in $\mathcal{O}$ bounded by γ would give either a compressing disc or an essential annulus in $W \setminus \mathcal{N}(P)$, contradicting that P is totally geodesic. Hence γ and the quotient of Λ_0 are

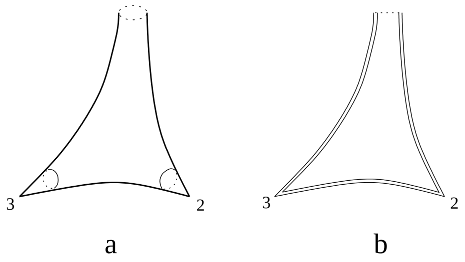

Figure 4: The possible quotients of P. Case b is non orientable, has mirror points (double lines), and corners.

not separated by the quotient of P, and therefore this quotient either bounds a handlebody or is parallel to the boundary. It cannot bound a handlebody, since it is totally geodesic. Hence P must be a boundary component, because two parallel totally geodesic submanifolds must be the same. $\square$

Consider now the manifold $N \setminus (\mathcal{K} \cup \Lambda \cup \bigcup_{i=1}^{3} C_i)$. This is a hyperbolic manifold obtained by gluing together two hyperbolic manifolds along their totally geodesic boundaries, which consist of two pairs of pants:

$$N \setminus (\mathcal{K} \cup \Lambda \cup \bigcup_{i=1}^{3} C_i) = \Big(H \setminus (\Lambda \cup \bigcup_{i=1}^{3} C_i)\Big) \cup \Big(N \setminus (\mathring{H} \cup \mathcal{K} \cup \bigcup_{i=1}^{3} C_i)\Big).$$

Since the hyperbolic structure of a pair of pants (with three cusps) is unique, both pairs of pants are compatible when gluing and they remain totally geodesic inside the resulting manifold. Take now the lift of $N \setminus (\mathcal{K} \cup \Lambda \cup \bigcup_{i=1}^{3} C_i)$ to the 3-sphere: we obtain the complement of a hyperbolic link L in $\mathbf{S}^3$ on which G acts freely, in particular G is a subgroup in $\mathrm{Isom}^+(\mathbf{S}^3 \setminus L)$.

Notice that we glue along totally geodesic pants, therefore we do not make any deformation to the structure of each piece, and we get:

Remark 1. *The shortest geodesic η in $H \setminus (\Lambda \cup \bigcup_{i=1}^{3} C_i)$ from Lemma 1 may also be chosen to be the shortest in $N \setminus (\mathcal{K} \cup \Lambda \cup \bigcup_{i=1}^{3} C_i)$, and its lift in $\mathbf{S}^3 \setminus L$ minimizes also the length spectrum.*

3 Showing that there are no more isometries

It remains to show that $G = \mathrm{Isom}^+(\mathbf{S}^3 \setminus L)$. Arguing by contradiction, let φ be an isometry of the complement of L which is not in G. Consider now $\tilde{\eta}$ the lift in $\mathbf{S}^3 \setminus L$ of the shortest geodesic η: it has as many components as the order of G, since G acts by freely permuting them. Choose one component of $\tilde{\eta}$, say η_0. Up to composing φ with an element of G we can assume that $\varphi(\eta_0) = \eta_0$. Note that η_0 is separated from the other components of $\tilde{\eta}$ by a unique lift $P_1 \cup P_2$ of both pairs of pants bounding $H \setminus (\Lambda \cup \bigcup_{i=1}^{3} C_i)$.

Let X denote the lift of $H \setminus (\Lambda \cup \bigcup_{i=1}^{3} C_i)$ bounded by $P_1 \cup P_2$.

Lemma 3. *We have* $\varphi(P_1 \cup P_2) \cap (P_1 \cup P_2) \neq \emptyset$.

Proof. Seeking a contradiction, assume that the intersection is empty. Since $P_1 \cup P_2$ separates and $\varphi(\eta_0) = \eta_0$, by volume reasons $P_1 \cup P_2$ must separate $\varphi(P_1)$ from $\varphi(P_2)$. As every end of $\varphi(P_1)$ meets one end of $\varphi(P_2)$ along a cusp, φ stabilizes the set of three cusps $C_1 \cup C_2 \cup C_3$, joining P_1 and P_2. Thus $\varphi(P_1)$ (or $\varphi(P_2)$) is a totally geodesic embedded pant in X, with the cusps $C_1 \cup C_2 \cup C_3$. By Lemma 2, $\varphi(P_i)$ must be either P_1 or P_2, hence a contradiction. $\square$

Lemma 4. *We have* $\varphi(P_1 \cup P_2) = P_1 \cup P_2$.

Proof. Since both $\varphi(P_1 \cup P_2)$ and $P_1 \cup P_2$ are totally geodesic, if they do not coincide, their intersection must be a union of disjoint simple geodesics, which are properly embedded. Figure 5 shows the possible form of non self-intersecting proper geodesics, depending on whether the ends are on the same or on different cusps. Observe that, since two distinct cusps cannot be sent by φ to the same one, the geodesics in the intersection look the same in $P_1 \cup P_2$ and in $\varphi(P_1 \cup P_2)$.

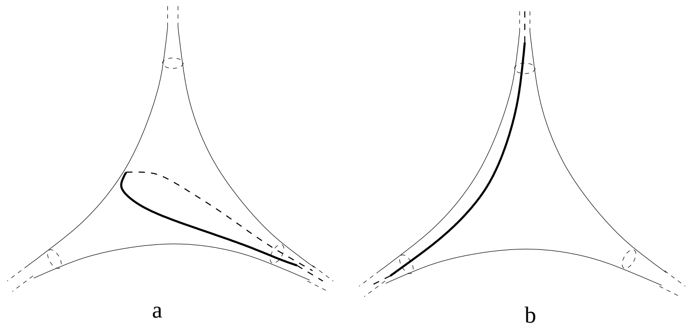

Figure 5: The two types of geodesic intersections

Assume that the intersection $P_i \cup \varphi(P_j)$ contains a geodesic as in Figure 5a, for some $i, j = 1, 2$. Notice that the intersection $\varphi(P_j) \cap (P_1 \cup P_2)$ does not contain more geodesics, because $P_1 \cup P_2$ separates. Consider now the half pant of $\varphi(P_j)$ contained inside X: it contains a cusp which ends either on a cusp inside X or on a cusp corresponding to a C_i different from that on which the geodesic of the intersection ends. In both cases we can find an annulus, properly embedded in X and joining two cusps. The annulus is illustrated in Figure 6a: the shaded region is a disc contained in a section of the rank 1 cusp. Since X is hyperbolic, the two cusps which support the annulus, must in fact be the same and the annulus must be boundary parallel. Since totally geodesic parallel surfaces must be the same, it follows that $P_i = \varphi(P_j)$.

We can thus assume that the intersection $\varphi(P_j) \cap (P_1 \cup P_2)$ contains a geodesic of the type pictured in Figure 5b. Since $P_1 \cup P_2$ separates, $\varphi(P_j) \cap (P_1 \cup P_2)$ must contain two other geodesics of the same type that divide $\varphi(P_j)$ into two triangles, as illustrated in Figure 6b. If the three geodesics are not contained in a single pant P_i (but in the union $P_1 \cup P_2$), they give an essential curve in the genus two surface $\partial H = P_1 \cup P_2 \cup \bigcup_{i=1}^{3} C_i$, which is the boundary of a compressing disc (half of $\varphi(P_j)$) (cf. Figure 7a). Hence we may assume that the geodesics are contained in one of the pairs of pants P_i and therefore they

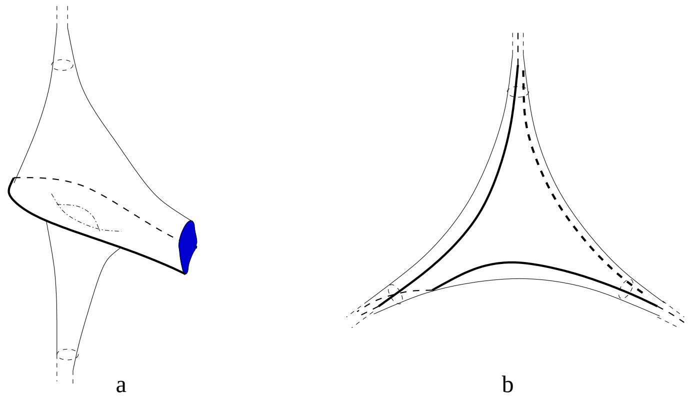

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Figure 6: An annulus and two ideal triangles

cut it into two ideal triangles, as in Figure 7b. The union of half of $\varphi(P_j)$ and half of P_i, along the three geodesics and three segments in $C_1 \cup C_2 \cup C_3$, gives an embedded 2-sphere. Irreducibility gives a parallelism between triangles in P_i and $\varphi(P_j)$, hence $P_i = \varphi(P_j)$.

Again for volume reasons, we see that the remaining pairs of pants must intersect. Repeating the argument once more we reach the desired conclusion. □

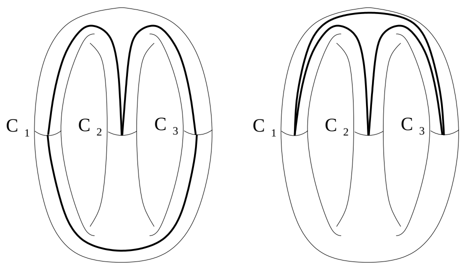

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Figure 7: The intersection $\varphi(P_j) \cap (P_1 \cup P_2)$, viewed in $\partial H = P_1 \cup P_2 \cup \bigcup_{i=1}^{3} C_i$.

Lemma 4 implies that φ induces an isometry of $X = H \setminus (\Lambda \cup \bigcup_{i=1}^{3} C_i)$. By Lemma 1 the restriction of φ to X is the identity. Since X has nonempty interior in $\mathbf{S}^3 \setminus L$ and φ is an isometry, we deduce that φ itself is the identity and the desired contradiction follows. This finishes the proof of Theorem 1. □

4 Rigidifying actions on homology spheres

With the previous construction in mind we can now prove the corollary.

Proof of Corollary 1. Consider a rational homology sphere M on which G acts freely [CL]. Repeating the previous construction, we can find a hyperbolic link

$\mathcal{L}$ in M which is G invariant and such that G is precisely the full orientation-preserving isometry group of the exterior of $\mathcal{L}$. The idea now is to do surgery in a G-equivariant way, so that we still have a rational homology sphere, and that the cores of the surgery tori are the shortest curves, so that the G-orbits of those curves are invariant by any isometry.

To do this surgery, we must be able to choose a G-equivariant meridian-longitude system (μ, λ) on each peripheral torus (i.e. so that the image of λ in $H^1(M \setminus \mathcal{L}; \mathbf{Q})$ is trivial). We specify how to adapt the construction of Theorem 1. First we remove a handlebody H from the quotient M/G and consider a hyperbolic anannular knot $\mathcal{K}$ in $M/G \setminus H$ in a trivial homotopy class, see [My2], so that it bounds a singular disc. This singular disc lifts to a family of G-equivariant singular discs, hence defining longitudes for the lifts of $\mathcal{K}$ in M.

The same construction with singular discs must be applied to the knots we remove from the interior of the handlebody H, but we need to justify that it is compatible with equivariance and the fact that we remove several curves:

- Recall from the proof of Lemma 1 that we remove a trivial knot γ from the quotient $\mathcal{O} = (H \setminus \bigcup_{i=1}^3 C_i)/\mathbf{Z}/2 \times \mathbf{D}_3$, we lift it $\tilde{\gamma} \subset H \setminus \bigcup_{i=1}^3 C_i$ and then we remove an equivariant hyperbolic anannular link Λ_0 from $H \setminus (\bigcup_{i=1}^3 C_i \cup \tilde{\gamma})$. We claim that Λ_0 can be choosen to project to a homotopically trivial knot in $\mathcal{O} \setminus \gamma$. To do that, we choose in H a fundamental domain for the action of $\mathbf{Z}/2 \times \mathbf{D}_3$, and choose a homotopically trivial knot in $\mathcal{O} \setminus \gamma$ that crosses transversally each side of the fundamental domain at least twice (so that the punctured sides of the fundamental domain have negative Euler characteristic). This gives a family of arcs in the fundamental domain, that can be homotoped relative to the boundary to a submanifold with hyperbolic and ananular exterior, by the main theorem in [My2]. By the gluing lemma of Myers, [My2, Lemma 2.1], the pieces of the fundamental domain match to give the link Λ_0 with the required properties.

- We justify the compatibility of singular discs when we remove more than one curve, that can intersect such a singular disc, or the meridian discs bounding the curves C_i (that we removed from ∂H). In this case we tube the discs along the curves we remove, in order to get a disjoint surface. This tubing is possible because, at each step, all curves we remove are homotopically trivial.

Finally, notice that the perturbation argument of Proposition 2, which is not used here, would be a problem for the existence of longitudes.

We now perform hyperbolic Dehn surgery with filling meridians $n\lambda + \mu$ on the components of $\mathcal{L}$, $n \gg 1$, in such a way that the following requirements are satisfied:

- the surgery is G-equivariant;
- the geodesics obtained after surgery have pairwise distinct lengths if they belong to different G-orbits;
- the lengths of these geodesics are the shortest ones.

Notice that the slopes are chosen so that the resulting manifold is a rational homology sphere. Since we choose homotopically trivial curves in the quotient, the G action permutes the componets of the link, hence G acts freely on the surgered manifold. The geodesics are then chosen to be of shortest lengths to ensure that all isometries must preserve the image of $\mathcal{L}$ after surgery, so that they must induce isometries of the complement of $\mathcal{L}$. The conclusion now follows at once. $\square$

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