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Objective combinatorics through decomposition spaces

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Abstract

This thesis provides general constructions in the context of decomposition spaces, generalising classical results from combinatorics to the homotopical setting. This requires developing general tools in the theory of decomposition spaces and new viewpoints, which are of general interest, independently of the applications to combinatorics.

In the first chapter, we summarise the homotopy theory and combinatorics of the 2-category of groupoids. We continue with a review of needed notions from the theory of ∞-categories. We then summarise the theory of decomposition spaces.

In the second chapter, we identify the structures that have incidence bi(co)modules: they are certain augmented double Segal spaces subject to some exactness conditions. We establish a Möbius inversion principle for (co)modules, and a Rota formula for certain more involved structures called Möbius bicomodule configurations. The most important instance of the latter notion arises as mapping cylinders of infinity adjunctions, or more generally of adjunctions between Möbius decomposition spaces, in the spirit of Rota's original formula.

In the third chapter, we present some tools for providing situations where the generalised Rota formula applies. As an example of this, we compute the Möbius function of the decomposition space of finite posets, and exploit this to derive also a formula for the incidence algebra of any directed restriction species, free operad, or more generally free monad on a finitary polynomial monad.

In the fourth chapter, we show that Schmitt's hereditary species induce monoidal decomposition spaces, and exhibit Schmitt's bialgebra construction as an instance of the general bialgebra construction on a monoidal decomposition space. We show furthermore that this bialgebra structure coacts on the underlying restriction-species bialgebra structure so as to form a comodule bialgebra. Finally, we show that hereditary species induce a new family of examples of operadic categories in the sense of Batanin and Markl.

In the fifth chapter, representing joint work with Joachim Kock, we introduce a notion of antipode for monoidal (complete) decomposition spaces, inducing a notion of weak antipode for their incidence bialgebras. In the connected case, this recovers the usual notion of antipode in Hopf algebras. In the non-connected case it expresses an inversion principle of more limited scope, but still sufficient to compute the Möbius function as $\mu = \zeta \circ S$, just as in Hopf algebras. At the level of decomposition spaces, the weak antipode takes the form of a formal difference of linear endofunctors $S_{\text{even}} - S_{\text{odd}}$, and it is a refinement of the general Möbius inversion construction of Gálvez–Kock–Tonks, but exploiting the monoidal structure.

Resum

Aquesta tesi proveeix construccions generals en el context d'espais de descomposici, generalitzant els resultats clàssics de la combinatòria al context homotòpic. Això requereix desenvolupar eines generals en la teoria d'espais de descomposició i noves perspectives, que siguin d'interès general, independentment de les aplicacions a la combinatòria.

Al primer capítol, resumim la teoria de l'homotopia i la combinatòria de la 2-categoria de grupoides. Continuem amb una revisió de les nocions necessàries de la teoria de categories d'ordre infinit. A continuació, resumim la teoria dels espais de descomposició.

Al segon capítol, identifiquem les estructures que tenen bi(co)mòduls d'incidència: són certs espais de Segal dobles augmentats subjectes a unes condicions d'exactitud. Establim un principi d'inversió de Möbius per a (co)mòduls i una fórmula de Rota per a certes estructures més implicades anomenades configuracions de bicomòduls de Möbius. La instància més important d'aquesta última noció sorgeix com cilindres d'aplicació d'adjuncions d'ordre infinit, o més generalment d'adjuncions entre espais de descomposició de Möbius, amb l'esperit de la fórmula original de Rota.

Al tercer capítol, presentem eines per proveir situacions en què s'aplica la fórmula generalitzada de Rota. Com a exemple, calculem la funció de Möbius de l'espai de descomposició dels conjunts parcialment ordenats finits i l'explotem per obtenir també una fórmula per a l'àlgebra d'incidència de qualsevol espècie de restricció dirigida, operad lliure, o més generalment monada lliure sobre una monada polinòmica finitària.

Al quart capítol, mostrem que les espècies hereditàries de Schmitt indueixen espais de descomposició monoidals i exhibim la construcció de biàlgebra de Schmitt com a instància de la construcció general de biàlgebra en un espai de descomposició monoidal. A més, mostrem que aquesta estructura de biàlgebra coactua sobre l'estructura de biàlgebra de les espècies restringides subjacent, per formar una biàlgebra en comòduls. Finalment, mostrem que les espècies hereditàries indueixen a una nova família d'exemples de categories operàdiques en el sentit de Batanin i Markl.

Al cinquè capítol, que representa un treball conjunt amb Joachim Kock, introduïm una noció d'antípoda per a espais de descomposició (complets) monoidals, que indueixen una noció d'antípoda feble per a les seves bialgebres d'incidència. En el cas connectat, recuperem la noció habitual d'antípoda per a les àlgebres de Hopf. En el cas no connectat expressa un principi d'inversió d'abast més limitat, però sempre suficient per calcular la funció de Möbius com $\mu = \zeta \circ S$, tal com per a les àlgebres de Hopf. Al nivell de les espais de descomposició, l'antípoda feble pren la forma d'una diferència formal d'endofunctors lineals $S_{\text{even}} - S_{\text{odd}}$, i és un refinament de la construcció general d'inversió de Möbius de Gálvez–Kock–Tonks, però explotant l'estructura monoidal.

Résumé

Cette thèse fournit des constructions générales dans le cadre des espaces de décomposition, généralisant des résultats classiques de combinatoire à un contexte homotopique. Cela nécessite de développer des outils généraux dans la théorie des espaces de décomposition et des nouvelles perspectives, qui présentent un intérêt aussi en dehors des applications à la combinatoire.

Dans le premier chapitre, nous synthétisons la théorie de l'homotopie et la combinatoire de la 2-catégorie des groupoïdes. Nous poursuivons avec une revue des notions nécessaires de la théorie des catégories d'ordre supérieur. Nous résumons ensuite la théorie des espaces de décomposition.

Dans le deuxième chapitre, nous identifions les structures ayant des bi(co)modules d'incidences : ce sont des espaces de Segal doubles augmentés satisfaisant des conditions d'exactitude. Nous établissons un principe d'inversion de Möbius pour les (co)modules et une formule à la Rota pour certaines structures que nous appelons configurations de bicomodule de Möbius. L'exemple le plus important de cette dernière notion provient de cylindres d'application d'adjonctions d'ordre infini, ou plus généralement d'adjonctions entre des espaces de décomposition de Möbius, dans l'esprit de la formule originale de Rota.

Dans le troisième chapitre, nous présentons des outils pour fournir des situations où la formule de Rota généralisée s'applique. En guise d'exemple, nous calculons la fonction de Möbius de l'espace de décomposition des ensemble partiellement ordonnés finis et l'utilisons pour obtenir une formule pour l'algèbre d'incidence de n'importe quelle espèce de structures de restriction dirigée, d'une opérade libre, ou plus généralement d'une monade libre sur une monade polynomiale finie.

Dans le quatrième chapitre, nous montrons que les espèces de structures héréditaires de Schmitt induisent des espaces de décomposition de Möbius, et nous exhibons la construction de bigèbre de Schmitt comme une occurrence de la construction générale de bigèbre associée à un espace de décomposition monoïdal. Nous montrons de plus que cette structure de bigèbre coagit sur la structure de bigèbre obtenue considérant l'espèce de structures restreintes sous-jacente, pour constituer une comodule bigèbre. Enfin, nous montrons que les espèces de structures héréditaires induisent une nouvelle famille d'exemples de catégories opéradiques, au sens de Batanin et Markl.

Dans le cinquième chapitre, représentant un travail en collaboration avec Joachim Kock, nous introduisons une notion d'antipode pour les espaces de décomposition (complets) monoïdaux, induisant une notion d'antipode faible pour leurs bigèbres d'incidence. Dans le cas connexe, nous retrouvons la notion usuelle d'antipode pour les algèbres de Hopf. Dans le cas non-connexe, cela explicite un principe d'inversion à portée plus restreinte, mais toujours suffisant pour calculer la fonction de Möbius par $\mu = \zeta \circ S$, comme pour les algèbres de Hopf. Au niveau des espaces de décomposition, l'antipode au sens faible prend la forme d'une différence formelle d'endofoncteurs linéaires $S_{\text{even}} - S_{\text{odd}}$ et est un raffinement de la construction générale d'inversion de Möbius due à Gálvez–Kock–Tonks, mais exploitant la structure monoïdale.

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It is most appropriate to begin by thanking my supervisor, Joachim Kock. This thesis would never have been possible without his help, and I am very grateful for the patient encouragement, invaluable assistance, enlightening discussions and insightful criticisms throughout all of these years.

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Contents

Introduction

Background and motivation

The art of counting is a fundamental aspect of mathematics, and is formalised into the branch of *enumerative combinatorics*. People have always been concerned with counting stuff and the systematic mathematical study of combinatorial problems goes back at least to the 18th century, with the work of Euler. Combinatorial problems arrive from many area of mathematics (algebra, geometry, probability, topology) and also from outside of the mathematics world. At first, some ad hoc solutions were developed to each problem, but from the 20th century, general methods appeared.

In enumerative combinatorics, we are interested in the enumeration of structures, or of configurations of objects. For example, how many ways one can choose 2-element sets from a set with 4 element? The answer is given by the binomial coefficient $\binom{n}{k}$. Looking at all such problems together can reveal algebraic structures, for example linear recursion, such as Pascal's formula

$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}.$$

More abstractly, one can realise that these numbers appear algebraically, as coefficients of polynomials and power series, and exploit this algebraic structure to obtain relationships between the numbers. This is one starting point for *algebraic combinatorics*. Conversely, given a counting problem which depends on a natural number, for example how many structures of a given type can be put on a set with n elements, one can define a power series, the *generating function* for the numbers, as first exploited by de Moivre in the 18th century, and then apply all the algebraic machinery available such as sums and products, but also derivatives and substitution. Remarkable combinatorial identities can be easily established in this way.

On the other hand, it has long been appreciated that bijective proofs contain more information, or represents deeper understanding, than just an algebraic manipulation. It is always interesting to find a new bijection or description of a known enumeration problem. (The Exercise 6.19 of the book [56] describes 66 different combinatorial interpretations of Catalan numbers. An addendum can be found on Stanley's website with 207 interpretations! [57])

Species

Joyal showed that many manipulations with power series and generating functions can be carried out directly on the combinatorial structures themselves, through the notion of *species* [34] (see also [9]). A species is a functor from the category of finite sets and bijections to the category of sets and arbitrary maps. To each finite set S, a species associates a set $F[S]$. The elements of $F[S]$ are F-structures on the set S. To each bijection $S \to T$ is associated a bijection $F[S] \to F[T]$. It is a functoriality condition allowing to transport the structure along bijections between finite sets. Examples of structures are graphs, trees, permutations, linear orders, endofunctions, etc. It is a way to clarify and unify the subject. For example, let us briefly describe an elegant bijective proof of Cayley's formula due to Joyal, which is nicely formalised using species.

Theorem (Cayley's formula)**.** *The number of trees on* n *vertices is* n^{n-2}.

A *tree* is a connected graph without cycles. We denote by T the species trees. A *vertebrate* is a pointed rooted tree (with two specified vertices, the head and the tail, possibly the same), and we denote by V the species of vertebrates. It is clear we have $|V_n| = n^2|T_n|$ since there are n possible choices for the head, and n possible choices for the tail. The goal is to prove $|V_n| = n^n$, and this is achieved by producing a bijection between the set of vertebrates on n vertices and the set of endofunctions of $\{1, ..., n\}$ (which has cardinality n^n). A vertebrate is a non-empty list of rooted trees, as illustrated in the following Figure 1. (There is an isomorphism of species: $V \simeq L^+ \circ R$, where L^+ is the species of non-empty linear orders, and R is the species of rooted trees.)

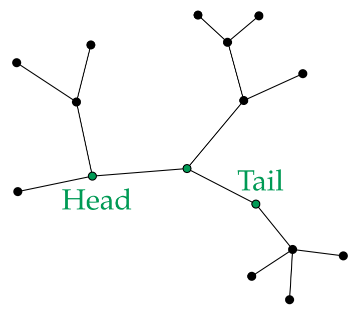


Figure 1: A vertebrate is a non-empty list of rooted trees.

After observing that there are as many linear orders as permutations on n elements, the Figure 2 below shows that an endofunction is given by a permutation of rooted trees, thus providing the bijection. (There is an isomorphism of species: $\mathrm{End} \simeq S \circ R$, where End is the species of endofuntions, S is the species of permutations, and R is the species of rooted trees.)

One key point of this pleasing proof is that the bijection between lists and permutations carries further structure, namely that of trees.

The theory of species is a starting point for the idea of *objective combinatorics*, advocated by Lawvere, where one tries to work directly with the combinatorial objects, instead of with algebraic structures generated by them.

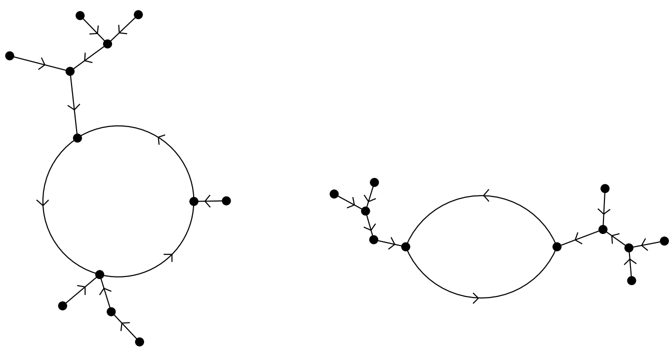


Figure 2: An endofunction is a permutation of trees.

Incidence coalgebras

Another algebraic technique such as convolutions essentially expresses the combinatorial viewpoint of breaking structures into smaller ones. It was realised by Rota [51] that the algebraic content of such techniques is that of *coalgebras*, now a second main ingredient in algebraic combinatorics. He showed that many such decomposition techniques admit interpretation in terms of *incidence coalgebras of posets*, and in particular that convolution products generally arise from coalgebras. Techniques such as *Möbius inversion* and exclusion-inclusion principles, and other overcounting-undercounting are examples of this more general notion. Underlying these coalgebras are vector spaces freely generated by (iso-classes of) combinatorial structures, and the comultiplication arises from the ability to decompose the structures. In particular, on the free vector space on the intervals of a locally finite poset, the comultiplication is given by [51]:

$$\Delta[x,y] = \sum_{x \leqslant m \leqslant y} [x,m] \otimes [m,y].$$

The theory of Möbius categories, developed by Leroux [45], generalises the theory for locally finite posets [51] and Cartier–Foata finite-decomposition monoids [18], as follows. Given a small category X, write X_0 for its set of objects and X_1 for its set of arrows. Let $\mathbb{Q}_{X_1}$ be the free vector space on X_1. We say a category X is *locally finite* if each morphism $f : x \to z$ in X admits only finitely many two-step factorisations $x \xrightarrow{g} y \xrightarrow{h} z$. This condition guarantees that the comultiplication on $\mathbb{Q}_{X_1}$, given by

$$\begin{aligned} \Delta : \mathbb{Q}_{X_1} &\to \mathbb{Q}_{X_1} \otimes \mathbb{Q}_{X_1} \\ f &\mapsto \sum_{hg=f} g \otimes h \end{aligned}$$

is well defined. The counit $\delta : \mathbb{Q}_{X_1} \to \mathbb{Q}$ is given by $\delta(\mathrm{id}_x) = 1$, and $\delta(f) = 0$ else.

The *incidence algebra* $\mathcal{I}_X$ is the linear dual, $(\mathrm{Lin}(\mathbb{Q}_{X_1}, \mathbb{Q}), *, \delta)$ with the convolution product:

$$(\alpha * \beta)(f) = \sum_{hg=f} \alpha(g)\,\beta(h),$$

where $\alpha, \beta \in \mathcal{I}_X$ and $f \in \mathbb{Q}_{X_1}$.

The *zeta function* $\zeta_X : \mathbb{Q}_{X_1} \to \mathbb{Q}$ is defined by $\zeta_X(f) = 1$ for all $f \in X_1$.

Define $\Phi_{\text{even}} : \mathbb{Q}_{X_1} \to \mathbb{Q}$ to be the number of even-length factorisations of a morphism, without identities, and $\Phi_{\text{odd}} : \mathbb{Q}_{X_1} \to \mathbb{Q}$ to be the number of odd-length factorisations, without identities. A category is *Möbius* [45] if it is locally finite and Φ_{even} and Φ_{odd} are finite.

Theorem (Content, Lemay, Leroux [19])**.** *If X is a Möbius category then the zeta function is invertible, and the inverse, called the Möbius function, is given by $\mu = \Phi_{\text{even}} - \Phi_{\text{odd}}$.*

Examples of Möbius categories are locally finite posets and monoids with the finite-decompositions property, and this theorem generalises similar theorems for these more specialised settings.

Algebraic combinatorics soon after Rota discovered the more powerful machinery of antipodes, when available. For example, in an incidence Hopf algebra, Möbius inversion amounts to precomposing with the antipode S, exhibiting in particular the Möbius function as $\mu = \zeta \circ S$. The work of Schmitt [52, 54] was seminal to the change of emphasis from Möbius inversion to antipodes and featured also the use of species to obtain a more objective approach. The recent work of Aguiar and Ardila [2] represents a striking example of the power of antipodes. This approach to combinatorial Hopf algebra is now a very active research area, see in particular the text book *Monoidal functors, species and Hopf algebras* by Aguiar and Mahajan [5].

Objective combinatorics

In the 21th century, an objective approach to Leroux's theory was taken up by Lawvere and Menni [44], who established an objective version of the Möbius inversion formula. The broad idea of the objective method is to work with objects instead of numbers. One can obtain, in this way, bijective proofs instead of algebraic proofs, leading to a deeper understanding. With this objective method, one can use linear algebra with coefficient in the category of sets, instead of rational numbers. A vector in the free vector space spanned by a set S is replaced by a family of sets indexed by S, that is an object of the slice category $\boldsymbol{Set}_{/S}$. Linear maps are replaced by spans, equalities are expressed by bijections. In this setting, there is a notion of cardinality and an algebraic identity is realised by the cardinality of a bijection of sets.

Decomposition spaces

Not all coalgebras in combinatorics arise from posets, monoids or categories. In a broader perspective important examples come from the Waldhausen S-construction in topology and Hall algebras of various flavours. In combinatorics, an important class of coalgebras are the coalgebras of restriction species in the sense of Schmitt [53] (see also [5, §8.7]), and the more general notion of directed restriction species of [28]. Examples of these notions are given by the chromatic Hopf algebra of graphs and the Butcher–Connes–Kreimer Hopf algebra of rooted trees, as well as many related structures such as matroids, posets (talking here about a Hopf algebra of all posets, not just a coalgebra of an individual poset).

The theory of Leroux has recently been generalised to ∞-categories and *decomposition spaces* by Gálvez, Kock, and Tonks [25, 26, 27]. It is a very general homotopical framework for incidence algebras and Möbius inversion. Decomposition spaces (the same thing as the unital 2-Segal spaces of Dyckerhoff–Kapranov [20]) are simplicial groupoids that satisfy an exactness condition weaker than the Segal condition: while the Segal condition essentially characterises categories (the ability to compose), the decomposition space axiom expresses the ability to decompose. Most of the above examples are decomposition spaces that do not satisfy the Segal condition. The Möbius-inversion formula, which classically is an equation in vector spaces, is lifted to an equivalence between ∞-groupoids defining spans, whose homotopy cardinality are the linear maps in question (the introductions of [25] and [29] contain further motivation).

Bialgebras and Hopf algebras, rather than just coalgebras, are obtained from *monoidal* decomposition spaces. In examples from combinatorics, the monoidal structure is often disjoint union. It is characteristic for the decomposition-space approach that the bialgebras obtained are (often filtered but) not connected in general. In particular they are not in general Hopf.

This theory is the starting point of the present thesis.

Contributions of the present thesis

This thesis furthers the theory of decomposition spaces by showing how some of the key techniques from classical combinatorics can be lifted to the setting of decomposition spaces. In each case, it requires to develop some general theory in order to accommodate the generalisation of the classical results. Three overall topics are treated.

- Rota's formula comparing Möbius functions across a Galois connection (Chapter 2 and Chapter 3)
- Schmitt's theory of hereditary species and their incidence bialgebras (Chapter 4)
- Classical antipode formulas for posets (mostly due to Schmitt) (Chapter 5)

For the Rota formula, the main contribution is a Rota formula for adjunctions between Möbius decomposition spaces (including ∞-adjunctions between ∞-categories). The general theory developed to support this result is the theory of bicomodules in the decomposition-space setting: they are certain augmented bisimplicial ∞-groupoids. We also provide situations where the generalised Rota formula can be applied.

For the hereditary species, the main contribution is to show that these constitute a class of examples of monoidal decomposition spaces, and that Schmitt's bialgebra construction is a special case of the general incidence bialgebra construction of monoidal decomposition spaces. Furthermore, we give an interpretation of hereditary species in terms of *comodule bialgebras* and *operadic categories* (in the sense of Batanin and Markl [8]).

For the antipode formulas (which is joint work with Joachim Kock), we use the ideas of the Gálvez–Kock–Tonks Möbius-inversion formula to establish an antipode formula for *any* monoidal decomposition space. One surprising feature of this is that it works also in the nonconnected case, meaning bialgebras that are not Hopf, leading to a notion of weak antipode.

In general we work at the level of ∞-groupoids, so that decomposition space means certain simplicial ∞-groupoids. However, the main example of the Rota formula concerns only a simplicial 1-groupoid (Chapter 3), and all of Chapter 4 is developed also in the setting of simplicial 1-groupoids, since this is the natural setting for Schmitt's hereditary species.

The three parts are essentially independent of each other, other than fitting together in the general project of developing objective combinatorics through decomposition spaces. The material of Chapter 5 was not actually planned, but developed very quickly through some discussions over the summer of 2018.

The material developed has also been organised into four papers, at different publication stages.

1. Incidence bicomodules, Möbius inversion, and a Rota formula for infinity adjunctions [14]

2. Möbius functions of directed restriction species and free operads, via the generalised Rota formula [15]

3. Hereditary species as monoidal decomposition spaces, comodule bialgebras, and operadic categories [16]

4. Antipodes of monoidal decomposition spaces [17]

The first has just been revised for possible publication in *Algebraic and Geometric Topology*, according to the referee's suggestions, and awaits the final decision. The second and third are ready to be submitted for publication. The fourth paper was accepted very quickly, and has already been published.

Outline

Groupoids, decomposition spaces, and combinatorics

In Chapter 1, we first review the combinatorics and homotopy theory of the 2-category of groupoids, with an emphasis on pullbacks, the main tool used throughout the present work. We continue with a short review of needed notions from the theory of ∞-categories. Finally, we summarise the basic theory of decomposition spaces.

Incidence bicomodules, Möbius inversion, and a Rota formula for infinity adjunctions

The original goal of the invertigations of Chapter 2, thought to be a routine exercise, was to generalise the Rota formula to ∞-adjunctions. It turned out a lot of machinery was required to do this in a satisfactory way, and developing this machinery ended up as a substantial contribution: they are general constructions in the theory of decomposition spaces, concerning bicomodules, which are of interest not just in combinatorics but also in representation theory, in connection with Hall algebras. The material of this chapter is is the main part of [14].

Rota formula for categories A classical formula due to Rota [51] compares the Möbius functions of two posets related by a *Galois connection*. The following generalisation of Rota's formula to Möbius categories is both natural and straightforward (but seems not to have been made before).

Theorem (Rota formula for Möbius categories)**.** *Let* X *and* Y *be Möbius categories, and let* $F : X \rightleftarrows Y : G$ *be an adjunction,* $F \dashv G$*. Then for all* $x \in X$*,* $y \in Y$*,*

$$\sum_{\substack{x' \in X \\ f:x \to x' \\ Fx'=y}} \mu_X(f) = \sum_{\substack{y' \in Y \\ g:y' \to y \\ Gy'=x}} \mu_Y(g).$$

The reader is not expected to read the following elementary proof, but only notice that it looks like an associativity formula for a convolution product, except that the arrows live in different categories.

Proof of the Rota formula.

$$\sum_{\substack{x' \in X \\ f:x \to x' \\ Fx'=y}} \mu_X(x \xrightarrow{f} x') \overset{(1)}{=} \sum_{\substack{x' \in X \\ f:x \to x' \\ h:Fx' \to y}} \mu_X(x \xrightarrow{f} x')\delta_Y(Fx' \xrightarrow{h} y)$$

$$\overset{(2)}{=} \sum_{\substack{x'\in X\\ f:x\to x'\\ h:Fx'\to y}} \mu_X(x \xrightarrow{f} x') \left(\sum_{\substack{y'\in Y\\ g:y'\to y\\ h':Fx'\to y'\\ \text{s.t. } h=gh'}} \zeta_Y(Fx' \xrightarrow{h'} y')\mu_Y(y' \xrightarrow{g} y) \right)$$

$$\overset{(3)}{=} \sum_{\substack{x'\in X, y'\in Y\\ f:x\to x'\\ h:Fx'\to y\\ g:y'\to y\\ h':Fx'\to y'\\ \text{s.t. } h=gh'}} \mu_X(x \xrightarrow{f} x')\zeta_Y(Fx' \xrightarrow{h'} y')\mu_Y(y' \xrightarrow{g} y)$$

$$\overset{(4)}{=} \sum_{\substack{x'\in X, y'\in Y\\ g:y'\to y\\ k:x\to Gy'\\ f:x\to x'\\ k':x'\to Gy'\\ \text{s.t. } k=k'f}} \mu_X(x \xrightarrow{f} x')\zeta_X(x' \xrightarrow{k'} Gy')\mu_Y(y' \xrightarrow{g} y) \text{ by adjunction}$$

$$\overset{(5)}{=} \sum_{\substack{y'\in Y\\ g:y'\to y\\ k:x\to Gy'}} \left(\sum_{\substack{x'\in X\\ f:x\to x'\\ k':x'\to Gy'\\ \text{s.t. } k=k'f}} \mu_X(x \xrightarrow{f} x')\zeta_X(x' \xrightarrow{k'} Gy') \right) \mu_Y(y' \xrightarrow{g} y)$$

$$\overset{(6)}{=} \sum_{\substack{y'\in Y\\ g:y'\to y\\ k:x\to Gy'}} \delta_X(x \xrightarrow{k} Gy')\mu_Y(y' \xrightarrow{g} y) \overset{(7)}{=} \sum_{\substack{y'\in Y\\ g:y'\to y\\ Gy'=x}} \mu_Y(y' \xrightarrow{g} y).$$

□

In the main result of the present paper, Theorem 2.3.8, we write this formula as

$$\mu_X \star_l \delta_Y = \delta_X \star_r \mu_Y,$$

with the following more conceptual proof

$$\mu_X \star_l \delta_Y \overset{(2)}{=} \mu_X \star_l (\zeta \star_r \mu_Y) \overset{(3-4-5)}{=} (\mu_X \star_l \zeta) \star_r \mu_Y \overset{(6)}{=} \delta_X \star_r \mu_Y,$$

referring to certain left and right convolution actions. The important insight here, which is due to Aguiar and Ferrer [4], is that the 'mixed arrows' which appear in the middle factors (those of the form $Fx \to y$, which in the crucial step of the proof are reinterpreted as $x \to Gy$ by adjunction) belong to a *bimodule*: they are acted upon from the left by arrows in the category X and from the right by arrows in Y (see Example 2.2.6 for an explicit description). The long complicated sums in the proof are thus condensed into convolution actions from the left and right, denoted $\star_l$ and $\star_r$. Aguiar and Ferrer [4] established a bimodule proof of the Rota formula in the setting of posets.

Two ingredients are necessary to make sense of the pleasing convolution proof above: one is to exhibit the data necessary to induce bicomodules and establish that adjunctions constitute an example. This already accounts for equalities (3-4-5) in the proof. The other is to establish a Möbius inversion principle for (co)modules (a notion which has not previously been considered in the literature, to the knowledge of the author), to account for the equalities (2) and (6).

In fact, as the notions and arguments become increasingly abstract and conceptual, it is natural to ask for further generalisation. In this work we take three considerable abstraction steps (beyond passing from posets to categories, which is already a fruitful step). First, we pass from categories and adjunctions to ∞-categories and ∞-adjunctions. Any ∞-adjunction defines a bicomodule in our sense. This step in itself is not so easy to justify from the viewpoint of combinatorics, but the homotopy content inherent in ∞-categories is important since already classical combinatorial structures have symmetries, and these can be handled more conveniently with groupoids than with sets, as advocated by Baez–Dolan [7], Gálvez-Kock-Tonks [29] and others. (This aspect will not be of importance in the present contribution, though.) Second, we pass from ∞-categories to decomposition spaces and introduce a notion of adjunction for them. This step has an important combinatorial motivation, because many combinatorial coalgebras admit a natural realisation as incidence coalgebras of decomposition spaces which are not posets or categories. An important example is the coalgebra of all finite posets, which is treated in detail in Chapter 3. The final abstraction step consists in noticing that the abstract Rota formula works equally well for certain bicomodules which do *not* come from adjunctions or ∞-adjunctions. In fact the example of Chapter 3 is of this type.

A word should be said about the objective aspect. Although most of constructions in this chapter belong to the ∞-groupoid level, it must be admitted that the final results are not fully objective. On one hand, the Möbius formula obtained for comodules is not directly realised as the homotopy cardinality of an equivalence: it has been found necessary here to take homotopy cardinality a little bit earlier in the constructions, so that the final arguments take place at the vector space level. This is due to the increased complexity compared to the plain Möbius-inversion formula of [26], where an even-odd splitting could be found for the single decomposition space involved. In the present situation, *two* decomposition spaces are involved, and the even-odd splitting at the objective level could not be found. Furthermore, the objective analogue of bicomodules given here is not fully satisfactory from the homotopy viewpoint. While it is shown to induce bicomodules up to homotopy, the *coherence* of this up-to-homotopy structure is not established in this work, and would seem to require considerable further efforts, in the line of coherence proofs given by [25] and [49]. Further discussion is included in the main text. The justification for not establishing coherence in the present contribution is that it is not necessary for the sake of taking cardinality, as required anyway in the final constructions for the Rota formula established.

In Section 2.1, following Walde [59] and Young [60], we first explain how to obtain a comodule in the context of decomposition spaces.

Proposition 2.1.1. *If $f : C \to X$ is a culf map between two simplicial ∞-groupoids such that C is Segal and X is a decomposition space, then the span*

$$C_0 \xleftarrow{d_1} C_1 \xrightarrow{(f_1,d_0)} X_1 \times C_0$$

induces on the slice ∞-category $\mathcal{S}_{/C_0}$ the structure of a left $\mathcal{S}_{/X_1}$-comodule (at the π_0 level), and the span

$$C_0 \xleftarrow{d_0} C_1 \xrightarrow{(d_1,f_1)} C_0 \times X_1.$$

induces on $\mathcal{S}_{/C_0}$ the structure of a right $\mathcal{S}_{/X_1}$-comodule (at the π_0 level).

The data needed to obtain a comodule is called a *comodule configuration*. In order to obtain a bicomodule structure, we first need an augmented bisimplicial ∞-groupoid Segal in each direction. We furthermore require this bisimplicial ∞-groupoid to be stable, see Section 2.1.3. This stability condition is a pullback condition on certain squares, and is a ∞-categorical reformulation of the notion of Bergner–Osorno–Ozornova–Rovelli–Scheimbauer [11], suitable for ∞-groupoids.

Theorem 2.1.10. *Let B be an augmented stable double Segal space, and such that the augmentation maps are culf. Suppose moreover $X := B_{\bullet,-1}$ and $Y := B_{-1,\bullet}$ are decomposition spaces. Then the spans*

$$B_{0,0} \xleftarrow{e_1} B_{1,0} \xrightarrow{(u,e_0)} X_1 \times B_{0,0}$$

and

$$B_{0,0} \xleftarrow{d_0} B_{0,1} \xrightarrow{(d_1,v)} B_{0,0} \times Y_1$$

induce on $\mathcal{S}_{/B_{0,0}}$ the structure of a bicomodule over $\mathcal{S}_{/X_1}$ and $\mathcal{S}_{/Y_1}$ (at the π_0 level).

An augmented bisimplicial ∞-groupoid satisfying the conditions of the theorem is called a *bicomodule configuration*.

In Section 2.2, we introduce the notion of correspondence of decomposition spaces: it is a decomposition space $\mathcal{M}$ with a map $\mathcal{M} \to \Delta^1$. We show that any correspondence of decomposition spaces gives rise to a bicomodule configuration. We then introduce the notion of *cartesian* and *cocartesian fibration* of decomposition spaces, adapting a homotopy-invariant definition for ∞-categories which can be found in [6]. They give rise to left and right pointed comodule configurations. We define an *adjunction* between decomposition spaces X and Y to be a simplicial map between decomposition spaces $p : \mathcal{M} \to \Delta^1$ which is both a cartesian and a cocartesian fibration, equipped with equivalences $X \simeq \mathcal{M}_{\{0\}}$ and $Y \simeq \mathcal{M}_{\{1\}}$. Adjunctions give rise to bicomodule configurations with two pointings.

In Section 2.3, we define left and right convolution actions $\star_l$ and $\star_r$ dual to the comodule structures. The following is a consequence of Theorem 2.1.10.

Corollary 2.3.1. *Given a bicomodule configuration, the left and right convolutions satisfy the associative law*

$$\alpha \star_l (\theta \star_r \beta) \simeq (\alpha \star_l \theta) \star_r \beta.$$

We then establish in Section 2.3.2 a Möbius inversion principle for complete comodules. Let $C \to Y$ be a right comodule configuration such that the simplicial ∞-groupoid C is augmented and with new bottom degeneracies $s_{-1} : C_{n-1} \to C_n$ which are sections to d_0. We say it is *complete* (Section 1.3.4) if the sections s_{-1} are monomorphisms.

For a complete decomposition space Y, let $\vec{Y}_n$ denote the full subgroupoid of simplices with all principal edges nondegenerate. The spans $Y_1 \xleftarrow{d_1^{n-1}} \vec{Y}_n \to 1$, where $\vec{Y}_n$ is the full subgroupoid of simplices with all principal edges nondegenerate, define linear functors, the *Phi functors* $\Phi_n : \mathcal{S}_{/Y_1} \to \mathcal{S}$. We also put $\Phi_{\text{even}} := \sum_{n \text{ even}} \Phi_n$, and $\Phi_{\text{odd}} := \sum_{n \text{ odd}} \Phi_n$.

The zeta functor $\zeta^C : \mathcal{S}_{/C_0} \to \mathcal{S}$ is the linear functor defined by the span $C_0 \xleftarrow{=} C_0 \to 1$, and $\delta^R : \mathcal{S}_{/C_0} \to \mathcal{S}$ is the linear functor given by the span $C_0 \xleftarrow{s_{-1}} C_{-1} \to 1$. We define δ^L similarly for left comodule configurations.

Theorem 2.3.5 and 2.3.6. *Given $C \to Y$ a complete right comodule configuration and $D \to X$ a complete left comodule configuration, then*

$$\zeta^C \star_r \Phi^Y_{\mathit{even}} \simeq \delta^R + \zeta^C \star_r \Phi^Y_{\mathit{odd}},$$

$$\Phi^X_{\mathit{even}} \star_l \zeta^D \simeq \delta^L + \Phi^X_{\mathit{odd}} \star_l \zeta^D.$$

In Section 2.3.3, we establish a Möbius inversion principle at the algebraic level. To this end, we need to impose some finiteness conditions in order to take homotopy cardinality. Define the *Möbius functions* as the homotopy cardinalities $|\mu^Y| := |\Phi^Y_{\text{even}}| - |\Phi^Y_{\text{odd}}|$ and $|\mu^X| := |\Phi^X_{\text{even}}| - |\Phi^X_{\text{odd}}|$.

Theorem 2.3.7. *Given $C \to Y$ a right Möbius comodule configuration and $D \to X$ a left Möbius comodule configuration,*

$$|\zeta^C| \star_r |\mu^Y| = |\delta^R|, \qquad |\mu^X| \star_l |\zeta^D| = |\delta^L|.$$

Finally we can extend the Rota formula to bicomodules with Möbius inversion for both comodules, called *Möbius bicomodule configurations*. Combining Proposition 2.3.1 and Theorem 2.3.7, we obtain the main theorem of the present chapter.

Theorem 2.3.8. *Given a Möbius bicomodule configuration B with $X := B_{\bullet,-1}$ and $Y := B_{-1,\bullet}$, we have*

$$|\mu^X| \star_l |\delta^R| = |\delta^L| \star_r |\mu^Y|,$$

where δ^R is the linear functor given by the span

$$B_{0,0} \leftarrow X_0 \to 1,$$

and δ^L is the linear functor given by the span

$$B_{0,0} \leftarrow Y_0 \to 1.$$

The motivating example, treated in Section 2.3.4, shows that any (co)cartesian fibration $p : \mathcal{M} \to \Delta^1$ such that $\mathcal{M}$ is a complete decomposition space gives rise to a complete left (or right) comodule configuration.

Theorem 2.3.11. *Given an adjunction of decomposition spaces in the form of a bicartesian fibration* $p : \mathcal{M} \to \Delta^1$*, suppose moreover that* $\mathcal{M}$ *is a Möbius decomposition space. Then the bicomodule configuration extracted from this data is Möbius. In particular, we have the Rota formula for the adjunction* p*:*

$$|\mu^X| \star_l |\delta^R| = |\delta^L| \star_r |\mu^Y|.$$

When specialised to the case of a classical adjunction between 1-categories, this is the classical Rota formula from page 7.

Möbius functions of directed restriction species and free operads, via the generalised Rota formula

Chapter 3 provides situations where the generalised Rota formula can be applied and treat in particular one example in detail: a certain bicomodule interpolating between the decomposition space of finite sets and the decomposition space of finite posets. The outcome is the formula $\mu(P) = (-1)^n$ for the Möbius function of a poset with n elements. This formula is well known (see for example [3]) but its derivation via a Rota formula is new and interesting, since the coalgebra of finite posets is not the incidence coalgebra of a locally finite poset or a Möbius category.

A key point in the construction is the notion of *abacus map*, a certain family of extra maps on a bisimplicial groupoid, which allows to modify the vertical top face maps artificially in an interesting way. This construction appears mysterious, but it is justified by the main example, the box product of the decomposition space of finite sets and the décalage of the decomposition space of finite posets: in this case the modification is precisely what allows to apply the generalised Rota formula to compute the Möbius function of any directed restriction species, starting with the case of the incidence algebra of finite posets. The coalgebra of finite posets is the incidence algebra of a decomposition space which is not a category, and the Möbius function is calculated via a bicomodule configuration (which is not an adjunction) with the decomposition space of finite sets. The construction also yields the Möbius function of the incidence algebra of any directed restriction species, including the Butcher–Connes–Kreimer Hopf algebra, and of the incidence bialgebra of any free operad, or more generally of any free monad on a finitary polynomial monad. The material of this chapter is is the main part of [15].

In Section 3.1, before defining the bicomodule configuration interpolating between the decomposition space **C** of finite posets and the decomposition space **I** of finite sets, we set up some general theory, introducing the notion of abacus map to modify a bisimplicial groupoid in a useful way. We furthermore identify conditions needed to obtain the required structure of bicomodule configuration. The constructions are applied to the box product as mentioned, to obtain a bicomodule configuration interpolating between the decomposition space of finite posets and the decomposition space of finite sets.

In Section 3.2, it is shown that this bicomodule configuration is Möbius. The verifications are actually elementary and amount essentially to computing some pullbacks of groupoids. Finally the generalised Rota formula can be applied rather easily, yielding a relationship between the two Möbius functions. Since the Möbius function for the coalgebra of finite sets is known, this gives a formula for the Möbius function for the incidence algebra of the decomposition space of finite posets.

Theorem 3.2.4. *The Möbius function of the incidence algebra of the decomposition space* $\mathbf{C}$ *of finite posets is*

$$\mu(P) = \begin{cases} (-1)^n & \textit{if } P \in \mathbf{C}_1 \textit{ is a discrete poset with } n \textit{ elements} \\ 0 & \textit{else.} \end{cases}$$

We show how the result extends almost verbatim to the incidence algebra of any directed restriction species in the sense of [28], via the decomposition space interpretation.

Corollary 3.2.5. *The Möbius function of the incidence algebra of the decomposition space* $\mathbf{R}$ *associated to a directed restriction species* $R : \mathbb{C}^{op} \to \boldsymbol{Grpd}$*, is*

$$\mu(Q) = \begin{cases} (-1)^n & \textit{if the underlying poset of } Q \in \mathbf{R}_1 \textit{ is discrete with } n \textit{ elements} \\ 0 & \textit{else.} \end{cases}$$

Corollary 3.2.6. *The Möbius function of the Butcher–Connes–Kreimer Hopf algebra of rooted forests is*

$$\mu(F) = \begin{cases} (-1)^n & \textit{if } F \textit{ consists of } n \textit{ isolated root nodes} \\ 0 & \textit{else.} \end{cases}$$

We also obtain a similar function for free operads, or more generally free monads on finitary polynomial monads.

Corollary 3.2.7. *The Möbius function of the incidence bialgebra of* P*-trees (for any finitary polynomial endofunctor* P*) is*

$$\mu(T) = \begin{cases} (-1)^n & \textit{if } T \textit{ consists of } n \ P\textit{-corollas and possibly isolated edges} \\ 0 & \textit{else.} \end{cases}$$

Hereditary species as monoidal decomposition spaces, comodule bialgebras, and operadic categories

Many important Hopf algebras in combinatorics are more asymmetric, having on one side of the comultiplication a monomial instead of a linear tensor factor. In the Segal case, the comultiplications with both tensor factors linear are typical for incidence coalgebras of categories, whereas the comultiplications with a monomial in the left-hand tensor factor are typical for operads. Indeed, incidence coalgebras and bialgebras of operads are another important class covered by the decomposition space framework, see [29] and [43].

Just as there are many linear-linear coalgebras that do not come from categories, there are important examples of multilinear-linear coalgebras that do not come from operads. An important class of such coalgebras is given by Schmitt's hereditary species [53]. These are structures that admit restriction (like restriction species) but also admit induction along quotient maps. Formally these are functors $\mathrm{H} : \mathbb{S}_p \to \boldsymbol{Set}$, where $\mathbb{S}_p$ denotes the category of finite sets and partially defined surjections. The induced comultiplication works by summing over all partitions of the underlying set, and then putting the monomial of all the blocks (with restricted structure) on the left and putting the quotient structure on the right. Schmitt [53] identified the properties needed for this to define a coassociative coalgebra (in fact a commutative bialgebra, and most often a Hopf algebra) and exhibited important examples, such as in particular the hereditary species of simple graphs.

Chapter 4 is the main part of [16]. We show that every hereditary species constitutes an example of a monoidal decomposition space, and that Schmitt's bialgebra construction is a special case of the general incidence bialgebra construction for decomposition spaces. These decomposition spaces are generally *not* Segal spaces (see Remark 4.2.7), and can therefore be seen as the first class of examples of decomposition spaces filling the missing entry in the following table.

	linear-linear	*multilinear-linear*
Segal-type	posets, monoids, categories	operads
non-Segal type	restriction species, S-construction, Hall algebras	

The construction is similar to the two-sided bar construction for operads, see [43].

Section 4.1 summarises Schmitt's hereditary species. In Section 4.2, we realise the hereditary species of finite sets as a Segal space **S** (hence a decomposition space) and establish some finiteness conditions.

Proposition 4.2.5 and 4.2.6. *The pseudosimplicial groupoid* **S** *is Segal, and is complete, locally finite, locally discrete, and of locally finite length.*

In Section 4.3, exploiting the hereditary species of finite sets, we show that every hereditary species H defines a monoidal decomposition space **H**, and hence a bialgebra at the groupoid-slice level, and we recover Schmitt's construction by taking homotopy cardinality.

Proposition 4.3.2 and 4.3.5. *For every hereditary species* H*, the simplicial groupoid* **H** *is a monoidal decomposition space. The incidence bialgebra obtained by taking homotopy cardinality coincides with the Schmitt bialgebra associated to* H.

Every hereditary species is also a restriction species, and the free algebra on its incidence coalgebra is therefore a bialgebra. In Section 4.4, we show that the incidence bialgebra of a hereditary species coacts on this bialgebra, so as constitute a *comodule bialgebra*.

Proposition 4.4.1. *The hereditary-species bialgebra* B *coacts on the restriction-species bialgebra* A*, so as to make* A *a left comodule bialgebra over* B*.*

Comodule bialgebras have been found important recently in numerical analysis [13] and in stochastic analysis [12], and there are general constructions based on operads and trees [22], [42]. The incidence comodule bialgebras of hereditary species introduced here constitute a new general class of comodule bialgebras, not related to trees.

In Section 4.5, we describe a different construction on hereditary species, showing that simple hereditary species induce operadic categories in the sense of Batanin and Markl [8]. Precisely we define a functor from (simple) hereditary species to operadic categories. This is interesting in its own right, as it constitutes a new family of examples of operadic categories. The construction is not directly related to decomposition spaces, but suggests that further connections are to be discovered.

Antipodes of monoidal decomposition spaces

In Chapter 5, with Joachim Kock [17], we upgrade the Gálvez–Kock–Tonks Möbius-inversion construction [26] to the construction of a kind of antipode in any *monoidal* (complete) decomposition space. Many of the constructions are quite similar; the main innovative idea is that there is a useful weaker notion of antipode for bialgebras even if they are *not* Hopf.

We introduce this notion and establish its main features (and limitations). Briefly, for X a monoidal (complete) decomposition space, the antipode is defined as a formal difference between linear endofunctors of $\mathcal{S}_{/X_1}$,

$$S := S_{\text{even}} - S_{\text{odd}},$$

given by multiplying principal edges of nondegenerate simplices. It cannot quite convolution-invert the identity endofunctor, as a true antipode should [58], but it can invert a modification of it, denoted Id':

$$\mathrm{Id}'(f) = \begin{cases} f & \text{if } f \text{ nondegenerate,} \\ \mathrm{id}_u & \text{if } f \text{ degenerate.} \end{cases}$$

Here u is the monoidal unit object, and we write id_u for $s_0 u$.

Precisely, our main theorem is the following inversion formula.

Theorem 5.1.4. *Given a monoidal complete decomposition space* X*, we have explicit equivalences*

$$S_{\text{even}} * \mathrm{Id}' \ \simeq\ e + S_{\text{odd}} * \mathrm{Id}' \qquad \textit{and} \qquad \mathrm{Id}' * S_{\text{even}} \ \simeq\ e + \mathrm{Id}' * S_{\text{odd}},$$

where $e := \eta \circ \epsilon$ *is the neutral element for convolution.*

Under the finiteness conditions satisfied by *Möbius* decomposition spaces, one can take homotopy cardinality (see Section 1.3.4) and form the difference $|S| := |S_{\mathrm{even}}| - |S_{\mathrm{odd}}|$ to arrive at the nicer-looking equation in the $\mathbb{Q}$-vector-space level convolution algebra:

$$|S| * |\mathrm{Id}'| \;=\; |\varepsilon| \;=\; |\mathrm{Id}'| * |S|.$$

The three main features justifying the weaker notion of antipode are:

1. If the monoidal decomposition space is connected, so that its incidence bialgebra is Hopf, then the homotopy cardinality of S is the usual antipode (cf. Proposition 5.1.6). (At the objective level of decomposition spaces, the construction of S is new also in the connected case.)

2. In any case, S computes the Möbius functor as

$$\mu \simeq \zeta \circ S$$

(cf. Corollary 5.2.1).

3. More generally, we establish an inversion formula for multiplicative functors (valued in any algebra) that send group-like elements to the unit (Theorem 5.2.3). The zeta functor is an example of this.

At the algebraic level of $\mathbb{Q}$-vector spaces, the weak antipode can be seen as a lift of the true antipode from the connected quotient of the bialgebra. When the bialgebra comes from the nerve of a category, this quotient is obtained by identifying all objects of the category. Recent developments have shown the utility of avoiding this reduction, which destroys useful information. For example, the Faà di Bruno formula for general operads [23], [43] crucially exploits the finer structure of the zeroth graded piece of the incidence bialgebra, and in the bialgebra version [40] of BPHZ renormalisation in perturbative quantum field theory, the zeroth graded piece of the bialgebra of Feynman graphs contains the terms of the Lagrangian (not visible in the quotient Hopf algebra usually employed).

1

Groupoids, ∞-groupoids, and decomposition spaces

1.1 The 2-category of groupoids

Our theoretical results are formulated in the setting of ∞-groupoids, and we freely use [46]. However, for the more specific applications to combinatorics (Chapters 3 and 4), we work with groupoids. The main results here, the prism lemma 1.1.1 and fibre lemma 1.1.5, could be derived from [46], but since they are fundamental, we prefer to give proofs.

1.1.1 Basics

A *groupoid* is a small category where all the arrows are invertible. A map of groupoids is just a functor. A homotopy between two maps of groupoids is just a natural transformation. Since the target category is a groupoid, all natural transformations are invertible. We denote by ***Grpd*** the 2-category of groupoids, maps and homotopies.

Groupoids generalise sets and groups: a set is a groupoid with only identity arrows, a group is a groupoid with only one object. Just as a group can be thought as a group of symmetries of one object, a groupoid is a collection of symmetries of possibly more than one object.

A map of groupoids $f\colon X \to Y$ is called a *homotopy equivalence* if there exist a map $g\colon Y \to X$ and homotopies $gf \simeq \mathrm{id}$ and $fg \simeq \mathrm{id}$. A functor is a homotopy equivalence if and only if it is essentially surjective and fully faithful, just as categories. It is the appropriate notion of sameness for groupoids and all the notions will be invariant under homotopy equivalences. We usually omit the term homotopy and say 'pullback', 'fibre', etc. instead of 'homotopy pullback', 'homotopy fibre', etc. The diagrams we consider are commutative up-to-homotopy, they come equipped with an (invertible) natural transformation.

A groupoid X is called *discrete* if it is equivalent to a set considered as a groupoid: the automorphism group $\mathrm{Aut}(x) := \mathrm{Hom}_X(x,x)$ is trivial for all x. It is *connected* if $\mathrm{Hom}_X(x,y)$ is non-empty for any $x, y \in X$. A groupoid is *contractible* if it is connected and discrete, that is homotopy equivalent to a point.

1.1.2 *Homotopy slices*

Given a groupoid I, the *homotopy slice* $\textbf{\textit{Grpd}}_{/I}$ is the 2-category of projective cones with base I: the objects are maps $X \to I$, arrows are triangles with a natural transformation ϕ:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] \arrow[rr, phantom, shift right=3, "\overset{\phi}{\Rightarrow}"] & & Y \arrow[dl, "q"] \\ & I, & \end{tikzcd}$$

and 2-cells are natural transformations

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\omega\Uparrow"] & Y \end{tikzcd}$$

commuting with the structure triangles:

$$\begin{tikzcd} X \arrow[rr, bend left, "f"] \arrow[rr, "f'"'] \arrow[rr, phantom, shift left=2, "\omega\Uparrow"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & I \arrow[u, phantom, "\overset{\phi'}{\Rightarrow}"] & \end{tikzcd} = \begin{tikzcd} X \arrow[rr, bend left, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & I. \arrow[u, phantom, "\overset{\phi}{\Rightarrow}"] & \end{tikzcd}$$

More generally, if $d\colon T \to \textbf{\textit{Grpd}}$ is any diagram, there is a 2-category $\textbf{\textit{Grpd}}_{/d}$ of projective cones with base d.

A *homotopy terminal object* in a 2-category $\mathscr{C}$ is an object t such that for any other object x, the groupoid $\mathrm{Hom}_{\mathscr{C}}(x, t)$ is contractible, i.e. equivalent to a point. More general homotopy limits are defined in the usual way using homotopy slices: the *homotopy limit* of a functor $d\colon T \to \textbf{\textit{Grpd}}$ is by definition a homotopy terminal object in the homotopy slice $\textbf{\textit{Grpd}}_{/d}$. Homotopy limits are unique up to equivalence.

1.1.3 *Homotopy pullbacks and homotopy fibres*

A *homotopy pullback* is a homotopy limit of a functor $d\colon \{\bullet \to \bullet \leftarrow \bullet\} \to \textbf{\textit{Grpd}}$. Explicitly, given groupoid maps $X \to S \leftarrow Y$, there is a 2-category whose objects are homotopies

$$\begin{tikzcd} Q \arrow[r] \arrow[d] \arrow[dr, phantom, "\nearrow"] & Y \arrow[d] \\ X \arrow[r] & S \end{tikzcd}$$

whose arrows are diagrams like

$$
\begin{tikzcd}
Q' \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] \arrow[drr, phantom, "\Uparrow"] \arrow[ddr, phantom, "\Rightarrow"] & & \\
& Q \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow"] & Y \arrow[d] \\
& X \arrow[r] & S
\end{tikzcd}
$$

and whose 2-cells are

$$
\begin{tikzcd}
Q' \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Uparrow"] & Q
\end{tikzcd}
$$

such that

$$
\begin{tikzcd}
Q' \arrow[rr, bend left] \arrow[rr, bend right] \arrow[rr, phantom, "\Uparrow"] \arrow[dr] \arrow[dr, phantom, shift left=4, "\Nearrow"] & & Q \arrow[dl] \\
& X &
\end{tikzcd}
=
\begin{tikzcd}
Q' \arrow[rr, bend left] \arrow[dr] \arrow[dr, phantom, shift left=4, "\Nearrow"] & & Q \arrow[dl] \\
& X &
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
Q' \arrow[rr, bend left] \arrow[rr, bend right] \arrow[rr, phantom, "\Downarrow"] \arrow[dr] \arrow[dr, phantom, shift left=4, "\Swarrow"] & & Q \arrow[dl] \\
& Y &
\end{tikzcd}
=
\begin{tikzcd}
Q' \arrow[rr, bend left] \arrow[dr] \arrow[dr, phantom, shift left=4, "\Swarrow"] & & Q \arrow[dl] \\
& Y. &
\end{tikzcd}
$$

A homotopy pullback is a terminal object in this 2-category, that is a commutative-up-to-homotopy square

$$
\begin{tikzcd}
P \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\Nearrow_{\phi}"] & Y \arrow[d, "g"] \\
X \arrow[r, "f"'] & S
\end{tikzcd}
$$

which is universal among such squares in a homotopy sense. This means that given any other such square

$$
\begin{tikzcd}
T \arrow[r, "q_2"] \arrow[d, "q_1"'] \arrow[dr, phantom, "\Nearrow_{\psi}"] & Y \arrow[d, "g"] \\
X \arrow[r, "f"'] & S
\end{tikzcd},
$$

there exists a morphism $u\colon T \to P$ and homotopies $\gamma_1\colon q_1 \Rightarrow p_1u$ and $\gamma_2\colon p_2u \Rightarrow q_2$ which are compatible with the given ones above, and given any two morphisms $u, v\colon T \to P$ and homotopies $\lambda_1\colon q_1 \Rightarrow p_1v$ and $\lambda_2\colon p_2v \Rightarrow q_2$ which are coherent, there exists a unique 2-cell $\chi\colon v \Rightarrow u$ such that $\gamma_2\chi = \lambda_2$ and $\lambda_1\chi = \gamma_1$. In other words for all ψ, $\mathrm{Hom}_{\mathbf{\mathit{Grpd}}_{/d}}(\psi, \phi)$ is contractible.

Given a map of groupoids $p\colon X \to S$ and an object $s \in S$, the *homotopy fibre* X_s of p over s is the homotopy pullback

$$\begin{tikzcd}
X_s \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "p"] \\
1 \arrow[r, "\ulcorner s \urcorner"'] & S,
\end{tikzcd}$$

where the map $\ulcorner s \urcorner$ chooses the element s.

A map of groupoids is a *monomorphism* when its fibres are either empty or contractible. If $f\colon X \to Y$ is a monomorphism, then there is a complement $Z := Y \backslash X$ such that $Y \simeq X + Z$; a monomorphism is essentially an equivalence from X onto some connected components of Y.

Lemma 1.1.1. *Given a prism diagram of groupoids*

$$\begin{tikzcd}
X_0 \arrow[r] \arrow[d] & X_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_2 \arrow[d] \\
Y_0 \arrow[r] \arrow[ur, phantom, "\Nearrow_{\psi_2}"] & Y_1 \arrow[r] \arrow[ur, phantom, "\Nearrow_{\psi_0}"] & Y_2
\end{tikzcd}$$

in which the right-hand square is a pullback. Then the outer rectangle is a pullback if and only of the left-hand square is.

Remark 1.1.2. We talk about a prism, it is a diagram consisting of three squares and two triangles. We have not drawn the square whose horizontal sides are composites of the horizontal arrows. The triangles are not drawn either. They all come equipped with homotopies.

Proof. Given any up-to-homotopy square,

$$\begin{tikzcd}
T \arrow[rrrd, bend left, "b"] \arrow[rdd, bend right, "a"'] & & & \\
 & & \arrow[phantom, "\Nearrow\phi"] & X_2 \arrow[d, "p_2"] \\
 & Y_0 \arrow[r, "g_2"'] & Y_1 \arrow[r, "g_0"'] & Y_2
\end{tikzcd}$$

we wish to construct a triple (u_1, α_1, β_1) consisting of a map $u_1 : T \to X_0$ and two triangles fitting it in, and show that this data is essentially unique.

Since the right-hand square ψ_0 is a pullback, there exists a map $u_0 : T \to X_1$ and 2-cells $\alpha_0 : g_2 a \Rightarrow p_1 u_0$ and $\beta_0 : f_0 u_0 \Rightarrow b$ such that

$$\begin{tikzcd}
T \arrow[rrd, dotted, "u_0"] \arrow[rrrd, bend left, "b"] \arrow[rdd, bend right, "a"'] & & & \\
 & \arrow[phantom, "\Uparrow\alpha_0"] & X_1 \arrow[r, "f_0"] \arrow[d, "p_1"] \arrow[u, phantom, "\beta_0\Nearrow"] & X_2 \arrow[d, "p_2"] \\
 & Y_0 \arrow[r, "g_2"'] & Y_1 \arrow[r, "g_0"'] \arrow[ur, phantom, "\Nearrow\psi_0"] & Y_2
\end{tikzcd}
=
\begin{tikzcd}
T \arrow[rrrd, bend left, "b"] \arrow[rdd, bend right, "a"'] & & & \\
 & & \arrow[phantom, "\Nearrow\phi"] & X_2 \arrow[d, "p_2"] \\
 & Y_0 \arrow[r, "g_2"'] & Y_1 \arrow[r, "g_0"'] & Y_2.
\end{tikzcd}$$

This data is essentially unique, but for the moment we only use existence. So we are choosing one of those triples. Essentially unique means for any other such $(u_0', \beta_0', \alpha_0')$, there exist a unique $\chi : u_0' \Rightarrow u_0$ such that

$$\begin{tikzcd}
T \arrow[dr, bend left, "u_0"] \arrow[dr, bend right, "u_0'"'] \arrow[drr, bend left=50, "b"] \arrow[dr, phantom, "\chi\nearrow"] & & \\
\arrow[r, phantom, "\beta_0\Uparrow"] & X_1 \arrow[r, "f_0"] & X_2
\end{tikzcd}
=
\begin{tikzcd}
T \arrow[dr, bend right, "u_0'"'] \arrow[drr, bend left=50, "b"] & \arrow[d, phantom, "\beta_0'\Uparrow"] & \\
& X_1 \arrow[r, "f_0"] & X_2
\end{tikzcd}$$

and

$$\begin{tikzcd}
T \arrow[rr, bend left, "u_0"] \arrow[dr, bend right, "u_0'"] \arrow[ddr, bend right=40, "a"'] \arrow[r, phantom, "\chi\Uparrow"] & & X_1 \arrow[dd, "p_1"] \\
& u_0' \arrow[ur] \arrow[d, phantom, "\alpha_0'\Uparrow"] & \\
& Y_0 \arrow[r, "g_2"'] & Y_1
\end{tikzcd}
=
\begin{tikzcd}
T \arrow[drr, bend left, "u_0"] \arrow[ddr, bend right=40, "a"'] & \arrow[d, phantom, "\alpha_0\Uparrow"] & \\
& & X_1 \arrow[d, "p_1"] \\
& Y_0 \arrow[r, "g_2"'] & Y_1.
\end{tikzcd}$$

Since the left-hand square ψ_2 is a pullback, there exists a map $u_2 : T \to X_0$ and 2-cells $\alpha_2 : a \Rightarrow p_0u_2$ and $\beta_2 : f_2u_2 \Rightarrow u_0$ such that

$$\begin{tikzcd}
T \arrow[dr, dotted, "u_2"] \arrow[drr, bend left, "u_0"] \arrow[ddr, bend right=40, "a"'] & \arrow[d, phantom, "\beta_2\nearrow"] & \\
\arrow[r, phantom, "\alpha_2\Rightarrow"] & X_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\nearrow\psi_2"] & X_1 \arrow[d] \\
& Y_0 \arrow[r] & Y_1
\end{tikzcd}
=
\begin{tikzcd}
T \arrow[drr, bend left, "u_0"] \arrow[ddr, bend right=40, "a"'] & \arrow[d, phantom, "\alpha_0\Uparrow"] & \\
& & X_1 \arrow[d] \\
& Y_0 \arrow[r] & Y_1.
\end{tikzcd}$$

Again, there is also a uniqueness statement, but for the moment we only need existence. The uniqueness statement is: for any other such $(u_2', \alpha_2', \beta_2')$ there exist a unique $\Xi : u_2' \Rightarrow u_2$ such that

$$\begin{tikzcd}
T \arrow[dr, bend left, "u_2"] \arrow[dr, bend right, "u_2'"'] \arrow[drr, bend left=50, "u_0"] \arrow[dr, phantom, "\Xi\nearrow"] & & \\
\arrow[r, phantom, "\beta_2\Uparrow"] & X_0 \arrow[r, "f_2"'] & X_1
\end{tikzcd}
=
\begin{tikzcd}
T \arrow[dr, bend right, "u_2'"'] \arrow[drr, bend left=50, "u_0"] & \arrow[d, phantom, "\beta_2'\Uparrow"] & \\
& X_0 \arrow[r, "f_2"'] & X_1
\end{tikzcd}$$

and

$$\begin{tikzcd}
T \arrow[dr, bend left, "u_2"] \arrow[dr, bend right, "u_2'"'] \arrow[ddr, bend right=40, "a"'] \arrow[dr, phantom, "\Xi\nearrow"] & \\
\arrow[r, phantom, "\Rightarrow\alpha_2'"] & X_0 \arrow[d, "p_0"] \\
& Y_0
\end{tikzcd}
=
\begin{tikzcd}
T \arrow[dr, bend left, "u_2"] \arrow[ddr, bend right=40, "a"'] & \\
\arrow[r, phantom, "\Rightarrow\alpha_2"] & X_0 \arrow[d, "p_0"] \\
& Y_0.
\end{tikzcd}$$

We want to show there exists a map $u_1 : T \to X_0$ and 2-cells $\beta_1 : f_0f_2u_1 \Rightarrow b$ and $\alpha_1 : a \Rightarrow p_0u_1$ such that

$$
\begin{tikzcd}
T \arrow[dr, dotted, "u_1"] \arrow[drr, bend left, "b"] \arrow[ddr, bend right, "a"'] \arrow[dr, phantom, shift right=3, "\alpha_1\Rightarrow"] & & \\
& X_0 \arrow[r, "f_1"] \arrow[d, "p_0"] \arrow[ur, phantom, "\Uparrow\beta_1"] \arrow[dr, phantom, "\psi_1"] & X_2 \arrow[d, "p_2"] \\
& Y_0 \arrow[r, "g_1"'] & Y_2
\end{tikzcd}
=
\begin{tikzcd}
T \arrow[drr, bend left, "b"] \arrow[ddr, bend right, "a"'] & & \\
& \arrow[r, phantom, "\Uparrow\phi"] & X_2 \arrow[d, "p_2"] \\
& Y_0 \arrow[r, "g_1"'] & Y_2
\end{tikzcd}
$$

and then we want to show that this data is essentially unique. Essentially unique means for any other such $(u_1', \alpha_1', \beta_1')$ there exist a unique $\omega : u_1' \Rightarrow u_1$ such that

$$
\begin{tikzcd}
T \arrow[dr, bend left, "u_1"] \arrow[dr, bend right, "u_1'"'] \arrow[dr, phantom, "\omega"] \arrow[drrr, bend left=40, "b"] & & & \\
& X_0 \arrow[r, "f_2"] \arrow[urr, phantom, "\Uparrow\beta_1"] & X_1 \arrow[r, "f_0"] & X_2
\end{tikzcd}
=
\begin{tikzcd}
T \arrow[dr, bend right, "u_1'"'] \arrow[drrr, bend left=40, "b"] & & & \\
& X_0 \arrow[r, "f_2"] \arrow[urr, phantom, "\Uparrow\beta_1'"] & X_1 \arrow[r, "f_0"] & X_2
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
T \arrow[dr, bend left, "u_1"] \arrow[dr, bend right, "u_1'"'] \arrow[dr, phantom, "\omega"] \arrow[ddr, bend right=50, "a"'] & \\
\arrow[r, phantom, "\Rightarrow\alpha_1'"] & X_0 \arrow[d, "p_0"] \\
& Y_0
\end{tikzcd}
=
\begin{tikzcd}
T \arrow[dr, bend left, "u_1"] \arrow[ddr, bend right=50, "a"'] & \\
\arrow[r, phantom, "\Rightarrow\alpha_1"] & X_0 \arrow[d, "p_0"] \\
& Y_0.
\end{tikzcd}
$$

The candidate is $(u_1, \alpha_1, \beta_1) = (u_2, \alpha_2, \beta_0\beta_2)$. The equality of 2-cells is verified by construction. It remains to show the uniqueness. This will follow from the uniqueness of the triples (u_0, α_0, β_0) and (u_2, α_2, β_2) used in the construction. Since in each case, the uniqueness says 'unique such that something', and since this 'something' is different for the three cases, the safest thing for matching up the uniquenesses is to define three solution sets with maps between them, and show that if two of those sets are singleton then so is the third. So suppose that we have a contendent triple $(u_1', \alpha_1', \beta_1')$.

For fixed $\psi_0, \psi_2, \psi_1, \phi, u_0, \alpha_0, \beta_0, u_2, \alpha_2, \beta_2, u_1', \alpha_1', \beta_1'$ as above (satisfying the equations stipulated in the constructions), consider the set

$$\Omega = \{\omega : u_1' \Rightarrow u_2 \mid \text{conditions}\}$$

that is, the set of all possible comparisons between our candidate triple and any other contendent. Here are the conditions:

$$
\begin{tikzcd}
T \arrow[dr, bend left, "u_2"] \arrow[dr, bend right, "u_1'"'] \arrow[dr, phantom, "\omega"] \arrow[drrr, bend left=40, "b"] & & & \\
& X_0 \arrow[r, "f_2"] \arrow[urr, phantom, "\Uparrow\beta_2\beta_0"] & X_1 \arrow[r, "f_0"] & X_2
\end{tikzcd}
=
\begin{tikzcd}
T \arrow[dr, bend right, "u_1'"'] \arrow[drrr, bend left=40, "b"] & & & \\
& X_0 \arrow[r, "f_2"] \arrow[urr, phantom, "\Uparrow\beta_1'"] & X_1 \arrow[r, "f_0"] & X_2
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
T \arrow[dr, bend left, "u_2"] \arrow[dr, "u_1'"'] \arrow[ddr, bend right=40, "a"'] \arrow[dr, phantom, bend left=20, "\omega\,\Uparrow"] & \\
\arrow[r, phantom, "\Rightarrow \alpha_1'"] & X_0 \arrow[d, "p_0"] \\
& Y_0
\end{tikzcd}
=
\begin{tikzcd}
T \arrow[dr, bend left, "u_2"] \arrow[ddr, bend right=40, "a"'] & \\
\arrow[r, phantom, "\Rightarrow \alpha_2"] & X_0 \arrow[d, "p_0"] \\
& Y_0 .
\end{tikzcd}
$$

Our aim is thus to establish that this set Ω is singleton.

Define another set as follows, still for the same fixed symbols as for Ω:

$$X = \{\chi : f_2 u_1' \Rightarrow u_0 \mid \text{conditions}\}$$

Here are the conditions:

$$
\begin{tikzcd}
T \arrow[drr, bend left=40, "b"] \arrow[dr, bend left, "u_0"] \arrow[dr, "f_2 u_1'"'] \arrow[dr, phantom, bend left=15, "\chi\,\Uparrow"] & \arrow[d, phantom, "\beta_0 \Uparrow"] & \\
& X_1 \arrow[r, "f_0"] & X_2
\end{tikzcd}
=
\begin{tikzcd}
T \arrow[drr, bend left=40, "b"] \arrow[dr, "f_2 u_1'"'] & \arrow[d, phantom, "\beta_1' \Uparrow"] & \\
& X_1 \arrow[r, "f_0"] & X_2
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
T \arrow[drr, bend left, "u_0"] \arrow[dr, dotted, "u_1'"] \arrow[ddr, bend right, "a"'] & \arrow[d, phantom, "\chi \Uparrow"] & \\
\arrow[r, phantom, "\alpha_1' \Rightarrow"] & X_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow \psi_2"] & X_1 \arrow[d] \\
& Y_0 \arrow[r] & Y_1
\end{tikzcd}
=
\begin{tikzcd}
T \arrow[drr, bend left, "u_0"] \arrow[ddr, bend right, "a"'] & & \\
& \arrow[phantom, "\alpha_0 \Uparrow"] & X_1 \arrow[d, "p_1"] \\
& Y_0 \arrow[r, "g_2"'] & Y_1 .
\end{tikzcd}
$$

There is a canonical map $\Omega \to X$: it takes ω to ω pasted with β_2. It follows immediately from the various conditions on ω that the pasted 2-cells satisfy the conditions characterising X.

For a fixed $\chi \in X$ we compute the fibre of this map, which is therefore a subset $\Omega_\chi \subset \Omega$. It has the following explicit description.

$$\Omega_\chi = \{\omega : u_1' \Rightarrow u_2 \mid \text{other conditions}\}$$

Here are those other conditions:

$$
\begin{tikzcd}
T \arrow[drr, bend left=40, "u_0"] \arrow[dr, bend left, "u_2"] \arrow[dr, "u_1'"'] \arrow[dr, phantom, bend left=15, "\omega\,\Uparrow"] & \arrow[d, phantom, "\beta_2 \Uparrow"] & \\
& X_0 \arrow[r, "f_2"'] & X_1
\end{tikzcd}
=
\begin{tikzcd}
T \arrow[drr, bend left=40, "u_0"] \arrow[dr, "u_1'"'] & \arrow[d, phantom, "\chi \Uparrow"] & \\
& X_0 \arrow[r, "f_2"'] & X_1
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
T \arrow[dr, bend left, "u_2"] \arrow[dr, "u_1'"'] \arrow[ddr, bend right=40, "a"'] \arrow[dr, phantom, bend left=20, "\omega\,\Uparrow"] & \\
\arrow[r, phantom, "\Rightarrow \alpha_1'"] & X_0 \arrow[d, "p_0"] \\
& Y_0
\end{tikzcd}
=
\begin{tikzcd}
T \arrow[dr, bend left, "u_2"] \arrow[ddr, bend right=40, "a"'] & \\
\arrow[r, phantom, "\Rightarrow \alpha_2"] & X_0 \arrow[d, "p_0"] \\
& Y_0 .
\end{tikzcd}
$$

Altogether, the three sets form a fibre sequence

$$\begin{tikzcd} \Omega_X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Omega \arrow[d] \\ 1 \arrow[r, "\ulcorner X \urcorner"'] & X. \end{tikzcd}$$

Now we can apply our uniqueness results: the pullback property of ψ_0 tells us that X is singleton. The pullback property of ψ_2 tells us that Ω_X is singleton. It follows that also Ω is singleton as required.

The proof of the converse statement follows the same ideas. We suppose we are given an enemy square to ψ_2. Paste it with ψ_0 to get altogether an enemy square to ψ_1; from here we get $u_1 : T \to X_0$ and two 2-cells α_1 and β_1. Here u_1 and α_1 are already our candidates for the required triple (u_2, α_2, β_2); it remains to get β_2. This cannot be constructed using only existence statement; we need here to invoke uniqueness from the universal property of ψ_0: β_2 will be the unique 2-cell mediating between $f_0 u_1$ and u_0. ☐

The following result is the groupoid analogue of the obvious result that for a map of sets $E \to B$, the domain is the disjoint union of the fibres.

Lemma 1.1.3. *For any map of groupoids* $p \colon X \to B$*, there is a canonical equivalence*

$$\sum_{b \in \pi_0(B)} \frac{X_b}{\mathrm{Aut}(b)} \xrightarrow{\sim} X.$$

Proof. Since we sum over the connected components of B, it is enough to treat the case where B is connected, so we assume this from now on. The statement is now $\frac{X_b}{\mathrm{Aut}(b)} \xrightarrow{\sim} X$. For the fibre X_b we use the standard explicit model where the objects are pairs $(x, \sigma \colon b \xrightarrow{\sim} px)$ and an arrow from $(y, \tau \colon b \xrightarrow{\sim} py)$ to $(x, \sigma \colon b \xrightarrow{\sim} px)$ is an arrow $f \colon y \to x$ such that $\sigma = pf \circ \tau$. The group $\mathrm{Aut}(b)$ acts canonically on X_b by

$$\begin{array}{rcl} \mathrm{Aut}(b) \times X_b & \longrightarrow & X_b \\ (\alpha, (x, \sigma)) & \longmapsto & \big(x, b \xrightarrow{\alpha} b \xrightarrow{\sigma} px\big). \end{array}$$

For the quotient $X_b / \mathrm{Aut}(b)$ we use the standard explicit model where the objects are those of X_b, and an arrow from (y, τ) to (x, σ) is a pair (α, f), where $\alpha \in \mathrm{Aut}(b)$ and $f \colon (y, \sigma) \to \alpha.(x, \sigma) = (x, b \xrightarrow{\alpha} b \xrightarrow{\sigma} px)$. By definition of arrows in X_b, this unpacks to a commutative diagram

$$\begin{tikzcd} b \arrow[r, "\tau"] \arrow[d, "\alpha"'] & py \arrow[d, "pf"] \\ b \arrow[r, "\sigma"'] & px, \end{tikzcd}$$

and in particular α is completely determined by f.

Now there is a canonical map

$$X_b / \mathrm{Aut}(b) \quad \longrightarrow \quad X$$

$$\begin{array}{rcl}(x, \sigma) & \longmapsto & x \\ (\alpha, f) & \longmapsto & f\end{array}$$

It is clear that this is (essentially) surjective. It is also full and faithful, since α is determined by f. $\square$

We write $\int^b(-)$ for $\sum_{b\in\pi_0(B)} \frac{(-)}{\mathrm{Aut}(b)}$. More generally, functoriality of sums and quotients give us the following lemma.

Lemma 1.1.4. *Given maps of groupoids* $Y \xrightarrow{u} X \to B$, *we have a commutative diagram*

$$\begin{tikzcd} \int^b Y_b \arrow[r, "\sim"] \arrow[d, "\int^b u_b"'] & Y \arrow[d, "u"] \\ \int^b X_b \arrow[r, "\sim"] & X. \end{tikzcd}$$

Lemma 1.1.5. *A square of groupoids*

$$\begin{tikzcd} P \arrow[r, "u"] \arrow[d] & Y \arrow[d] \\ X \arrow[r, "f"'] \arrow[ur, phantom, "\nearrow"] & S \end{tikzcd}$$

is a pullback if and only if for each $x \in X$ *the induced comparison map* $u_x \colon P_x \to Y_{fx}$ *is an equivalence.*

Proof. It is clear that if the square is a pullback, then all the fibre maps u_x are equivalences, by transitivity of pullback. Conversely, assume that all the fibre maps u_x are equivalences, and consider the canonical map

$$\begin{tikzcd} P \arrow[rr, "v"] \arrow[dr] & & X \times_S Y \arrow[dl] \\ & X & \end{tikzcd}$$

induced by the universal property of the standard explicit fibre product. (The triangle commutes strictly.) We want to prove that this map v is an equivalence, and by Lemma 1.1.4 it is enough to establish that its x-fibres $v_x \colon P_x \to (X \times_S Y)_x$ are equivalences. But transitivity of pullback gives $(X \times_S Y)_x \simeq Y_{fx}$, as seen in the following diagram:

$$\begin{tikzcd} (X \times_S Y)_x \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \times_S Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d] \\ 1 \arrow[r, "\ulcorner x \urcorner"'] & X \arrow[r, "f"'] & S \end{tikzcd}$$

so the maps v_x are equivalences if and only if the u_x are, which we have assumed. Hence also v is an equivalence (by Lemma 1.1.4). $\square$

1.1.4 *Cardinality*

A groupoid X is called *locally finite* if each automorphism group is finite, and is called *finite* if furthermore it has only finitely many components. We denote by ***grpd*** the 2-category of finite groupoids. Note that every set is locally finite. A morphism of groupoids is called *finite* when all its fibres are finite.

The *homotopy cardinality of a finite groupoid* [7] (sometimes called *groupoid cardinality*) is defined to be the nonnegative rational number given by the formula

$$|X| := \sum_{x \in \pi_0 X} \frac{1}{|\mathrm{Aut}(x)|}.$$

This is independent of the choice of the x in the same connected component since an arrow between two choices induces an isomorphism of vertex groups. Homotopy equivalent groupoids have the same cardinality.

Example 1.1.6. If X is a finite set considered as a groupoid, then the groupoid cardinality coincides with the set cardinality. If G is a group considered as a one-object groupoid, then the groupoid cardinality is the inverse of the order of the group.

Remark 1.1.7. The groupoid cardinality is a standard construction in physics and combinatorics: one can sum over the different isomorphism classes of objects and for each object divide out by the order of its symmetry group.

The *homotopy cardinality of a finite map of groupoids* $X \xrightarrow{p} S$ is

$$|X \xrightarrow{p} S| := \sum_{x \in \pi_0 S} \frac{|X_s|}{|\mathrm{Aut}(s)|} \delta_s \in \mathbb{Q}_{\pi_0 S},$$

where X_s is the homotopy fibre, and $\mathbb{Q}_{\pi_0 S}$ is the vector space spanned by iso-classes, denoted by the formal symbol δ_s for $s \in \pi_0 S$. Remark that the cardinality of the basis object $1 \xrightarrow{\ulcorner s \urcorner} S$ in $\boldsymbol{grpd}_{/S}$ is the basis vector δ_s in $\mathbb{Q}_{\pi_0 S}$.

1.2 *The ∞-category of ∞-groupoids*

While most of the examples are formulated in the 2-category of groupoids, our theoretical results are in the setting of ∞-categories, which is a natural generalisation, and is for most of our purposes as usable as the theory of category thanks to Joyal [35, 36] and Lurie [46].

We review here the needed notions from the theory of ∞-categories and we give a glimpse of homotopy linear algebra [24].

1.2.1 *Infinity-groupoids, functor category, and diagrams*

Our ∞-categories are quasi-categories, these are simplicial sets satisfying the weak Kan condition: every inner horn admits a filler (not necessarily unique). The theory of quasi-categories has been substantially developed by Joyal [35, 36] and Lurie [46]. There are other models of ∞-categories, for example Segal categories, or complete Segal spaces, see [10]. A model-independent abstract formulation is being developed in the work in progress [50].

An ∞-groupoid is an ∞-category in which all morphisms are invertible. They are precisely Kan complexes: simplicial sets in which every horn admits a filler (and not only the inner ones). We work in the ∞-category of ∞-groupoids, denoted $\mathcal{S}$, following the notation of [25]. We sometimes use the word space instead of ∞-groupoid. ∞-groupoids have an analogous role to sets in the 1-category theory.

Defining ∞-categories by describing the simplices in all dimensions, and verify filler conditions is more difficult than in the 1 or 2-category setting. Instead, we obtain new ∞-categories from already existing ones and constructions that guarantees we obtain ∞-categories. Between two objects X, Y of a ∞-category $\mathscr{C}$, there is a mapping space $\mathrm{Map}_{\mathscr{C}}(X, Y)$ which is an ∞-groupoid. Between two ∞-categories, there is a functor ∞-category $\mathrm{Fun}(\mathscr{C}, \mathscr{D})$, whose objects are the ∞-functors from $\mathscr{C}$ to $\mathscr{D}$, morphisms are the corresponding homotopies, etc. A commutative diagram of shape I in an ∞-category is an object in the functor ∞-category $\mathrm{Fun}(I, \mathscr{C})$. For example, a commutative triangle is an object in $\mathrm{Fun}(\Delta^2, \mathscr{C})$, a commutative square is an object in $\mathrm{Fun}(\Delta^1 \times \Delta^1, \mathscr{C})$.

1.2.2 *Pullbacks, fibres*

Given an ∞-category $\mathscr{C}$ and a square $\sigma \colon \Delta^1 \times \Delta^1 \to \mathscr{C}$, denoted

$$\begin{tikzcd} X' \arrow[r, "p'"] \arrow[d, "q'"'] & X \arrow[d, "q"] \\ Y' \arrow[r, "p"'] & Y. \end{tikzcd}$$

There is an isomorphism of simplicial sets $\Delta^1 \times \Delta^1 \simeq \Delta^0 \star \Lambda^2_2$ (cone over Λ^2_2) and it makes sense to ask whether or not σ is a limit diagram (a cone which is universal in the ∞-categorical sense). If σ is a limit, we say σ is a *pullback square*, and write $X' = X \times_Y Y'$.

Lemma 1.2.1 ([46, Lemma 4.4.2.1]). *Given a prism diagram of ∞-groupoids*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X'' \arrow[d] \\ Y \arrow[r] & Y' \arrow[r] & Y'' \end{tikzcd}$$

in which the right-hand square is a pullback. Then the outer rectangle is a pullback if and only if the left-hand square is.

Remark 1.2.2. We talk about a prism, it is a $\Delta^1 \times \Delta^2$-diagram, so consisting of three squares and two triangles. We have not drawn the square whose horizontal sides are composites of the horizontal arrows. The triangles are not drawn either, they are the fillers that exist by the axioms of ∞-categories.

Given a map of ∞-groupoids $p : X \to S$ and an object $s \in S$, the *fibre* X_s of p over s is the pullback

$$\begin{tikzcd} X_s \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "p"] \\ 1 \arrow[r, "\ulcorner s \urcorner"'] & S. \end{tikzcd}$$

A map of ∞-groupoids is a *monomorphism* when its fibres are either empty or contractible. If $f \colon X \to Y$ is a monomorphism, then there is a complement $Z := Y \backslash X$ such that $Y \simeq X + Z$; a monomorphism is essentially an equivalence from X onto some connected components of Y.

1.2.3 *Homotopy linear algebra*

Recall that the objects of the slice ∞-category $\mathcal{S}_{/I}$ are maps of ∞-groupoids with codomain I. For the terminal object $*$, we have $\mathcal{S}_{/*} \simeq \mathcal{S}$, as in the slice category in ordinary category theory. For every ∞-groupoid I, we have the following fundamental equivalence (follows from [46, Theorem 2.2.1.2]):

$$\mathcal{S}_{/I} \simeq \mathrm{Fun}(I, \mathcal{S})$$

which takes $X \to I$ to the functor sending i to the fibre X_i. Pullback along a morphism $f \colon J \to I$, defines an functor $f^* \colon \mathcal{S}_{/I} \to \mathcal{S}_{/J}$. This functor is right adjoint to the functor $f_! \colon \mathcal{S}_{/J} \to \mathcal{S}_{/I}$ given by post-composing with f.

A *span* is a pair of ∞-groupoid maps with common domain $I \xleftarrow{p} M \xrightarrow{q} J$. It induces a functor between the slices by pullback and post-composition

$$\mathcal{S}_{/I} \xrightarrow{p^*} \mathcal{S}_{/M} \xrightarrow{q_!} \mathcal{S}_{/J}.$$

A functor is *linear* if it is homotopy equivalent to a functor induced by a span. The following Beck-Chevalley rule holds for ∞-groupoids: for any pullback square

$$\begin{tikzcd} J \arrow[r, "f"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & I \arrow[d, "q"] \\ V \arrow[r, "g"'] & U, \end{tikzcd}$$

the functors $p_! f^*, g^* q_! \colon \mathcal{S}_{/I} \to \mathcal{S}_{/V}$ are naturally homotopy equivalent (see [31] for the technical details regarding coherence of these equivalences). By the Beck-Chevalley rule, the composition of two linear functor is linear.

We denote by ***LIN*** the symmetric monoidal ∞-category who objects are slice ∞-categories $\mathcal{S}_{/I}$ and morphisms are linear functors, with the tensor product induced by the cartesian product:

$$\mathcal{S}_{/I} \otimes \mathcal{S}_{/J} := \mathcal{S}_{/I \times J},$$

with neutral object $\mathcal{S} \simeq \mathcal{S}_{/1}$.

The ∞-category $\mathcal{S}_{/I}$ plays the role of the vector space with basis I. The presheaf category $\mathcal{S}^I$ can be considered the linear dual of the slice category $\mathcal{S}_{/I}$ since

$$\boldsymbol{LIN}(\mathcal{S}_{/I}, \mathcal{S}) \simeq \mathcal{S}_{/I} \simeq \mathcal{S}^I.$$

A span $I \leftarrow M \rightarrow J$ defines both a linear functor $\mathcal{S}_{/I} \rightarrow \mathcal{S}_{/J}$ and the dual linear functor $\mathcal{S}^J \rightarrow \mathcal{S}^I$.

For an extended treatment of linear functors and homotopy linear algebra, we refer to [24].

1.2.4 *Cardinality*

An ∞-groupoid X is *locally finite* if at each base point x the homotopy groups $\pi_i(X, x)$ are finite for $i \geqslant 1$ and are trivial for i sufficiently large. It is called *finite* if furthermore it has only finitely many components. We denote by $\mathcal{F}$ (following the notation of [26]) the ∞-category of finite ∞-groupoids. A map is *finite* if each fibre is finite. A pullback of any homotopy finite map is again finite. A span $I \xleftarrow{p} M \xrightarrow{q} J$ and the corresponding linear functor $\mathcal{S}_{/I} \rightarrow \mathcal{S}_{/J}$ are *finite* if the map p is finite.

Proposition 1.2.3 ([24, proposition 4.3])**.** *Let* I, J, M *be locally finite ∞-groupoids and* $I \xleftarrow{p} M \xrightarrow{q} J$ *a finite span. Then the induced finite linear functor* $\mathcal{S}_{/I} \rightarrow \mathcal{S}_{/J}$ *restricts to* $\mathcal{F}_{/I} \rightarrow \mathcal{F}_{/J}$.

The *cardinality* [7] of a finite ∞-groupoid X is the alternating product of the cardinalities of the homotopy groups

$$|X| = \sum_{x \in \pi_0(X)} \prod_{k=1}^{\infty} |\pi_k(X, x)|^{(-1)^k}.$$

For a locally finite ∞-groupoid S, there is a notion of cardinality $|{-}|\colon \mathcal{F}_{/S} \rightarrow \mathbb{Q}_{\pi_0 S}$ sending a basis element $\ulcorner s \urcorner$ to the basis element $\delta_s = |\ulcorner s \urcorner|$.

1.3 *Decomposition spaces*

1.3.1 *Segal spaces and decomposition spaces*

We consider the functor ∞-category

$$\mathrm{Fun}\,(\boldsymbol{\Delta}^{\mathrm{op}}, \mathcal{S})$$

whose objects are *simplicial ∞-groupoids*, that is functors from the ∞-category $\boldsymbol{\Delta}^{\mathrm{op}}$ to the ∞-category $\mathcal{S}$.

A simplicial ∞-groupoid X is called a *Segal space* if the following squares are pullbacks, for all $n > 0$:

$$\begin{tikzcd} X_{n+1} \arrow[r, "d_0"] \arrow[d, "d_{n+1}"'] \arrow[dr, phantom, "\lrcorner", very near start] & X_n \arrow[d, "d_n"] \\ X_n \arrow[r, "d_0"'] & X_{n-1}. \end{tikzcd}$$

The simplex category $\mathbf{\Delta}$ has an active-inert factorisation system. A morphism $[m] \to [n]$ is *active* (also called *generic*) if it preserves endpoints: $g(0) = 0$, $g(m) = n$. A morphism is *inert* (also called *free*) if it is distance preserving: $f(i+1) = f(i) + 1$, for $0 \leqslant i \leqslant m-1$. The active maps are generated by the codegeneracy maps and the inner coface maps, and the inert maps are generated by the outer coface maps $d^\perp := d^0$ and $d^\top := d^n$.

A *decomposition space* $X\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{S}$ is a simplicial ∞-groupoid such that the image of any pushout diagram in $\mathbf{\Delta}$ of an active map g along an inert map f is a pullback of ∞-groupoids. It is enough to check that the following squares are pullbacks, where $0 \leqslant k \leqslant n$:

$$\begin{tikzcd}
X_{n+1} \arrow[r, "s_{k+1}"] \arrow[d, "d_\perp"'] \arrow[dr, phantom, "\lrcorner", very near start] & X_{n+2} \arrow[d, "d_\perp"] \\
X_n \arrow[r, "s_k"'] & X_{n+1},
\end{tikzcd}
\qquad
\begin{tikzcd}
X_{n+2} \arrow[d, "d_\perp"] & X_{n+3} \arrow[l, "d_{k+2}"'] \arrow[d, "d_\perp"] \arrow[dl, phantom, "\llcorner", very near start] \\
X_{n+1} & X_{n+2}, \arrow[l, "d_{k+1}"]
\end{tikzcd}$$

$$\begin{tikzcd}
X_{n+1} \arrow[r, "s_k"] \arrow[d, "d_\top"'] \arrow[dr, phantom, "\lrcorner", very near start] & X_{n+2} \arrow[d, "d_\top"] \\
X_n \arrow[r, "s_k"'] & X_{n+1},
\end{tikzcd}
\qquad
\begin{tikzcd}
X_{n+2} \arrow[d, "d_\top"] & X_{n+3} \arrow[l, "d_{k+1}"'] \arrow[d, "d_\top"] \arrow[dl, phantom, "\llcorner", very near start] \\
X_{n+1} & X_{n+2}. \arrow[l, "d_{k+1}"]
\end{tikzcd}$$

The notion of decomposition space was introduced by Gálvez-Carrillo, Kock, and Tonks [25], and independently by Dyckerhoff and Kapranov [20] under the name unital 2-Segal space. The equivalence of the two notions follows from the pullback formulation of 2-Segal spaces given in Proposition 2.3.2 of [20]. It is precisely the condition required to obtain a counital coassociative comultiplication on $\mathcal{S}_{/X_1}$, see also [49] for the exact role played by the decomposition-space condition. Since the motivation in the present paper comes from combinatorics, we follow the terminology of [25]; for a survey motivated by combinatorics, see [29].

Proposition 1.3.1 ([20, Proposition 2.3.3], [25, Proposition 3.5]). *Every Segal space is a decomposition space.*

There are plenty of examples of decomposition spaces which are not Segal, e.g. Schmitt's Hopf algebra of graphs, which is a running example in [29].

1.3.2 *Incidence coalgebras and culf functors*

For any decomposition space X, we get an incidence coalgebra [20], [25]. The span $X_1 \xleftarrow{d_1} X_2 \xrightarrow{(d_2,d_0)} X_1 \times X_1$ defines a linear functor, the *comultiplication*:

$$\begin{aligned}
\Delta\colon \mathcal{S}_{/X_1} &\to \mathcal{S}_{/X_1 \times X_1} \\
(T \xrightarrow{t} X_1) &\mapsto (d_2, d_0)_! \circ d_1^*(t).
\end{aligned}$$

The span $X_1 \xleftarrow{s_0} X_0 \xrightarrow{z} 1$ defines a linear functor, the *counit*:

$$\delta\colon \mathcal{S}_{/X_1} \to \mathcal{S}$$

$$(T \xrightarrow{t} X_1) \mapsto z_! \circ s_0^*(t).$$

The up-to-coherent-homotopy coassociativity follows from the decomposition space axioms, see [25, §5 and 7] or [49, §4.3] for a proof. We obtain a coalgebra $(\mathcal{S}_{/X_1}, \Delta, \delta)$ called the *incidence coalgebra*.

The incidence coalgebra associated to a decomposition space X is a comonoid object in the symmetric monoidal ∞-category ***LIN***.

If X is a decomposition space, the coalgebra structure on $\mathcal{S}_{/X_1}$ therefore induces an algebra structure on $\mathcal{S}^{X_1}$. In details, the *convolution product* of two linear functors $F, G \colon \mathcal{S}_{/X_1} \to \mathcal{S}$, given by the spans $X_1 \leftarrow M \to 1$ and $X_1 \leftarrow N \to 1$, is the composite of their tensor product $F \otimes G$ with the comultiplication:

$$F * G \colon \mathcal{S}_{/X_1} \xrightarrow{\Delta} \mathcal{S}_{/X_1} \otimes \mathcal{S}_{/X_1} \xrightarrow{F \otimes G} \mathcal{S} \otimes \mathcal{S} \simeq \mathcal{S},$$

where the tensor product $F \otimes G$ is given by the span $X_1 \times X_1 \leftarrow M \times N \to 1$. The neutral element for convolution is

$$\delta \colon \mathcal{S}_{/X_1} \to \mathcal{S}$$

defined by the span $X_1 \xleftarrow{s_0} X_0 \to 1$.

A map $f \colon X \to Y$ of simplicial spaces is *cartesian* on an arrow $[n] \to [k]$ in $\boldsymbol{\Delta}$ if the naturality square for F with respect to this arrow is a pullback. It is called a *right fibration* if it is cartesian on $d_\perp$ and on all active maps, and is called a *left fibration* if it is cartesian on $d_\top$ and on all active maps.

A simplicial map $f \colon X \to Y$ is *conservative* if it is cartesian with respect to codegeneracy maps

$$\begin{tikzcd} X_n \arrow[r, "s_i"] \arrow[d, "f_n"'] \arrow[dr, phantom, "\lrcorner", very near start] & X_{n+1} \arrow[d, "f_{n+1}"] \\ Y_n \arrow[r, "s_i"'] & Y_{n+1} \end{tikzcd}, \qquad 0 \leqslant i \leqslant n.$$

It is *ulf* (unique lifting of factorisations) if it is cartesian with respect to inner coface maps

$$\begin{tikzcd} X_{n+1} \arrow[d, "f_{n+1}"'] \arrow[dr, phantom, "\lrcorner", very near start] & X_{n+2} \arrow[l, "d_{i+1}"'] \arrow[d, "f_{n+2}"] \\ Y_{n+1} & Y_{n+2} \arrow[l, "d_{i+1}"'] \end{tikzcd}, \qquad 0 \leqslant i \leqslant n.$$

We write *culf* for conservative and ulf, that is cartesian on all active maps. The culf functors induce coalgebra homomorphisms between the incidence algebras. They play an essential role in [25] and [26] as a natural notion of morphism between decomposition spaces, but the present work deals also with general simplicial maps.

Proposition 1.3.2 ([25, Lemma 4.6]). *If X is a decomposition space and $f \colon Y \to X$ is a culf map, then also Y is a decomposition space.*

1.3.3 *Finiteness and cardinality*

A decomposition space X is *locally finite* if X_1 is locally finite and both s_0 and d_1 are finite maps [26, §7.4]. A decomposition space is *locally discrete* if the fibres of s_0 and d_1 are discrete groupoids [29, §1.4].

For any locally finite decomposition space X, we can take the cardinality of the linear functors $\delta\colon \mathcal{F}_{/X_1} \to \mathcal{F}$ and $\Delta\colon \mathcal{F}_{/X_1} \to \mathcal{F}_{/X_1 \times X_1}$ to obtain a coalgebra structure

$$\mathbb{Q}_{\pi_0 X_1} \xrightarrow{|\delta|} \mathbb{Q}$$

$$\mathbb{Q}_{\pi_0 X_1} \xrightarrow{|\Delta|} \mathbb{Q}_{\pi_0 X_1} \otimes \mathbb{Q}_{\pi_0 X_1}$$

called the *numerical incidence coalgebra* of X, see [26, §7.7].

1.3.4 *Completeness and Möbius condition*

A decomposition space is called *complete* if $s_0\colon X_0 \to X_1$ is a monomorphism [26, §2]. Since s_0 is a monomorphism, we can identify X_0 with a ∞-subgroupoid of X_1. We denote X_a its complement: $X_1 \simeq X_0 + X_a$. More generally, recall the word notation from [26]: consider the alphabet with three letters $\{0, 1, a\}$; 0 indicates degenerate edges $s_0(x) \in X_1$, a denotes edges specified to be nondegenerate, and 1 denotes unspecified edges. For w a word of length n in this alphabet, define

$$X^w = \prod_{i \in w} X_i \subset (X_1)^n.$$

This inclusion is full since $X_a \subset X_1$ is full by completeness.

Denote by X_w the ∞-groupoid of n-simplices whose principal edges have the types indicated in the word w, that is the full subgroupoid of X_n given by the following pullback

$$\begin{tikzcd} X_w \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_n \arrow[d] \\ X^w \arrow[r] & (X_1)^n. \end{tikzcd}$$

We define $\vec{X}_n = X_{a\ldots a} \subset X_n$ to be the full subgroupoid of simplices with all principal edges nondegenerate. It is the complement of the union of the essential images of the degeneracy maps $s_i\colon X_{n-1} \to X_n$, that is

$$\vec{X}_n = X_n \setminus \bigcup_{i=0}^{n-1} \mathrm{Im}(s_i).$$

By definition $\vec{X}_0 = X_0$.

For a complete decomposition space, the spans $X_1 \xleftarrow{d_1^{n-1}} \vec{X}_n \to 1$ define linear functors, the *Phi functors*

$$\Phi_n\colon \mathcal{S}_{/X_1} \to \mathcal{S}.$$

We also put $\Phi_{\text{even}} := \sum_{n \text{ even}} \Phi_n$, and $\Phi_{\text{odd}} := \sum_{n \text{ odd}} \Phi_n$.

The incidence algebra of a decomposition space contains the *zeta functor*

$$\zeta\colon \mathcal{S}_{/X_1} \to \mathcal{S}$$

given by the span $X_1 \xleftarrow{=} X_1 \to 1$.

Theorem 1.3.3 ([26, Theorem 3.8]). *For a complete decomposition space, the following Möbius inversion holds:*

$$\zeta * \Phi_{even} \simeq \delta + \zeta * \Phi_{odd}$$
$$\simeq \Phi_{even} * \zeta \simeq \delta + \Phi_{odd} * \zeta.$$

This is however not enough to allow the Möbius inversion formula to descend to the vector space level. A complete decomposition space X is of *locally finite length* [26] if every edge $f \in X_1$ has finite length, that is, the fibres $(\vec{X}_n)_f$ of $d_1^{(n)}\colon \vec{X}_n \to X_1$ over f are empty for n sufficiently large.

Proposition 1.3.4 ([28]). *If X is a decomposition space of locally finite length (resp. locally discrete) and $f\colon Y \to X$ is a culf map, then also Y is a decomposition space of locally finite length (resp. locally discrete). This is also the case for locally finite, but we must check moreover that Y_1 is locally finite.*

A *Möbius decomposition space* [26] is a decomposition space which is locally finite and of locally finite length; the fibre $(\vec{X}_n)_f$ is finite (eventually empty). It follows that the map

$$\sum_n d_1^{n-1}\colon \sum_n \vec{X}_n \to X_1$$

is finite; by proposition 1.2.3, the Phi functors descend to

$$\Phi_n\colon \mathcal{F}_{/X_1} \to \mathcal{F}$$

and we can take cardinality to obtain functions $|\Phi_n|\colon \mathbb{Q}_{\pi_0 X_1} \to \mathbb{Q}$.

Finally, we can take cardinality of the abstract Möbius inversion formula of 1.3.3, see [26] for a complete exposition.

Theorem 1.3.5 ([26, Theorem 8.9]). *If X is a Möbius decomposition space, then the cardinality of the zeta functor, $|\zeta|\colon \mathbb{Q}_{\pi_0 X_1} \to \mathbb{Q}$, is convolution invertible with inverse $|\mu| := |\Phi_{even}| - |\Phi_{odd}|$:*

$$|\zeta| * |\mu| = |\delta| = |\mu| * |\zeta|.$$

1.3.5 *Monoidal decomposition spaces*

A *monoidal structure* [25, §9] on a decomposition space X is given by the data of simplicial maps $\eta : 1 \to X$ and $\otimes : X \times X \to X$ required to be culf, and satisfying the standard associative and unital laws of monoidal structures. This monoidal structure induces a monoid structure on the incidence coalgebra

$\mathcal{S}_{/X_1}$ in the ∞-category of coalgebras, and hence a bialgebra structure. In combinatorics, the monoidal structure is often given by disjoint union, and in particular the resulting bialgebra structure is then commutative.

A monoidal decomposition space (or its incidence bialgebra) is called *connected* when X_0 is contractible (that is, $X_0 \simeq 1$). When X is Möbius and connected, its cardinality is a connected bialgebra, and therefore a Hopf algebra [58]. However, many important incidence bialgebras are not connected.

2

Incidence bicomodules, Möbius inversion, and a Rota formula for infinity adjunctions

2.1 *Bicomodules*

2.1.1 *Comodules*

The theory of modules in the context of decomposition spaces has been developed by Walde [59], and independently by Young [60], both in the context of Hall algebras. They call them relative 2-Segal spaces. Here we give a conceptual way to reformulate their definitions using linear functors.

Given a map between two simplicial ∞-groupoids $f : C \to X$, the span

$$C_0 \xleftarrow{d_1} C_1 \xrightarrow{(f_1,d_0)} X_1 \times C_0$$

defines a linear functor $\gamma_l : \mathcal{S}_{/C_0} \to \mathcal{S}_{/X_1} \otimes \mathcal{S}_{/C_0}$, and the span

$$C_0 \xleftarrow{d_0} C_1 \xrightarrow{(d_1,f_1)} C_0 \times X_1$$

defines a linear functor $\gamma_r : \mathcal{S}_{/C_0} \to \mathcal{S}_{/C_0} \otimes \mathcal{S}_{/X_1}$.

Proposition 2.1.1. *Let* $f : C \to X$ *be a map between two simplicial* ∞*-groupoids. Suppose moreover that* C *is Segal,* X *is a decomposition space and the map* $f : C \to X$ *is culf, then the span*

$$C_0 \xleftarrow{d_1} C_1 \xrightarrow{(f_1,d_0)} X_1 \times C_0$$

induces on the slice ∞*-category* $\mathcal{S}_{/C_0}$ *the structure of a left* $\mathcal{S}_{/X_1}$*-comodule (at the* π_0 *level), and the span*

$$C_0 \xleftarrow{d_0} C_1 \xrightarrow{(d_1,f_1)} C_0 \times X_1.$$

induces on $\mathcal{S}_{/C_0}$ *the structure of a right* $\mathcal{S}_{/X_1}$*-comodule (at the* π_0 *level).*

The data needed to obtain a comodule is called a *comodule configuration*, that is a culf map from a Segal space to a decomposition space.

Remark 2.1.2. The relevance of the Segal condition on C and the culf condition on f can be explained individually as follows. It is standard that for a category C, the coalgebra of arrows C_1 coacts on C_0: the coaction (from the right) is given by $b \mapsto \sum_{f:a\to b} a \otimes f$. Coassociativity of this coaction is equivalent to the Segal condition. Now a culf map $C \to X$ defines a coalgebra homomorphism, and in this way, also X_1 coacts on C_0, by “corestriction of coscalars”.

Remark 2.1.3. The proposition is stated only at the π_0-level. This means that we establish only the comodule structure up to homotopy, but do not establish the *coherence* of this up-to-homotopy structure. A stronger result, a partial coherence result, is given by [59] and [60], who establish the coherence at the 1-truncated level (rather than the 0-truncated level established here). It is most likely that full coherence can be established by exploiting the techniques employed by [25] and [49]. While only a small bit of the axioms are used to establish the proposition as stated, the full decomposition-space axioms and the culf condition are expected to be required for the fully coherent result, and this is why these conditions have been included in the definition of comodule configuration.

Proof. We want to prove that the map γ_l is a left $\mathcal{S}_{/X_1}$-coaction. The desired diagram, commutative up to homotopy

$$\begin{tikzcd}
\mathcal{S}_{/C_0} \arrow[r, "\gamma_l"] \arrow[d, "\gamma_l"'] & \mathcal{S}_{/X_1\times C_0} \arrow[d, "\mathrm{Id}\otimes\gamma_l"] \\
\mathcal{S}_{/X_1\times C_0} \arrow[r, "\Delta\otimes\mathrm{Id}"'] & \mathcal{S}_{/X_1\times X_1\times C_0}
\end{tikzcd}$$

is induced by the solid spans in the diagram

$$\begin{tikzcd}
C_0 & C_1 \arrow[l, "d_1"'] \arrow[r, "{(f_1,d_0)}"] & X_1\times C_0 \\
C_1 \arrow[u, "d_1"] \arrow[d, "{(f_1,d_0)}"'] & C_2 \arrow[l, dashed, "d_1"] \arrow[r, dashed, "{(d_2f_1,d_0)}"'] \arrow[u, dashed, "d_2"] \arrow[d, dashed, "{(f_1,d_0d_0)}"] \arrow[ur, phantom, very near start, "\urcorner"] \arrow[dl, phantom, very near start, "\llcorner"] & X_1\times C_1 \arrow[u, "\mathrm{Id}\otimes d_1"'] \arrow[d, "\mathrm{Id}\otimes(f_1,d_0)"] \\
X_1\times C_0 & X_2\times C_0 \arrow[l, "d_1\otimes\mathrm{Id}"] \arrow[r, "{(d_2,d_0)\otimes\mathrm{Id}}"'] & X_1\times X_1\times C_0 .
\end{tikzcd}$$

The coassociativity (at the π_0 level) will follow from Beck-Chevalley equivalences if we have the two pullbacks indicated in the diagram. The upper right-hand square is a pullback if and only if its composite with the second projection is a pullback. This composite outer square is a pullback because C satisfies the Segal condition. Similarly, the lower left-hand square is a pullback if its composite with the first projection is a pullback. This composite outer square is a pullback because $f : C \to X$ is culf. □

Example 2.1.4 (Décalage [32]). Given a simplicial space X, the *lower décalage* $\mathrm{Dec}_\perp(X)$ is the simplicial space obtained by deleting X_0, all d_0 face maps and s_0 degeneracy maps. The original d_0 maps induce a simplicial map

$d_\perp : \mathrm{Dec}_\perp(X) \to X$, called the décalage map. Similarly, the *upper décalage* $\mathrm{Dec}_\top(X)$ is the simplicial space obtained by deleting X_0, all last face maps $d_\top$ and last degeneracy maps $s_\top$. The original $d_\top$ maps induce a simplicial map $d_\top : \mathrm{Dec}_\top X \to X$.

It is well known that $\mathrm{Dec}_\perp(X)$ is a Segal space and the décalage map is culf (see [25, Proposition 4.9]). Hence we have a comodule configuration. The resulting comodule is the incidence coalgebra of X as a (right) comodule over itself.

For categories, given a functor $f : C \to D$, define the *mapping cylinder* (or *collage* in [37]) M_f to be the category where objects are either objects of C or objects of D and

$$\mathrm{Hom}_{M_f}(x,y) = \begin{cases} \mathrm{Hom}_C(x,y) & \text{if } x,y \in C, \\ \mathrm{Hom}_D(x,y) & \text{if } x,y \in D, \\ \mathrm{Hom}_D(f(x),y) & \text{if } x \in C, y \in D, \\ \varnothing & \text{else .} \end{cases}$$

There exists a unique $p : M_f \to \Delta^1$ such that $p^{-1}(0) = C$ and $p^{-1}(1) = D$. This is moreover a cocartesian fibration, the cocartesian lift for $x \in C$ being given by $\mathrm{Id}_{f(x)} \in \mathrm{Map}_{M_f}(x, f(x))$. The shape of a comodule configuration is that of $(M_{\mathrm{id}})^{\mathrm{op}}$, where M_{id} is the mapping cylinder of the identity of $\boldsymbol{\Delta}$. In other words, a comodule configuration is a functor from $(M_{\mathrm{id}})^{\mathrm{op}}$ to $\mathcal{S}$ (satisfying certain conditions).

Let $\boldsymbol{\Delta}_{\mathrm{bot}}$ be the simplex category of finite linear orders with a specified bottom element, and bottom-preserving monotone maps. Consider the mapping cylinder M_j of the functor $j : \boldsymbol{\Delta} \to \boldsymbol{\Delta}_{\mathrm{bot}}$ freely adding a bottom element. Presheaves on M_j are diagrams of the following shape.

$$\begin{tikzcd}
 & X_0 \arrow[r, shift left=2] \arrow[r, shift right=2] & X_1 \arrow[l] \arrow[r, shift left=3] \arrow[r] \arrow[r, shift right=3] & X_2 \arrow[l, shift left=1.5] \arrow[l, shift right=1.5] & \cdots \\
C_{-1} \arrow[r, bend right, "s_{-1}"'] & C_0 \arrow[l, "u"'] \arrow[u, "v"'] \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[r, bend right, "s_{-1}"'] & C_1 \arrow[l] \arrow[u, "v"'] \arrow[r, shift left=3] \arrow[r] \arrow[r, shift right=3] \arrow[r, bend right, "s_{-1}"'] & C_2 \arrow[l, shift left=1.5] \arrow[l, shift right=1.5] \arrow[u, "v"'] & \cdots
\end{tikzcd}$$

This is the shape of what we call a *right pointed comodule configuration*: it is a comodule configuration $C \to X$ such that the Segal space C is augmented, and with new bottom sections $s_{-1} : C_{n-1} \to C_n$. The importance of the pointing (the extra bottom degeneracy maps) is that it makes possible to formulate the notion of completeness and the condition locally finite length, see 2.3.3 below; it guarantees the existence of a filtration on the associated comodule (see [26, §6] for a similar argument), which is of independent interest.

Example 2.1.5. The comodule configuration obtained from the lower décalage of a decomposition space X is also right pointed, the augmentation map is given by $d_1 : X_1 \to X_0$, and the extra bottom sections by s_0.

2.1.2 *Augmented bisimplicial infinity-groupoids*

We shall establish conditions under which left and right comodule structures define a bicomodule. The main objects of interest are augmented bisimplicial ∞-groupoids subject to conditions, which are formulated in terms of pullbacks. We consider the functor ∞-category

$$\mathrm{Fun}\,(\boldsymbol{\Delta}^{\mathrm{op}} \times \boldsymbol{\Delta}^{\mathrm{op}}, \mathcal{S})$$

whose objects are *bisimplicial ∞-groupoids*, that is functors from the ∞-category $\boldsymbol{\Delta}^{\mathrm{op}} \times \boldsymbol{\Delta}^{\mathrm{op}}$ to the ∞-category $\mathcal{S}$.

A *double Segal space* is a bisimplicial ∞-groupoid satisfying the Segal condition for each restriction $\boldsymbol{\Delta}^{\mathrm{op}} \times \{[n]\} \to \mathcal{S}$ (the columns) and $\{[n]\} \times \boldsymbol{\Delta}^{\mathrm{op}} \to \mathcal{S}$ (the rows).

Let $\boldsymbol{\Delta}_+$ be the augmented simplex category of all finite ordinals and order-preserving maps. An *augmented* bisimplicial ∞-groupoid B has in addition ∞-groupoids $B_{i,-1}$ and $B_{-1,i}$ of (-1)-simplices. We consider the functor ∞-category

$$\mathrm{Fun}\,(\boldsymbol{\Delta}_+^{\mathrm{op}} \times \boldsymbol{\Delta}_+^{\mathrm{op}} \backslash \{(-1,-1)\}, \mathcal{S})$$

whose objects are *augmented bisimplicial ∞-groupoids*.

Remark 2.1.6. The shape of an augmented bisimplicial ∞-groupoid is $(\boldsymbol{\Delta}_{/\Delta^1})^{\mathrm{op}}$. We denote $[i,j]$ the object given by the map $\Delta^{i+1+j} \to \Delta^1$ sending the $i+1$ first vertices to 0 and the others to 1. We allow i or j to be equal to -1 but not both. Maps $[i,j] \to [k,l]$ are given by the inclusions respecting the horizontal map. For example, the object $[2,1]$ can be drawn as follows

$$\begin{tikzcd}
 & \cdot \\
\cdot \arrow[r] & \cdot \arrow[u] \\
\cdot \arrow[u] & \\
\cdot \arrow[u] &
\end{tikzcd}$$

where the horizontal maps lie over the map in Δ^1.

We can draw $[i,j]$ as a column of i+1 black dots followed by j+1 white dots. Maps send black dots to black dots and white dots to white dots, without crossing.

We use the following notation for an augmented bisimplicial ∞-groupoid. We denote $d_k : B_{i,j} \to B_{i,j-1}$ and $e_l : B_{i,j} \to B_{i-1,j}$ the face maps, and $s_k : B_{i,j-1} \to B_{i,j}$ and $t_l : B_{i-1,j} \to B_{i,j}$ the degeneracy maps; u and v are the augmentation maps.

$$
\begin{tikzcd}
 & B_{-1,0} \arrow[r, shift left=2, leftarrow, "d_1"] \arrow[r, "s_0"] \arrow[r, shift right=2, leftarrow, "d_0"'] & B_{-1,1} \arrow[r, shift left=3, leftarrow, "d_2"] \arrow[r, shift left=1, "s_1"] \arrow[r, leftarrow, "d_1"] \arrow[r, shift right=1, "s_0"] \arrow[r, shift right=3, leftarrow, "d_0"'] & B_{-1,2} & \cdots \\
 & & & & \\
B_{0,-1} & B_{0,0} \arrow[l, "u"'] \arrow[uu, "v"] \arrow[r, shift left=2, leftarrow, "d_1"] \arrow[r, "s_0"] \arrow[r, shift right=2, leftarrow, "d_0"'] & B_{0,1} \arrow[uu, "v"] \arrow[r, shift left=3, leftarrow, "d_2"] \arrow[r, shift left=1, "s_1"] \arrow[r, leftarrow, "d_1"] \arrow[r, shift right=1, "s_0"] \arrow[r, shift right=3, leftarrow, "d_0"'] & B_{0,2} \arrow[uu, "v"] & \cdots \\
B_{1,-1} \arrow[u, shift left=2, "e_0"] \arrow[u, shift right=2, "e_1"'] \arrow[u, leftarrow, "t_0"] & B_{1,0} \arrow[l, "u"'] \arrow[u, shift left=2, "e_0"] \arrow[u, shift right=2, "e_1"'] \arrow[u, leftarrow, "t_0"] \arrow[r, shift left=2, leftarrow, "d_1"] \arrow[r, "s_0"] \arrow[r, shift right=2, leftarrow, "d_0"'] & B_{1,1} \arrow[u, shift left=2, "e_0"] \arrow[u, shift right=2, "e_1"'] \arrow[u, leftarrow, "t_0"] \arrow[r, shift left=3, leftarrow, "d_2"] \arrow[r, shift left=1, "s_1"] \arrow[r, leftarrow, "d_1"] \arrow[r, shift right=1, "s_0"] \arrow[r, shift right=3, leftarrow, "d_0"'] & B_{1,2} \arrow[u, shift left=2, "e_0"] \arrow[u, shift right=2, "e_1"'] \arrow[u, leftarrow, "t_0"] & \cdots \\
\vdots & \vdots & \vdots & \vdots &
\end{tikzcd}
$$

An augmented double Segal space satisfies that rows and columns are Segal. If we suppose that augmentations are culf and $B_{\bullet,-1}$ and $B_{-1,\bullet}$ are decomposition spaces, we can apply Proposition 2.1.1 to obtain comodules: the span

$$B_{0,0} \xleftarrow{e_1} B_{1,0} \xrightarrow{(u,e_0)} B_{1,-1} \times B_{0,0}$$

induces on $\mathcal{S}_{/B_{0,0}}$ the structure of a left comodule over $\mathcal{S}_{/B_{1,-1}}$, and the span

$$B_{0,0} \xleftarrow{d_0} B_{0,1} \xrightarrow{(d_1,v)} B_{0,0} \times B_{-1,1}$$

induces on $\mathcal{S}_{/B_{0,0}}$ the structure of a right comodule over $\mathcal{S}_{/B_{-1,1}}$.

2.1.3 *Stability*

We say a bisimplicial ∞-groupoid is *stable* if the following squares are pullbacks:

$$
\begin{tikzcd}
B_{i-1,j-1} & B_{i-1,j} \arrow[l, "d_k"'] \\
B_{i,j-1} \arrow[u, "e_l"] & B_{i,j}, \arrow[l, "d_k"] \arrow[u, "e_l"'] \arrow[ul, phantom, "\ulcorner" very near start]
\end{tikzcd}
\qquad
\begin{tikzcd}
B_{i-1,j-1} \arrow[r, "s_k"] & B_{i-1,j} \\
B_{i,j-1} \arrow[u, "e_l"] \arrow[r, "s_k"'] \arrow[ur, phantom, "\urcorner" very near start] & B_{i,j}, \arrow[u, "e_l"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
B_{i-1,j-1} \arrow[d, "t_l"] & B_{i-1,j} \arrow[l, "d_k"'] \arrow[d, "t_l"] \arrow[dl, phantom, "\llcorner" very near start] \\
B_{i,j-1} & B_{i,j}, \arrow[l, "d_k"]
\end{tikzcd}
\qquad
\begin{tikzcd}
B_{i-1,j-1} \arrow[r, "s_k"] \arrow[d, "t_l"] \arrow[dr, phantom, "\lrcorner" very near start] & B_{i-1,j} \arrow[d, "t_l"] \\
B_{i,j-1} \arrow[r, "s_k"'] & B_{i,j},
\end{tikzcd}
$$

for all values of the indices except for $d_\perp$ along $e_\top$ and $d_\top$ along $e_\perp$.

Remark 2.1.7. A bisimplicial ∞-groupoid is stable if it satisfies all the following properties:

- $s_k : B_{i,j-1} \to B_{i,j}$ is a cartesian natural transformation, for all $0 \leqslant k \leqslant j-1$;
- d_k, $k \neq \top, \perp$, is a cartesian natural transformation;

- $d_\top$ is a left fibration;
- $d_\perp$ is a right fibration.

Remark 2.1.8. Bergner, Osorno, Ozornova, Rovelli, and Scheimbauer introduced the notion of stable double category (bisimplicial *set*) in [11]: they define a double category to be stable if every square is uniquely determined by its span of source morphisms and, independently by its cospan of target morphisms. The present definition is a categorical reformulation of their notion suitable for ∞-groupoids. The motivation for the terminology is the following example. Let C be a stable ∞-category (see [47, Chapter 1]). Define a double Segal space B where $B_{0,0}$ is the ∞-groupoid of objects of C, where $B_{0,1}$ is the ∞-groupoid of arrows of C (as in the fat nerve), and $B_{1,1}$ is the ∞-groupoid of pullback squares (equivalently, pushout squares). More generally, $B_{m,n}$ is the ∞-groupoid of $(\Delta^m \times \Delta^n)$-diagrams in C for which all the rectangles are pullbacks (and hence pushouts). This is a stable bisimplicial ∞-groupoid (which of course is a double Segal space). This is almost by definition: since we only took pullback and pushout squares, they are determined by their sources by pushout or their targets by pullback, in the sense of our definition.

Lemma 2.1.9. *Let B be a double Segal space. Suppose we have the two following pullbacks:*

$$\begin{tikzcd}
B_{0,0} & B_{0,1} \arrow[l, "d_0"'] \\
B_{1,0} \arrow[u, "e_0"] & B_{1,1} \arrow[l, "d_0"] \arrow[u, "e_0"'] \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}
\qquad
\begin{tikzcd}
B_{0,0} & B_{0,1} \arrow[l, "d_1"'] \\
B_{1,0} \arrow[u, "e_1"] & B_{1,1}, \arrow[l, "d_1"] \arrow[u, "e_1"'] \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

then the double Segal space is stable.

Proof. First, the second pullback implies that every square with top face maps $d_\top : B_{i,j+1} \to B_{i,j}$ is a pullback. Indeed, in the cube

$$\begin{tikzcd}
& B_{0,0} & & B_{0,1} \arrow[ll, "d_\top"'] \\
B_{0,1} \arrow[ur, "d_\perp"] & & B_{0,2} \arrow[ll, "d_\top"'] \arrow[ur, "d_\perp"] & \\
& B_{1,0} \arrow[uu, "e_\top"] & & B_{1,1}, \arrow[ll, "d_\top"'] \arrow[uu, "e_\top"'] \\
B_{1,1} \arrow[uu, "e_\top"] \arrow[ur, "d_\perp"] & & B_{1,2} \arrow[ll, "d_\top"] \arrow[uu, "e_\top"] \arrow[ur, "d_\perp"'] &
\end{tikzcd}$$

the top and bottom squares are pullbacks because every row is Segal, and the back square is a pullback by hypothesis. Thus the rectangle consisting of bottom and back is a pullback because bottom and back squares are; front is a pullback because top and rectangle are. By induction, suppose the squares

$$\begin{tikzcd} B_{i-1,j-1} & B_{i-1,j} \arrow[l, "d_\top"'] \\ B_{i,j-1} \arrow[u, "e_\top"] & B_{i,j} \arrow[l, "d_\top"] \arrow[u, "e_\top"'] \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

are pullback, we can form cubes with the top and bottom faces pullbacks thanks to the Segal condition, and the back square is a pullback by hypothesis. This proves that every square involving top face maps are pullbacks. Starting with the first pullback, we prove in the same way that every square involving bottom face maps are pullbacks.

Now we want to prove that the following square is a pullback, for $0 < k < i$,

$$\begin{tikzcd} B_{i-1,j-1} & B_{i-1,j} \arrow[l, "d_\top"'] \\ B_{i,j-1} \arrow[u, "e_k"] & B_{i,j}. \arrow[l, "d_\top"] \arrow[u, "e_k"'] \end{tikzcd}$$

If $k = i - r$, we postcompose vertically with $(e_\top)^r$:

$$\begin{tikzcd} B_{k-1,j-1} & B_{k-1,j} \arrow[l, "d_\top"'] \\ B_{i-1,j-1} \arrow[u, "(e_\top)^r"] & B_{i-1,j} \arrow[l, "d_\top"] \arrow[u, "(e_\top)^r"'] \\ B_{i,j-1} \arrow[u, "e_k"] & B_{i,j} \arrow[l, "d_\top"] \arrow[u, "e_k"'] \end{tikzcd}$$

Then the vertical composite is equivalent to $e_\top \circ (e_\top)^r$ (by face map identities), so both the rectangle and the upper square are pullbacks by assumption, and therefore by Lemma 1.2.1, the lower square is a pullback too, as required.

We can do the same proof with bottom face maps. We can also replace $d_\top$ in the new previous pullback squares and obtain the remaining pullbacks involving the face maps.

For squares with face and degeneracy maps, we use the following strategy: in the diagram

$$\begin{tikzcd} B_{00} \arrow[r, "s_0"] & B_{01} \arrow[r, "d_\perp"] & B_{00} \\ B_{10} \arrow[u, "e_\perp"] \arrow[r, "s_0"'] & B_{11} \arrow[u, "e_\perp"'] \arrow[r, "d_\perp"'] & B_{10}, \arrow[u, "e_\perp"'] \end{tikzcd}$$

the map s_0 is a section of d_1, then the long edge is an identity. The right-hand square is a pullback (it is one of the two pullback in the hypothesis). Thus the left-hand square is a pullback. We can proceed in the same way for the other degeneracy maps.

There remains the case of squares involving only degeneracy:

$$\begin{tikzcd} B_{ij} \arrow[r, "s_0"] \arrow[d, "t_k"'] & B_{i,j+1} \arrow[d, "t_k"] \\ B_{i+1,j} \arrow[r, "s_0"'] & B_{i+1,j+1}. \end{tikzcd}$$

We again glue on the right a square with face map such that the long edge is an identity and use once again the Lemma 1.2.1. □

2.1.4 *Bicomodules*

Theorem 2.1.10. *Let* B *be an augmented stable double Segal space, and such that the augmentation maps are culf. Suppose moreover* $X := B_{\bullet,-1}$ *and* $Y := B_{-1,\bullet}$ *are decomposition spaces. Then the spans*

$$B_{0,0} \xleftarrow{e_1} B_{1,0} \xrightarrow{(u,e_0)} X_1 \times B_{0,0}$$

and

$$B_{0,0} \xleftarrow{d_0} B_{0,1} \xrightarrow{(d_1,v)} B_{0,0} \times Y_1$$

induce on $\mathcal{S}_{/B_{0,0}}$ *the structure of a bicomodule over* $\mathcal{S}_{/X_1}$ *and* $\mathcal{S}_{/Y_1}$ *(at the* π_0 *level).*

A bisimplicial ∞-groupoid satisfying the conditions of the theorem is called a *bicomodule configuration*.

Remark 2.1.11. In analogy with Remark 2.1.2, the notion of bicomodule configuration can be broken up into steps. First, for any double Segal space B, since the zeroth column $B_{\bullet,0}$ is a Segal space, $\mathcal{S}_{/B_{0,0}}$ is a left comodule over $\mathcal{S}_{/B_{1,0}}$, and similarly $\mathcal{S}_{/B_{0,0}}$ is a right comodule over $\mathcal{S}_{/B_{0,1}}$. It is now the stability of B that expresses the bicomodule condition. From here, a culf augmentation on the left to a decomposition space X induces a coalgebra homomorphism, and a culf augmentation on the right to a decomposition space Y induces another coalgebra homomorphism, and coextension of coscalars along these coalgebra homomorphisms makes $\mathcal{S}_{/B_{0,0}}$ an $\mathcal{S}_{/X_1}$-$\mathcal{S}_{/Y_1}$ bicomodule. This viewpoint might well be useful in the proof of full coherence.

Remark 2.1.12. As for the Proposition 2.1.1, the theorem is stated at the π_0-level. It is most likely that the full coherence can be established using the techniques employed in [25] and [49]. It is expected that all the stability pullbacks are required for the fully coherent result. For the present purposes, we are going to take homotopy cardinality anyway, and for that, coherence is not essential.

Proof. The left and right comodule structures were established in Proposition 2.1.1. The desired homotopy coherent diagram

$$\begin{tikzcd} \mathcal{S}_{/B_{0,0}} \arrow[r, "\gamma_l"] \arrow[d, "\gamma_r"'] & \mathcal{S}_{/B_{1,-1}\times B_{0,0}} \arrow[d, "\mathrm{Id}\otimes\gamma_r"] \\ \mathcal{S}_{/B_{0,0}\times B_{-1,1}} \arrow[r, "\gamma_l\otimes\mathrm{Id}"'] & \mathcal{S}_{/B_{1,-1}\times B_{0,0}\times B_{-1,1}} \end{tikzcd}$$

is induced by the solid spans in the diagram

$$\begin{tikzcd}
B_{0,0} & B_{1,0} \arrow[l, "e_1"'] \arrow[r, "{(u,e_0)}"] & B_{1,-1} \times B_{0,0} \\
B_{0,1} \arrow[u, "d_0"] \arrow[d, "{(d_1,v)}"'] & B_{1,1} \arrow[u, dashed, "d_0"'] \arrow[l, dashed, "e_1"] \arrow[r, dashed, "{(ud_0,e_0)}"'] \arrow[d, dashed, "{(d_1,ve_1)}"] \arrow[ur, phantom, "\urcorner", very near start] \arrow[dl, phantom, "\llcorner", very near start] & B_{1,-1} \times B_{0,1} \arrow[u, "\mathrm{Id} \otimes d_0"'] \arrow[d, "\mathrm{Id} \otimes (d_1,v)"] \\
B_{0,0} \times B_{-1,1} & B_{1,0} \times B_{-1,1} \arrow[l, "e_1 \otimes \mathrm{Id}"] \arrow[r, "{(u,e_0) \otimes \mathrm{Id}}"'] & B_{1,-1} \times B_{0,0} \times B_{-1,1}.
\end{tikzcd}$$

The homotopy commutativity of the squares follows one again from the new augmentation simplicial identities. The upper-right hand square is a pullback if and only if its composite with the second projection is a pullback and, similarly, the lower-left hand square is a pullback if and only if its composite with the first projection is a pullback. These composite outer squares are pullbacks due to the stability condition. $\square$

Example 2.1.13. In analogy with Example 2.1.4, given a decomposition space X, let B be the total decalage of X. (Its zeroth column is $\mathrm{Dec}_\top(X)$ and its zeroth row is $\mathrm{Dec}_\bot(X)$. With its natural augmentation maps, this becomes a bicomodule configuration, realising the coalgebra of X as a bicomodule over itself.

2.2 *Correspondences, fibrations, and adjunctions*

2.2.1 *Decomposition space correspondences*

A *correspondence* is by definition a decomposition space $\mathcal{M}$ with a map to the 1-simplex Δ^1. We consider the slice ∞-category $\boldsymbol{Cat}_{\infty/\Delta^1}$. It contains in particular $\boldsymbol{\Delta}_{/\Delta^1}$, whose objects are $[i,j]$, see Remark 2.1.6. There is now a natural notion of nerve in this context. Given a correspondence $p : \mathcal{M} \to \Delta^1$, the relative nerve $N_{\Delta^1} : \boldsymbol{Cat}_{\infty/\Delta^1} \to \mathrm{Fun}((\boldsymbol{\Delta}_{/\Delta^1})^{\mathrm{op}}, \mathcal{S})$ of p is the augmented bisimplicial ∞-groupoid given by $B_{i,j} := N_{\Delta^1}(p)_{i,j} = \mathrm{Map}_{/\Delta^1}([i,j], p)$, where $[i,j]$ is given by the map $\Delta^{i+1+j} \to \Delta^1$ sending the $i+1$ first vertices to 0 and the others to 1. It is allowed for i or j to be equal to -1 but not both.

From the nerve definition, the following square is a standard mapping-space fibre sequence for slices:

$$\begin{tikzcd}
B_{i,j} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Map}(\Delta^{i+1+j}, \mathcal{M}) \arrow[d, "\text{post } p"] \\
1 \arrow[r, "\ulcorner {[i,j]} \urcorner"'] & \mathrm{Map}(\Delta^{i+1+j}, \Delta^1).
\end{tikzcd}$$

Proposition 2.2.1. *Given a decomposition space correspondence* $p : \mathcal{M} \to \Delta^1$*, the bisimplicial* ∞*-groupoid* B *described above enjoys the following properties:*

1. *it is Segal in both directions;*
2. *it is stable;*
3. *it is augmented;*
4. *these augmentations are culf.*

To prove these properties, we will use the following lemmas.

Lemma 2.2.2. *Given a diagram such that top and bottom are two fibre sequences*

$$\begin{tikzcd}
F \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & B \arrow[d, "q"] \\
F' \arrow[r] & E' \arrow[r] & B'.
\end{tikzcd}$$

If q *is an equivalence, then the left-hand square is a pullback.*

Proof. In the following cube, the front and back squares are pullbacks by assumption; the bottom one is since q is an equivalence.

$$\begin{tikzcd}
F \arrow[rr] \arrow[dr] \arrow[dd] & & E \arrow[dr] \arrow[dd] & \\
& F' \arrow[rr] \arrow[dd] & & E' \arrow[dd] \\
1 \arrow[rr] \arrow[dr] & & B \arrow[dr, "q"] & \\
& 1 \arrow[rr] & & B'.
\end{tikzcd}$$

The rectangle consisting of the back square and the bottom square is a pullback by Lemma 1.2.1, since both squares are pullbacks. Thus the rectangle consisting of the top square and the front one is. Applying one more time Lemma 1.2.1, we conclude the top square is a pullback. ☐

Lemma 2.2.3. *Given a diagram such that horizontal maps form fibre sequences*

$$\begin{tikzcd}
F_4 \arrow[rr] \arrow[dr] \arrow[dd] & & E_4 \arrow[rr] \arrow[dr] \arrow[dd] & & B_4 \arrow[dr] \arrow[dd, "u"] & \\
& F_3 \arrow[rr] \arrow[dd] & & E_3 \arrow[rr] \arrow[dd] & & B_3 \arrow[dd, "v"] \\
F_2 \arrow[rr] \arrow[dr] & & E_2 \arrow[rr] \arrow[dr] & & B_2 \arrow[dr] & \\
& F_1 \arrow[rr] & & E_1 \arrow[rr] & & B_1.
\end{tikzcd}$$

Suppose the vertical middle square (involving $E_i, 1 \leqslant i \leqslant 4$) is a pullback, and suppose u and v are equivalences, then the left vertical square is a pullback.

Proof. By Lemma 2.2.2, since u and v are equivalences, the front and back squares of the left cube of the diagram are pullbacks. We conclude by applying Lemma 1.2.1 twice, as in the proof of Lemma 2.2.2. □

Proof of Proposition 2.2.1. (1) Segal in both directions means: for any i, the squares

$$\begin{tikzcd} B_{n+1,i} \arrow[r, "e_0"] \arrow[d, "e_{n+1}"'] \arrow[dr, phantom, "\lrcorner", very near start] & B_{n,i} \arrow[d, "e_n"] \\ B_{n,i} \arrow[r, "e_0"'] & B_{n-1,i}, \end{tikzcd} \qquad \begin{tikzcd} B_{i,n+1} \arrow[r, "d_0"] \arrow[d, "d_{n+1}"'] \arrow[dr, phantom, "\lrcorner", very near start] & B_{i,n} \arrow[d, "d_n"] \\ B_{i,n} \arrow[r, "d_0"'] & B_{i,n-1}. \end{tikzcd}$$

are pullbacks.

The ∞-groupoids $B_{n,i}$, for $n \geqslant 0$ are also given by the following fibre sequences:

$$\begin{tikzcd} B_{n,i} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Map}(\Delta^{n+1+i}, \mathcal{M}) \arrow[d, "R_{n+1+i}"] \\ 1 \arrow[r, "\ulcorner \mathrm{id} \urcorner"'] & \mathrm{Map}(\Delta^1, \Delta^1), \end{tikzcd}$$

the right-hand map R_{n+1+j} sends $\sigma \in \mathrm{Map}(\Delta^{n+1+j}, \mathcal{M})$ to $p \circ \sigma \circ \rho_{n+1}$ where $\rho_{n+1} : \Delta^1 \to \Delta^{n+1+j}$ maps the arrow in Δ^1 to the $(n+1)$st edge of Δ^{n+1+j}, that is $\rho_{n+1} = (d^\perp)^n (d^\top)^j$. Indeed, in the diagram

$$\begin{tikzcd} B_{i,j} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Map}(\Delta^{i+1+j}, \mathcal{M}) \arrow[d, "\mathrm{post}\ p"] \\ 1 \arrow[r, "\ulcorner \beta_{i,j} \urcorner"'] \arrow[d] & \mathrm{Map}(\Delta^{i+1+j}, \Delta^1) \arrow[d, "\mathrm{pre}\ \rho_{i+1}"] \\ 1 \arrow[r, "\ulcorner \mathrm{id} \urcorner"'] & \mathrm{Map}(\Delta^1, \Delta^1), \end{tikzcd}$$

the bottom square is a pullback, because the fibre of the right bottom map is contractible, thus the whole rectangle is a pullback by Lemma 1.2.1.

Using the Lemma 2.2.2, we only have to check that the front square in the cube

$$\begin{tikzcd}[row sep=small, column sep=small] B_{n+1,i} \arrow[rr] \arrow[dr] \arrow[dd] & & B_{n+1,i} \arrow[dr] \arrow[dd] & \\ & \mathrm{Map}(\Delta^{n+1+1+i}, \mathcal{M}) \arrow[rr] \arrow[dd] & & \mathrm{Map}(\Delta^{n+1+i}, \mathcal{M}) \arrow[dd] \\ B_{n,i} \arrow[rr] \arrow[dr] & & B_{n-1,i} \arrow[dr] & \\ & \mathrm{Map}(\Delta^{n+1+i}, \mathcal{M}) \arrow[rr] & & \mathrm{Map}(\Delta^{n+i}, \mathcal{M}) \end{tikzcd}$$

is a pullback, and apply Lemma 1.2.1. But the squares

$$\begin{tikzcd}
\mathcal{M}_{n+1+1+i} \arrow[r, "d_\perp"] \arrow[d, "d_{n+1}"'] & \mathcal{M}_{n+1+i} \arrow[d, "d_n"] \\
\mathcal{M}_{n+1+i} \arrow[r, "d_\perp"'] & \mathcal{M}_{n+i}
\end{tikzcd}$$

are pullbacks because $\mathcal{M}$ is a decomposition space, $d_\perp$ is an inert map and d_{n+1} and d_n are always inner coface maps thus active maps. For the remaining squares, we use that the squares

$$\begin{tikzcd}
\mathcal{M}_{i+1+n+1} \arrow[r, "d_{i+1}"] \arrow[d, "d_\top"'] & \mathcal{M}_{i+1+n} \arrow[d, "d_\top"] \\
\mathcal{M}_{i+1+n} \arrow[r, "d_{i+1}"'] & \mathcal{M}_{i+1+n-1}
\end{tikzcd}$$

are also pullbacks because $\mathcal{M}$ is a decomposition space.

(2) To establish the stability condition, by Lemma 2.1.9 it is enough to prove that the two following squares are pullbacks:

$$\begin{tikzcd}
B_{0,0} & B_{0,1} \arrow[l, "d_0"'] \\
B_{1,0} \arrow[u, "e_0"] & B_{1,1}, \arrow[l, "d_0"] \arrow[u, "e_0"'] \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}
\qquad
\begin{tikzcd}
B_{0,0} & B_{0,1} \arrow[l, "d_1"'] \\
B_{1,0} \arrow[u, "e_1"] & B_{1,1}. \arrow[l, "d_1"] \arrow[u, "e_1"'] \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

We can prove this with the same strategy used above. The decomposition space axioms used here are that the following squares are pullbacks

$$\begin{tikzcd}
\mathcal{M}_0 \arrow[r, "s_0"] & \mathcal{M}_1 \\
\mathcal{M}_1 \arrow[u, "d_\perp"] \arrow[r, "s_1"'] \arrow[ur, phantom, "\urcorner", very near start] & \mathcal{M}_2, \arrow[u, "d_\perp"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{M}_0 \arrow[r, "s_0"] & \mathcal{M}_1 \\
\mathcal{M}_1 \arrow[u, "d_\top"] \arrow[r, "s_0"'] \arrow[ur, phantom, "\urcorner", very near start] & \mathcal{M}_2. \arrow[u, "d_\top"']
\end{tikzcd}$$

(3) and (4) The augmentations $Y_j := B_{-1,j}$ and $X_i := B_{i,-1}$ are also given by the following fibre sequences

$$\begin{tikzcd}
Y_j \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Map}(\Delta^j, \mathcal{M}) \arrow[d, "S"] \\
1 \arrow[r, "\ulcorner d_0 \urcorner"'] & \mathrm{Map}(\Delta^0, \Delta^1),
\end{tikzcd}
\qquad
\begin{tikzcd}
X_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Map}(\Delta^i, \mathcal{M}) \arrow[d, "T"] \\
1 \arrow[r, "\ulcorner d_1 \urcorner"'] & \mathrm{Map}(\Delta^0, \Delta^1),
\end{tikzcd}$$

where the map S sends σ to $p \circ \sigma \circ (d^\top)^j$, and the map T sends σ to $p \circ \sigma \circ (d^\perp)^i$. Since the following squares commute

$$\begin{tikzcd}
\mathrm{Map}(\Delta^{i+1+j},\mathcal{M}) \arrow[r, "(d_\perp)^{i+1}"] \arrow[d, "R_{i+1+j}"'] & \mathrm{Map}(\Delta^{j},\mathcal{M}) \arrow[d, "S"] \\
\mathrm{Map}(\Delta^1,\Delta^1) \arrow[r, "d_\perp"'] & \mathrm{Map}(\Delta^0,\Delta^1),
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Map}(\Delta^{i+1+j},\mathcal{M}) \arrow[r, "(d_\top)^{j+1}"] \arrow[d, "R_{i+1+j}"'] & \mathrm{Map}(\Delta^{i},\mathcal{M}) \arrow[d, "T"] \\
\mathrm{Map}(\Delta^1,\Delta^1) \arrow[r, "d_\top"] & \mathrm{Map}(\Delta^0,\Delta^1),
\end{tikzcd}$$

it is enough to define maps $B_{i,j} \to Y_j$ and $B_{i,j} \to X_i$.

The augmentation maps are culf: we need to prove that the back squares of the following cubes are pullbacks:

$$\begin{tikzcd}
Y_0 \arrow[rr, "s_0"] \arrow[dr] & & Y_1 \arrow[dr] & \\
& \mathrm{Map}(\Delta^0,\mathcal{M}) \arrow[rr, "s_0"] & & \mathrm{Map}(\Delta^1,\mathcal{M}) \\
B_{0,0} \arrow[uu, "d_0"] \arrow[rr, "s_0"] \arrow[dr] & & B_{0,1} \arrow[uu, "d_0"'] \arrow[dr] & \\
& \mathrm{Map}(\Delta^1,\mathcal{M}) \arrow[uu, "d_0"] \arrow[rr, "s_1"] & & \mathrm{Map}(\Delta^2,\mathcal{M}) \arrow[uu, "d_0"'],
\end{tikzcd}$$

$$\begin{tikzcd}
Y_1 \arrow[dr] & & Y_2 \arrow[ll, "d_1"'] \arrow[dr] & \\
& \mathrm{Map}(\Delta^1,\mathcal{M}) & & \mathrm{Map}(\Delta^2,\mathcal{M}) \arrow[ll, "d_1"'] \\
B_{0,1} \arrow[uu, "d_0"] \arrow[dr] & & B_{0,2} \arrow[ll, "d_1"'] \arrow[uu, "d_0"'] \arrow[dr] & \\
& \mathrm{Map}(\Delta^2,\mathcal{M}) \arrow[uu, "d_0"] & & \mathrm{Map}(\Delta^3,\mathcal{M}). \arrow[ll, "d_2"] \arrow[uu, "d_0"']
\end{tikzcd}$$

We can apply the Lemma 2.2.3 since the front square is a pullback because $\mathcal{M}$ is a decomposition space. ☐

To summarise, given a decomposition space correspondence $p : \mathcal{M} \to \Delta^1$, we get a bicomodule configuration and then $\mathcal{S}_{/B_{0,0}}$ is a bicomodule by Theorem 2.1.10.

2.2.2 *Cocartesian and cartesian fibrations of decomposition spaces*

Ayala and Francis [6] formulate a homotopy-invariant definition of cartesian and cocartesian fibrations so it can be equally well formulated in any model for ∞-categories. We adapt here those definitions to decomposition spaces.

Let $p : X \to Y$ be a simplicial map between decomposition spaces. A morphism $\Delta^1 \xrightarrow{<s \xrightarrow{a} t>} X$ is p-*cocartesian* if the diagram of coslices of decomposition spaces

$$\begin{tikzcd} {}^{a/}X \arrow[r] \arrow[d] & {}^{s/}X \arrow[d] \\ {}^{pa/}Y \arrow[r] & {}^{ps/}Y \end{tikzcd}$$

is a pullback, where the coslice ${}^{s/}X$ is given by pullback of lower décalage $\mathrm{Dec}_\perp(X)$:

$$\begin{tikzcd} ({}^{s/}X)_n \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Dec}_\perp(X)_n \arrow[d, "(d_\top)^{n+1}"] \\ 1 \arrow[r, "\ulcorner s \urcorner"'] & X_0, \end{tikzcd}$$

similarly the coslice ${}^{a/}X$ is given by pullback of $\mathrm{Dec}_\perp(\mathrm{Dec}_\perp(X))$

$$\begin{tikzcd} ({}^{a/}X)_n \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Dec}_\perp(\mathrm{Dec}_\perp(X))_n \arrow[d, "(d_\top)^{n+1}"] \\ 1 \arrow[r, "\ulcorner a \urcorner"'] & X_1, \end{tikzcd}$$

and the functor ${}^{a/}X \to {}^{s/}X$ is given by $\mathrm{Dec}_\perp(d_\perp)$, where the simplicial map $d_\perp : \mathrm{Dec}_\perp(X) \to X$ is given by the original d_0.

The functor $p : X \to Y$ is a *cocartesian fibration* if any diagram of solid arrows

$$\begin{tikzcd} \Delta^0 \arrow[r] \arrow[d, "d_1"'] & X \arrow[d, "p"] \\ \Delta^1 \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

admits a p-cocartesian diagonal filler.

Similarly, a morphism $\Delta^1 \xrightarrow{<s \xrightarrow{a} t>} X$ is p-*cartesian* if the diagram of slice decomposition spaces

$$\begin{tikzcd} X_{/a} \arrow[r] \arrow[d] & X_{/t} \arrow[d] \\ Y_{/pa} \arrow[r] & Y_{/pt} \end{tikzcd}$$

is a pullback, where the slice $X_{/t}$ is given by pullback of the upper décalage $\mathrm{Dec}_\top(X)$, the slice $X_{/a}$ is given by pullback of $\mathrm{Dec}_\top(\mathrm{Dec}_\top(X))$, and the functor $X_{/a} \to X_{/t}$ is given by $\mathrm{Dec}_\top(d_\top) = d_{\top-1}$, where $d_\top : \mathrm{Dec}_\top(X) \to X$ is given by the original $d_\top$.

The functor $p : X \to Y$ is a *cartesian fibration* if any diagram of solid arrows

$$\begin{tikzcd} \Delta^0 \arrow[r] \arrow[d, "d_0"'] & X \arrow[d, "p"] \\ \Delta^1 \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

admits a p-cartesian diagonal filler.

Bisimplex category with diagonal maps

We define $\overline{[i,j]} : M_{\phi_{i,j}} \to \Delta^1$ to be the canonical projection from the mapping cylinder of $\phi_{i,j} := (d^\top)^j : \Delta^i \to \Delta^{i+j}$; it is a cocartesian fibration. They assemble into a category, denoted $\overline{\mathbf{\Delta}_{/\Delta^1}}$, of shape like $\mathbf{\Delta}_{/\Delta^1}$, but with extra diagonal maps $d : \overline{[i-1,j]} \to \overline{[i,j-1]}$ given by inclusion. These satisfy new simplicial identities: $\sigma_k d = d\sigma_{k+1}$, $0 \leqslant k \leqslant j$, where σ_k are degeneracy maps "on j" (horizontal) and d are diagonal maps, and with face maps: $d\delta_{k+1} = \delta_k d$, $0 \leqslant k \leqslant j$, where δ_k are horizontal face maps. Similarly for degeneracy maps τ_k "on i" (vertical), $\tau_k d = d\tau_k$, $0 \leqslant k < i$ and $d\epsilon_k = \epsilon_k d$, $0 \leqslant k < i$, where ϵ_k are vertical face maps. For example, we can draw $\overline{[2,1]}$ as follows

$$\begin{tikzcd} & \cdot \\ \cdot \arrow[r] & \cdot \arrow[u] \\ \cdot \arrow[r] \arrow[u] & \cdot \arrow[u] \\ \cdot \arrow[r] \arrow[u] & \cdot \arrow[u] \end{tikzcd}$$

where the horizontal maps lie over the map in Δ^1. It is like a cocartesian version of the earlier drawing of Remark 2.1.6.

Remark 2.2.4. We can draw $\overline{[i,j]}$ as a column of $i+1$ black dots followed by $j+1$ white dots. Where arrows in $\mathbf{\Delta}_{/\Delta^1}$ send black dots to black dots and white dots to white dots (without crossing), in $\overline{\mathbf{\Delta}_{/\Delta^1}}$ we allow moreover to map white dots to black dots.

There is a natural notion of nerve in the context of cocartesian fibrations over Δ^1: given a cocartesian fibration $p : \mathcal{M} \to \Delta^1$ between decomposition spaces, define the cocartesian nerve $N_{\text{cocart}} : \mathit{Cat}^{\text{cocart}}_{\infty/\Delta^1} \to \text{Fun}((\overline{\mathbf{\Delta}_{/\Delta^1}})^{\text{op}}, \mathcal{S})$ by $N_{\text{cocart}}(p)_{\overline{i,j}} := \text{Map}^{\text{cocart}}_{/\Delta^1}(\overline{[i,j]}, \mathcal{M})$, the mapping space preserving cocartesian arrows.

Similarly to 2.2.1, we get a bicomodule configuration and $\mathcal{S}_{/B_{0,0}}$ is a bicomodule over $\mathcal{S}_{/X_1}$ and $\mathcal{S}_{/Y_1}$. We have here moreover diagonal maps $B_{i,j-1} \to B_{i-1,j}$ and new sections $s_{-1} : B_{i,j-1} \to B_{i,j}$, for $i \geqslant 0$ given by the composition with a diagonal map. That is $\mathcal{S}_{/B_{0,0}}$ is pointed as a right comodule over $\mathcal{S}_{/Y_1}$.

$$\begin{tikzcd}
 & B_{-1,0} \arrow[r, shift left=2, leftarrow] \arrow[r] \arrow[r, shift right=2, leftarrow] & B_{-1,1} \arrow[r, shift left=3, leftarrow] \arrow[r, shift left=1] \arrow[r, shift right=1, leftarrow] \arrow[r, shift right=3] & B_{-1,2} & \cdots \\
B_{0,-1} \arrow[ur] \arrow[u, phantom] & B_{0,0} \arrow[l] \arrow[u] \arrow[ur] \arrow[r, shift left=2, leftarrow] \arrow[r] \arrow[r, shift right=2, leftarrow] & B_{0,1} \arrow[u] \arrow[ur] \arrow[r, shift left=3, leftarrow] \arrow[r, shift left=1] \arrow[r, shift right=1, leftarrow] \arrow[r, shift right=3] & B_{0,2} \arrow[u] & \cdots \\
B_{1,-1} \arrow[u, shift left=2] \arrow[u, leftarrow] \arrow[u, shift right=2] \arrow[ur] & B_{1,0} \arrow[l] \arrow[u, shift left=2] \arrow[u, leftarrow] \arrow[u, shift right=2] \arrow[ur] \arrow[r, shift left=2, leftarrow] \arrow[r] \arrow[r, shift right=2, leftarrow] & B_{1,1} \arrow[u, shift left=2] \arrow[u, leftarrow] \arrow[u, shift right=2] \arrow[ur] \arrow[r, shift left=3, leftarrow] \arrow[r, shift left=1] \arrow[r, shift right=1, leftarrow] \arrow[r, shift right=3] & B_{1,2} \arrow[u, shift left=2] \arrow[u, leftarrow] \arrow[u, shift right=2] & \cdots \\
\vdots & \vdots & \vdots & \vdots &
\end{tikzcd}$$

We now adapt Lurie's definition of adjunction of ∞-categories [46] to decomposition spaces.

An *adjunction* between decomposition spaces X and Y is a simplicial map between decomposition spaces $p : \mathcal{M} \to \Delta^1$ which is both a cartesian and a cocartesian fibration together with equivalences $X \simeq \mathcal{M}_{\{0\}}$ and $Y \simeq \mathcal{M}_{\{1\}}$.

Proposition 2.2.5. *Given an adjunction $p : \mathcal{M} \to \Delta^1$, the bisimplicial ∞-groupoid B described above is a bicomodule configuration. Moreover $\mathcal{S}_{/B_{0,0}}$ is pointed as a right comodule over $\mathcal{S}_{/Y_1}$, and as a left comodule over $\mathcal{S}_{/X_1}$.*

Proof. By Proposition 2.2.1, the bisimplicial ∞-groupoid B is a bicomodule configuration. The pointings are given by the cartesian and cocartesian conditions. □

Adjunctions between ∞-categories are examples of adjunctions between decomposition spaces.

Example 2.2.6. To illustrate these abstract concepts, let us spell out the bicomodule configuration associated to an ordinary adjunction of 1-categories $F\colon X \rightleftarrows Y\colon G$. The ∞-groupoid $B_{0,0}$ is now just the set of arrows $Fx \to y$, which by adjunction correspond to arrows $x \to Gy$, and $B_{1,0}$ is the set of composable pair $Fx \to Fx' \to y$ (which is the same as $x \to x' \to Gy$). In general $B_{i,j}$ is the set of chains of composable arrows $Fx_0 \to \cdots \to Fx_i \to y_j \to \cdots \to y_0$. These chains can be drawn as in the picture page 38, where the horizontal arrow is a 'mixed' arrow $Fx \to y$. This drawing can be filled as in the picture page 49, and thus giving a right pointing by the following rearrangement:

$$\begin{tikzcd}
Fx' \arrow[r] & y & & & y \\
 & & \mapsto & & \\
Fx \arrow[uu] & & & Fx \arrow[r] & Fx' \arrow[uu]
\end{tikzcd}$$

Similarly the functor $G : Y \to X$ induces a left pointing on the equivalent (by adjunction) augmented simplicial set of chains $x_0 \to \cdots \to x_i \to Gy_j \to \cdots \to Gy_0$.

2.3 *Möbius inversion for comodules and a Rota formula*

2.3.1 *Right and left convolutions*

We introduce left and right convolution actions as dual to the comodule structures. Explicitly, given a right comodule configuration $C \to Y$, we get a right comodule $\mathcal{S}_{/C_0}$ over $\mathcal{S}_{/Y_1}$. The *right convolution* $\theta \star_r \beta$ of the two functors $\theta : \mathcal{S}_{/C_0} \to \mathcal{S}$ and $\beta : \mathcal{S}_{/Y_1} \to \mathcal{S}$, given by the spans $C_0 \leftarrow M \to 1$ and $Y_1 \leftarrow N \to 1$, is the composite of $\theta \otimes \beta$ with the right coaction γ_r:

$$\theta \star_r \beta : \mathcal{S}_{/C_0} \xrightarrow{\gamma_r} \mathcal{S}_{/C_0} \otimes \mathcal{S}_{/Y_1} \xrightarrow{\theta \otimes \beta} \mathcal{S},$$

where the tensor product $\theta \otimes \beta$ is given by the span $C_0 \times Y_1 \leftarrow M \times N \to 1$.

Similarly, given a left comodule configuration, we can define the *left convolution* $\alpha \star_l \theta$ of $\alpha : \mathcal{S}_{/X_1} \to \mathcal{S}$ and $\theta : \mathcal{S}_{/C_0} \to \mathcal{S}$:

$$\alpha \star_l \theta : \mathcal{S}_{/C_0} \xrightarrow{\gamma_l} \mathcal{S}_{/X_1} \otimes \mathcal{S}_{/C_0} \xrightarrow{\alpha \otimes \theta} \mathcal{S}.$$

If we have a bicomodule configuration, then the following associativity formula expresses the compatibility of coactions from Theorem 2.1.10.

Corollary 2.3.1. *Given a bicomodule configuration, the convolutions defined above satisfy*

$$\alpha \star_l (\theta \star_r \beta) \simeq (\alpha \star_l \theta) \star_r \beta.$$

2.3.2 *Möbius inversion for (co)modules*

Let $C \to Y$ be a comodule configuration. The zeta functor

$$\zeta^C : \mathcal{S}_{/C_0} \to \mathcal{S}$$

is the linear functor defined by the span

$$C_0 \xleftarrow{=} C_0 \to 1.$$

Let $C \to Y$ be a right pointed comodule configuration. The augmented simplicial ∞-groupoid C is an object of the functor ∞-category

$$\mathrm{Fun}\,(\boldsymbol{\Delta}_{\mathrm{bot}}^{\mathrm{op}}, \mathcal{S})$$

where $\boldsymbol{\Delta}_{\mathrm{bot}}$ is the simplex category of finite linear orders with a specified bottom element, and with monotone maps preserving the bottom element. The forgetful functor $\boldsymbol{\Delta}_{\mathrm{bot}} \to \boldsymbol{\Delta}$ is right adjoint to the functor $j : \boldsymbol{\Delta} \to \boldsymbol{\Delta}_{\mathrm{bot}}$ adding a bottom element.

Remark 2.3.2. In the situation where Y is Segal and $C = \mathrm{Dec}_\perp Y$, we can take C_{-1} to be Y_0, with d_0 as augmentation map. By [46, Lemma 6.1.3.16], this is a colimit diagram.

A right pointed comodule configuration $f : C \to Y$ is *complete* if the new degeneracies $s_{-1} : C_{n-1} \to C_n$ are monomorphisms. Since s_{-1} is a monomorphism, we can identify C_{-1} with a ∞-subgroupoid of C_0. We denote by C_b its complement: $C_0 = C_{-1} + C_b$. Denote by C_{vw} the ∞-groupoid of simplices whose principal edges have the type indicated in the word vw, where $v \in \{-1, 0, b\}$ and w is a word in the alphabet $\{0, 1, a\}$, that is the full ∞-subgroupoid of C_n given by the pullback

$$\begin{tikzcd}
C_{vw} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C_n \arrow[d] \\
C_v \times Y^w \arrow[r] & C_0 \times (Y_1)^n,
\end{tikzcd}$$

where $n = |w| \geqslant 0$. The principal edges of the ∞-groupoid C_n consist of an element in C_0 given by $(d_\top)^n$, and n edges in Y_1, the principal edges of the image of C_n by f. In this situation, we define $\vec{C}_n = C_{ba\ldots a} \subset C_n$ to be the full subgroupoid of simplices with all principal edges nondegenerate. It is given by the pullback diagram

$$\begin{tikzcd}
C_{ba\ldots a} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C_n \arrow[d] \\
C_b \times Y^{a\ldots a} \arrow[r] & C_0 \times Y_1^n.
\end{tikzcd}$$

Define

$$\delta^R : \mathcal{S}_{/C_0} \to \mathcal{S}$$

to be the linear functor given by the span

$$C_0 \xleftarrow{s_{-1}} C_{-1} \to 1$$

and define the *right Phi functors*

$$\Phi_n^R : \mathcal{S}_{/C_0} \to \mathcal{S}$$

to be the linear functors given by the spans

$$C_0 \leftarrow \vec{C}_n \to 1.$$

If $n = -1$, $\vec{C}_{-1} = C_{-1}$ (by convention) and Φ^R_{-1} is the linear functor δ^R.

Lemma 2.3.3. *For every word w in the alphabet $\{0, 1, a\}$, the following square is a pullback:*

$$\begin{tikzcd}
C_{0w} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C_1 \arrow[d, "(d_\top, f)"] \\
C_0 \times Y_w \arrow[r] & C_0 \times Y_1.
\end{tikzcd}$$

Proof. Let $n = |w|$. The square is the top rectangle of the diagram

$$\begin{tikzcd}
C_{0w} \arrow[r] \arrow[d] & C_n \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & C_1 \arrow[d, "(d_\top, f)"] \\
C_0 \times Y_w \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & C_0 \times Y_n \arrow[r] \arrow[d] & C_0 \times Y_1 \\
C_0 \times Y^w \arrow[r] & C_0 \times Y_1^n .
\end{tikzcd}$$

The bottom square and left-hand rectangle are pullbacks by definition of Y_w and C_{0w}, hence the top left-hand square is a pullback. The right-hand square is a pullback because the augmentation map $C \to Y$ is culf. Hence the top rectangle, which is the desired square, is a pullback. $\square$

Given a complete decomposition space Y, we denote $\Phi_n^Y : \mathcal{S}_{/Y_1} \to \mathcal{S}$ the usual Phi functors.

Proposition 2.3.4. *The right Phi functors satisfy*

$$\zeta^C \star_r \Phi_n^Y \simeq \Phi_{n-1}^R + \Phi_n^R.$$

Proof. Compute the convolution action $\zeta^C \star_r \Phi_n^Y$ by Lemma 2.3.3 as:

$$\begin{tikzcd}
C_0 & & \\
C_1 \arrow[u] \arrow[d] & C_{0a\ldots a} \arrow[l] \arrow[ul] \arrow[d] \arrow[dr] \arrow[dl, phantom, "\llcorner"] & \\
C_0 \times Y_1 & C_0 \times \vec{Y_n} \arrow[l] \arrow[r] & 1.
\end{tikzcd}$$

But $C_{0a\ldots a} \simeq C_{-1a\ldots a} + C_{ba\ldots a} \simeq \vec{C}_{n-1} + \vec{C}_n$. This is an equivalence of ∞-groupoids over C_0 and the resulting span is $\Phi_{n-1}^R + \Phi_n^R$. $\square$

Denote

$$\Phi_{\text{even}}^Y := \sum_{n \text{ even}} \Phi_n^Y, \quad \Phi_{\text{odd}}^Y := \sum_{n \text{ odd}} \Phi_n^Y.$$

The previous proposition implies the following Möbius inversion formula.

Theorem 2.3.5. *Given $C \to Y$ a complete right pointed comodule configuration,*

$$\zeta^C \star_r \Phi_{\textit{even}}^Y \simeq \delta^R + \zeta^C \star_r \Phi_{\textit{odd}}^Y.$$

Proof. The two linear functors are equivalent to the sum of the right Phi functors:

$$\zeta^C \star_r \Phi_{\text{even}}^Y \simeq \Phi_{-1}^R + \Phi_0^R + \Phi_1^R + \cdots \simeq \delta^R + \zeta^C \star_r \Phi_{\text{odd}}^Y.$$

$\square$

We can also define a *left pointed* comodule configuration $D \to X$, with new top sections instead of bottom: we consider instead the mapping cylinder of $\boldsymbol{\Delta} \to \boldsymbol{\Delta}_{\text{top}}$, where $\boldsymbol{\Delta}_{\text{top}}$ is the simplex category of finite linear orders with a specified top element, and with monotone maps preserving the top element. A left pointed comodule configuration is *complete* if the new degeneracies $t_{\top+1} : D_{n-1} \to D_n$ are monomorphisms. Similarly, we define the *left Phi functors* and δ^L using $t_{\top+1}$ and $e_\top$ and we obtain the following formula.

Theorem 2.3.6. *Given* $D \to X$ *a complete left pointed comodule configuration,*

$$\Phi^X_{even} \star_l \zeta^D \simeq \delta^L + \Phi^X_{odd} \star_l \zeta^D.$$

2.3.3 *Möbius bicomodule configurations and the Rota formula*

In order to take homotopy cardinality to recover the usual Möbius inversions, we need to impose some finiteness conditions. We adapt the approach of [26] summarised in the preliminaries.

A *right Möbius comodule configuration* is a complete right pointed comodule configuration $C \to Y$ such that the decomposition space Y is Möbius and the augmented comodule is Möbius, that is

- C is locally finite: the ∞-groupoid C_0 is locally finite and both s_{-1} and d_0 are finite maps;
- C is of locally finite length: every edge has a finite length, that is for all $a \in C_0$, the fibres of $d_0^{(n)} : \vec{C}_n \to C_0$ over a are empty for n sufficiently large.

Under these conditions, the Phi functors descend to

$$\Phi^R_n : \mathcal{F}_{/C_0} \to \mathcal{F}$$

and we can now take the cardinality of the "Möbius formulas" (Theorems 2.3.5 and 2.3.6).

Similarly we define a *left Möbius comodule configuration* to be a complete left pointed comodule configuration $D \to X$ such that the decomposition space X is Möbius and the augmented comodule is Möbius, using $t_{\top+1}$ and $d_\top$.

Theorem 2.3.7. *Given* $C \to Y$ *a right Möbius comodule configuration and* $D \to X$ *a left Möbius comodule configuration,*

$$|\zeta^C| \star_r |\mu^Y| = |\delta^R|, \qquad |\mu^X| \star_l |\zeta^D| = |\delta^L|,$$

where $|\mu^Y| := |\Phi^Y_{even}| - |\Phi^Y_{odd}|$ *and* $|\mu^X| := |\Phi^X_{even}| - |\Phi^X_{odd}|$.

A *Möbius bicomodule configuration* is a bicomodule configuration with two pointings such that both left and right comodule configurations are Möbius. It hence has extra degeneracy maps in both directions, extra bottom degeneracy maps in horizontal direction and extra top degeneracy maps in vertical direction.

Note that given a Möbius bicomodule configuration B, the zeta functors are defined only on the ∞-groupoid $B_{0,0}$ and then are the same for the two comodules. In both cases it is given by the span

$$B_{0,0} \xleftarrow{=} B_{0,0} \to 1.$$

Theorem 2.3.8. *Given a Möbius bicomodule configuration B with $X := B_{\bullet,-1}$ and $Y := B_{-1,\bullet}$, we have*

$$|\mu^X| \star_l |\delta^R| = |\delta^L| \star_r |\mu^Y|,$$

where δ^R is the linear functor given by the span

$$B_{0,0} \leftarrow X_0 \to 1,$$

and δ^L is the linear functor given by the span

$$B_{0,0} \leftarrow Y_0 \to 1.$$

Proof. Using the Möbius formulas at the algebraic level from Theorem 2.3.7, and the associativity of the convolution actions from Proposition 2.3.1, we compute

$$\begin{aligned} |\mu^X| \star_l |\delta^R| &= |\mu^X| \star_l (|\zeta| \star_r |\mu^Y|) \\ &= (|\mu^X| \star_l |\zeta|) \star_r |\mu^Y| \\ &= |\delta^L| \star_r |\mu^Y|. \end{aligned}$$

□

2.3.4 *Möbius bicomodule configurations from adjunctions of Möbius decomposition spaces*

We saw in Section 2.2.2 that given a cocartesian fibration $p : \mathcal{M} \to \Delta^1$ between decomposition spaces, we obtain a right comodule configuration B, with diagonal maps $B_{i,j-1} \to B_{i-1,j}$ and new sections $s_{-1} : B_{i,j-1} \to B_{i,j}$, for $i \geqslant 0$ given by the composition with a diagonal map.

Lemma 2.3.9. *Given a cocartesian fibration $p : \mathcal{M} \to \Delta^1$ between decomposition spaces, suppose moreover that $\mathcal{M}$ is complete. Then the associated right pointed comodule configuration is complete.*

Proof. The new sections will be monomorphisms if the following square is a pullback:

$$\begin{tikzcd} B_{i,j-1} \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & B_{i,j-1} \arrow[d, "s_{-1}"] \\ B_{i,j-1} \arrow[r, "s_{-1}"'] & B_{i,j}. \end{tikzcd}$$

By assumption, $\mathcal{M}$ is a complete decomposition space, hence all degeneracy maps are monomorphisms, and we can apply Lemma 2.2.3, to obtain the desired pullbacks. □

Instantiating the general definitions from Section 2.3.2, the zeta functor

$$\zeta : \mathcal{S}_{/B_{0,0}} \to \mathcal{S}$$

is given by the span

$$B_{0,0} \xleftarrow{=} B_{0,0} \to 1,$$

and the functor

$$\delta^R : \mathcal{S}_{/B_{0,0}} \to \mathcal{S}$$

is defined by the span

$$B_{0,0} \xleftarrow{s_{-1}} B_{0,-1} \to 1.$$

The right comodule configuration being complete, we get a Möbius inversion formula (Theorem 2.3.5):

$$\zeta \star_r \Phi^Y_{\text{even}} \simeq \delta^R + \zeta \star_r \Phi^Y_{\text{odd}},$$

where $Y := B_{-1,\bullet}$.

Similarly, given a cartesian fibration $p : \mathcal{M} \to \Delta^1$ between decomposition spaces, we obtain a left pointed comodule configuration.

Lemma 2.3.10. *Given a cartesian fibration $p : \mathcal{M} \to \Delta^1$ between decomposition spaces, suppose moreover that $\mathcal{M}$ is complete. Then the left pointed comodule configuration is complete.*

The functor

$$\delta^L : \mathcal{S}_{/B_{0,0}} \to \mathcal{S}$$

is given by the span

$$B_{0,0} \leftarrow B_{-1,0} \to 1.$$

This leads to the Möbius inversion formula

$$\Phi^X_{\text{even}} \star_l \zeta \simeq \delta^L + \Phi^X_{\text{odd}} \star_l \zeta.$$

Given an adjunction between decomposition spaces, that is a simplicial map $\mathcal{M} \to \Delta^1$ which is both cartesian and cocartesian, and suppose that $\mathcal{M}$ is complete, then we just obtained two Möbius inversion formulas.

Theorem 2.3.11. *Given an adjunction of decomposition spaces in the form of a bicartesian fibration $p : \mathcal{M} \to \Delta^1$, suppose moreover that $\mathcal{M}$ is a Möbius decomposition space. Then the bicomodule configuration extracted from this data is Möbius. In particular, we have the Rota formula for the adjunction p:*

$$|\mu^X| \star_l |\delta^R| = |\delta^L| \star_r |\mu^Y|.$$

Proof. First observe that $B_{0,0}$, and in fact all $B_{i,j}$, are locally finite since they are given by pullback (see page 43) of locally finite spaces. Second, note that $e_\top \colon B_{i+1,j} \to B_{i,j}$ is induced in the same way from the face map $d_i \colon \mathcal{M}_{i+2+j} \to \mathcal{M}_{i+1+j}$, which is an inner face map, and is therefore finite since $\mathcal{M}$ is Möbius. Similarly $d_0 \colon B_{i,j+1} \to B_{i,j}$ is obtained from $d_{i+1} \colon \mathcal{M}_{i+2+j} \to \mathcal{M}_{i+1+j}$ which is also an inner face map. Finally the fibres of $e_\top^{(n)}$ are empty for n sufficiently large because the fibres of $d_{i-n+1} \circ \cdots \circ d_i \colon \mathcal{M}_{i+2+j} \to \mathcal{M}_{i+2-n+j}$ are empty for n sufficiently large since $\mathcal{M}$ is Möbius. Similarly, the fibres of $d_0^{(n)}$ are empty for n sufficiently large. $\square$

3

Möbius functions of directed restriction species

3.1 Bisimplicial groupoids, abacus maps, and bicomodule configurations

We want to define a bicomodule configuration interpolating between the decomposition space $\mathbf{C}$ of finite posets and the decomposition space $\mathbf{I}$ of finite sets, in order to relate the Möbius functions of the incidence algebras of these decomposition spaces. As explained in the introduction, we shall achieve this by modifying the box product $\mathbf{I} \square \operatorname{Dec}_\perp \mathbf{C}$, and we introduce the notion of abacus map for a bisimplicial groupoid for this purpose. The modification is necessary in order to be able to define an extra vertical degeneracy map, in turned required to establish the Möbius property.

Abacus maps

Let B be a bisimplicial groupoid with horizontal face and degeneracy maps denoted by d_k and s_k, and vertical face and degeneracy maps denoted by e_k and t_k. A family f of maps $f_{i,j} : B_{i+1,j} \to B_{i,j+1}$ is called an *abacus map* if

- for all i, the map $f_{i,\bullet} : B_{i+1,\bullet} \to \operatorname{Dec}_\perp(B_{i,\bullet})$ is simplicial (between rows),
- for all j, the map $f_{\bullet,j} : \operatorname{Dec}_\top(B_{\bullet,j}) \to B_{\bullet,j+1}$ is simplicial (between columns) except for the top face map,
- $d_\perp f_{i,j} t_\top = \mathrm{id}$, where $d_\perp$ is the horizontal bottom face map, $t_\top$ is the vertical top degeneracy map.

Theorem 3.1.1. *Let B be a bisimplicial groupoid, and f an abacus map. Define new vertical top degeneracy maps $\tilde{e}_\top := d_\perp f_{i,j}$. Then the groupoids $B_{i,j}$ with the new $\tilde{e}_\top$ form a bisimplicial groupoid, denoted $\tilde{B}$ (for which f is still an abacus map).*

Proof. Firstly, let us prove that the new $\tilde{e}_\top$ is simplicial between rows: since the map $f_{i,\bullet}$ is simplicial, and by the face-map identities for the rows, we have

$$\tilde{e}_\top d_k = d_\perp f_{i,j} d_k = d_\perp d_{k+1} f_{i,j+1} = d_k d_\perp f_{i,j+1} = d_k \tilde{e}_\top,$$

and similarly for s_k.

Secondly, let us check that the columns are simplicial. The simplicial identities involving $\tilde{e}_\top = \tilde{e}_i : B_{i,j} \to B_{i-1,j}$ are:

1. $t_k \tilde{e}_\top = \tilde{e}_\top t_k$, for $k < i$;
2. $e_k \tilde{e}_\top = \tilde{e}_\top e_k$, for $k < i$;
3. $\tilde{e}_\top t_\top = \mathrm{id}$.

To verify the first two identities, we use the commutativity of horizontal maps against vertical maps, and that $f_{\bullet,j}$ is simplicial except for the top maps. The third identity is exactly the last condition in the definition of an abacus map. □

We say an abacus map is *perfect* if $f_{i,j} e_{\top-1} = d_\perp f_{i,j+1} f_{i+1,j}$ and $e_\top = d_\perp f_{i,j}$ for all i, j. We get the following proposition, whose proof is straightforward.

Proposition 3.1.2. *Let* B *be a bisimplicial groupoid and* f *a perfect abacus map. Then the map* $f_{\bullet,j} : \mathrm{Dec}_\top(\tilde{B}_{\bullet,j}) \to \tilde{B}_{\bullet,j+1}$ *simplicial, and the construction of Theorem 3.1.1 is idempotent.*

Given an augmented bisimplicial groupoid B, we say an abacus map is *left augmented* if there are maps $f_{i,-1} : B_{i+1,-1} \to B_{i,0}$ such that $f_{i,\bullet} : B_{i+1,\bullet} \to \mathrm{Dec}_\perp(B_{i,\bullet})$ is augmented simplicial for all $i \geqslant 0$, that is the following diagram commutes

$$\begin{tikzcd} & B_{i,0} & B_{i,1} \arrow[l, "d_1"'] \\ B_{i+1,-1} \arrow[ur, "f_{i,-1}"] & B_{i+1,0} \arrow[l, "u"] \arrow[ur, "f_{i,0}"'] & \end{tikzcd}$$

and moreover $uf_{i,-1} = e_\top$:

$$\begin{tikzcd} B_{i,-1} & B_{i,0} \arrow[l, "u"'] \\ B_{i+1,-1} \arrow[u, "e_\top"] \arrow[ur, "f_{i,-1}"'] & \end{tikzcd}$$

We say an abacus map is *right augmented* if there are maps $f_{-1,j} : B_{0,j} \to B_{-1,j+1}$ such that $f_{\bullet,j} : \mathrm{Dec}_\top(B_{\bullet,j}) \to B_{\bullet,j+1}$ is augmented simplicial except for the top face map for all $j \geqslant 0$, that is the following diagram commutes

$$\begin{tikzcd} & B_{-1,j} \\ B_{0,j} \arrow[ur, "f_{-1,j}"] & B_{0,j+1} \arrow[u, "v"'] \\ B_{1,j} \arrow[u, "e_0"] \arrow[ur, "f_{0,j}"'] & \end{tikzcd}$$

and moreover $v = d_0 f_{-1,0}$:

$$\begin{tikzcd} B_{-1,j} & B_{-1,j+1} \arrow[l, "d_0"'] \\ B_{0,j} \arrow[u, "v"] \arrow[ur, "f_{-1,j}"'] & \end{tikzcd}$$

We say an abacus map is *augmented* if it is left and right augmented.

Lemma 3.1.3. *Given an augmented bisimplicial groupoid* B *and a left augmented abacus map. Then the augmentation map* $u : \tilde{B}_{\bullet,0} \to \tilde{B}_{\bullet,-1}$ *is simplicial.*

Proof. We only need to check the commutativity with the new top face map, which is a direct verification: $u\tilde{e}_\top = ud_0 f_{i,0} = ud_1 f_{i,0} = uf_{i,-1}u = e_\top u$. □

Lemma 3.1.4. *Given an augmented bisimplicial groupoid* B *and a right augmented abacus map. Then* $v : \tilde{B}_{0,\bullet} \to \tilde{B}_{-1,\bullet}$ *is an augmentation map.*

Proof. We only need to verify it coequalises e_0 and $\tilde{e}_1$. We have $ve_0 = d_0 f_{-1,j} e_0 = d_0 v f_{0,j} = v d_0 f_{0,j} = v\tilde{e}_1$. □

Remark 3.1.5. A augmented perfect abacus map produces an example of a cocartesian nerve, see Section 2.2.2, that is a map $N_{\mathrm{cocart}} : \boldsymbol{Cat}^{\mathrm{cocart}}_{\infty/\Delta^1} \to \mathrm{Fun}((\overline{\boldsymbol{\Delta}_{/\Delta^1}})^{\mathrm{op}}, \mathcal{S})$ where $N_{\mathrm{cocart}}(p)_{\overline{i,j}}$ is a mapping space preserving cocartesian arrows, and $\overline{\boldsymbol{\Delta}_{/\Delta^1}}$ is a category of shape like $\boldsymbol{\Delta}_{/\Delta^1}$, but with extra diagonal maps. This ensures $\boldsymbol{Grpd}_{/B_{0,0}}$ is pointed as a right comodule over $\boldsymbol{Grpd}_{/B_{1,-1}}$.

Example 3.1.6 (Bisimplicial groupoid associated to a functor)**.** Given a functor $F : X \to Y$ between categories, we consider the groupoid $B_{i,j}$ whose objects consist of $(i+j+1)$-tuples of composable morphisms such that the i first morphisms are given as images of morphisms in X. An object in $B_{i,j}$ can be pictured as follows:

$$\begin{tikzcd} Fx \arrow[r, "Ff"] & Fx' \arrow[r] & \dots \arrow[r] & Fx^{(i)} \arrow[dlll] & \\ y^{(j)} \arrow[r] & y^{(j-1)} \arrow[r] & \dots \arrow[r] & y. & \end{tikzcd}$$

The groupoids $B_{i,j}$ assemble into a (augmented) bisimplicial groupoid, where horizontal face and degeneracy maps are given by face and degeneracy maps of the nerve of Y, and similarly, vertical face and degeneracy maps are given by face and degeneracy maps of the nerve of X. This is moreover a bicomodule configuration. There is an (augmented) perfect abacus map sending

$$\begin{tikzcd} Fx \arrow[r] & Fx' \arrow[dl] \\ y & \end{tikzcd} \quad \text{to} \quad \begin{tikzcd} & Fx \arrow[dl] \\ Fx' \arrow[r] & y. \end{tikzcd}$$

Bicomodule configurations

Given two simplicial groupoids X and Y, their *box product* [38] is the bisimplicial groupoid $X \square Y$ given by the groupoids $X_i \times Y_j$, with horizontal and vertical face and degeneracy maps induced by those of X and Y. We shall be concerned rather with the box product

$$B := X \square \operatorname{Dec}_\perp Y,$$

and use the following notation: the horizontal maps $\underline{d}_k : B_{i,j} \to B_{i,j-1}$ and $\underline{s}_k : B_{i,j} \to B_{i,j+1}$ are given by:

$$\underline{d}_k = \mathrm{id}_{X_i} \times d_{k+1}, \qquad \underline{s}_k = \mathrm{id}_{X_i} \times s_{k+1}, \qquad 0 \leqslant k \leqslant j,$$

where d_k and s_k are the face and degeneracy maps of Y. The vertical maps $\underline{e}_k : B_{i,j} \to B_{i-1,j}$ and $\underline{t}_k : B_{i,j} \to B_{i+1,j}$ are given by:

$$\underline{e}_k = e_k \times \mathrm{id}_{Y_{j+1}}, \qquad \underline{t}_k = t_k \times \mathrm{id}_{Y_{j+1}}, \qquad 0 \leqslant k \leqslant i,$$

where e_k and t_k are the face and degeneracy maps of X. Note that since the second factor is given by décalage, there is also an extra bottom degeneracy map given by $\underline{s}_{-1} = \mathrm{id}_{X_i} \times s_0$. There are also augmentation maps $u : B_{\bullet,0} \to X_\bullet \times Y_0$ given by $\mathrm{id}_X \times d_1$, and $v : B_{0,\bullet} \to X_0 \times Y$ given by the décalage map.

The following trivial lemma will be invoked several times.

Lemma 3.1.7. *The following square of groupoids is a pullback:*

$$\begin{tikzcd} X \times Y \arrow[r, "f \times \mathrm{id}_Y"] \arrow[d, "\mathrm{id}_X \times g"'] & X' \times Y \arrow[d, "\mathrm{id}_{X'} \times g"] \\ X \times Y' \arrow[r, "f \times \mathrm{id}_{Y'}"'] & X' \times Y'. \end{tikzcd}$$

Proposition 3.1.8. *Suppose f is an augmented abacus map for $B := X \square \operatorname{Dec}_\perp Y$, such that $f_{\bullet,j}$ is a right fibration (that is cartesian on bottom face maps $\underline{e}_\perp$), and $f_{i,\bullet}$ is a left fibration (that is cartesian on top face maps $\underline{d}_\top$). Then the modified augmented bisimplicial groupoid $\tilde{B}$ with the new $\underline{\tilde{e}}_\top$ obtained from the abacus map as in Theorem 3.1.1 form an augmented bisimplicial groupoid which is double Segal, stable, and such that the augmentation maps are culf.*

The bisimpliciality follows from Theorem 3.1.1. We split the rest of the proof into the following Lemmas 3.1.9-3.1.11.

Lemma 3.1.9. *Suppose $f_{\bullet,j}$ is a right fibration, that is cartesian on bottom face maps $\underline{e}_\perp$. Then for every $i \geqslant 0$, the simplicial groupoid $\tilde{B}_{i,\bullet}$ is Segal, and for every $j \geqslant 0$ the simplicial groupoid $\tilde{B}_{\bullet,j}$ is Segal.*

Proof. The simplicial groupoid $\tilde{B}_{i,\bullet}$ is Segal since it is the product with the groupoid X_i of the décalage of the decomposition space Y. The simplicial groupoid $\tilde{B}_{\bullet,j}$ is Segal if, for all $j \geqslant 0$, and $n \geqslant 1$, the following square is a pullback

$$\begin{tikzcd} \tilde{B}_{n+1,j} \arrow[r, "\tilde{e}_{n+1}"] \arrow[d, "e_0"'] & \tilde{B}_{n,j} \arrow[d, "e_0"] \\ \tilde{B}_{n,j} \arrow[r, "\tilde{e}_n"'] & \tilde{B}_{n-1,j}. \end{tikzcd}$$

It is the outer square in the following diagram:

$$\begin{tikzcd} \tilde{B}_{n+1,j} \arrow[r, "f_{n,j}"] \arrow[d, "e_0"'] & \tilde{B}_{n,j+1} \arrow[r, "d_0"] \arrow[d, "e_0"'] & \tilde{B}_{n,j} \arrow[d, "e_0"] \\ \tilde{B}_{n,j} \arrow[r, "f_{n-1,j}"'] & \tilde{B}_{n-1,j+1} \arrow[r, "d_0"'] & \tilde{B}_{n-1,j}. \end{tikzcd}$$

The right-hand square is a pullback by Lemma 3.1.7. The left-hand square is a pullback because $f_{\bullet,j}$ is a right fibration. □

Lemma 3.1.10. *The augmentation maps* $u : \tilde{B}_{\bullet,0} \to X \times Y_0$ *and* $v : \tilde{B}_{0,\bullet} \to X_0 \times Y$ *are culf.*

Proof. Note that u is a simplicial map by Lemma 3.1.3 and that v is an augmentation map by Lemma 3.1.4. It is enough to check the two following squares are pullbacks [25, Lemma 4.3]:

$$\begin{tikzcd} X_1 \times Y_0 & \tilde{B}_{1,0} \arrow[l, "u"'] \\ X_2 \times Y_0 \arrow[u, "e_1 \times \mathrm{id}_{Y_0}"] & \tilde{B}_{2,0}, \arrow[l, "u"] \arrow[u, "e_1"'] \end{tikzcd} \qquad \begin{tikzcd} X_0 \times Y_1 & X_0 \times Y_2 \arrow[l, "d_1"'] \\ \tilde{B}_{0,1} \arrow[u, "v"] & \tilde{B}_{0,2}. \arrow[l, "d_1"] \arrow[u, "v"'] \end{tikzcd}$$

By definition, the left-hand one is the following square

$$\begin{tikzcd} X_1 \times Y_0 & X_1 \times Y_1 \arrow[l, "\mathrm{id}_{X_1} \times d_1"'] \\ X_2 \times Y_0 \arrow[u, "e_1 \times \mathrm{id}_{Y_0}"] & X_2 \times Y_1, \arrow[l, "\mathrm{id}_{X_2} \times d_1"] \arrow[u, "e_1 \times \mathrm{id}_{Y_1}"'] \end{tikzcd}$$

which is a pullback by Lemma 3.1.7. The right-hand square is a pullback since the décalage map of a decomposition space is culf. □

Lemma 3.1.11. *Suppose* $f_{i,\bullet}$ *is a left fibration, that is cartesian on top face maps* $d_\top$*. Then the bisimplicial groupoid* $\tilde{B}$ *is stable.*

Proof. Since the bisimplicial groupoid $\tilde{B}$ is a double Segal space by Lemma 3.1.9, the stability can be established checking only the two following squares are pullbacks by Lemma 2.1.9:

$$\begin{tikzcd} \tilde{B}_{0,0} & \tilde{B}_{0,1} \arrow[l, "d_0"'] \\ \tilde{B}_{1,0} \arrow[u, "e_0"] & \tilde{B}_{1,1}, \arrow[l, "d_0"] \arrow[u, "e_0"'] \end{tikzcd} \qquad \begin{tikzcd} \tilde{B}_{0,0} & \tilde{B}_{0,1} \arrow[l, "d_\top"'] \\ \tilde{B}_{1,0} \arrow[u, "\tilde{e}_\top"] & \tilde{B}_{1,1}. \arrow[l, "d_\top"] \arrow[u, "\tilde{e}_\top"'] \end{tikzcd}$$

The first square is a pullback by Lemma 3.1.7. The second square is the outer rectangle of the following diagram:

$$\begin{tikzcd}
\tilde{B}_{0,0} & \tilde{B}_{0,1} \arrow[l, "d_\top"'] \\
\tilde{B}_{0,1} \arrow[u, "d_0"] & \tilde{B}_{0,2} \arrow[l, "d_\top"'] \arrow[u, "d_0"'] \\
\tilde{B}_{1,0} \arrow[u, "f_{0,0}"] & \tilde{B}_{1,1}. \arrow[l, "d_\top"] \arrow[u, "f_{0,1}"']
\end{tikzcd}$$

The top square is a pullback since Y is a decomposition space. The bottom one is a pullback because $f_{i,\bullet}$ is a left fibration. □

Layered sets and posets

We now specialise to the situation where $X = \mathbf{I}$ the decomposition space of layered finite sets and $Y = \mathbf{C}$ the decomposition space of layered finite posets.

We refer to [28] for the following material. An n-*layering* of a finite poset P is a monotone map $l\colon P \to \underline{n}$, where $\underline{n} = \{1, \dots, n\}$ are the objects of the skeleton of the category of finite ordered sets (possibly empty) and monotone maps. The fibres $P_i = l^{-1}(i)$, $i \in \underline{n}$ are called layers, and can be empty. The objects of the groupoid $\mathbf{C}_n$ of n-layered finite posets are monotone maps $l\colon P \to \underline{n}$ and the morphisms are triangles

$$\begin{tikzcd}
P \arrow[rr, "\simeq"] \arrow[dr] & & P' \arrow[dl] \\
& \underline{n}, &
\end{tikzcd}$$

where $P \to P'$ is a monotone bijection. They assemble into a simplicial groupoid $\mathbf{C}$. The face maps are given by joining layers, or deleting an outer layer for the top and bottom face maps. The degeneracy maps are given by inserting empty layers.

Proposition 3.1.12 ([28, Proposition 6.12, Lemma 6.13])**.** *The simplicial groupoid* $\mathbf{C}$ *of layered finite posets is a decomposition space (but not a Segal space), and is complete, locally finite, locally discrete, and of locally finite length.*

The incidence coalgebra of $\mathbf{C}$ has comultiplication given by the span

$$\mathbf{C}_1 \xleftarrow{d_1} \mathbf{C}_2 \xrightarrow{(d_2, d_0)} \mathbf{C}_1 \times \mathbf{C}_1,$$

where d_1 joins the two layers, and d_2 and d_0 return the two layers. The comultiplication of a poset is thus obtained by summing over admissible cuts (a 2-layering of the poset) and taking tensor product of the two layers.

Similarly, let $\mathbf{I}_n$ denote the groupoid of all layerings of finite sets. Again these groupoids assemble into a simplicial groupoid, denoted $\mathbf{I}$.

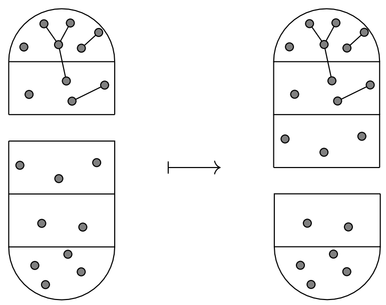
Proposition 3.1.13 ([28, Proposition 4.3, Lemma 4.4]). *The simplicial groupoid* $\mathbf{I}$ *is a Segal space, and hence a decomposition space, which is complete, locally finite, locally discrete, and of locally finite length.*

The simplicial groupoid $\mathbf{C}$ is the decomposition space corresponding to the terminal directed restriction species, finite posets and convex maps, while $\mathbf{I}$ is the decomposition space corresponding to the terminal restriction species, finite sets and injections. The incidence coalgebra of $\mathbf{I}$ is the binomial coalgebra [28, §2.4] with well-known Möbius function $(-1)^n$ for a set with n elements.

We can now apply the abacus construction introduced above.

Lemma 3.1.14. *The bisimplicial groupoid* $B := \mathbf{I} \square \mathrm{Dec}_\perp \mathbf{C}$ *has an augmented abacus map with* $f_{i,j} : B_{i+1,j} \to B_{i,j+1}$ *given by moving the last layer of the set into a new first layer of the poset.*

Proof. Remark that the augmentation column is $\mathbf{I}$ and the augmentation row is $\mathbf{C}$. The groupoid $B_{i,j}$ consists of pairs of layerings $(S \to \underline{i}, P \to \underline{j+1})$ where S is a finite set, and P is a finite poset. The map $f_{i,j}$ sends $(S \xrightarrow{a} \underline{i}, P \xrightarrow{b} \underline{j+1})$ to $(a^{-1}(\underline{i-1}) \to \underline{i-1}, (a^{-1}(i) + P) \to \underline{1+j+1})$. The following picture represents the map $f_{2,1}$ sending an element in the groupoid $B_{3,1}$ to an element in the groupoid $B_{2,2}$, where layers are numbered from bottom to top.



The verification of the abacus map axioms is straightforward. $\square$

Lemma 3.1.15. *The map* $f_{\bullet,j}$ *is a right fibration and the map* $f_{i,\bullet}$ *is a left fibration.*

Proof. The map $f_{\bullet,j}$ is a right fibration: for each $(S, P) \in B_{n,j}$, the fibres of $e_0 : B_{n+1,j} \to B_{n,j}$ along (S, P) and $e_0 : B_{n,j+1} \to B_{n-1,j+1}$ along $f_{n-1,j}(S, P)$ consist both of triples $(A, P' \xrightarrow{\alpha} P, S' \xrightarrow{\beta} S)$, where A is a finite set and α and β are monotone bijections:

$$\begin{tikzcd} F_{(S,P)} \arrow[r] \arrow[d] & B_{n+1,j} \arrow[r, "f_{n,j}"] \arrow[d, "e_0"] & B_{n,j+1} \arrow[d, "e_0"] \\ 1 \arrow[r, "\ulcorner (S,P) \urcorner"'] & B_{n,j} \arrow[r, "f_{n-1,j}"'] & B_{n-1,j+1}. \end{tikzcd}$$

The map $f_{i,\bullet}$ is a left fibration: for each $(S, P) \in B_{i+1,j}$, the fibres of $d_\top : B_{i+1,j+1} \to B_{i+1,j}$ along (S, P), and of $d_\top : B_{i,j+2} \to B_{i,j+1}$ along $f_{i,j+1}(S, P)$ are equivalent since they consist both of pairs $(S' \to \underline{i+1}, P' \xrightarrow{b} \underline{j+2})$, such that $S' \simeq S$, and $b^{-1}(\{0, \ldots, j+1\}) \simeq P$. $\square$

The main object of interest will be the augmented bisimplicial groupoid obtained by applying Theorem 3.1.1 to B. We denote it $\mathbf{B}$. By unpacking the general construction we get the following explicit description: the groupoid $\mathbf{B}_{i,j}$ consists of pairs of layerings $(S \to \underline{i}, P \to \underline{j+1})$ where S is a finite set, and P is a finite poset. For example, $\mathbf{B}_{0,0}$ is the groupoid of 1-layered finite posets. The horizontal face maps (taking place only on the $(j+1)$-layered finite poset part) are given by:

- $d_k : \mathbf{B}_{i,j} \to \mathbf{B}_{i,j-1}$ joins the layers $(k+1)$ and $(k+2)$ of the poset, for all $j > 0$ and $0 \leqslant k \leqslant j-1$;
- $d_\top = d_j : \mathbf{B}_{i,j} \to \mathbf{B}_{i,j-1}$ deletes the last layer.

Horizontal degeneracy maps are given by inserting empty layers: $s_k : \mathbf{B}_{i,j} \to \mathbf{B}_{i,j+1}$ inserts an empty $(k+2)$nd layer in the poset, for all $j \geqslant 0$ and $0 \leqslant k \leqslant j$.

The vertical face maps are given by:

- $e_\perp = e_0 : \mathbf{B}_{i,j} \to \mathbf{B}_{i-1,j}$ deletes the first layer of the set, for all $i > 0$;
- $e_k : \mathbf{B}_{i,j} \to \mathbf{B}_{i-1,j}$ joins the layers k and $k+1$ of the set, for all $0 < k < i$.

According to the modification, the top vertical face map is given by $e_\top = d_0 \circ f_{i,j}$.

- $e_\top = e_i : \mathbf{B}_{i,j} \to \mathbf{B}_{i-1,j}$ joins the last layer of the set and the first layer of the poset into the first layer of the poset.

In this way, the top vertical map keeps some information about the last layer of the set, instead of just throwing it away. Vertical degeneracy maps are given by inserting empty layers: $t_k : \mathbf{B}_{i,j} \to \mathbf{B}_{i+1,j}$ inserts an empty $(k+1)$st layer to the set, for all $0 \leqslant k \leqslant i$. The augmentation maps are $u : \mathbf{B}_{i,0} \to \mathbf{I}_i$ deleting the whole 1-layered poset and $v : \mathbf{B}_{0,j} \to \mathbf{C}_j$ deleting the first layer of the poset. It should be noted that the row $\mathbf{B}_{0,\bullet}$ is the lower décalage of $\mathbf{C}$, that v is the décalage map given by the original d_0, and that u is the augmentation map that décalage always have.

Proposition 3.1.16. *With augmentations maps u and v, the bisimplicial groupoid $\mathbf{B}$ is a bicomodule configuration.*

Proof. The result follows from Lemmas 3.1.14-3.1.15 and Proposition 3.1.8. □

3.2 *Möbius functions*

Möbius function of the decomposition space of finite posets

The augmented bisimplicial groupoid $\mathbf{B}$ of layered sets and posets is a bicomodule configuration between $\mathbf{I}$ and $\mathbf{C}$. By Theorem 2.1.10, the spans

$$\mathbf{B}_{0,0} \xleftarrow{e_1} \mathbf{B}_{1,0} \xrightarrow{(u,e_0)} \mathbf{I}_1 \times \mathbf{B}_{0,0}$$

and
$$\mathbf{B}_{0,0} \xleftarrow{d_0} \mathbf{B}_{0,1} \xrightarrow{(d_1,\nu)} \mathbf{B}_{0,0} \times \mathbf{C}_1$$
induce on $\boldsymbol{Grpd}_{/\mathbf{B}_{0,0}}$ the structure of a bicomodule over $\boldsymbol{Grpd}_{/\mathbf{I}_1}$ and $\boldsymbol{Grpd}_{/\mathbf{C}_1}$.

In order to be in position to apply the generalised Rota formula from Theorem 2.3.8, we first need more structure to define Möbius functions, and then some finiteness conditions to take homotopy cardinality.

Lemma 3.2.1. *The comodule configuration $\mathbf{B}_{0,\bullet} \to \mathbf{C}$ is complete.*

Proof. The right pointing is given by the extra degeneracy map $s_{-1} = \mathrm{id}_{\mathbf{I}_i} \times s_0$ that the décalage always have, and is thus a section to d_0. It is also a monomorphism since s_0 is a monomorphism (the decomposition space $\mathbf{C}$ of layered finite posets is complete). □

The right completeness of Lemma 3.2.1 holds for any bisimplicial groupoid of the form $X \,\square\, \mathrm{Dec}_\perp Y$ such that that X is a Segal space and Y a complete decomposition space. In contrast, the left completeness of the following lemma requires further structure, namely the extra top degeneracy map which we can define in this specific example (and which is a section to the new top face map but not to the old).

Lemma 3.2.2. *The comodule configuration $\mathbf{B}_{\bullet,0} \to \mathbf{I}$ is complete.*

Proof. We provide a left pointing: define a new extra (vertical) degeneracy map $t_{\top+1} : \mathbf{B}_{i,j} \to \mathbf{B}_{i+1,j}$ for $i \geqslant -1$ in the following way: we move the discrete part of the bottom layer of the poset into a new top layer of the set. It is a section to $e_\top$. It is also a monomorphism: the fibre $F_{(S,P)}$ of $t_{\top+1}$ over $(S \xrightarrow{a} \underline{i{+}1}, P \xrightarrow{b} \underline{j{+}1})$ is given by the pullback

$$\begin{tikzcd} F_{(S,P)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{B}_{i,j} \arrow[d, "t_{\top+1}"] \\ 1 \arrow[r, "\ulcorner (S,P) \urcorner"'] & \mathbf{B}_{i+1,j}. \end{tikzcd}$$

If P is a $(j{+}1)$-layered poset such that the bottom layer has an non empty discrete part, then the fibre is empty. Otherwise, the fibre consists of pairs (S', P') such that $S' \simeq a^{-1}(\{1, \ldots i\})$, the discrete part dP'_1 of the bottom layer of the poset P' is isomorphic to the last layer of the set S, that is $dP'_1 \simeq a^{-1}(\{i+1\})$, and $P' - dP'_1 \simeq P$. There can only be one morphism, and the fibre is then contractible. □

Proposition 3.2.3. *The bicomodule configuration $\mathbf{B}$ is Möbius.*

Proof. The groupoid $\mathbf{B}_{0,0}$ of 1-layered finite posets is locally finite since each finite poset has only finitely many automorphisms. The maps s_{-1} and $t_{\top+1}$ are finite (since they are monomorphisms as seen above). The maps d_0 and $e_\top$ are finite because each finite poset has only a finite number of 2-layerings.

The groupoid $\vec{\mathbf{B}}_{0,n}$ consists of simplices which are not in the image of a degeneracy map, including the new one s_{-1}. In the present poset case, a simplex is an $(n+1)$-layered poset with no empty layers. Similarly, a simplex in $\vec{\mathbf{B}}_{n,0}$ is an n-layered set and a poset, such that the set does not have empty layers and the poset does not have an empty discrete part. Finally, for all $P \in \mathbf{B}_{0,0}$, the fibres of $d_0^{(n)} : \vec{\mathbf{B}}_{0,n} \to \mathbf{B}_{0,0}$ and $e_\top^{(n)} : \vec{\mathbf{B}}_{n,0} \to \mathbf{B}_{0,0}$ are empty for n big enough since layered posets in $\vec{\mathbf{B}}_{0,n}$, or layered sets in $\vec{\mathbf{B}}_{n,0}$, do not have empty layers. $\square$

The augmented bisimplicial groupoid of layered sets and posets is a Möbius bicomodule configuration. The main contribution of the present paper is the fact that the following formula can be derived from the generalised Rota formula of Theorem 2.3.8.

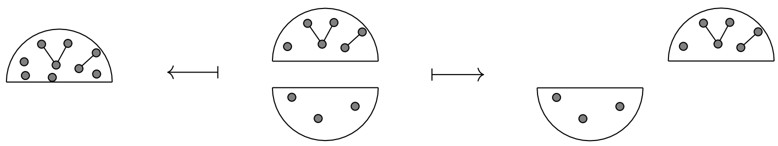
Theorem 3.2.4. *The Möbius function of the incidence algebra of the decomposition space* $\mathbf{C}$ *of finite posets is*

$$\mu(P) = \begin{cases} (-1)^n & \text{if } P \in \mathbf{C}_1 \text{ is a discrete poset with } n \text{ elements} \\ 0 & \text{else.} \end{cases}$$

Proof. The left coaction $\gamma_l : \boldsymbol{Grpd}_{/\mathbf{B}_{00}} \to \boldsymbol{Grpd}_{/\mathbf{I}_1} \otimes \boldsymbol{Grpd}_{/\mathbf{B}_{0,0}}$ is given by the span

$$\mathbf{B}_{0,0} \xleftarrow{e_1} \mathbf{B}_{1,0} \xrightarrow{(u,e_0)} \mathbf{I}_1 \times \mathbf{B}_{0,0},$$

where u deletes the last layer, e_0 deletes the first layer, and e_1 joins the two layers. The following picture represents elements in the corresponding groupoids.

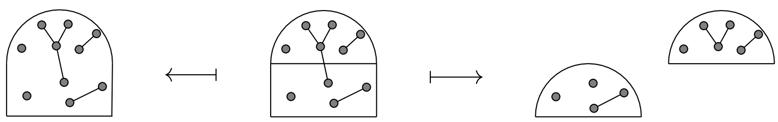


This left coaction γ_l splits a 1-layered poset into a 1-layered set and a 1-layered poset.

The right coaction $\gamma_r : \boldsymbol{Grpd}_{/\mathbf{B}_{00}} \to \boldsymbol{Grpd}_{/\mathbf{B}_{0,0}} \otimes \boldsymbol{Grpd}_{/\mathbf{C}_1}$ is given by the span

$$\mathbf{B}_{0,0} \xleftarrow{d_0} \mathbf{B}_{0,1} \xrightarrow{(d_1,\nu)} \mathbf{B}_{0,0} \times \mathbf{C}_1,$$

where d_1 deletes the last layer, ν deletes the first layer, and d_0 joins the two layers. The following picture represents elements in the corresponding groupoids.



This right coaction γ_r splits a 1-layered poset into two 1-layered posets.

Computing the right-hand side of the formula of Theorem 2.3.8, we obtain

$$(|\delta^L| \star_r |\mu^Y|)(P) = |\mu^Y|(P), \text{ for all poset } P \in \mathbf{B}_{0,0}.$$

Indeed, $|\delta^L|$ is different from 0 only if evaluated on the empty poset, and we are in this situation only if the 2-layered poset in $\mathbf{B}_{0,1}$ consists of an empty first layer, and a second layer with the whole original poset. The left-hand side gives

$$(|\mu^X| \star_l |\delta^R|)(P) = \begin{cases} |\mu^X|(P) & \text{if the poset } P \text{ is discrete} \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, $|\delta^R|$ is different from 0 only for the empty set, and we are in this situation only if the 2-layered poset in $\mathbf{B}_{1,0}$ consists of an empty second layer, and a first discrete layer (that is a set) in $\mathbf{I}_1$. We conclude by recalling that the Möbius function μ of a set S with n elements is given by $\mu(S) = (-1)^n$. □

Möbius function of any directed restriction species

We have treated the case of the decomposition space of finite posets, corresponding to the terminal directed restriction species. A *directed restriction species* is a groupoid-valued presheaf on the category $\mathbb{C}$ of finite posets and convex maps. Every directed restriction species $R : \mathbb{C}^{op} \to \boldsymbol{Grpd}$ defines a decomposition space $\mathbf{R}$ (and hence a coalgebra); we refer to [28] for all details. This decomposition space comes equipped with a culf functor $\mathbf{R} \to \mathbf{C}$ [28, Lemma 7.6]. It follows that $\mathbf{R}$ is complete, locally finite, locally discrete, and of locally finite length, and is in particular a Möbius decomposition space. Examples of directed restrictions species include rooted forests and directed graphs of various kinds [28]. Having computed the Möbius function for the decomposition space $\mathbf{C}$ in the previous section, we can now obtain it for all directed restriction species.

Corollary 3.2.5. *The Möbius function of the incidence algebra of the decomposition space $\mathbf{R}$ associated to a directed restriction species $R : \mathbb{C}^{op} \to \boldsymbol{Grpd}$ is*

$$\mu(Q) = \begin{cases} (-1)^n & \textit{if the underlying poset of } Q \in \mathbf{R}_1 \textit{ is discrete with } n \textit{ elements} \\ 0 & \textit{else.} \end{cases}$$

Proof. Once an expression has been found for the Möbius function of the decomposition space $\mathbf{C}$, it can be pulled back to $\mathbf{R}$ along the culf functor $\mathbf{R} \to \mathbf{C}$ to obtain the corresponding expression for the Möbius function of $\mathbf{R}$, as done in [29]. □

Note that an ordinary restriction species in the sense of Schmitt [53] is a special case of a directed restriction species, namely one supported on discrete posets [28]. The Möbius function then reduces to the well-known formula $(-1)^n$ for an underlying set with n elements, see [29, §3.3.10].

Corollary 3.2.6. *The Möbius function of the Butcher–Connes–Kreimer Hopf algebra of rooted forests is*

$$\mu(F) = \begin{cases} (-1)^n & \textit{if } F \textit{ consists of } n \textit{ isolated root nodes} \\ 0 & \textit{else.} \end{cases}$$

Proof. Rooted forests form an example of directed restriction species [28, §7.12]: a rooted forest has an underlying poset, with induced rooted-forest structure on convex subposets. The resulting bialgebra is the Butcher–Connes–Kreimer Hopf algebra [28, §2.2]. □

Finally, we obtain the Möbius function of the incidence bialgebra of P-trees, for any finitary polynomial endofunctor $P : \boldsymbol{Grpd}_{/I} \to \boldsymbol{Grpd}_{/I}$, that is, given by a diagram of groupoids $I \to E \xrightarrow{p} B \to I$ such that the fibres of $E \xrightarrow{p} B$ are finite. A P-tree is a tree with edges decorated in I, and nodes decorated in B; we refer to [23] for a precise definition and examples. Note that allowing the nodeless tree, the notion of forests do not form a directed restriction species [28, §7.12].

Corollary 3.2.7. *The Möbius function of the incidence bialgebra of P-trees (for any finitary polynomial endofunctor P) is*

$$\mu(T) = \begin{cases} (-1)^n & \textit{if } T \textit{ consists of } n \ P\textit{-corollas and possibly isolated edges} \\ 0 & \textit{else.} \end{cases}$$

Proof. The free monad F on a finitary polynomial endofunctor P is the polynomial monad represented by

$$A \leftarrow T'_P \xrightarrow{q} T_P \to A,$$

where T_P is the groupoid of P-trees, T'_P is the set of isomorphism classes of P-trees with a marked leaf, the left map returns the decoration of the marked leaf, the right map returns the decoration of the root, and the middle map forgets the mark. Operads can be seen as certain polynomial monads [43, §2.6]. The operations are the P-corollas. The factorisations of operations correspond to cuts in trees. The bialgebra of P-trees is the incidence bialgebra of the free monad on P, meaning the incidence bialgebra of the two-sided bar construction on the free monad on P, see [43].

The *core* of a P-tree is the combinatorial tree obtained by forgetting the P-decoration, the leaves, and the root edge [39]. It defines a culf functor from the bar-construction of a free monad to the decomposition space of trees [43]. In the same way as the proof of Corollary 3.2.5, we pull back the expression of Corollary 3.2.6 along the culf functor to conclude. □

4

Hereditary species as monoidal decomposition spaces, comodule bialgebras, and operadic categories

4.1 Hereditary species

Let $\mathbb{B}$ denote the category of finite sets and bijections, let $\mathbb{I}$ denote the category of finite sets and injective maps, and let $\mathbb{S}$ denote the category of finite sets and surjective maps. We denote by $\mathbb{S}_p$ the category of finite sets and partially defined surjections. A partially defined surjection $V \to W$ consists of a subset $U \subset V$ and a surjection $U \to W$. More formally, the arrows in $\mathbb{S}_p$ are given by equivalence classes of spans

$$\begin{tikzcd} & U \arrow[dl, tail, "i"'] \arrow[dr, two heads, "p"] & \\ V & & W \end{tikzcd}$$

where i is injective and p is surjective, and where two such spans are equivalent if they are isomorphic as spans. Partially defined surjections are composed by pullback composition of spans (in the category of sets). This is meaningful since both injections and surjections are stable under pullbacks in the category of sets. Note that the empty set is included here. Note also that the category $\mathbb{S}_p$ contains the category $\mathbb{S}$ as a subcategory (the spans in which the injection leg is an identity map) and also contains the category $\mathbb{I}^{op}$ as a subcategory (the spans in which the surjection leg is an identity map).

Species Recall that a *species* [34] is a functor $F\colon \mathbb{B} \to \textbf{\textit{Set}}$, $V \mapsto F[V]$. An element of $F[V]$ is called an F-structure on the finite set V. A *restriction species* [53] is a functor $R\colon \mathbb{I}^{op} \to \textbf{\textit{Set}}$. An R-structure on a set V thus restricts to any subset $U \subset V$. Schmitt [53] further defines a *hereditary species* to be a functor $H\colon \mathbb{S}_p \to \textbf{\textit{Set}}$. An element $G \in H[V]$ is called a H-*structure* on the set V. A hereditary species is thus covariantly functorial (not only in bijections but also) in surjections, and also contravariantly functorial in injections (that is, is a restriction species). This means that a H-structure on a set V induces also a H-structure on any quotient set and on any subset. Furthermore, these functorialities are compatible in the sense that for any pullback square

$$\begin{tikzcd} U \arrow[d, two heads, "p"'] & U' \arrow[l, tail, "i"'] \arrow[d, two heads, "p'"] \arrow[dl, phantom, "\lrcorner"] \\ V & V' \arrow[l, tail, "j"] \end{tikzcd}$$

we have

$$H[p'] \circ H[i] = H[j] \circ H[p].$$

This 'Beck-Chevalley' law is a consequence of the fact that H must respect the composition of spans.

If π is a partition of V, and $\rho_{V,\pi}\colon V \to \pi$ is the canonical surjection, the *quotient* G/π is the H-structure on the set π defined by

$$G/\pi = H[\rho_{V,\pi}](G).$$

The *restriction* $G|\pi$ is defined to be the family

$$G|\pi = \{G|B\}_{B\in\pi}.$$

A morphism of hereditary species is a natural transformation of functors. We denote by ***HSp*** the category of hereditary species and natural transformations.

Example 4.1.1. For a graph G with vertex set V, and π a partition of V, we define $G|\pi$ to be the family of graphs whose vertex sets are blocks of π and with an edge between two elements of the same block if there is an edge in G with both incident vertices in the block. We define G/π to be the graph with vertex set π and with an edge between two vertices if there is a edge in G between the corresponding blocks.

Suppose that τ is a finer partition than σ (denoted $\tau \leqslant \sigma$), that is each block of σ is a union of blocks of τ. We denote σ/τ the partition of the set τ induced by σ.

The following proposition is a consequence of the functorialities in surjections and injections, and the Beck-Chevalley law.

Proposition 4.1.2 ([53, Proposition 4.1]). *If τ and σ are partitions of V such that $\tau \leqslant \sigma$, then the following identities hold:*

$$\begin{aligned} [(G|\sigma)|\tau] &= [G|\tau], \\ [(G/\tau)|(\sigma/\tau)] &= [(G|\sigma)/\tau], \\ [(G/\tau)/(\sigma/\tau)] &= [G/\sigma], \end{aligned}$$

where $[G]$ is the isomorphism class of G.

Bialgebra To any hereditary species, Schmitt associates a commutative bialgebra structure B on the vector space spanned by all isomorphism classes of families of *non-empty* H-structures. As an algebra it is free commutative. The comultiplication of a H-structure G on the set V is defined by:

$$\Delta(G) = \sum_{\sigma\in\Pi(V)} G|\sigma \otimes G/\sigma.$$

The counit is defined by

$$\varepsilon(G) = \begin{cases} 1 & \text{if every member in the family is a singleton,} \\ 0 & \text{otherwise.} \end{cases}$$

Note the importance of disallowing empty structures. With the empty H-structure we would have $\Delta(\varnothing) = (\,) \otimes \varnothing$, and the comultiplication would not be counital on the right. While not counital on the right, we shall see later that it is still a left comodule over B.

Hereditary species of non-empty sets We consider the hereditary species of non-empty finite sets. The comultiplication of a finite set V is defined by summing over all partitions of V and putting on the left the family of blocks of the partition and on the right the set whose elements are blocks of the partition:

$$\Delta(V) = \sum_{\pi \in \Pi(V)} (V_1, \dots, V_k) \otimes \pi.$$

4.2 *A decomposition space of surjections*

We work with groupoids instead of sets, to take into account symmetries (see [7] and [29]) and define a *hereditary species* to be a functor

$$H\colon \mathbb{S}_p \to \textbf{\textit{Grpd}}.$$

In particular, a hereditary species is (covariantly) functorial in surjections and contravariantly functorial in injections, and these two functorialities interact via the Beck-Chevalley rule.

Partitions and surjections The groupoid of partitions (whose objects are sets with a partition into blocks, and arrows are bijections between sets preserving blocks) is equivalent to the groupoid of surjections (arrows are pairs of compatible bijections). More precisely, a partition of a set V can be given by a surjection $V \twoheadrightarrow P$: a block of the partition of V is the preimage of an element of P. Refinement of partitions is rendered conveniently in terms of composition of surjections. Precisely, the poset of partitions of V under refinement is equivalent to the coslice $\mathbb{S}_{V/}$: to say that $\rho \leqslant \pi$ is precisely to say that there is a commutative triangle of the corresponding surjections

$$\begin{tikzcd} & V \arrow[dl, two heads] \arrow[dr, two heads] & \\ P \arrow[rr, two heads] & & Q. \end{tikzcd}$$

More generally, given n composable surjections from V, we get n partitions of V. A pair of composable surjections $V \overset{f}{\twoheadrightarrow} P \overset{g}{\twoheadrightarrow} Q$ induces surjective maps between the fibres of gf and g over each element of Q.

Fat nerve of the category of finite sets and surjections The *fat nerve* [25] of $\mathbb{S}$ is the simplicial groupoid

$$\begin{aligned} \mathbf{N}\mathbb{S} \colon \mathbf{\Delta}^{\mathrm{op}} &\to \boldsymbol{Grpd} \\ [n] &\mapsto \mathrm{Map}([n], \mathbb{S}). \end{aligned}$$

Explicitly, $(\mathbf{N}\mathbb{S})_0$ is the groupoid of finite sets and bijections, $(\mathbf{N}\mathbb{S})_1$ is the groupoid whose objects are surjections and maps consist of a bijection between the sources, and a compatible bijection between the targets. The objects of the groupoid $(\mathbf{N}\mathbb{S})_n$ are n composable surjections, and maps are $(n+1)$-uplets of compatible bijections. The inner face maps are given by composition, the outer face maps by forgetting the first, or the last set in the chain. The degeneracy maps are given by inserting identity maps.

The skeleton $\mathbb{S}_{\mathrm{ord}}$ of $\mathbb{S}$ consisting of ordinal numbers and surjections is a full subcategory of $\mathbb{S}$, and $\mathbf{N}\mathbb{S}_{\mathrm{ord}} \simeq \mathbf{N}\mathbb{S}$.

Symmetric monoidal category monad The symmetric monoidal category monad $\mathsf{S} \colon \boldsymbol{Grpd} \to \boldsymbol{Grpd}$ [43, §2.5] is the monad represented by the polynomial

$$1 \leftarrow \mathbb{B}' \to \mathbb{B} \to 1$$

where $\mathbb{B}$ is the groupoid of finite ordinals and bijections (not required to be monotone), and $\mathbb{B}'$ is the groupoid of finite pointed ordinals and basepoint-preserving bijections. It sends a groupoid X to

$$\int^{n \in \mathbb{B}} \mathrm{Map}(\mathbb{B}'_n, X) \simeq \int^{n \in \mathbb{B}} X^{\underline{n}},$$

where $\underline{n}$ denotes the fibre over n, and the integral sign is a homotopy sum [24]:

$$\int^k X = \sum_k \frac{X}{\mathrm{Aut}\, k}.$$

Given a groupoid X, on objects $\mathsf{S}X$ is the groupoid whose objects are finite lists of objects of X, and a morphism $(a_1, \ldots, a_n) \to (b_1, \ldots, b_m)$ consists of a bijection $\underline{n} \to \underline{m}$ and morphisms $a_i \to b_{\sigma(i)}$ in X. The algebras over S are symmetric monoidal categories. The unit sends an element l to the list with one element (l), and the multiplication concatenates the lists.

Simplicial groupoid of surjections Consider the hereditary species of non-empty sets. Associated to it we construct a simplicial groupoid $\mathbf{S}$. Later this simplicial groupoid $\mathbf{S}$ will be the base ingredient in the construction of a simplicial groupoid $\mathbf{H}$ associated to each hereditary species H. The simplicial groupoid $\mathbf{H}$ will be a symmetric monoidal decomposition space, and therefore define a commutative bialgebra, which will be shown to be the Schmitt bialgebra construction. The basis elements of this bialgebra (i.e. the objects of the groupoid $\mathbf{H}_1$), will be *families* of non-empty H-structures, not individual non-empty H-structures. Similarly for $\mathbf{S}$, the basis elements, the objects of $\mathbf{S}_1$, will be *families* of non-empty finite sets, not just individual non-empty finite sets. Including families rather than just individual structures is necessary in order to have a well-defined comultiplication, since the left-hand tensor factor will be a monomial rather than a linear factor, as explained in the introduction. At the same time, working with families gives immediately the algebra structure (which is commutative free). Nevertheless, it will be technically important to consider also individual structures, which we regard as *connected* families.

Accordingly, we first describe a simplicial-groupoid-with-missing-top-face-maps, which we call $\mathbf{C}$ for 'connected'. We first consider the groupoid $\mathbf{C}_j$ of $(j-1)$ composable surjections between non-empty finite sets. The objects of $\mathbf{C}_2$ are surjections, the objects of $\mathbf{C}_1$ are non-empty finite sets, and $\mathbf{C}_0$ is the terminal groupoid, that is equivalent to a point. Face maps are given by:

- d_0 forgets the first set in the chain of surjections;
- $d_i \colon \mathbf{C}_j \to \mathbf{C}_{j-1}$ compose the ith and $(i+1)$st surjection, for $0 < i < j-1$;
- d_{j-1} forgets the last set in the chain of surjections.

The degeneracy maps $s_i \colon \mathbf{C}_j \to \mathbf{C}_{j+1}$, for $0 \leqslant i \leqslant j-1$ are given by inserting an identity arrow at object number i. The degeneracy map $s_\top \colon \mathbf{C}_j \to \mathbf{C}_{j+1}$ is given by appending with the map whose target is the terminal set 1.

Remark 4.2.1. The map to 1 is a surjection since the sets were required non-empty. With possibly empty sets, it would not be possible to define the top degeneracy map.

To obtain top face maps, it is necessary to introduce families: the top face map of a surjections chain must be the family of surjections chain shorter by one, obtained as the fibre over each element in the last set. We define $\mathbf{S}$ to be the symmetric monoidal category monad S applied to $\mathbf{C}$. All the face maps (except the missing top ones) and all the degeneracy maps are just S applied to the face and degeneracy maps of $\mathbf{C}$. The missing top face map $d_\top$ is now given by fibres: given $(j-1)$ composable surjections, for each element k of the last target set, we can form the fibres over k of the different source sets. It also induces surjective maps between these different fibres. We end up with a family (indexed by elements of the last target set) of $(j-2)$ composable surjections between the fibres. Note that the fibres are non-empty since we only consider surjections, not arbitrary maps.

Remark 4.2.2. Simplicially, the multi aspect is localised to the top face maps. This is already a feature well known from operads [43]: the 'domain' (given by the simplicial map d_1) of a single operation of an operad is not a single object but a family of objects. In the present situation, beyond operads, a new feature is that the top face maps do not satisfy the simplicial identities on the nose, only up to coherent homotopy, making altogether the simplicial groupoids pseudosimplicial (as already allowed for in the homotopy setting in which the theory of decomposition spaces is staged). This is caused by symmetries acting on blocks of partitions, and hence on factors in monomials, and seems to be an unavoidable nuisance, except in fully rigid situations such as L-species with monotone surjections.

Proposition 4.2.3. *The groupoids* $\mathbf{S}_j$ *and the degeneracy and face maps given above form a pseudosimplicial groupoid* $\mathbf{S}$*.*

Proof. The only pseudosimplicial identity is $d_\top \circ d_\top \simeq d_\top \circ d_{\top-1}$. The other simplicial identities are strict and straightforward to check. The ones involving top face and top degeneracy maps are:

$$\begin{array}{lll} d_i \circ d_\top = d_\top \circ d_i & s_i \circ d_\top = d_\top \circ s_i & d_\top \circ s_\top = \mathrm{id} \\ d_\top \circ d_\perp = d_\perp \circ d_\top & d_i \circ s_\top = s_\top \circ d_i & s_\top \circ s_i = s_i \circ s_\top \end{array}$$

It remains to verify the functor is pseudosimplicial for the top face map $d_\top$. Given two composable surjections $V \overset{f}{\twoheadrightarrow} P \overset{g}{\twoheadrightarrow} Q$, we can consider the family $\{V_p\}_{p\in P}$ of fibres of f over elements of P, and we can also consider the family of families $\left\{\{V_p\}_{p\in P_q}\right\}_{q\in Q}$. There is a canonical isomorphism $\left\{\{V_p\}_{p\in P_q}\right\}_{q\in Q} \to \{V_p\}_{p\in P}$. We want to show the following square is commutative:

$$\begin{tikzcd} ((V_k)_l)_m \arrow[r] \arrow[d] & (V_k)_l \arrow[d] \\ (V_k)_p \arrow[r] & V_p. \end{tikzcd}$$

The isomorphisms are compatible with the injections of fibres into V, by the universal property of pullback: given $p \in P$ such that $g(p) = q$, the following square is a pullback by definition:

$$\begin{tikzcd} V_p \arrow[r] \arrow[d] & V \arrow[d] \\ 1 \arrow[r, "\ulcorner p \urcorner"'] & P. \end{tikzcd}$$

We also have the following commutative diagram, given by pullbacks

$$\begin{tikzcd} (V_q)_p \arrow[r] \arrow[d] & V_q \arrow[r] \arrow[d] & V \arrow[d] \\ 1 \arrow[r, "\ulcorner p \urcorner"'] & P_q \arrow[r] \arrow[d] & P \arrow[d] \\ & 1 \arrow[r, "\ulcorner q \urcorner"'] & Q. \end{tikzcd}$$

Hence by the universal property of pullbacks, there exists a unique isomorphism $(V_q)_p \to V_p$ such that the following triangle commutes:

$$\begin{tikzcd} (V_q)_p \arrow[d, "\simeq"'] \arrow[dr] & \\ V_p \arrow[r] & V. \end{tikzcd}$$

In a similar way, we obtain the three other isomorphisms between the fibres compatible with the injections into V. Since all four isomorphisms in the square are compatible with the injections into V, the commutativity of the square is ensured by the fact that each $V_i \to V$ is a monomorphism. □

Considering only non-empty ordinals $\underline{n}$ instead of all non-empty finite sets, we get an equivalent pseudosimplicial groupoid, which we denote $\mathbf{S}_{\mathrm{ord}}$. The objects of $(\mathbf{S}_{\mathrm{ord}})_1$ are lists of non-empty ordinals, and a map between two lists $(\underline{n}_1, \dots, \underline{n}_p)$ and $(\underline{m}_1, \dots, \underline{m}_{p'})$ consists of a bijection $\sigma\colon p \to p'$ and, for all $i \in p$, a map $\rho_i\colon \underline{n}_i \to \underline{m}_{\sigma(i)}$.

Proposition 4.2.4. *We have a natural equivalence* $\mathbf{N}\mathrm{S}_{ord} \simeq \mathbf{S}_{ord}$.

Proof. The functor $\mathbf{N}\mathrm{S}_{\mathrm{ord}} \to \mathbf{S}_{\mathrm{ord}}$ sends a surjection $\underline{n} \twoheadrightarrow \underline{p}$ to the list of fibres $(\underline{n}_1, \dots, \underline{n}_p)$, and a map $(\underline{n} \xrightarrow{\rho} \underline{n}', \underline{p} \xrightarrow{\sigma} \underline{p}')$ between two surjections $\underline{n} \twoheadrightarrow \underline{p}$ and $\underline{n}' \twoheadrightarrow \underline{p}'$ to the map $(\underline{n}_1, \dots, \underline{n}_p) \xrightarrow{(\sigma, \rho_1, \dots, \rho_p)} (\underline{n}'_1, \dots, \underline{n}'_{p'})$. Note that the surjection $\varnothing \twoheadrightarrow \varnothing$ gives the empty list, which is allowed. Similarly, it sends a family of n composable surjections to the list of $(n-1)$ composable surjections given by the fibres. The maps are given in a similar way.

There is also a functor $\mathbf{S}_{\mathrm{ord}} \to \mathbf{N}\mathrm{S}_{\mathrm{ord}}$. Given a list of non-empty ordinals $(\underline{n}_1, \dots, \underline{n}_p)$, we obtain a surjection $\sum_{i \in \underline{p}} \underline{n}_i \twoheadrightarrow \underline{p}$ from the disjoint union of the elements of the list to the indexing set. (This map is a surjection because all the n_i are non-empty.) Given a map between two lists $(\underline{n}_1, \dots, \underline{n}_p)$ and $(\underline{m}_1, \dots, \underline{m}_{p'})$, we obtain a map $\sum_{i \in \underline{p}} \underline{n}_i \to \sum_{i \in \underline{p}'} \underline{m}_i$ by sending $\underline{n}_i$ to $\underline{m}_{\sigma(i)}$. Similarly, given a list of $(j-1)$ composable surjections, we obtain j composable surjections by disjoint union, and the last one is given as before, using the target sets of the surjections.

It is easy to check they form an equivalence since the disjoint union of fibres of a surjection is isomorphic to the source set of this surjection. □

Proposition 4.2.5. *The pseudosimplicial groupoid* $\mathbf{S}$ *is Segal, and hence a decomposition space.*

Proof. It follows from the equivalence $\mathbf{N}\mathrm{S}_{\mathrm{ord}} \simeq \mathbf{S}_{\mathrm{ord}} \simeq \mathbf{S}$ since the fat nerve of a small category is always Segal [25]. □

Proposition 4.2.6. *The Segal space* $\mathbf{S}$ *is complete, locally finite, locally discrete, and of locally finite length.*

Proof. The map s_0 is a monomorphism because the fibre is empty if the first surjection is not the identity, and is contractible else. The groupoid $\mathbf{S}_1$ is locally finite, because elements of the families are non-empty finite sets, and each finite set has only a finite number of automorphisms. We have seen s_0 is finite and discrete, the map d_1 is also finite and discrete: the fibre of d_1 over $(n-1)$ composable surjections $f \in \mathbf{S}_n$ is either empty or the finite discrete groupoid of n composable surjections where the first one is the identity and the other surjections are given by f. Finally, $\mathbf{S}$ is of locally finite length (every edge f has finite length): the degenerate simplices are families where one of the surjections is an identity, or the last set is a singleton. The fibre of f has no nondegenerate simplices for n greater than the total number of elements of the source sets of the family. □

Remark 4.2.7. The decomposition space $\mathbf{H}$ is not usually a Segal space. The base case of the Segal condition stipulates that the square

$$\begin{tikzcd} \mathbf{H}_2 \arrow[r, "d_2"] \arrow[d, "d_0"'] & \mathbf{H}_1 \arrow[d, "d_0"] \\ \mathbf{H}_1 \arrow[r, "d_1"'] & \mathbf{H}_0 \end{tikzcd}$$

is a pullback. This would mean that one should be able to reconstruct a H-structure on a surjection $V \twoheadrightarrow P$ by knowing it on P and on the fibres $V_1, \ldots, V_p$. In other words, one could substitute the H-structures on $V_1, \ldots, V_p$ into the elements of P of another H-structure, as if those elements were 'input slots' of the operation of an operad.

Consider for example the case of simple graphs (Example 4.1.1). Given p simple graphs with vertex sets $V_1, \ldots, V_p$ and another graph with p vertices, there is no canonical prescription for substituting the p graphs into those vertices.

Since in every degree, the groupoid is given by applying S, the decomposition space $\mathbf{S}$ is automatically a symmetric monoidal decomposition space. The associated incidence bialgebra has the property that the comultiplication applied to a connected element gives a monomial in the left-hand tensor factor and a connected element (linear factor) in the right-hand tensor factor. That's the immediate conclusion of the fact that d_2 requires S whereas d_0 does not. This observation can be formalised at the simplicial level by the following result.

Recall from Proposition 2.1.1 that the decomposition space analog of (left) comodule is given by a simplicial map $f\colon C \to X$ between two simplicial groupoids such that C is Segal, X is a decomposition space and the map $f\colon C \to X$ is culf. Then the span

$$C_0 \xleftarrow{d_1} C_1 \xrightarrow{(f_1, d_0)} X_1 \times C_0$$

induces on the slice category $\boldsymbol{Grpd}_{/C_0}$ the structure of a left $\boldsymbol{Grpd}_{/X_1}$-comodule.

Lemma 4.2.8. *The slice category* $\boldsymbol{Grpd}_{/\mathbf{C}_1}$ *is a left comodule over* $\boldsymbol{Grpd}_{/\mathbf{S}_1}$*.*

Proof. Note that $\mathbf{C}$ is lacking top face maps, but $\mathrm{Dec}_\top\, \mathbf{C}$ is a genuine simplicial groupoid as required by the notion of comodule. Since $\mathrm{Dec}_\top(\mathbf{C})$ is a Segal space (the décalage of a decomposition space is always a Segal space [25, Proposition 4.9]), and $\mathbf{S}$ is a decomposition space, we just need to exhibit a culf map $\mathrm{Dec}_\top(\mathbf{C}) \to \mathbf{S}$, which is given by $d_\top$. Note that it is essentially the unit for the monad S, therefore it is cartesian, and in particular culf. □

In fact, we can consider all (possibly empty) finite sets, and still obtain a comodule structure. (The surjection $\varnothing \to \varnothing$ will be sent to the empty family.)

Lemma 4.2.9. *The slice category* $\textbf{\textit{Grpd}}_{/\mathbf{NS}_0}$ *is a left comodule over* $\textbf{\textit{Grpd}}_{/\mathbf{S}_1}$.

Proof. The fat nerve of a category is always a Segal space [25, §2.14]. We need to exhibit a culf map $\mathbf{NS} \to \mathbf{S}$, which is given by sending a surjection to the family of fibres (which are non-empty), as the map $d_\top$ of the decomposition space $\mathbf{S}$. □

4.3 *Hereditary species and decomposition spaces*

We can now add a hereditary structure on the source of each member of the family. Given a hereditary species H, define $\mathbf{H}_1$ to be the groupoid of families of non-empty H-structures. More formally, $\mathbf{H}_1$ is defined as families of the Grothendieck construction of the underlying ordinary species $H\colon \mathbb{B}_+ \to \textbf{\textit{Grpd}}$, where $\mathbb{B}_+$ denote the category of non-empty finite sets and bijections. The groupoid $\mathbf{H}_n$ is given by the pullback

$$\begin{tikzcd} \mathbf{H}_n \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{H}_1 \arrow[d] \\ \mathbf{S}_n \arrow[r, "\text{source}"'] & \mathbf{S}_1 . \end{tikzcd}$$

Inner face maps and degeneracy maps are defined by pullback. For example, in the following diagram, the right-hand square and the rectangle are pullbacks by definition

$$\begin{tikzcd} \mathbf{H}_3 \arrow[r] \arrow[d] & \mathbf{H}_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{H}_1 \arrow[d] \\ \mathbf{S}_3 \arrow[r, "d_2"'] & \mathbf{S}_2 \arrow[r, "d_1"'] & \mathbf{S}_1 . \end{tikzcd}$$

Thus the left-hand square is a pullback and it induces a map $d_2\colon \mathbf{H}_3 \to \mathbf{H}_2$. For the bottom face maps we use that H is functorial in surjections: given an object in $\mathbf{H}_n$, that is a chain of surjections $V_1 \twoheadrightarrow V_2 \twoheadrightarrow \ldots \twoheadrightarrow V_n$ with a H-structure on V_1, we get a H-structure on V_2 by functoriality, and thus an object $V_2 \twoheadrightarrow \ldots \twoheadrightarrow V_n$ of $\mathbf{H}_{n-1}$. For the top face maps, we use (contravariant) functoriality in injections, this is restriction to each of the fibres, and produces thus a family, even if the input is a single chain of surjections.

Proposition 4.3.1. *The groupoids* $\mathbf{H}_j$ *form a simplicial groupoid* $\mathbf{H}$.

Proof. Checking the simplicial identities requires precisely the three functorialities of the notion of hereditary species (covariant in surjections, contravariant in injections, and Beck–Chevalley condition). For example, the maps $d_\top$ and $d_\perp$ are given by the following pullbacks

$$\begin{tikzcd} \mathbf{H}_{n+1} \arrow[r, "d_\top"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{H}_n \arrow[r, "d_\perp"] \arrow[d, "f"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{H}_{n-1} \arrow[d, "g"] \\ \mathbf{S}_{n+1} \arrow[r, "d_\top"'] & \mathbf{S}_n \arrow[r, "d_\perp"'] & \mathbf{S}_{n-1} \end{tikzcd}$$

and by Beck–Chevalley:

$$\begin{aligned}(d_\perp)_! f^* d_\top &\simeq g^*((d_\perp)_! d_\top)\\ (d_\perp)! d_\top &\simeq g^*(d_\perp d_\top)\\ d_\perp d_\top &\simeq g^*(d_\top d_\perp)\\ d_\perp d_\top &\simeq d_\top d_\perp.\end{aligned}$$

□

Proposition 4.3.2. *The simplicial groupoid* **H** *is a symmetric monoidal decomposition space.*

Proof. This follows from Proposition 1.3.2 because there is a culf map to the decomposition space **S** (in fact even a Segal space) by construction. The symmetric monoidal structure is obtained by concatenation of families. □

Lemma 4.3.3. *The groupoid* $\mathbf{H}_1$ *is locally finite.*

Proof. The objects of the groupoid $\mathbf{H}_1$ are families of non-empty H-structures, and elements of the families are non-empty finite sets. Since a finite set has only a finite number of automorphism, the automorphisms groups are finite. □

Proposition 4.3.4. *The decomposition space* **H** *is complete, locally finite, locally discrete, and of locally finite length.*

Proof. It follows from Lemmas 1.3.4 and 4.3.3, and Proposition 4.2.6 since there is a functor $\mathbf{H} \to \mathbf{S}$ which is culf by construction. □

Proposition 4.3.5. *The incidence bialgebra* B *obtained by taking homotopy cardinality of* $\boldsymbol{Grpd}_{/\mathbf{H}_1}$ *coincides with the Schmitt bialgebra associated to a hereditary species* H.

Proof. The comultiplication is given by pullback along d_1 followed by composition with (d_2, d_0). Given a H-structure G as a morphism $1 \xrightarrow{\ulcorner G \urcorner} \mathbf{H}_1$, the pullback along d_1 is the groupoid $(\mathbf{H}_2)_G$ of families of surjections with the source sets given by G (thus having H-structure). Composing with (d_2, d_0) amounts to returning for each surjection $V \twoheadrightarrow P$ the family of fibres over the element of P (having H-structure given by restriction), and the target set P (having H-structure given by quotient). We can then take cardinality since **H** is locally finite. Since **H** is locally discrete, $(\mathbf{H}_2)_G$ is discrete and the homotopy cardinality counts isomorphisms classes, giving Schmitt's comultiplication. □

Comodule structure Given a hereditary species H, define $\mathbf{M}_0$ to be the groupoid of (possibly empty) H-structures. More formally, $\mathbf{M}_0$ is defined as the Grothendieck construction of the underlying ordinary species $H\colon \mathbb{B} \to$ ***Grpd***. The groupoid $\mathbf{M}_n$ is given by the pullback

$$\begin{tikzcd} \mathbf{M}_n \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathbf{M}_0 \arrow[d] \\ \mathbf{N}\mathbb{S}_n \arrow[r, "\text{source}"'] & \mathbf{N}\mathbb{S}_0. \end{tikzcd}$$

The objects of the groupoid $\mathbf{M}_1$ are surjections with a H-structure on the source; the objects of $\mathbf{M}_n$ are composable surjections with a H-structure on the source. In the same way as for $\mathbf{H}$, the face and degeneracy maps are defined by pullback.

Lemma 4.3.6. *The groupoids* $\mathbf{M}_n$ *form a Segal space* $\mathbf{M}$.

Proof. It is easy to check that they assemble into a simplicial groupoid, using the three functorialities of the notion of hereditary species. The simplicial groupoid is equivalent to the fat nerve of the Grothendieck construction of the underlying species $H\colon \mathbb{S} \to \textit{\textbf{Grpd}}$. Whereas $\mathbf{H}$ is defined as a chain of surjections with a H-structure on the source, in the Grothendieck construction, an n-simplex is a chain with an H-structure on *each* set, and with specified isomorphisms with the H-structures pushed forward along the surjections. The presence of these specified isomorphisms readily shows that the two simplicial groupoids are level-wise equivalent. Thus $\mathbf{M}$ is Segal since the fat nerve of a small category is always Segal [25]. □

Lemma 4.3.7. *The slice category* $\textit{\textbf{Grpd}}_{/\mathbf{M}_0}$ *is a left comodule over* $\textit{\textbf{Grpd}}_{/\mathbf{H}_1}$.

Proof. The culf map $f\colon \mathbf{M} \to \mathbf{H}$ is given by taking fibres, as the map $d_\top$ of $\mathbf{H}$. Thus the span

$$\mathbf{M}_0 \xleftarrow{d_1} \mathbf{M}_1 \xrightarrow{(f_1,d_0)} \mathbf{H}_1 \times \mathbf{M}_0$$

induces on $\textit{\textbf{Grpd}}_{/\mathbf{M}_0}$ the structure of a comodule over $\textit{\textbf{Grpd}}_{/\mathbf{H}_1}$. □

4.4 *The incidence comodule bialgebra of a hereditary species*

We have constructed, via a symmetric monoidal decomposition space, the incidence bialgebra B of a hereditary species H. It is the vector space spanned by all families of non-empty H-structures. But every hereditary species is in particular a restriction species, by precomposition with the inclusion $\mathbb{I}^{op} \to \mathbb{S}_p$. Therefore there is another coalgebra, linearly spanned by the (possibly empty) H-structures themselves. The free algebra on this coalgebra is therefore the bialgebra A linearly spanned by the families of H-structures. So now we have two different bialgebra structures on closely related vector spaces, and the two share the same multiplication. The main result of this section relates these two structures.

Proposition 4.4.1. *The hereditary-species bialgebra* B *coacts on the restriction-species bialgebra* A*, so as to make* A *a left comodule bialgebra over* B.

The proof is a nice illustration of the objective method: after unpacking the definitions, the proof consists in computing a few pullbacks. Let us first recall some definitions and set notation.

Hereditary species and decomposition spaces Given a hereditary species $H\colon \mathbb{S}_p \to \textbf{\textit{Grpd}}$, we get a decomposition space $\mathbf{H}$ where an n-simplex is a family of $(n-1)$ composable surjections between non-empty finite sets, with a H-structure on each source set. The comultiplication $\textbf{\textit{Grpd}}_{/\mathbf{H}_1} \to \textbf{\textit{Grpd}}_{/\mathbf{H}_1 \times \mathbf{H}_1}$ is given by the span

$$\mathbf{H}_1 \xleftarrow{d_1} \mathbf{H}_2 \xrightarrow{(d_2, d_0)} \mathbf{H}_1 \times \mathbf{H}_1,$$

where d_1 returns the family of source sets, d_0 returns the family of target sets, and d_2 returns the family of families of fibres over each element of the target sets. Let B denote the homotopy cardinality, i.e. the numerical incidence bialgebra of $\mathbf{H}$. Note that B is commutative.

Restriction species and decomposition spaces Since $\mathbb{I}^{op}$ is a subcategory of $\mathbb{S}_p$, every hereditary species H induces a restriction species : $\mathbb{I}^{op} \to \textbf{\textit{Grpd}}$. Since the hereditary species H is fixed throughout this section, we denote the underlying restriction species simply by R. Recall from [28] that every restriction species R induces a decomposition space $\mathbf{R}$ where an n-simplex is an n-layered set with an R-structure on the underlying set. The comultiplication $\Delta\colon \textbf{\textit{Grpd}}_{/\mathbf{R}_1} \to \textbf{\textit{Grpd}}_{/\mathbf{R}_1 \times \mathbf{R}_1}$ is given by the span

$$\mathbf{R}_1 \xleftarrow{d_1} \mathbf{R}_2 \xrightarrow{(d_2, d_0)} \mathbf{R}_1 \times \mathbf{R}_1,$$

where d_1 joins the two layers of the 2-simplex, and d_2 and d_0 return the first and second layers respectively. Note that $\mathbf{R}_1 = \mathbf{M}_0$ and by Lemma 4.3.7 the slice category $\textbf{\textit{Grpd}}_{/\mathbf{R}_1}$ is a left $\textbf{\textit{Grpd}}_{/\mathbf{H}_1}$-comodule.

Comodule bialgebra For background on comodule bialgebras, see [1, §3.2] and [48]. Let B be a commutative bialgebra. Recall that a *(left) B-comodule bialgebra* is a bialgebra object in the braided monoidal category of left B-comodules. For any coalgebra B we have the category of left B-comodules. A left B-comodule is a vector space M equipped with a coaction $\gamma\colon M \to B \otimes M$ satisfying the usual axioms. So far only the coalgebra structure of B is needed. The algebra structure of B comes in to provide a monoidal structure on the category of left B-comodules. It is given as follows. If M and N are left B-comodules, then $M \otimes N$ is given a left B-comodule structure by the composite map

$$M \otimes N \to B \otimes M \otimes B \otimes N \xrightarrow{\omega} B \otimes M \otimes N,$$

where the map ω is given by first swapping the two middle tensor factors, and then using the multiplication of B in the two now adjacent B-factors. It follows from the bialgebra axioms that this is a valid left B-coalgebra structure. This defines the monoidal structure on the category of left B-comodules. The unit object for this monoidal structure is the B-comodule $\mathbb{Q}$ (with structure map the unit of B). Finally, it is easy to check that the braiding of the underlying tensor product of vector spaces lifts to a braiding on the category of B-comodules. This requires the multiplication of B to be commutative.

We now have a braided monoidal structure on the category of left B-comodules, and it makes sense to consider bialgebras in here. For reference, let us recall that a bialgebra in the braided monoidal category of left B-comodules is a B-comodule M together with structure maps

$$\Delta_M : M \to M \otimes M \qquad \varepsilon_M : M \to \mathbb{Q}$$
$$\mu_M : M \otimes M \to M \qquad \eta_M : \mathbb{Q} \to M$$

which are all required to be B-comodule maps and to satisfy the usual bialgebra axioms. We shall be concerned in particular with the requirement that Δ and ε be B-comodule maps:

$$\begin{tikzcd} M \arrow[rr, "\Delta_M"] \arrow[dd, "\gamma"'] & & M \otimes M \arrow[d, "\gamma\otimes\gamma"] \\ & & B \otimes M \otimes B \otimes M \arrow[d, "\omega"] \\ B \otimes M \arrow[rr, "B\otimes\Delta_M"'] & & B \otimes M \otimes M \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "\varepsilon_M"] \arrow[d, "\gamma"'] & \mathbb{Q} \arrow[d, "\eta_B"] \\ B \otimes M \arrow[r, "B\otimes\varepsilon"'] & B \end{tikzcd}$$

To simplify the proof of Proposition 4.4.1, we shall invoke the following general result.

Lemma 4.4.2. *If* M *is a comodule coalgebra over* B*, then the free algebra* SM *is naturally a comodule bialgebra over* B*.*

Proof. If $\gamma : M \to B \otimes M$ is the coaction for M, then the new coaction $\overline{\gamma} : SM \to B \otimes SM$ is given by extending multiplicatively, and using the algebra structure of B:

$$SM \xrightarrow{S(\gamma)} S(B \otimes M) \longrightarrow SB \otimes SM \xrightarrow{\mu_B \otimes \mathrm{Id}_{SM}} B \otimes SM.$$

Here the middle map is the oplax-monoidal structure of S. If $\Delta_M : M \to M \otimes M$ is the comultiplication of M, then the new comultiplication $\overline{\Delta}_M : SM \to SM \otimes SM$ is given by extending multiplicatively in the usual way:

$$SM \xrightarrow{S(\Delta_M)} S(M \otimes M) \longrightarrow SM \otimes SM.$$

It is now straightforward to check that $\overline{\Delta}_M$ and the new free multiplication are B-comodule maps for $\overline{\gamma}$. □

Proposition 4.4.1 now follows from the following result, together with Lemma 4.4.2.

Proposition 4.4.3. *Let* A *be the incidence coalgebra of the ordinary restriction species underlying* H*. Then* A *is naturally a left* B*-comodule coalgebra (where* B *is the incidence bialgebra of the hereditary species as in Section 4.3).*

Proof. The underlying vector space of A is the homotopy cardinality of the comodule of Lemma 4.3.7. It remains to check that the structure maps Δ and ε of the incidence coalgebra of the ordinary restriction species are B-comodule maps. We need to check that the two above squares are commutative.

The composition of comultiplications is given by composition of spans. We need to exhibit a commutative diagram as follows, such that the bottom left-hand square, and the top right-hand squares are pullbacks:

$$\begin{tikzcd}
\mathbf{R}_1 & \mathbf{R}_2 \arrow[l, "d_1"'] \arrow[r, "(d_2,d_0)"] & \mathbf{R}_1 \times \mathbf{R}_1 \\
\mathbf{M}_1 \arrow[u, "d_1"] \arrow[ddd, "(f,d_0)"'] & X_3 \arrow[u, dashed, "\bar{d}_1"] \arrow[l, dashed, "\bar{d}_2"'] \arrow[r, dashed, "\bar{d}_3"'] \arrow[ddd, dashed, "(g,\bar{d}_0)"] \arrow[ur, phantom, "\lrcorner"] \arrow[ddl, phantom, "\llcorner"] & \mathbf{M}_1 \times \mathbf{M}_1 \arrow[u, "d_1 \otimes d_1"'] \arrow[d] \\
 & & \mathbf{H}_1 \times \mathbf{R}_1 \times \mathbf{H}_1 \times \mathbf{R}_1 \arrow[dd] \\
 & & \\
\mathbf{H}_1 \times \mathbf{R}_1 & \mathbf{H}_1 \times \mathbf{R}_2 \arrow[l, "\mathrm{id} \otimes d_1"] \arrow[r, "\mathrm{id} \otimes (d_2,d_0)"'] & \mathbf{H}_1 \times \mathbf{R}_1 \times \mathbf{R}_1 .
\end{tikzcd}$$

The objects of $\mathbf{R}_1$ are H-structures. The groupoid X_3 consists of pairs of composable maps $V \twoheadrightarrow P \to 2$, such that the first one is a surjection, and with a H-structure on V. The map $\overline{d}_0$ sends $(V \twoheadrightarrow P \to \underline{2})$ to $(P \to \underline{2})$, the map $\overline{d}_1$ sends it to $(V \to \underline{2})$, the map $\overline{d}_2$ to $(V \twoheadrightarrow P)$, the map $\overline{d}_3$ to the pair of surjections between the fibres $(V_1 \twoheadrightarrow P_1, V_2 \twoheadrightarrow P_2)$, and the map g to the family $\{V_i\}_{i \in P}$ of fibres of the surjection $V \twoheadrightarrow P$ over all the elements of P.

Recall that objects of $\mathbf{R}_2$ are maps of sets $V \to \underline{2}$, with a H-structure on the source. Objects of $\mathbf{H}_1$ are surjections with a H-structure on the source.

It is straightforward to verify the four squares are commutative, using the functoriality of H and the Beck-Chevalley rule. The structure on $\mathbf{H}_1 \times \mathbf{R}_1 \times \mathbf{R}_1$ is obtained as follow. On $\mathbf{H}_1$, the structure is given by restriction on the fibres, and the different paths give equivalent output since H is contravariantly functorial in injections. For the two $\mathbf{R}_1$, the structure is given by quotient (functoriality in surjections) then restriction taking the left then down path, or by restriction then quotient taking the top then right path; this gives equivalent output by the Beck-Chevalley rule.

The lower left-hand square is a pullback. Indeed after projecting away $\mathbf{H}_1$, it is enough to verify, by Lemma 1.2.1, that the bottom square of the following diagram is a pullback:

$$\begin{tikzcd}
\mathbf{M}_1 \arrow[d, "(f,d_0)"] \arrow[dd, bend right=60, "d_0"'] & X_3 \arrow[l, "\overline{d}_2"'] \arrow[d, "(g,\overline{d}_0)"'] \arrow[dd, bend left=60, "\overline{d}_0"] \\
\mathbf{H}_1 \times \mathbf{R}_1 \arrow[d] & \mathbf{H}_1 \times \mathbf{R}_2 \arrow[l, "\mathrm{id} \times d_1"] \arrow[d] \\
\mathbf{R}_1 & \mathbf{R}_2 . \arrow[l, "d_1"]
\end{tikzcd}$$

The fibre of $\overline{d}_0$ over a element $P \to \underline{2}$ of $\mathbf{R}_2$, or the fibre of $\overline{d}_0$ over the element $d_1(P \to \underline{2}) = P$ consist both of pairs with a surjection onto P: $V \twoheadrightarrow P$ and the map $P \to \underline{2}$. Thus by Lemma 1.1.5, the bottom square is a pullback

The top right-hand square is also a pullback, using Lemma 1.1.5 one more time: the fibre of $\overline{d}_1$ over a object $V \to \underline{2}$ of $\mathbf{R}_2$ consists of pairs of composable maps, where the first one is a surjection from V, the second one is a map to $\underline{2}$, such that the composition is $V \to \underline{2}$. The fibre of $\overline{d}_1$ over the object $(d_2, d_0)(V \to \underline{2})$ is a pair of surjections with sources V_1 and V_2. This is equivalent to the fibre of $\overline{d}_0$ over $P \to \underline{2}$ since we can take the disjoint union of $V_1 \twoheadrightarrow P_1$ and $V_2 \twoheadrightarrow P_2$ to get a surjection, and we obtain a map to $\underline{2}$ by sending elements of P_1 to 1, and elements of P_2 to 2.

For the counit condition, it is easy to verify that the following diagram is commutative and the two marked squares are pullbacks

$$\begin{tikzcd}
\mathbf{R}_1 & \mathbf{R}_0 \arrow[l, "s_0"'] \arrow[r] & 1 \\
\mathbf{M}_1 \arrow[u, "d_1"] & \mathbf{R}_0 \arrow[l, dashed] \arrow[r, dashed] \arrow[u, dashed] \arrow[ur, phantom, "\urcorner"] \arrow[dl, phantom, "\llcorner"] & 1 \arrow[u] \arrow[d] \\
\mathbf{H}_1 \times \mathbf{R}_1 & \mathbf{H}_1 \times \mathbf{R}_0 \arrow[l, "\mathrm{id} \otimes s_0"] \arrow[r] & \mathbf{H}_1 .
\arrow[from=2-1, to=3-1, "{(f,d_0)}"']
\arrow[from=2-2, to=3-2, dashed]
\end{tikzcd}$$

Indeed, the bottom left pullback is the groupoid of surjections $V \twoheadrightarrow P$ with a H-structure on V, such that the induced H-structure on P is an empty H-structure. This implies that both P and V are empty, so it is just any H-structure on the empty set, which is $\mathbf{R}_0$.

□

Example 4.4.4. The hereditary species of simple graphs, described in Example 4.1.1, induces a comodule bialgebra. The first comultiplication is given by the hereditary structure as in Section 4.1. The secondary comultiplication is given by plain restriction species structure. By Proposition 4.4.1, this is a comodule bialgebra. It has been studied deeply by Foissy [21]. It is interesting here to see this example as an instance of a general construction.

4.5 *Hereditary species and operadic categories*

As we have seen, hereditary species are operad-like without being operads, in the sense that they admit a kind of two-sided bar construction, which is not in general a Segal space. A relationship between operadic categories and decomposition spaces was established recently be Garner, Kock, and Weber [30]. They show that certain unary operadic categories are decomposition spaces. The following construction shows that certain *non-unary* operadic categories are decomposition spaces, namely those that come from hereditary species.

Since we are going to verify the axioms of operadic category in detail, we list them here, following the formulation of [30].

Let $\mathbb{F}$ denote the category whose objects are the sets $n = \{1, \ldots, n\}$ for $n \in \mathbb{N}$ and whose maps are arbitrary functions. We denote by $1 \in 1$ the unique element in the terminal object. In any category with terminal object 1, we write $\tau_X \colon X \to 1$ for the unique map from an object to the terminal.

Given a function $\varphi \colon m \to n$ in $\mathbb{F}$ and $i \in n$, there is a unique monotone injection

$$\varepsilon_{\varphi,i} \colon \varphi^{-1}(i) \rightarrowtail m \tag{1}$$

in $\mathbb{F}$ whose image is $\{ j \in m \colon \varphi(j) = i \}$; the object $\varphi^{-1}(i)$ is called the *fibre of* φ *at* i. Often the map φ is clear from the context, and we write simply

$$\varepsilon_i \colon m_i \rightarrowtail m.$$

If we are given two maps in $\mathbb{F}$, $\ell \xrightarrow{\psi} m \xrightarrow{\varphi} n$, then we denote by ψ_i^φ the unique map comparing the fibres, given by the universal property of pullback:

$$\begin{tikzcd} \ell_i \arrow[r, "\psi_i^\varphi"] \arrow[d, tail, "\varepsilon_i"'] \arrow[dr, phantom, "\lrcorner", very near start] & m_i \arrow[r] \arrow[d, tail, "\varepsilon_i"'] \arrow[dr, phantom, "\lrcorner", very near start] & \{i\} \arrow[d, tail] \\ \ell \arrow[r, "\psi"'] & m \arrow[r, "\varphi"'] & n \end{tikzcd} \tag{2}$$

and call it the *fibre map of* ψ *with respect to* φ *at* i. We usually omit the ψ-decoration.

Operadic categories An *operadic category* [8] is given by the following data:

(D1) a category $\mathcal{C}$ endowed with chosen local terminal objects (i.e. a chosen terminal object in each connected component);

(D2) a *cardinality functor* $|{-}| \colon \mathcal{C} \to \mathbb{F}$;

(D3) for each object $X \in \mathcal{C}$ and each $i \in |X|$ a *fibre functor*

$$\phi_{X,i} : \mathcal{C}/X \to \mathcal{C}$$

whose action on objects and morphisms we denote as follows:

$$\begin{tikzcd} Y \arrow[r, "f"] & X & \mapsto & f^{-1}(i) \end{tikzcd}$$

$$\begin{tikzcd} Z \arrow[rr, "g"] \arrow[dr, "fg"'] & & Y \arrow[dl, "f"] & \mapsto & g_i^f \colon (fg)^{-1}(i) \to f^{-1}(i)\,, \\ & X \end{tikzcd}$$

referring to the object $f^{-1}(i)$ as the *fibre of* f *at* i, and the morphism $g_i^f \colon (fg)^{-1}(i) \to f^{-1}(i)$ as the *fibre map of* g *with respect to* f *at* i;

all subject to the following axioms, where in (A5), we write εj for the image of $j \in |f|^{-1}(i)$ under the map $\varepsilon_{|f|,i} \colon |f|^{-1}(i) \to |Y|$ of Equation (1):

(A1) if X is a local terminal then $|X| = 1$;

(A2) for all $X \in \mathcal{C}$ and $i \in |X|$, the object $(\mathrm{id}_X)^{-1}(i)$ is chosen terminal;

(A3) for all $f \in \mathcal{C}/X$ and $i \in |X|$, we have $|f^{-1}(i)| = |f|^{-1}(i)$, while for all $g\colon fg \to f$ in $\mathcal{C}/X$ and $i \in |X|$, we have $|g^f_i| = |g|^{|f|}_i$;

(A4) for $Y \in \mathcal{C}$, we have $\tau_Y^{-1}(1) = Y$, and for $g\colon Z \to Y$, we have $g_1^{\tau_Y} = g$;

(A5) for $g\colon fg \to f$ in $\mathcal{C}/X$, $i \in |X|$ and $j \in |f|^{-1}(i)$, we have that $(g^f_i)^{-1}(j) = g^{-1}(\varepsilon j)$, and given also $h\colon fgh \to fg$ in $\mathcal{C}/X$, we have $(h^{fg}_i)^{g^f_i}_j = h^g_{\varepsilon j}$.

Example 4.5.1. The terminal operadic category is the category $\mathbb{F}$ of finite ordered sets and arbitrary maps. The cardinality functor is the identity, the fibres are the 'true' fibres (as in Equation (1)).

Example 4.5.2. For the present purposes the key example is the category $\mathbb{S}_{\mathrm{ord}}$ of finite ordinals and arbitrary surjections. The cardinality functor is the inclusion functor $\mathbb{S}_{\mathrm{ord}} \to \mathbb{F}$. The fibre functor is the same as that from $\mathbb{F}$, but note that these fibres are not true fibres in the strict sense of the word, because they are not given by pullback. Indeed, the category of surjections does not have pullbacks. And the 'inclusion of a fibre' is not a morphism in the category. It is important nevertheless that many constructions with surjections can be interpreted as taking place in $\mathbb{F}$. The axioms are easily verified.

The construction We now work with ***Set***-valued species as in the classical theory. This is needed to achieve the strictness characteristic for operadic categories. We also need to assume that the hereditary species have the property that $H[1] = 1$. Schmitt [53] calls such hereditary species *simple*. This is true for example for simple graphs.

Given a simple hereditary species $H\colon \mathbb{S}_p \to$ ***Set***, we consider first its Grothendieck construction. It is a left fibration (discrete opfibration) $\int H \to \mathbb{S}_p$. The objects of $\int H$ are pairs (n, x) where $n \in \mathbb{S}_p$ and $x \in H[n]$. We will denote such an object X. The morphisms in $\int H$ are described in the usual way. They have an underlying span as in $\mathbb{S}_p$. We are interested in a subcategory, namely the subcategory obtained by pullback along the inclusion $\mathbb{S}_{\mathrm{ord}} \to \mathbb{S}_p$ (from the category of finite ordinals and genuine surjections, not all partial surjections). We denote this category by $\mathcal{H}$. Its objects are $X = (n, x)$ as before, and an arrow from $Y = (m, y)$ to $X = (n, x)$ is given by a genuine surjection $s\colon m \twoheadrightarrow n$ such that $H[s](y) = x$.

We now work towards equipping $\mathcal{H}$ with the structure of operadic category. The category $\mathcal{H}$ is clearly connected. So to choose local terminal objects is to choose a global terminal object. By our assumption $H[1] = 1$, there is a unique such, namely $(1, 1)$, easily seen to be terminal in $\mathcal{H}$.

We define the cardinality functor to be the composite functor $\mathcal{H} \to \mathbb{S}_{\mathrm{ord}} \to \mathbb{F}$.

We define the fibre functor, for each $X = (n, x) \in \mathcal{H}$ and each $i \in n$, to be the assignment

$$\begin{array}{ccc} \mathcal{H}_{/X} & \longrightarrow & \mathcal{H} \\ f\colon Y \to X & \longmapsto & Y_i := (|f|^{-1}(i), H[\varepsilon_i](y)). \end{array}$$

Since this will be needed in all the checks, let us spell this out in more detail. We assume $X = (n, x)$ and $Y = (m, y)$, and the morphism $f\colon Y \to X$ is given by an underlying surjection $|f|\colon m \twoheadrightarrow n$ such that $H[|f|](y) = x$. The fibre Y_i is defined to be the pair (m_i, y_i), where $m_i = |f|^{-1}(i)$ is the fibre of the map in $\mathbb{F}$:

$$\begin{tikzcd} m_i \arrow[r, two heads] \arrow[d, tail, "\varepsilon_i"'] \arrow[dr, phantom, "\lrcorner", very near start] & \{i\} \arrow[d, tail] \\ m \arrow[r, two heads, "|f|"'] & n \end{tikzcd}$$

and y_i is defined as $y_i = H[\varepsilon_i](y)$, the restriction of the H-structure y along the injection $\varepsilon_i\colon m_i \rightarrowtail m$.

We must also provide the assignment on arrows. So given

$$Z \xrightarrow{g} Y \xrightarrow{f} X$$

considered as a morphism in $\mathcal{H}_{/X}$ from fg to f, we need to provide a morphism

$$Z_i \to Y_i.$$

If we let $Z = (\ell, z)$ then we have $H[|g|](z) = y$. The morphism must be constituted by a surjection $g_i\colon \ell_i \twoheadrightarrow m_i$, such that $H[g_i](z_i) = y_i$, where $z_i = H[\varepsilon_i](z)$ is the point in $H[\ell_i]$ representing Z_i (that is, restriction of the H-structure z along the injection $\varepsilon_i\colon \ell_i \rightarrowtail \ell$), and $y_i = H[\varepsilon_i](y)$ is the point in $H[m_i]$ representing Y_i. For the surjection $g_i\colon \ell_i \twoheadrightarrow m_i$ to be valid, we need to check that $H[g_i](z_i) = y_i$. But this is precisely the pull-push formula for hereditary species on the pullback square from (2):

$$\begin{tikzcd} \ell_i \arrow[r, two heads, "g_i"] \arrow[d, tail, "\varepsilon_i"'] \arrow[dr, phantom, "\lrcorner", very near start] & m_i \arrow[d, tail] \\ \ell \arrow[r, two heads, "g"'] & m. \end{tikzcd}$$

(This shows that the assignment extends to arrows. The check that this assignment on arrows respects composition and identity arrows is routine, and depends on transitivity of pullbacks in the skeletal category $\mathbb{F}$.)

We have now exhibited all the data required for an operadic category.

Proposition 4.5.3. *The structures on $\mathcal{H}$ given above satisfy the axioms for an operadic category.*

Since this is a new class of operadic categories, not considered previously, and since the operadic category axioms can be a bit subtle, we include the details of the checks.

Proof. (A1) The chosen terminal $(1, 1)$ clearly has cardinality 1.

(A2) We must check that all fibres of an identity map $\mathrm{id}_X\colon (n, x) \to (n, x)$ are the chosen terminal. By definition, for $i \in n$, the fibre is $(1, ?)$ where 1 is the $\mathbb{F}$-fibre of the identity $n \to n$, and ? can be no other than $1 \in H[1]$.

(A3) We need to compute the cardinality of a fibre $Y_i = (m_i, y_i)$ of a morphism $Y \to X$ and $i \in |X| = n$. But by construction this is m_i, the fibre of the underlying surjection $m \twoheadrightarrow n$. We must also verify that for a triangle $Z \to Y \to X$, the cardinality of the fibre map (over $i \in |X| = n$) is the fibre map in $\mathbb{F}$. But this is clear from the definition of fibre map: it was defined to have as underlying surjection $\ell_i \twoheadrightarrow m_i$, the fibre map in $\mathbb{F}$.

(A4) We must check that for any object $Y = (m, y)$, the fibre of the unique map $(m, y) \to (1, 1)$ has unique fibre Y. For the underlying map in $\mathbb{F}$ this is clear: the unique fibre of $m \twoheadrightarrow 1$ is m. And the new point must be $H[\mathrm{id}](y) = y$, so altogether we find Y again as required. We must also check that given $g\colon Z \to Y$ (given by $(\ell, z) \to (m, y)$), then the fibre map $g_1^\tau\colon Z_1 \to Y_1$ over the unique point in $(1, 1)$ coincides with g itself. On the $\mathbb{F}$-level, this is clear, as we get $\ell \twoheadrightarrow m$ again. The points $z_i \in H[\ell]$ and $y_i \in H[m]$ are given, by construction of the fibre functor, by contravariant functoriality in the injections (fibre inclusions in $\mathbb{F}$) $\ell_i \rightarrowtail \ell$ and $m_i \rightarrowtail m$. But these are the identity maps, so $z_i = z$ and $y_i = y$ as required. Note that axiom (A4) just says that the fibre functor $\phi_{1,1}\colon \mathcal{C}_{/1} \to \mathcal{C}$ must coincide with the canonical projection functor.

(A5) Given morphisms $Z \xrightarrow{g} Y \xrightarrow{f} X$ and elements $i \in |X|$ and $j \in |f|^{-1}(i) = m_i$, we need to establish that $(g_i^f)^{-1}(j) = g^{-1}(\varepsilon j)$. In detail, if the objects and maps are given by

$$(\ell, z) \xrightarrow{g} (m, y) \xrightarrow{f} (n, x)$$

and we have $i \in |X| = n$ and $j \in m_i$, then we first form the diagram of pullbacks in $\mathbb{F}$:

$$\begin{tikzcd}
(\ell_i)_j \arrow[r, two heads] \arrow[d, tail, "\varepsilon_j"'] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[dd, tail, bend right=60, "\varepsilon_{ij}"'] & \{j\} \arrow[d, tail] & \\
\ell_i \arrow[r, two heads, "|g|_i"] \arrow[d, tail, "\varepsilon_i"'] \arrow[dr, phantom, "\lrcorner", very near start] & m_i \arrow[r, two heads] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & \{i\} \arrow[d, tail] \\
\ell \arrow[r, two heads, "|g|"'] & m \arrow[r, two heads, "|f|"'] & n.
\end{tikzcd}$$

Note that the set $(\ell_i)_j$ has two interpretations: it is at the same time the fibre of $|g|$ over εj, and the fibre of $|g|_i$ over j. This shows that the two objects $(g_i^f)^{-1}(j)$ and $g^{-1}(\varepsilon j)$ have the same underlying set. We just need to check their H-structures are the same. According to the definitions, the point in $(g_i^f)^{-1}(j)$ is given by $H[\varepsilon_j](z_i)$, where $z_i = H[\varepsilon_i](z)$. On the other hand, the point in $g^{-1}(\varepsilon j)$ is given by $H[\varepsilon_{ij}](z)$. But these two are the same, by contravariant functoriality of H in injections:

$$H[\varepsilon_j](z_i) = H[\varepsilon_j]\big(H[\varepsilon_i](z)\big) = H[\varepsilon_i \circ \varepsilon_j](z) = H[\varepsilon_{ij}](z).$$

For the second part of (A5), given morphisms $W \xrightarrow{h} Z \xrightarrow{g} Y \xrightarrow{f} X$ and elements $i \in |X|$ and $j \in |f|^{-1}(i) = m_i$, we need to establish that $(h_i^{fg})_j^{g_i^f} = h_{\varepsilon j}^g$. These morphisms have the same source and target thanks to the first item in (A5). More precisely the second part of the axiom can be formulated as saying that this square commutes:

$$\begin{tikzcd}
((gh)_i^f)^{-1}(j) \arrow[r, "(h_i^{fg})^{g_i^f}"] \arrow[d, equal] & (g_i^f)^{-1}(j) \arrow[d, equal] \\
(gh)^{-1}(\varepsilon j) \arrow[r, "h_{\varepsilon j}^g"'] & g^{-1}(\varepsilon j).
\end{tikzcd}$$

Checking this is only a question of unpacking. At the level of underlying sets, we have the pullback diagram

$$\begin{tikzcd}
(k_i)_j \arrow[r, two heads] \arrow[d, tail, "\varepsilon_j"'] \arrow[dr, phantom, "\lrcorner", very near start] & (\ell_i)_j \arrow[r, two heads] \arrow[d, tail, "\varepsilon_j"'] \arrow[dr, phantom, "\lrcorner", very near start] & \{j\} \arrow[d, tail] & \\
k_i \arrow[r, two heads, "|h|_i"] \arrow[d, tail, "\varepsilon_i"'] \arrow[dr, phantom, "\lrcorner", very near start] & \ell_i \arrow[r, two heads, "|g|_i"] \arrow[d, tail, "\varepsilon_i"'] \arrow[dr, phantom, "\lrcorner", very near start] & m_i \arrow[r, two heads] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & \{i\} \arrow[d, tail] \\
k \arrow[r, two heads, "|h|"'] & \ell \arrow[r, two heads, "|g|"'] & m \arrow[r, two heads, "|f|"'] & n.
\end{tikzcd}$$

The point is that the surjection $(k_i)_j \twoheadrightarrow (\ell_i)_j$ has two interpretations, namely as the j-fibre map of $|h|_i$ or as the εj-fibre map of $|h|$. But this is precisely to say that the two morphisms $(h_i^{fg})_j^{g_i^f}$ and $h_{\varepsilon j}^g$ have the same underlying surjection. But they also have the same source (and the same target), by the first part of A5. It follows that they agree, because the underlying map is a surjection and the projection $\mathcal{H} \to \mathbb{S}_{\mathrm{ord}}$ is a discrete opfibration by construction.

$\square$

Operadic functors A functor $F\colon \mathcal{C} \to \mathcal{D}$ between operadic categories is called an *operadic functor* if it strictly preserves local terminal objects, strictly commutes with the cardinality functors to $\mathbb{F}$, and preserves fibres and fibre maps in the sense that

$$F(f^{-1}(i)) = (Ff)^{-1}(i) \qquad \text{and} \qquad F(g_i^f) = (Fg)_i^{Ff}$$

for all $g\colon fg \to f$ in $\mathcal{C}/X$ and $i \in |X|$. We denote by ***OpCat*** the category of operadic categories and operadic functors.

Proposition 4.5.4. *The construction given above is the object part of a functor* $\boldsymbol{HSp}_{\mathrm{simple}} \to \boldsymbol{OpCat}$.

Proof. A morphism of hereditary species is by definition a natural transformation $F\colon H' \Rightarrow H$, or equivalently a morphism of discrete opfibrations over $\mathbb{S}_p$. Clearly this defines also a morphism $F\colon \mathcal{H}' \to \mathcal{H}$ of discrete opfibrations over $\mathbb{S}_{\mathrm{ord}}$, and in particular a functor. We just need to check that this functor is operadic. It is clear that it preserves the chosen terminal objects. It is also clear that it is compatible with cardinality, since a morphism of discrete opfibrations over $\mathbb{S}_{\mathrm{ord}}$ obviously induces a functor over $\mathbb{F}$. To check compatibility with fibres, consider a morphism $f\colon (m, y) \to (n, x)$ in $\mathcal{H}'$ and an element $i \in n$. The fibre over i is by definition $(m_i, H'[\varepsilon_i](y))$, and applying F to that gives

$$F(f^{-1}(i)) = F\big(m_i, H'[\varepsilon_i](y)\big) = \big(m_i, F(H'[\varepsilon_i](y)\big) = \big(m_i, H[\varepsilon_i](F(y))\big),$$

the last equality by naturality of F with respect to the arrow $\varepsilon_i\colon m_i \rightarrowtail m$. But the last object is precisely the fibre of $F(f)$ over i, as required.

We also have to show that given $(\ell, z) \xrightarrow{g} (m, y) \xrightarrow{f} (n, x)$ and an element $i \in n$, we have

$$F(g^f_i) = (F(g))^{Ff}_i.$$

This is well typed in view of the first part of the proof. More precisely the assertion is that this diagram commutes:

$$\begin{tikzcd}
F(fg)^{-1}(i) \arrow[r, "F(g^f_i)"] \arrow[d, equal] & F(f^{-1}(i)) \arrow[d, equal] \\
F(fg)^{-1}(i) \arrow[r, "F(g)^{Ff}_i"'] & F(f)^{-1}(i).
\end{tikzcd}$$

Checking this is only a question of unpacking. Both morphisms have the same underlying surjection, namely $|F(g)|_i\colon \ell_i \twoheadrightarrow m_i$. Since they also have the same source (and target), the fact that $\mathcal{H} \to \mathbb{S}_{\mathrm{ord}}$ is a discrete opfibration ensures that they are also equal as morphisms in $\mathcal{H}$, as required. □

Example 4.5.5. Let $H\colon \mathbb{S}_p \to \boldsymbol{Set}$ be the hereditary species of simple graphs (see Example 4.1.1). The associated operadic category is the category $\mathcal{H}$ whose objects are simple graphs (with vertex set some ordinal n), and whose morphisms are graph contractions $G \twoheadrightarrow Q$. This means it is a surjective map on vertices, and also on edges, and edges are allowed to map to a vertex. The chosen terminal graph is the one-vertex graph. The cardinality of a graph is the set of vertices, and the fibre of a contraction $G \twoheadrightarrow Q$ over some vertex in Q is the preimage of that vertex (the graph contracted onto the vertex).

5

Antipodes of monoidal decomposition spaces

In the present chapter, with Joachim Kock, we upgrade the Gálvez–Kock–Tonks Möbius-inversion construction [26] to the construction of a kind of antipode in any *monoidal* (complete) decomposition space. Many of the constructions are quite similar; the main innovative idea is that there is a useful weaker notion of antipode for bialgebras even if they are *not* Hopf.

5.1 *Antipodes for monoidal complete decomposition spaces*

Convolution. Let X be a monoidal decomposition space. For $F, G : \mathcal{S}_{/X_1} \to \mathcal{S}_{/X_1}$ two linear endofunctors, the *convolution product* $F * G : \mathcal{S}_{/X_1} \to \mathcal{S}_{/X_1}$ is given by first comultiplying, then composing with the tensor product $F \otimes G$, and finally multiplying. If F and G are given by the spans $X_1 \leftarrow M \to X_1$ and $X_1 \leftarrow N \to X_1$, then $F * G$ is given by the composite of spans

$$\begin{tikzcd}
X_1 & & & \\
X_2 \arrow[u] \arrow[d] & M * N \arrow[l] \arrow[ul] \arrow[d] \arrow[drr] \arrow[dl, phantom, "\llcorner", very near start] & & \\
X_1 \times X_1 & M \times N \arrow[l] \arrow[r] & X_1 \times X_1 \arrow[r, "\mu"'] & X_1.
\end{tikzcd}$$

The neutral element for convolution in $\textbf{\textit{LIN}}(\mathcal{S}_{/X_1}, \mathcal{S}_{/X_1})$ is $e := \eta \circ \epsilon$. By composition of spans, it is given by the span

$$X_1 \xleftarrow{s_0} X_0 \xrightarrow{w} X_1,$$

where w denotes the composite $X_0 \to \mathbf{1} \xrightarrow{\eta} X_1$.

The antipode. Define the linear endofunctor $S_n : \mathcal{S}_{/X_1} \to \mathcal{S}_{/X_1}$ by the span

$$X_1 \xleftarrow{g} \vec{X}_n \xrightarrow{p} X_1 \times \ldots \times X_1 \xrightarrow{\mu_n} X_1, \tag{3}$$

where g returns the 'long edge' of a simplex, and p returns its n principal edges.

In the case $n = 0$, we have $g = s_0$ and $(X_1)^0 = \mathbf{1}$ and $\mu_0 = \eta$, whence S_0 coincides with the neutral element:

$$S_0 = e.$$

Note also that the functor S_1 is given by the span

$$X_1 \xleftarrow{i} \vec{X}_1 \xrightarrow{i} X_1. \tag{4}$$

Lemma 5.1.1. *We have*

$$S_n \simeq (S_1)^{*n}.$$

We will use the following lemma, see 1.3.4 for notation.

Lemma 5.1.2 ([26, Lemma 3.5])**.** *Let* X *be a complete decomposition space. Then for any words* v, v' *in the alphabet* $\{0, 1, a\}$*, the square*

$$\begin{tikzcd}
X_2 \arrow[d] & X_{vv'} \arrow[l] \arrow[d] \\
X_1 \times X_1 & X_v \times X_{v'} \arrow[l]
\end{tikzcd}$$

is a pullback.

Proof. The case $n = 0$ is trivial since S_0 is neutral. In the convolution $S_n * S_1$, the main pullback is given by the Lemma 5.1.2:

$$\begin{tikzcd}
X_1 & & & \\
X_2 \arrow[u, "d_1"] \arrow[d, "{(d_2,d_0)}"'] & \vec{X}_{n+1} \arrow[ul, "g"'] \arrow[l, "d_1 \circ \cdots \circ d_{n-1}"'] \arrow[d, "{(d_\top, {d_\perp}^n)}"] \arrow[drr, "\mu \circ p"] \arrow[dl, phantom, "\lrcorner" very near start] & & \\
X_1 \times X_1 & \vec{X}_n \times \vec{X}_1 \arrow[l, "g \times i"] \arrow[r, "p \times \mathrm{id}"'] & (\vec{X}_1)^n \times \vec{X}_1 \arrow[r, "\mu"'] & X_1.
\end{tikzcd}$$

Commutativity of the upper triangle is precisely the face-map description of g. The lower triangle commutes since ${d_\perp}^n$ returns itself the last principal edge. □

Put

$$S_{\mathrm{even}} := \sum_{n \text{ even}} S_n, \qquad S_{\mathrm{odd}} := \sum_{n \text{ odd}} S_n.$$

Note that the sum of linear functors is given by the sum (disjoint union) of the middle objects of the respesenting spans. Hence S_{even} is given by the span

$$X_1 \longleftarrow \sum_{n \text{ even}} \vec{X}_n \longrightarrow X_1,$$

where the left leg returns the long edge of a simplex, and the right leg returns the monoidal product of the principal edges. Similarly of course with S_{odd}.

The *antipode* S is defined as the formal difference

$$S := S_{\mathrm{even}} - S_{\mathrm{odd}}.$$

The difference cannot be formed at the objective level where there is no minus sign available, but it does make sense after taking homotopy cardinality to arrive at $\mathbb{Q}$-vector spaces. For this to be meaningful, certain finiteness conditions must be imposed: X should be *Möbius*, which means locally finite and of locally finite length, see Sections 1.3.3 and 1.3.4. We shall continue to work with S_{even} and S_{odd} individually.

The idea of an antipode is that it should be convolution inverse to the identity functor, i.e. $S * \text{Id}$ should be $\eta \circ \epsilon$. This is not in general true for monoidal decomposition spaces. We show instead that S inverts the following modified identity functor.

The linear functor $\text{Id}' : \mathcal{S}_{/X_1} \to \mathcal{S}_{/X_1}$ is given by the span

$$X_1 \xleftarrow{=} X_0 + \vec{X}_1 \xrightarrow{w|i} X_1,$$

where i is the inclusion $\vec{X}_1 \subset X_1$, and $w : X_0 \xrightarrow{p} \mathbf{1} \xrightarrow{\eta} X_1$ is the constant map with value id_u, the identity at the monoidal unit object u. In other words,

$$\text{Id}' \simeq S_0 + S_1.$$

On elements,

$$\text{Id}'(f) = \begin{cases} f & \text{if } f \text{ nondegenerate,} \\ \text{id}_u, & \text{if } f \text{ degenerate.} \end{cases}$$

Lemma 5.1.3. *The linear functors S_n satisfy*

$$S_n * \text{Id}' \;\simeq\; S_n + S_{n+1} \;\simeq\; \text{Id}' * S_n.$$

Proof. Since $\text{Id}' \simeq S_0 + S_1$, the result follows from $S_n * S_1 \simeq S_{n+1} \simeq S_1 * S_n$ (which is a consequence of Lemma 5.1.1), and $S_n * S_0 \simeq S_n \simeq S_0 * S_n$ (S_0 is neutral for convolution). □

Theorem 5.1.4. *Given a monoidal complete decomposition space X, we have explicit equivalences*

$$S_{\text{even}} * \text{Id}' \;\simeq\; e + S_{\text{odd}} * \text{Id}' \qquad \textit{and} \qquad \text{Id}' * S_{\text{even}} \;\simeq\; e + \text{Id}' * S_{\text{odd}}.$$

Proof. It follows from Lemma 5.1.3 that all four functors are equivalent to $\sum_{n \geqslant 0} S_n$. □

Finiteness conditions and homotopy cardinality. If the monoidal complete decomposition space X is locally finite (meaning that X_1 is locally finite and $X_0 \xrightarrow{s_0} X_1 \xleftarrow{d_1} X_2$ are finite maps, see Section 1.3.3), then we can take homotopy cardinality (see Section 1.3.4) to obtain the incidence bialgebra at the $\mathbb{Q}$-vector space level, and obtain also linear endomorphisms

$$|S_n| : \mathbb{Q}_{\pi_0 X_1} \to \mathbb{Q}_{\pi_0 X_1}.$$

If X is furthermore Möbius, the sums involved in the definitions of S_{even} and S_{odd} are finite, and the difference $|S| = |S_{\text{even}}| - |S_{\text{odd}}|$ is a well-defined linear endomorphism of $\mathbb{Q}_{\pi_0 X_1}$, and we arrive at the following weak antipode formula:

Proposition 5.1.5. *If X is a Möbius monoidal decomposition space, then we have*

$$|S| * |\text{Id}'| = |e| = |\text{Id}'| * |S|$$

in $\mathbb{Q}_{\pi_0 X_1}$, the homotopy cardinality of the incidence bialgebra of X.

Connectedness and the usual notion of antipode. We say a monoidal decomposition space is *connected* if X_0 is contractible. In this situation, X_0 contains only the monoidal unit, so that the maps w and s_0 coincide, and hence $\mathrm{Id}' \simeq \mathrm{Id}$. (Indeed, note that the identity endofunctor $\mathrm{Id} : \mathcal{S}_{/X_1} \to \mathcal{S}_{/X_1}$ is given by the span $X_1 \xleftarrow{=} X_1 \xrightarrow{=} X_1$, and that $s_0|i : X_0 + \vec{X}_1 \to X_1$ is an equivalence.) We then get the following stricter inversion result, yielding the usual notion of antipode in Hopf algebras, after taking homotopy cardinality:

Proposition 5.1.6. *If* X *is a connected monoidal complete decomposition space, then*

$$S_{\mathrm{even}} * \mathrm{Id} \;\simeq\; e + S_{\mathrm{odd}} * \mathrm{Id} \qquad \textit{and} \qquad \mathrm{Id} * S_{\mathrm{even}} \;\simeq\; e + \mathrm{Id} * S_{\mathrm{odd}}.$$

If moreover X *is Möbius, we get*

$$|S| * |\mathrm{Id}| = |e| = |\mathrm{Id}| * |S|.$$

Relationship with classical antipode formulae. If X is the nerve of a Möbius category $\mathscr{C}$, then the comultiplication formula reads

$$\Delta(f) = \sum_{b \circ a = f} a \otimes b.$$

The decomposition space X becomes monoidal if $\mathscr{C}$ is *monoidal extensive* [25, §9], meaning that it has a monoidal structure $(\mathscr{C}, \otimes, k)$ with natural equivalences

$$\mathscr{C}/x \times \mathscr{C}/y \xrightarrow{\sim} \mathscr{C}/(x \otimes y), \qquad\qquad \mathbf{1} \xrightarrow{\sim} \mathscr{C}/k.$$

In combinatorics, extensive monoidal structures most often arise as disjoint union.

Spelling out the the general antipode formula in the case of a monoidal extensive category gives

$$S(f) = \sum_{k \geqslant 0} (-1)^k \sum_{\substack{a_k \circ \cdots \circ a_1 = f \\ a_i \neq \mathrm{id}}} a_1 \cdots a_k.$$

When $\mathscr{C}$ is just a locally finite hereditary poset (with intervals regarded as arrows), this is Schmitt's antipode formula for the reduced incidence Hopf algebra of the poset [52].

Schmitt's formula works more generally for hereditary families of poset intervals, meaning classes of poset intervals that are closed under taking subintervals and cartesian products [54]. Our general formula covers that case as well. The intervals of such a family do not necessarily come from a single poset (or even a Möbius category). One can prove that such a family always forms a monoidal decomposition space, the most important case being the family of *all* (finite) poset intervals [27].

Other classical antipode formulae are readily extracted. For example, from the general formula $S_{n+1} \simeq S_n * S_1$ (see Lemma 5.1.1), one finds

$$S_{\mathrm{even}} \simeq S_0 + S_{\mathrm{odd}} * S_1, \qquad S_{\mathrm{odd}} \simeq S_{\mathrm{even}} * S_1,$$

whence the recursive formula

$$S \simeq S_0 - S * S_1,$$

valid after taking homotopy cardinality. Spelling this out in the case of the nerve of a monoidal extensive Möbius category yields the familiar formula

$$S(f) = S_0(f) - \sum_{\substack{b \circ a = f \\ b \neq \mathrm{id}}} S(a) \cdot b,$$

which also goes back to Schmitt [52], in the poset case.

5.2 *Inversion in convolution algebras*

Möbius inversion. The Möbius inversion formula of Theorem 1.3.3 is recovered easily from Theorem 5.1.4. Recall that the *zeta functor* is the linear functor $\zeta : \mathcal{S}_{/X_1} \to \mathcal{S}$ defined by the span $X_1 \xleftarrow{=} X_1 \to \mathbf{1}$.

First we define

$$\Phi_n := \zeta \circ S_n.$$

By composition of spans, Φ_n is given by

$$X_1 \xleftarrow{g} \vec{X}_n \longrightarrow \mathbf{1}$$

in accordance with [26] (see Section 1.3.4). We also get

$$\Phi_{\text{even}} := \zeta \circ S_{\text{even}} = \sum_{n \text{ even}} \Phi_n, \qquad \Phi_{\text{odd}} := \zeta \circ S_{\text{odd}} = \sum_{n \text{ odd}} \Phi_n.$$

The following is now an immediate consequence of Theorem 5.1.4.

Corollary 5.2.1 ([26] Theorem 3.8)**.** *For a monoidal complete decomposition space, the Möbius inversion principle holds, expressed by the explicit equivalences*

$$\Phi_{\text{even}} * \zeta \simeq \epsilon + \Phi_{\text{odd}} * \zeta \quad \textit{and} \quad \zeta * \Phi_{\text{even}} \simeq \epsilon + \zeta * \Phi_{\text{odd}}.$$

This proof is a considerable simplification compared to the proof given in [26], but note that it crucially depends on the monoidal structure. The theorem of [26] is more general in that it works also in the absence of a monoidal structure.

More general inversion. One advantage of the antipode over the Möbius inversion formula is that it gives a uniform inversion principle, rather than just inverting the zeta function. At the $\mathbb{Q}$-vector space level, the result $|\mu| = |\zeta| \circ |S|$ is readily generalised as follows. Let B_X denote the homotopy cardinality of the incidence bialgebra of a monoidal Möbius decomposition space X.

Lemma 5.2.2. *For any $\mathbb{Q}$-algebra A with unit η_A, consider the convolution algebra $(\mathrm{Lin}(B_X, A), *, \eta_A \epsilon)$. If $\phi : B_X \to A$ is multiplicative and sends all group-like elements to η_A, then ϕ is convolution invertible with inverse $\phi \circ S$.*

Proof. Indeed, 'multiplicative' ensures that $\phi \circ (S * \mathrm{Id}') = (\phi \circ S) * (\phi \circ \mathrm{Id}')$, and the condition on group-like elements ensures that $\phi \circ \mathrm{Id}' = \phi$ (and that $\phi \circ \eta_B = \eta_A$). □

The *connected quotient* H_X is defined as $H_X := B_X/J_X$, where

$$J_X = \langle s_0 x - s_0 u \mid x \in X_0 \rangle,$$

which is a Hopf ideal [58] since the elements $s_0 x$ are group-like. (Here u denotes the monoidal unit.) It is clear that H_X is connected, hence a Hopf algebra. Now the conditions on ϕ in Lemma 5.2.2 amount precisely to saying that ϕ vanishes on the Hopf ideal J_X, and hence factors through the quotient Hopf algebra H_X:

$$\begin{tikzcd} B_X \arrow[rr, "\phi"] \arrow[dr] & & A \\ & H_X \arrow[ur, dotted, "\overline{\phi}"'] & \end{tikzcd}$$

From this perspective, the weak antipode of B_X does not invert anything that could not have been inverted with classical technology, namely by the true antipode in H_X. The point of the weak antipode is that it is defined already at the objective level of decomposition spaces, without the need of quotienting. We shall establish the following objective version of Lemma 5.2.2.

Theorem 5.2.3. *Let X be a monoidal complete decomposition space, and let A be a monoidal ∞-groupoid—this makes $\mathcal{S}_{/A}$ an algebra in **LIN**. Consider the convolution algebra $(\boldsymbol{LIN}(\mathcal{S}_{/X_1}, \mathcal{S}_{/A}), *, \eta_A \epsilon)$. If a linear functor $\phi : \mathcal{S}_{/X_1} \to \mathcal{S}_{/A}$ is multiplicative and contracts degenerate elements, then ϕ is convolution invertible with inverse $\phi \circ S$.*

The main task is to define the notions involved. Throughout, we let X denote a monoidal complete decomposition space, and A a monoidal ∞-groupoid. A linear functor $\phi : \mathcal{S}_{/X_1} \to \mathcal{S}_{/A}$ given by a span

$$X_1 \xleftarrow{u} F \xrightarrow{v} A$$

is called *multiplicative* if it is a span of monoidal functors with u culf. This means that we have commutative diagrams

$$\begin{tikzcd} X_1 \times X_1 \arrow[d, "\mu_1"'] & F \times F \arrow[l, "u \times u"'] \arrow[r, "v \times v"] \arrow[d, "\mu_F"] \arrow[dl, phantom, "\lrcorner"] & A \times A \arrow[d, "\mu_A"] \\ X_1 & F \arrow[l, "u"] \arrow[r, "v"'] & A \end{tikzcd} \qquad \begin{tikzcd} \mathbf{1} \arrow[d, "\eta_1"'] & \mathbf{1} \arrow[l, "="'] \arrow[r, "="] \arrow[d, "\eta_F"] \arrow[dl, phantom, "\lrcorner"] & \mathbf{1} \arrow[d, "\eta_A"] \\ X_1 & F \arrow[l, "u"] \arrow[r, "v"'] & A. \end{tikzcd} \tag{5}$$

Commutativity of the diagrams expresses of course that the functors u and v are monoidal. culfness amounts to the pullback conditions indicated, which are required because we need to do pull-push along these squares.

A linear functor $\phi : \mathcal{S}_{/X_1} \to \mathcal{S}_{/A}$ given by a span

$$X_1 \xleftarrow{u} F \xrightarrow{v} A$$

is said to *contract degenerate elements* if the following condition holds:

$$\begin{tikzcd}
X_0 \arrow[d, "s_0"'] \arrow[dr, phantom, "\llcorner"] & X_0 \arrow[l, "="'] \arrow[r, "p"] \arrow[d, "s_F"] & \mathbf{1} \arrow[d, "\eta_A"] \\
X_1 & F \arrow[l, "u"] \arrow[r, "v"'] & A.
\end{tikzcd} \tag{6}$$

Two conditions are expressed by this: the first is that u pulled back along s_0 gives the identity map. (The map s_F is defined by this pullback.) The second condition says that $v \circ s_F$ factors through the unit. Altogether, the conditions express the idea of mapping all degenerate elements to the unit object of A.

Lemma 5.2.4. *If ϕ contracts degenerate elements (Equation (6)), then it is unital (Equation (5) RHS).*

Proof. In the diagram

$$\begin{tikzcd}
\mathbf{1} \arrow[d, "\eta_0"'] \arrow[dr, phantom, "\llcorner"] & \mathbf{1} \arrow[l, "="'] \arrow[r, "="] \arrow[d, "\eta_0"] & \mathbf{1} \arrow[d, "="] \\
X_0 \arrow[d, "s_0"'] \arrow[dr, phantom, "\llcorner"] & X_0 \arrow[l, "="'] \arrow[r, "p"] \arrow[d, "s_F"] & \mathbf{1} \arrow[d, "\eta_A"] \\
X_1 & F \arrow[l, "u"] \arrow[r, "v"'] & A,
\end{tikzcd}$$

the bottom squares are (6), and the outline diagram is (5) RHS, since the composite vertical arrows are η_1, η_F, and η_A. $\square$

Lemma 5.2.5. *If a linear functor $\phi : \mathcal{S}_{/X_1} \to \mathcal{S}_{/A}$ is multiplicative, then $\phi \circ -$ distributes over convolution. Precisely, for any linear endofunctors $\alpha, \beta : \mathcal{S}_{/X_1} \to \mathcal{S}_{/X_1}$, we have*

$$\phi \circ (\alpha * \beta) \simeq (\phi \circ \alpha) * (\phi \circ \beta).$$

Note that $*$ on the left refers to convolution of endofunctors, while $*$ on the right refer to convolution in $\boldsymbol{LIN}(\mathcal{S}_{/X_1}, \mathcal{S}_{/A})$.

Proof. The left-hand side $\phi \circ (\alpha * \beta)$ is computed by the pullbacks

$$\begin{tikzcd}[column sep=small]
 & & & P \arrow[ddll] \arrow[dr] \arrow[d, phantom, "\vee"] & & & & \\
 & & & & (M \times N) \underset{X_1 \times X_1}{\times} (F \times F) \arrow[dl, "pr_1"'] \arrow[dr] \arrow[d, phantom, "\vee"] & & & \\
 & X_2 \arrow[dl] \arrow[dr] & & M \times N \arrow[dl, "a \times b"'] \arrow[dr] & & F \times F \arrow[dl] \arrow[dr] \arrow[d, phantom, "\vee"] & & \\
X_1 & & X_1 \times X_1 & & X_1 \times X_1 \arrow[dr] & & F \arrow[dl, "u"'] \arrow[dr, "v"] & \\
 & & & & & X_1 & & A.
\end{tikzcd}$$

The right-hand side $(\phi \circ \alpha) * (\phi \circ \beta)$ is computed by the pullback

$$\begin{tikzcd}
& & P \arrow[dl] \arrow[dr] \arrow[d, phantom, "\vee"] & & & \\
& X_2 \arrow[dl] \arrow[dr] & {} & (M \underset{X_1}{\times} F) \times (N \underset{X_1}{\times} F) \arrow[dl, "f"'] \arrow[dr] & & \\
X_1 & & X_1 \times X_1 & & A \times A \arrow[dr] & \\
& & & & & A.
\end{tikzcd}$$

Here f is the map $(a \circ \mathrm{pr}_1) \times (b \circ \mathrm{pr}_1)$. These two composed spans agree since clearly

$$(M \times N) \underset{X_1 \times X_1}{\times} (F \times F) \simeq (M \underset{X_1}{\times} F) \times (N \underset{X_1}{\times} F)$$

(and $f \simeq (a \times b) \circ \mathrm{pr}_1$). ☐

Lemma 5.2.6. *If a linear functor $\phi : \mathcal{S}_{/X_1} \to \mathcal{S}_{/A}$ contracts degenerate elements, then we have*

$$\phi \circ \mathrm{Id}' \simeq \phi.$$

Proof. Let ω denote the endofunctor defined by the span $X_1 \xleftarrow{s_0} X_0 \xrightarrow{s_0} X_1$. Since $\mathrm{Id}' = S_0 + S_1$ and $\mathrm{Id} = \omega + S_1$, it is enough to establish

$$\phi \circ S_0 \simeq \phi \circ \omega.$$

The left-hand side $\phi \circ S_0$ is computed by the pullbacks

$$\begin{tikzcd}
& & & X_0 \arrow[dl, "="'] \arrow[dr, "p"] \arrow[d, phantom, "\vee"] & & & & \\
& & X_0 \arrow[ddddll, "s_0"'] \arrow[dr] & {} & \mathbf{1} \arrow[dl, "="'] \arrow[dr, "\eta_0"] \arrow[d, phantom, "\vee"] \arrow[ddrrr, bend left, "\eta_A"] & & & \\
& & & \mathbf{1} \arrow[dr, "\eta_0"'] & {} & X_0 \arrow[dl, "="'] \arrow[dr, "s_F"] \arrow[d, phantom, "\vee"] & & \\
& & & & X_0 \arrow[dr, "s_0"'] & {} & F \arrow[dl, "u"'] \arrow[dr, "v"] & \\
X_1 & & & & & X_1 & & A.
\end{tikzcd}$$

The right-hand side $\phi \circ \omega$ is computed by the pullback

$$\begin{tikzcd}
& & X_0 \arrow[dl, "="'] \arrow[dr, "s_F"] \arrow[d, phantom, "\vee"] & & \\
& X_0 \arrow[dl, "s_0"'] \arrow[dr, "s_0"] & {} & F \arrow[dl, "u"'] \arrow[dr, "v"] & \\
X_1 & & X_1 & & A,
\end{tikzcd}$$

and the composite $\nu \circ s_F$ is again $\eta_A \circ p$ by hypothesis. □

Proof of Theorem 5.2.3. We need to show that $\phi \circ S$ is convolution inverse to ϕ. With the preparations made, this is now direct:

$$(\phi \circ S) * \phi \overset{5.2.6}{\simeq} (\phi \circ S) * (\phi \circ \mathrm{Id}') \overset{5.2.5}{\simeq} \phi \circ (S * \mathrm{Id}') \overset{5.1.4}{\simeq} \phi \circ \eta_1 \circ \epsilon \overset{5.2.4}{\simeq} \eta_A \circ \epsilon.$$

□

Remark 5.2.7. The more general Möbius inversion principle of Lemma 5.2.2 and Theorem 5.2.3 is of interest for two reasons. Firstly, in the connected case, the general Möbius inversion principle, which we here derived from the antipode, but which can be formulated without reference to S, is actually *equivalent* to the existence of the antipode. Indeed, if one takes A to be X_1 itself (so that at the cardinality level one uses B_X as the algebra A), and takes ϕ to be the identity map, then the resulting inverse is the antipode.

Secondly, the extra generality serves to highlight the tight analogy between Möbius inversion and abstract Hopf-algebraic renormalisation in perturbative quantum field theory, as explained in [41]. The ϕ are then the (regularised) Feynman rules (which are inherently multiplicative, and can be arranged to send group-like elements to 1). In this generality, the passage from Möbius inversion to renormalisation consists just in adding a Rota–Baxter operator to the formulae (see [41] for details). (The result is then no longer an inverse but rather a counter-term.)

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Colophon

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