Incidence Hopf algebras

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Abstract

This is a brief introduction to incidence algebras and Möbius inversion, starting with the classical theory, passing through the seminal work of Rota, and the generalisation to categories by Leroux, and finishing with the notion of decomposition space, where the notions of incidence algebra and Möbius inversion find their natural generality.

This first four sections contain standard material, most of which can be found in Stanley [11].

1 The classical Möbius function

1.1 Arithmetic functions and Dirichlet series. Write

$$\mathbb{N}^\times = \{1, 2, 3, \dots\}$$

for the set of positive natural numbers. An *arithmetic function* is just a function

$$f : \mathbb{N}^\times \to \mathbb{C}$$

(meant to encode some arithmetic feature of each number n). To each arithmetic function f one associates a *Dirichlet series*

$$F(s) = \sum_{n \geq 1} \frac{f(n)}{n^s}$$

thought of as a function defined on some open set of the complex plane. Analytic number theory is to a large extent the study of arithmetic functions in terms of their associated Dirichlet series.

1.2 The zeta function. The most famous example is the *zeta function*

$$\begin{aligned} \zeta : \mathbb{N}^\times &\longrightarrow \mathbb{C} \\ n &\longmapsto 1. \end{aligned}$$

The associated Dirichlet series is the *Riemann zeta function*

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}.$$

Another famous expression for the Riemann zeta function is given by Euler's product expansion formula

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - \frac{1}{p^s}}$$

where the product is over all primes. (This formula was found by Euler in 1737, so one should not think that Dirichlet or Möbius (in the 1830s) were the first to study these issues!) There are many different proofs of this formula, and it is not as complicated as it looks. We shall see below that it is an easy consequence of a basic fact about incidence algebras.

1.3 Classical Möbius inversion. The classical Möbius inversion principle says that

$$\begin{array}{lrcl} \text{if} & f(n) & = & \sum_{d|n} g(d) \\ \text{then} & g(n) & = & \sum_{d|n} f(d)\mu(n/d), \end{array}$$

where μ is the *Möbius function*

$$\mu(n) = \begin{cases} 0 & \text{if } n \text{ contains a square factor} \\ (-1)^r & \text{if } n \text{ is the product of } r \text{ distinct primes.} \end{cases}$$

1.4 Example: Euler's totient function. To see an example of the classical Möbius inversion, consider another important arithmetic function, *Euler's totient function*

$$\phi(n) = \#\{1 \leq k \leq n \mid (k, n) = 1\}.$$

It is not difficult to see that we have the relation

$$n = \sum_{d|n} \phi(d),$$

so by Möbius inversion we get a formula for ϕ:

$$\phi(n) = \sum_{d|n} d\ \mu(n/d).$$

1.5 Dirichlet convolution. What is really going on here is that there is a convolution product for arithmetic functions, called *Dirichlet convolution*:

$$(f * g)(n) = \sum_{i \cdot j = n} f(i)g(j)$$

which corresponds precisely to (pointwise) product of Dirichlet series. The neutral element for this convolution product is the delta function

$$\delta(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{else.} \end{cases}$$

Now the Möbius inversion formula reads more conceptually

$$f = g * \zeta \qquad \Rightarrow \qquad g = f * \mu,$$

and the content is this:

1.6 Proposition. *The Möbius function is the convolution inverse of the zeta function.*

1.7 Example (continued). To come back to the example with Euler's totient, let ι denote the arithmetic function $\iota(n) = n$. Its associated Dirichlet series is

$$\sum_{n \geq 1} \frac{n}{n^s} = \zeta(s-1).$$

Restating the Möbius inversion formula for ϕ in terms of Dirichlet convolution yields

$$\iota = \phi * \zeta \qquad \Rightarrow \qquad \phi = \iota * \mu,$$

so that the Dirichlet series associated to ϕ is

$$\frac{\zeta(s-1)}{\zeta(s)}.$$

2 Incidence algebras and Rota–Möbius inversion

2.1 Intervals and locally finite posets. Let $(P, \leq)$ be a poset. An *interval* in P is a nonempty subposet of the form

$$[x, y] := \{z \in P : x \leq z \leq y\}.$$

A poset is called *locally finite* if all its intervals are finite. An example of a locally finite poset is $(\mathbb{N}, \leq)$. Another example is $(\mathbb{N}^\times, \mid\ )$, the poset of positive integers under the divisibility relation. Let $\mathrm{int}(P)$ denote the set of all intervals in P.

2.2 Incidence coalgebras. Let k be field, fixed throughout; it plays no role. The free vector space on the set $\mathrm{int}(P)$ becomes a coalgebra with comultiplication defined by

$$\Delta([x, y]) := \sum_{z \in [x,y]} [x, z] \otimes [z, y]$$

and counit

$$\delta([x, y]) := \begin{cases} 1 & \text{if } x = y \\ 0 & \text{else.} \end{cases}$$

(Checking the coalgebra axioms is straightforward.)

2.3 Convolution algebras. If (C, Δ, ε) is a coalgebra, and (A, m, u) is an algebra, then $\mathrm{Hom}_k(C, A)$ becomes an algebra under the convolution product defined by sending $\phi, \psi \in \mathrm{Hom}_k(C, A)$ to the composite

$$C \xrightarrow{\Delta} C \otimes C \xrightarrow{\phi\otimes\psi} A \otimes A \xrightarrow{m} A,$$

that is, in Sweedler notation:

$$(\phi * \psi)(x) = \sum_{(x)} \phi(x')\psi(x'').$$

The unit for the convolution product is

$$u \circ \varepsilon.$$

2.4 The incidence algebra of a locally finite poset P is the convolution algebra of the incidence coalgebra (with values in the ground field). The elements can be viewed as all the functions $\mathrm{int}(P) \to k$, and the multiplication is given by

$$(\phi * \psi)([x, y]) = \sum_{z\in[x,y]} \phi([x, z])\psi([z, y]).$$

and the unit is δ.

2.5 Reduced incidence algebra. The *reduced incidence algebra* is the subalgebra of the incidence algebra consisting of the functions ϕ with the property that if two intervals $[x, y]$ and $[x', y']$ are isomorphic as posets, then $\phi([x, y]) = \phi([x', y'])$. Equivalently, the reduced incidence algebra can be described as the convolution algebra of the coalgebra given by isomophism classes of intervals in P.

2.6 Example. Consider the poset $(\mathbb{N}, \leq)$ of natural numbers with the usual order. Then there is an interval $[m, n]$ whenever $m \leq n$. The incidence algebra consists of functions assigning a scalar to every such pair, so it amounts to an infinite upper-triangular matrix (indexed from 0 to ∞), where the (i, j) entry holds the value on the interval $[i, j]$ (if $i \leq j$ and zero otherwise). Given two such matrices ϕ and ψ, their convolution product is given by

$$(\phi * \psi)([i, j]) = \sum_{i\leq k\leq j} \phi([i, k])\psi([k, j])$$

That's nothing but the ij entry in the matrix product! (noting that k might as well run from 0 to ∞: for $k \leq i$ we are below the diagonal in the first matrix, and for $k \geq j$ we are below the diagonal in the second matrix). And of course the function δ is the infinite diagonal matrix.

If instead we look at the reduced incidence algebra, we first observe that every interval $[m, n]$ is isomorphic as a poset to the interval $[0, n - m]$. So the reduced incidence algebra consists of functions on $\mathbb{N}$, i.e. infinite sequences $(a_0, a_1, a_2, \dots)$. The convolution product is given by

$$(a * b)_n = (a * b)([0, n]) = \sum_{k\in[0,n]} a([0, k])b([k, n]) = \sum_{i+j=n} a_i b_j,$$

so we can identify the incidence algebra with the algebra of formal power series (interpreting the sequence $(a_0, a_1, \dots)$ as $\sum a_i x^i$).

2.7 The zeta function. In any incidence algebra, the *zeta function* is defined as

$$\begin{aligned} \zeta : \operatorname{int} P &\longrightarrow k \\ [x, y] &\longmapsto 1. \end{aligned}$$

(Note that this function is constant on the intervals, but of course not constant on the vector space generated by the intervals!)

2.8 Theorem (Rota). *ϕ is invertible if and only if $\phi([x, x])$ is invertible in k for all $x \in P$. In particular, the zeta function is invertible; its inverse is called the Möbius function:*

$$\mu := \zeta^{-1}.$$

We have

$$\mu([x, y]) = \begin{cases} 1 & \text{if } x = y \\ -\sum_{z\in[x,y]} \mu([x, z]) & \text{if } x < y. \end{cases}$$

This is a recursive definition by length of intervals.

2.9 Corollary. *(Möbius inversion in general incidence algebras.) We have*

$$f = g * \zeta \qquad \Rightarrow \qquad g = f * \mu,$$

In other words,

$$\begin{aligned} \text{if} \quad f([x, y]) &= \sum_{z\in[x,y]} g([x, z]) \\ \text{then} \quad g([x, y]) &= \sum_{z\in[x,y]} f([x, z])\mu([z, y]), \end{aligned}$$

2.10 Möbius inversion in reduced incidence algebras. In important cases, the poset P has a initial element 0 and the property that every interval in it is isomorphic to one that starts in 0. In this case, we use the following shorthand notation in the reduced incidence algebra:

$$f(y) := f([0, y])$$

and then we can write

$$\begin{array}{lrcl} \text{if} & f(y) & = & \sum_{z \le y} g(z) \\ \text{then} & g(y) & = & \sum_{z \le y} f(z)\mu([z,y]), \end{array}$$

recovering the formula for the two main examples above.

2.11 Example. In the reduced incidence algebra of the poset $(\mathbb{N}, \le)$, the Möbius function is

$$\mu([m,n]) = \begin{cases} 1 & \text{if } m = n \\ -1 & \text{if } n = m+1 \\ 0 & \text{else.} \end{cases}$$

Hence the inversion principle says in this case

$$\begin{array}{lrcl} \text{if} & f(n) & = & \sum_{k \le n} g(k) \\ \text{then} & g(n) & = & f(n) - f(n-1). \end{array}$$

In other words, convolution with the Möbius function is Newton's (backward) finite-difference operator. So convolution with μ acts as 'differentiation' while convolution with ζ acts as 'integration'.

If we interpret the sequences $a : \mathbb{N} \to \mathbb{C}$ as formal power series, then the zeta function is the geometric series, while the Möbius function is $1 - x$.

2.12 Example. We should note of course that Möbius inversion in the reduced incidence algebra of the divisibility poset $(\mathbb{N}^\times, |)$ is just the classical Möbius inversion principle.

3 The product rule

3.1 Products of posets and tensor product of incidence algebras. The product of two posets P and Q is again a poset, with $(p,q) \le (p',q')$ iff $p \le p'$ and $q \le q'$. If P and Q are intervals then so is $P \times Q$, and if P and Q are locally finite posets, then so is $P \times Q$. It is easy to see that we have a natural bijection of sets

$$\mathrm{int}(P \times Q) = \mathrm{int}(P) \times \mathrm{int}(Q)$$

and it induces a natural coalgebra isomophism between the corresponding incidence coalgebras In turn, this yields an algebra isomorphism between the incidence algebras

$$I(P \times Q) = I(P) \otimes I(Q).$$

The convolution product on $I(P \times Q)$ is the tensor product of the convolution products, i.e.

$$(\phi_1 \otimes \phi_2) *_{P_1 \times P_2} (\psi_1 \otimes \psi_2) = (\phi_1 *_1 \psi_1) \otimes (\phi_2 *_2 \psi_2).$$

In particular we can write

$$\phi \otimes \phi' = (\phi \otimes \delta') * (\delta \otimes \psi'),$$

where δ and δ' are the respective neutral elements for the convolutions.

3.2 Zeta function and Möbius function in a tensor product incidence algebra. If P_1 and P_2 are (locally finite) posets, then clearly the zeta function in the incidence algebra $I(P_1 \times P_2) = I(P_1) \otimes I(P_2)$ is just the tensor product of the two zeta functions:

$$\zeta_{P_1 \times P_2} = \zeta_1 \otimes \zeta_2,$$

and it follows that the same is true for the Möbius functions:

$$\mu_{P_1 \times P_2} = \mu_1 \otimes \mu_2.$$

We note that this can also be written as

$$(\mu_1 \otimes \delta_2) * (\delta_1 \otimes \mu_2).$$

3.3 The classical Möbius function revisited. We can compute the classical Möbius function by observing that the poset $(\mathbb{N}^\times, |)$ is a product of copies of the poset $(\mathbb{N}, \leq)$. Indeed, for each prime p consider the subposet $\{p^k\}_{k \in \mathbb{N}} \subset (\mathbb{N}^\times, |)$ conisting of all the powers of p. Clearly this poset is isomorphic to $(\mathbb{N}, \leq)$. By unique factorisation into primes, we have an isomorphism

$$(\mathbb{N}^\times, |) \simeq \prod_{p \text{ prime}} (\{p^k\}, |) \simeq \prod_{p \text{ prime}} (\mathbb{N}, \leq).$$

More explicitly, the number $n = \prod_p p^{k_p}$ corresponds to the element $(k_p)_{p \text{ prime}} \in \prod_p (\mathbb{N}, \leq)$. Therefore, the classical Möbius function can be written

$$\mu_{\text{clas.}} = \bigotimes_p \mu_{\leq}.$$

Here $\mu_{\leq}$ denotes of course the Möbius function of the incidence algebra of $(\mathbb{N}, \leq)$, which we already noted is given by

$$\mu_{\leq}(k) = \begin{cases} 1 & \text{if } k = 0 \\ -1 & \text{if } k = 1 \\ 0 & \text{if } k > 1. \end{cases}$$

Hence, for $n = p_1^{k_1} \cdots p_r^{k_r}$, the classical Möbius function gives

$$\mu_{\text{clas.}}(n) = \begin{cases} 0 & \text{if any } k_i > 1 \\ (-1)^r & \text{else.} \end{cases}$$

as claimed in the introductory section.

3.4 Euler's product formula revisited. In fact, Euler's product formula for the Riemann zeta function is just a translation of the product formula

$$\mu_{\text{clas.}} = \bigotimes_p \mu_{\leq}.$$

To see this, let us analyze what the p-factor $\mu_{\leq}$ looks like as a function on all of $\mathbb{N}^{\times}$, or more precisely, what is the function

$$\delta_{\leq} \otimes \cdots \otimes \delta_{\leq} \otimes \mu_{\leq} \otimes \delta_{\leq} \cdots$$

with the Möbius function only in the p-factor: it is the function whose value on 1 is 1, whose value on p is -1 (corresponding to exponent $k = 1$), and whose value elsewhere is 0. In other words, it is the arithmetic function

$$\delta - \delta_p,$$

where

$$\delta_p(n) = \begin{cases} 1 & \text{if } n = p \\ 0 & \text{else.} \end{cases}$$

Altogether, the Möbius function for $(\mathbb{N}^{\times}, |)$ is

$$\mu_{\text{clas.}} = \bigotimes_p \mu_{\leq} \;=\; \mathop{\ast}_p (\delta - \delta_p).$$

To get Euler's product formula, just note that the Dirichlet series corresponding to δ_p is $\frac{1}{p^s}$, so altogether we have

$$\frac{1}{\zeta(s)} = \prod_p \big(1 - \frac{1}{p^s}\big);$$

the inverse is Euler's product formula.

3.5 Example: powersets — the inclusion-exclusion principle. Let X be a fixed finite set, and consider the powerset of X, i.e. the set $\mathscr{P}(X)$ of all subsets of X. It is a poset under the inclusion relation $\subset$. An interval in $\mathscr{P}(X)$ is given by a pair of nested subsets of X, say $T \subset S$. It is clear that this interval isomorphic to the interval $\emptyset \subset S{-}T$.

If the cardinality of X is n, then clearly $\mathscr{P}(X)$ is isomorphic as a poset to $\mathbf{2}^n$, where $\mathbf{2} = \mathscr{P}(1) = \{\emptyset, \{*\}\}$ denotes the powerset of the terminal set $1 = \{*\}$. The poset $\mathbf{2}$ in turn can be viewed as the interval $[0, 1] \subset (\mathbb{N}, \leq)$, so it is clear that its Möbius function is just $(-1)^n$, $n \in \{0, 1\}$. By the product rule, the Möbius function on $\mathscr{P}(X)$ is therefore given by

$$\mu(T \subset S) = (-1)^{|S-T|}.$$

This is the inclusion-exclusion principle. As an example of this, consider the problem of counting *derangements*, i.e. permutations without fixpoints. Since every permutation of a set S determines a subset T of points which are actually moved, we can write

$$\operatorname{perm}(S) = \sum_{T \subset S} \operatorname{der}(T)$$

(with the evident notation). Hence by Möbius inversion, we find the formula for derangements

$$\operatorname{der}(S) = \sum_{T \subset S} (-1)^{|S-T|} \operatorname{perm}(T),$$

and hence, if S is of cardinality n, we can write (with obvious variation in the notation):

$$\operatorname{der}(n) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} k!\,,$$

which is a typical inclusion-exclusion formula.

Similarly, the number of surjections from N to S can be computed by first observing that

$$\operatorname{map}(N, S) = \sum_{T \subset S} \operatorname{surj}(N, T),$$

and therefore, by Möbius inversion,

$$\operatorname{surj}(N, S) = \sum_{T \subset S} (-1)^{|S-T|} \operatorname{map}(N, T),$$

so if N has cardinality n and S cardinality s,

$$\operatorname{surj}(n, s) = \sum_{k=0}^{s} (-1)^{s-k} \binom{s}{k} k^n\,.$$

(Corollary: for $n < s$, we have $0 = \sum_{k=0}^{s} (-1)^{s-k} \binom{s}{k} k^n$.)

4 Euler characteristic

5 Möbius categories

5.1 Cartier–Foata monoids. Motivated by the combinatorics of words, Cartier and Foata [1] (1969) studied Möbius inversion in certain monoids, the monoids having the finite dceomposition property: each element can be written in only finitely many ways as a product of non-identity elements. Examples of such monoids are free monoids (word monoids) and various variation such as free-commutative and free-partially commutative and so on. One such example is the monoid of positive integers under multiplication. For each such monoid there is a coalgebra which takes an element to all its factorisations, and Cartier and Foata showed that the Möbius inversion principle holds in the corresponding incidence algebra. For the monoid of positive integers under multiplication, this is another construction of classical Möbius inversion.

5.2 Möbius categories of Leroux. Leroux [10] introduced the common generalisation of locally finite posets and Cartier–Foata monoids: *Möbius categories.* A category is *Möbius* if every arrow admits only finitely many non-trivial decompositions (of any length). For a Möbius category $\mathbb{C}$, the set $\mathbb{C}_1$ of arrows generates a coalgebra where the comultiplication of an arrow is the set of all its (length-2) factorisations.

Several equivalent characterisations were identified by Leroux: with terminology introduced by Lawvere and Menni [9], call a category pre-Möbius if for each arrow there are only finitely many factorisations. This property is enough to guarantee a well-defined comultiplication. Leroux characterised also Möbius categories as the pre-Möbius categories such that identities are indecomposable and only identities are cancellable.

Other characterisations were provided by Lawvere and Menni:

5.3 Universal Möbius inversion. Lawvere in the 1970s (unpublished but cited both by Joyal [8] 1981 and I think by Joni–Rota [7]) showed that all Möbius inversion formulae (in Möbius categories) are induced by a single master inversion formular in a Hopf algebra of Möbius intervals. A Möbius interval is a Möbius category with an initial and a terminal object. (Lecture notes exist from talks at the Sydney Combinatorics Seminar May 1988.)

6 Decomposition spaces

This section is to contain a brief introduction to decomposition spaces (at least in groupoids), following [2], [3], [4], [5], [6].

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