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Dear Giordano,

I promised to send you a few mathematical remarks.
In fact I was already planning to send them to Paul.

Dear Paul,

Since your talk in Valladolid, I have had on my to-do list to study your
spectrum-and-supports paper more closely.  Recently I finally found the
occasion, in connection with some talks in our seminar by Henning Krause on
support varieties.  I wanted to make a few remarks, just some tiny
footnotes to your work.  I don't know if they can be of any practical value
(I haven't really got anything to say about triangulated categories), but
you might find interest in them anyway:

1a) The universal property of spectrum-and-supports is a general
    phenomenon;
1b) it was discovered by Joyal in 1970.
2a) The topology on Balmer's spectrum is in fact the Hochster dual
    of the Zariski topology, and
2b) Hochster duality (1969) becomes a triviality in the language
    of locales/frames.

Remarks 1a and 2a are also made by Buan, Krause and Solberg.
1b and 2b represent standard stuff in pointless topology: everything I
write can be found in some form in Peter Johnstone's book "Stones Spaces"
(Cambridge 1982).

I am sorry the text I send you is so long: I have included the variations
distributive-lattice/frame/topological-space, which is at the core of the
whole yoga, and I also felt like revisiting some very familiar constructions
just to have them available in this language.  I hope it is not too boring.

I welcome any comments you may have.

Cheers,
Joachim.

Cc: Henning Krause

PS: I hope what I write is correct, but I often make mistakes.
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Remarks on spectra, supports, and Hochster duality

Joachim Kock, 2007-12-03

1 Classification and equivalences

Classification results claim a bijective correspondence between two sets, and often an equivalence between two categories. The best results not only state that an equivalence exists, but rather say that certain naturally defined functors are in fact (quasi)-inverse to each other. Very often such a pair of functors is naturally a restriction of an adjoint pair of functors defined on larger categories.

The general situation is this: given an adjoint pair $F \dashv G$,

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G$$

with unit and counit

$$\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow G \circ F \qquad\qquad \varepsilon : F \circ G \Rightarrow \mathrm{Id}_{\mathcal{D}},$$

define full subcategories

$$\mathrm{Fix}(\eta) := \{X \in \mathcal{C} \mid \eta_X \text{ is iso}\} \qquad\qquad \mathrm{Fix}(\varepsilon) := \{Y \in \mathcal{D} \mid \varepsilon_Y \text{ is iso}\}.$$

Then the adjunction restricts to an adjoint equivalence between these fixpoint categories:

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "F"] & \mathcal{D} \arrow[l, shift left, "G"] \\ \mathrm{Fix}(\eta) \arrow[u, phantom, "\cup" rotate=90] \arrow[r, shift left] \arrow[r, phantom, "\sim"] & \mathrm{Fix}(\varepsilon) \arrow[u, phantom, "\cup"] \arrow[l, shift left] \end{tikzcd}$$

Furthermore, if the image of F is contained in $\mathrm{Fix}(\varepsilon)$, then also the image of G in contained in $\mathrm{Fix}(\eta)$, and vice versa. In this situation, $\mathrm{Fix}(\eta)$ is a reflective subcategory of $\mathcal{C}$, the left adjoint to the inclusion $\mathrm{Fix}(\eta) \subset \mathcal{C}$ being $G \circ F$, and $\mathrm{Fix}(\varepsilon)$ is a coreflective subcategory of $\mathcal{D}$, the right adjoint to the inclusion $\mathrm{Fix}(\varepsilon) \subset \mathcal{D}$ being $F \circ G$. The task of establishing an equivalence then amounts to identifying the fixpoint categories.

2 Frames

2.1 Distributive lattices. A *distributive lattice* is a poset with meet $\wedge$ and join $\vee$, such that meet distributes over join:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z);$$

furthermore we require a bottom element 0 (neutral for join) and a top element 1 (neutral for meet). We should also explicitly require $x \wedge 0 = 0$. (It follows automatically that also join distributes over meet, and $x \vee 1 = 1$.)

If X is a topological space, the set of open subsets, denoted $\mathcal{O}(X)$, is a distributive lattice: union is join and intersection is meet. In fact it is a *complete* lattice: also infinite joins and infinite meets exist. The infinite joins are just infinite unions; the infinite meet is given as the interior of the infinite intersection.

In this situation the infinite distributive law holds:

$$x \wedge (\bigvee_{i \in I} y_i) = \bigvee_{i \in I} (x \wedge y_i)$$

The dual also holds, but is of no importance. In fact, the categorical symmetry of the structure is regarded as an accidental circumstance.

2.2 Frames and locales. [Ehresmann and Bénabou ∼1954.] A *frame* is a complete lattice in which finite meets distribute over arbitrary joins:

$$x \wedge (\bigvee_{i \in I} y_i) = \bigvee_{i \in I} (x \wedge y_i)$$

In other words they are distributive lattices satisfying the axioms for open sets in a topology. The opposite of a frame is called a *locale*. The words 'frame' and 'locale' both refer to the same objects, but the notion of morphism differs: a *frame morphism* is required to preserve infinite joins and finite meets but not necessarily infinite meets — just like the inverse image of a continuous map — while a morphism of locales is required to preserve infinite meets and finite joins (but not necessarily infinite joins). Locales behave more geometrically, frames are more algebraic. Henceforth we stick to frames.

2.3 'Pointless topology'. There is an obvious functor from topological spaces to frames,

$$\begin{array}{rcl} \boldsymbol{Top} & \longrightarrow & \boldsymbol{Frm}^{\mathrm{op}} \\ X & \longmapsto & \mathcal{O}(X), \end{array}$$

(contravariant since it sends a continuous map the inverse image of open sets). This functor has a right adjoint, the *points functor*, which assigns to a frame its set of points, as we now explain. Let $*$ denote the singleton topological space. Just as a point of a topological space X amounts to a continuous map $* \to X$, a *point* of a frame F is a frame map $F \to \{0, 1\} = \mathcal{O}(*)$. The set of points of F, denoted $\mathrm{pt}(F)$, has a natural topology: the open subsets of $\mathrm{pt}(F)$ are those of the form $\{p : F \to \{0, 1\} \mid p(U) = 1\}$ for some element U of F.

The topological spaces that occur as spaces of points of frames are precisely the sober spaces. *Sober* means that every irreducible closed subset has a unique generic point. (Most topological spaces occurring in nature are sober. In particular, all Hausdorff spaces are sober.)

The frames that appear as frames of open sets of topological spaces are called *spatial*: they are characterised as those frames having enough points. This means that for any two elements [think: any two open sets], there is a point that separates them [think: a point contained in one of the open sets but not the other].

Altogether, the adjunction restricts to an equivalence of categories

$$\begin{array}{ccc} \boldsymbol{Top} & \rightleftarrows & \boldsymbol{Frm}^{\mathrm{op}} \\ \cup & & \cup \\ \boldsymbol{SobTop} & \underset{\sim}{\rightleftarrows} & \boldsymbol{SpFrm}^{\mathrm{op}} \end{array}$$

between sober spaces and spatial frames.

3 Prime spectra

3.1 Zariski spectrum and 'Nullstellensatz'. Let R be a commutative ring. Denote by $\mathrm{Spec}(R)$ the set of prime ideals in R. For any set S denote by $\mathcal{P}(S)$ the power set, the set of all subsets of S. This is a poset, and hence a category.

We start with the poset map

$$\begin{array}{rcl} Z : \mathcal{P}(R) & \longrightarrow & \mathcal{P}(\mathrm{Spec}(R))^{\mathrm{op}} \\ S & \longmapsto & \{\mathfrak{p} \mid \mathfrak{p} \supset S\}. \end{array}$$

[Thinking of elements $f \in R$ as functions on $\mathrm{Spec}(R)$, $Z(S)$ is the set of common zeros of the functions $f \in S$.] It is clear that we have

$$Z\big(\bigcup_{\lambda\in\Lambda} S_\lambda\big) = \bigcap_{\lambda\in\Lambda} Z(S_\lambda)$$

This implies that Z has a right adjoint; this right adjoint is given by

$$\begin{aligned} I : \mathcal{P}(\mathrm{Spec}(R))^{\mathrm{op}} &\longrightarrow \mathcal{P}(R) \\ Y &\longmapsto \{f \in R \mid Z(f) \supset Y\}. \end{aligned}$$

[We think of $I(Y)$ as the set of functions vanishing on Y.] (Here is the proof of the adjunction:

$$S \subset I(Y) \;\Leftrightarrow\; \forall f \in S : Z(f) \supset Y \;\Leftrightarrow\; \bigcap_{f\in S} Z(f) \supset Y \;\Leftrightarrow\; Z(S) \supset Y.)$$

Since I is a right adjoint it preserves limits: $I\big(\bigcup_{\lambda\in\Lambda} Y_\lambda\big) = \bigcap_{\lambda\in\Lambda} I(Y_\lambda)$, and in particular we have

$$I(Y) = \bigcap_{\mathfrak{p}\in Y} \mathfrak{p}.$$

This shows that $I(Y)$ is always an ideal, in fact a radical ideal. The unit for the adjunction $Z \dashv I$ is 'taking radical':

$$\eta : S \hookrightarrow I(Z(S)) = \bigcap_{\mathfrak{p}\supset S} \mathfrak{p} = \sqrt{\langle S\rangle} = \{f \mid \exists n \in \mathbb{N} : f^n \in \langle S\rangle\}.$$

Hence we find that $\mathrm{Fix}(\eta)$ is the set of radical ideals.

On the other side of the adjunction we describe the fixpoints for ε. Define a subset of the form $Z(S)$ to be a *closed set*, and check that this really defines a topology on $\mathrm{Spec}(R)$: we already noted that these sets are closed under arbitrary intersections (and in particular $Z(\emptyset)$ is the whole space $\mathrm{Spec}(R)$). Next we observe that $Z(R) = \emptyset$ because an ideal is a proper subset, and finally we find that $Z(A) \cup Z(B) = Z(AB)$ by using the prime property $fg \in \mathfrak{p} \Rightarrow [f \in \mathfrak{p} \text{ or } g \in \mathfrak{p}]$. (As an example of the argument, $Z(fg) = \{\mathfrak{p} \mid \mathfrak{p} \ni fg\} = \{\mathfrak{p} \mid \mathfrak{p} \ni f \text{ or } \mathfrak{p} \ni g\} = Z(f) \cup Z(g)$.)

Now we can interpret the counit for the adjunction $Z \dashv I$ as 'taking closure':

$$\varepsilon : U \hookrightarrow \overline{U} := Z(I(U))$$

The universal property implied by the adjointness is that every closed set containing U also contains $\overline{U}$. Indeed, the statement $Z(S) \supset U$ is equivalent by adjunction to $S \subset I(U)$, and applying Z we get $Z(S) \supset Z(I(U))$. We see that the $\mathrm{Fix}(\varepsilon)$ is the set of closed subsets.

Altogether the adjunction restricts to an isomorphism of posets

$$\begin{array}{ccc} \mathcal{P}(R) & \rightleftarrows & \mathcal{P}(\mathrm{Spec}(R))^{\mathrm{op}} \\ \cup & & \cup \\ \mathrm{RadId}(R) & \simeq & \mathrm{Closed}(\mathrm{Spec}(R))^{\mathrm{op}} \end{array}$$

This can be seen as a classification of radical ideals in R. It can also be called a Nullstellensatz. (Or perhaps the identification of the unit could be called the Nullstellensatz.)

We can reformulate this by looking at open sets instead, setting $D(S) := \complement Z(S)$. The isomorphism reads:

$$\mathrm{RadId}(R) \simeq \mathcal{O}(\mathrm{Spec}(R)).$$

Since we know that the space $\mathrm{Spec}(R)$ is sober, we don't need to work with the topological space of points: we can stick with the frame $\mathcal{O}(\mathrm{Spec}(R)) = \mathrm{RadId}(R)$. (This viewpoint is crucial in constructive algebraic geometry: the prime spectrum is not constructive because it relies on Zorn's lemma, whereas $\mathrm{RadId}(R)$ makes sense in any topos.)

3.2 Variation: the Zariski spectrum of a distributive lattice. An *ideal* in a distributive lattice K is a subset I such that $[a \in I \text{ and } x \leq a] \Rightarrow x \in I$ (i.e. I is a down-set), and which is closed under finite joins [Stone 1936]. The notion of a *filter* [due to Cartan] is the dual notion. An ideal is *prime* is its complement is a filter. This expresses the usual idea $a \wedge b \in I \Leftrightarrow [a \in I \text{ or } b \in I]$. Let $\mathrm{Spec}(K)$ denote the set of prime ideals in K. Mimicking the constructions above, with the obvious notion of Zariski topology on $\mathrm{Spec}(K)$, we get an adjunction restricting to an isomorphism of posets

$$\begin{array}{ccc} \mathcal{P}(K) & \rightleftarrows & \mathcal{P}(\mathrm{Spec}(K))^{\mathrm{op}} \\ \cup & & \cup \\ \mathrm{Idl}(K) & \simeq & \mathrm{Closed}(\mathrm{Spec}(K))^{\mathrm{op}} \end{array}$$

(There is no need for a notion of radical ideal in a distributive lattice, since $a \wedge a = a$.) It follows in particular, that $\mathrm{Idl}(K)$ is a frame.

4 Universal property of the Zariski spectrum

The viewpoint that $\mathrm{RadId}(R)$ is the essence of the Zariski spectrum is due to Joyal ∼1970. He also identified its universal property, which grosso modo says that 'RadId is the best frame approximation to a ring':

Definition. Let R be a commutative ring. A *support* for R (with values in a frame) is a pair (F, d) where F is a frame and d is a map $d : R \to F$ satisfying

$$\begin{array}{ll} d(1) = 1 & d(fg) = d(f) \wedge d(g) \\ d(0) = 0 & d(f+g) \leq d(f) \vee d(g). \end{array}$$

A *morphism of supports* is a frame map compatible with the map from R.

The *Zariski support* is the frame of radical ideals, with 'taking radical':

$$\begin{array}{rcl} R & \longrightarrow & \mathrm{RadId}(R) \\ f & \longmapsto & \sqrt{(f)}. \end{array}$$

Theorem. *The Zariski support is the initial support.* In other words, for any support $d : R \to F$, there is a unique frame map $u : \mathrm{RadId}(R) \to F$ making this diagram commute:

$$\begin{tikzcd} R \arrow[rr, "d"] \arrow[dr, "\sqrt{}"'] & & F \\ & \mathrm{RadId}(R) \arrow[ur, dashed, "\exists! u"'] & \end{tikzcd}$$

(The map u is defined as $J \mapsto \bigvee\{d(f) \mid f \in J\}$.)

Remarks. The axioms reflect our intuition about what the support of a real-valued function is: the locus where the function does not vanish. The inclusion (instead of equality) in the last condition reflects the fact that two functions might cancel each other at some points, hence the support of their sum may be slightly smaller than the union of the supports. (In algebraic geometry, this discrepancy is much more pronounced: with $R = k[x, y]$, the support of $x+y$ is the complement of the diagonal line whereas the union of the supports of x and y is the complement of the origin.) Often in analysis one takes the support to be the closure. This has the effect of turning the inclusion into an equality. However, this artefact is not useful in the Zariski topology, where every open set of an integral scheme is dense.

Reformulation. *Definition.* A *support* (with values in a topological space) is a pair (X, d) where X is a topological space and d is a map $d : R \to \mathcal{O}(X)$ satisfying $d(1) = X$, $d(0) = \emptyset$, $d(fg) = d(f) \cap d(g)$, and finally $d(f + g) \subset d(f) \cup d(g)$. A morphism of supports is a continuous map whose inverse-image map on open sets is compatible with the map from R. *Theorem.* $(\mathrm{Spec}(R), D)$ *is the terminal support.* (Here $D(f) = \{\mathfrak{p} \mid f \notin \mathfrak{p}\}$.) In other words, for any support (X, d), there is a unique continuous map $v : X \to \mathrm{Spec}(R)$ making this diagram commute:

$$\begin{tikzcd} R \arrow[rr, "d"] \arrow[dr, "D"'] & & \mathcal{O}(X) \\ & \mathcal{O}(\mathrm{Spec}\, R) \arrow[ur, "v^{-1}"'] & \end{tikzcd}$$

(Note that X is not required to be sober. The map $d : R \to \mathcal{O}(X)$ only refers to the frame of open sets of X, and to give a map $v : X \to \mathrm{Spec}(R) = \mathrm{pt}(\mathrm{RadId}(R))$ is equivalent by adjunction to giving $v^{-1} : \mathrm{RadId}(R) \to \mathcal{O}(X)$, so it makes no difference for the result whether X is sober or not.)

Variation. Joyal also gave the following finitary version, using only 'compact elements'. *Definition.* A *support* (with values in a distributive lattice) is a pair (L, d) where L is a distributive lattice L and d is a map $d : R \to L$ satisfying $d(1) = 1$, $d(0) = 0$, $d(fg) = d(f) \wedge d(g)$, and finally $d(f + g) \leq d(f) \vee d(g)$. *Theorem. The lattice of finitely generated radical ideals, denoted* $\mathrm{fgRadId}(R)$ *is the initial support.* In other words, for any support $d : R \to L$, there is a unique lattice map $\mathrm{fgRadId}(R) \to L$ making this diagram commute:

$$\begin{tikzcd} R \arrow[rr, "d"] \arrow[dr, "\sqrt{}"'] & & L \\ & \mathrm{fgRadId}(R) \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

The relation between the frame version of the theorem and the distributive-lattice version is explained by the following discussion.

5 Coherent frames and spectral spaces

The association $R \mapsto \mathrm{Spec}(R)$ is clearly functorial. It is natural to ask which topological spaces arise in this way. We already observed that $\mathrm{Spec}(R)$ is sober (i.e. every irreducible closed set has a unique generic point). The second crucial property of the Zariski topology is that it is quasi-compact. [This fact is standard; let me just point out that the reason is that infinite joins in the lattice of ideals in ***Ring*** are given by sums of ideal in ***Ring***, and sums are finite. (In contrast, in settings without addition, as commutative monoids, it is not automatic that the Zariski topology is quasi-compact, and this must be added as a separate axiom.)] Moreover, $\mathrm{Spec}(R)$ has a basis of quasi-compact opens (namely the basic open sets $D(f)$).

These two properties in fact characterise spectra of rings:

5.1 Hochster's Theorem (1969). *Definition.* A space is *spectral* if it is sober, and if the quasi-compact open sets form a sub-lattice that is a basis for the topology. (In particular the space itself is therefore quasi-compact). *Theorem. Every spectral space is homeomorphic to* $\mathrm{Spec}(R)$ *for some commutative ring* R*.*

(This is a difficult theorem. The construction of a ring from a spectral space is not (and cannot be) functorial; it is functorial only for certain maps (e.g. spectral epis). Note also that not every ringed spectral space is isomorphic as a ringed space to $(\mathrm{Spec}(R), \widetilde{R})$. For example $\mathbb{P}^n$ is spectral: it is possible to construct a ring R such that $\mathbb{P}^n$ is homeomorphic to $\mathrm{Spec}(R)$, but this homeomorphism cannot be an isomorphism of ringed spaces.)

5.2 Coherent frames. The frame-theoretic counterpart of 'spectral space' is 'coherent frame'. An element c in a frame F is called *compact* (or *finite*) when

$$c \leq \bigvee S \quad \Rightarrow \quad \exists \text{ finite } T \leq S \text{ with } c \leq \bigvee T$$

Let $K(F)$ denote the set of compact elements. A frame F is called *coherent* (or compactly generated) if $K(F)$ is a sublattice (and in particular includes 1), and $K(F)$ generates F. A frame map is called coherent if it takes compact elements to compact elements; denote by ***CohFrm*** the category of coherent frames and coherent maps.

5.3 Coherent frames and distributive lattices. *Theorem.* [Non-trivial.] *The functor* K : ***CohFrm*** $\to$ ***DistrLat*** *is an equivalence of categories. The inverse functor assigns to a distributive lattice* L *the lattice of ideals in* L*, denoted* $\mathrm{Idl}(L)$. The assignment $L \mapsto \mathrm{Idl}(L)$ (outlined in 3.2) can also be described as the cocompletion of L: taking ideals introduces freely the infinite joins, because although the join of infinitely many elements may not exist as an element in L, these elements still generate an ideal.

The following example is archetypical: let R be a commutative ring. In the frame $\mathrm{RadId}(R)$ of radical ideals in R, the compact elements are precisely the finitely generated radical ideals. The functor RadId factors like this:

$$\begin{tikzcd} \boldsymbol{Ring} \arrow[rr, "\mathrm{RadId}"] \arrow[dr, "\mathrm{fgRadId}"'] & & \boldsymbol{CohFrm} \\ & \boldsymbol{DistrLat} \arrow[ur, "\mathrm{Idl}"'] & \end{tikzcd}$$

5.4 Stone's representation theorems. It is not difficult to show that every coherent frame is spatial. Under the equivalence $\boldsymbol{SobTop} \simeq \boldsymbol{SpFrm}^{\mathrm{op}}$, the coherent frames correspond precisely to the spectral spaces, and coherent maps correspond to spectral maps (i.e. for which the inverse image of a quasi-compact open is quasi-compact).

The composite of these two equivalences constitute the modern version of Stone's representation theorem for distributive lattices (1939): *There is an equivalence of categories between distributive lattices and spectral spaces.*

(One can restrict further to get an equivalence between the category of boolean algebras and Hausdorff spectral spaces – these are nowadays called Stone spaces. This is Stone's representation theorem for boolean algebras (1936).)

To summarise these adjunctions and equivalences:

$$\begin{array}{ccccc}
\boldsymbol{Top} & \rightleftarrows & \boldsymbol{Frm}^{\mathrm{op}} & & \\
\cup & & \cup & & \\
\boldsymbol{SobTop} & \simeq & \boldsymbol{SpFrm}^{\mathrm{op}} & & \\
\cup & & \cup & & \\
\boldsymbol{SpecSp} & \simeq & \boldsymbol{CohFrm}^{\mathrm{op}} & \simeq & \boldsymbol{DistrLat}^{\mathrm{op}} \\
\cup & & & & \cup \\
\boldsymbol{Stone} & & \simeq & & \boldsymbol{Bool}^{\mathrm{op}}
\end{array}$$

6 Hochster duality

Let X be a spectral space. The *Hochster dual* of X, denoted $X^\vee$ is the topological space with the same underlying set of points as X, but whose closed sets are generated by the quasi-compact open sets of X. In other words, the new closed sets are the arbitrary intersections of the original quasi-compact open sets.

Hochster's second theorem: $X^\vee$ *is spectral again, and* $X^{\vee\vee} \simeq X$.

In particular, given any ring R, there exists another ring S with an inclusion reversing bijection between the primes of R and the primes of S.

Since only sober spaces are involved in Hochster's second theorem, it can be formulated in terms of frames, and in this formulation it becomes a triviality: starting from a spectral space X, we consider instead the coherent frame $F = \mathcal{O}(X)$ of open sets in X. Let K denote the distributive lattice of compact elements (i.e. quasi-compact open sets in X). The frame $F = \mathrm{Idl}(K)$ is the cocompletion of K, i.e. obtained by formally adding arbitrary joins of elements in K, just as we generate a topology from the set of quasi-compact open sets in X by taking arbitrary unions of those. Now K as a distributive lattice has a dual lattice $K^\vee$ which is also distributive. Now we get another frame $F^\vee := \mathrm{Idl}(K^\vee)$ which is again coherent and therefore corresponds to a spectral space which is just $X^\vee$. It is clear that doing this twice gives back the original frame or space.

(The only thing that is not completely trivial in this setting is the fact that X and $X^\vee$ have the same points. But the points of X are the prime elements of $F = \mathrm{Idl}(K)$, and in general the prime elements of a lattice of ideals are just the prime ideals, so the points of X are the prime ideals of K. These correspond one-to-one to prime filters of K, which are the prime ideals of $K^\vee$, which finally are the points of $X^\vee$.)

[There is a third natural topology on a spectral space X, also introduced and studied by Hochster, called the *patch topology*, whose open sets are generated by the open sets from both X and $X^\vee$. In the abstract setting, this amounts to the boolean completion of the lattice of compact elements, i.e. it is a Stone space.]

7 Balmer's spectrum of a tensor triangulated category

Balmer has introduced the spectrum of a tensor triangulated category and shown that it enjoys a universal property as recipient of supports. His topology on $\mathrm{Spec}(\mathcal{K})$ is the Hochster dual of the Zariski topology. We first formulate Balmer's results in terms of the Zariski topology. Then we recover his original formulations by playing with Hochster duality. The Zariski topology formulation may appear more natural. Balmer's version is justified by the applications to algebraic geometry and representation theory.

Let $(\mathcal{K}, \otimes, 1)$ be a tensor triangulated category with 0 and $\oplus$, and where $\otimes$ distributes over $\oplus$. An *tensor ideal* is a thick subcategory $A \subset \mathcal{K}$ such that $[s \in \mathcal{K}, f \in A] \Rightarrow s \otimes f \in A$. The thickness condition is important:

$$f \oplus g \in A \Rightarrow [f \in A \text{ and } g \in A].$$

An ideal $\mathfrak{p}$ is *prime* if $f \otimes g \in \mathfrak{p} \Rightarrow [f \in \mathfrak{p} \text{ or } g \in \mathfrak{p}]$. Let $\mathrm{Spec}(\mathcal{K})$ denote the set of prime ideals in $\mathcal{K}$.

7.1 The Zariski topology. Just as in §3, there is an adjunction

$$\mathcal{P}(\mathcal{K}) \underset{I}{\overset{Z}{\rightleftarrows}} \mathcal{P}(\mathrm{Spec}(\mathcal{K}))^{\mathrm{op}}$$

where $Z(S) = \{\mathfrak{p} \mid \mathfrak{p} \supset S\}$ and $I(Y) = \{f \in \mathcal{K} \mid Z(f) \supset Y\}$. We identify the fixpoint sets: The subsets $Z(S)$ are the closed subsets of the *Zariski topology* of $\mathrm{Spec}(\mathcal{K})$ — here the prime property is crucial for establishing that the union of two closed subsets is again closed. Just as in §3, for formal reasons, there is an inclusion-reversing bijection between radical tensor ideals in $\mathcal{K}$ and closed subsets of $\mathrm{Spec}(\mathcal{K})$. (The notion of radical tensor ideal is the standard one: the radical of a tensor ideal is the intersection of all prime ideals containing it, and an ideal is radical if it coincides with its own radical.)

Now $\mathrm{Spec}(\mathcal{K})$ is a spectral space. In other words, the frame $\mathrm{RadId}(\mathcal{K}) = \mathcal{O}(\mathrm{Spec}(\mathcal{K}))$ is coherent. This frame enjoys a universal property, discovered by Balmer:

7.2 Universal property of support. *Definition.* An *open support* on $\mathcal{K}$ is a pair (F, d) where F is a frame and $d : \mathcal{K} \to F$ is a map satisfying

$$\begin{aligned} d(0) &= 0 \\ d(1) &= 1 \\ d(f \oplus g) &= d(f) \vee d(g) \\ d(f \otimes g) &= d(f) \wedge dg) \\ d(\Sigma f) &= d(g) \\ f \to g \to h \to \Sigma f &\Rightarrow d(f) \leq d(g) \vee d(h) \end{aligned}$$

Theorem. The function

$$\begin{array}{ccc} \mathcal{K} & \longrightarrow & \mathrm{RadId}(\mathcal{K}) \\ f & \longmapsto & \sqrt{(f)} \end{array}$$

is the initial open support.

The result can be reformulated in terms of topological spaces, by regarding instead $\mathrm{Spec}(\mathcal{K})$ and the function $D := \complement Z$. This pair is then terminal among pairs (X, d) consisting of a topological space X and an open support $d : \mathcal{K} \to \mathcal{O}(X)$. The unique comparison map is a continuous map $v : X \to \mathrm{Spec}(\mathcal{K})$ such that $v^{-1} : \mathcal{O}(\mathrm{Spec}(\mathcal{K})) \to \mathcal{O}(X)$ is compatible with the support functions.

It is useful to consider also the compact version: An open support on $\mathcal{K}$ with values in a distributive lattice is a pair (L, d) where L is a distributive lattice and $d : \mathcal{K} \to L$ is a map satisfying the six axioms above. The notion of finitely generated radical tensor ideal is clear. *Theorem. The function*

$$\begin{array}{ccc} \mathcal{K} & \longrightarrow & \mathrm{fgRadId}(\mathcal{K}) \\ a & \longmapsto & \sqrt{(a)} \end{array}$$

is the initial open support with values in a distributive lattice.

7.3 Balmer's topology. Balmer's topology on $\mathrm{Spec}(\mathcal{K})$ is given by taking the closed sets to be

$$B(S) := \{\mathfrak{p} \mid S \cap \mathfrak{p} = \emptyset\}.$$

It is clear that this set of subsets is stable under arbitrary intersections. It is also stable under union: we have $B(S_1) \cup B(S_2) = B(S_1 \oplus S_2)$ thanks to the *thickness condition.* Note that the prime condition is not essential here!

This topology is the Hochster dual of the Zariski topology: clearly

$$B(S) = \bigcap_{f \in S} B(f) = \bigcap_{f \in S} \{\mathfrak{p} \mid \mathfrak{p} \not\ni f\} = \bigcap_{f \in S} D(f),$$

the arbitrary intersections of the quasi-compact open sets of the Zariski topology. It is quite remarkable that a uniform description of the closed sets of the Hochster dual topology exists in this case: it is due to the thickness condition, whose analogue for commutative rings would be an absurdity.

We now run through some more Hochster duality yoga.

7.4 Reformulation of the classification of radical tensor ideals. Put $\mathrm{supp}(f) := D(f) = \complement Z(f)$. For any $S \in \mathcal{P}(\mathcal{K})$, put

$$\mathrm{supp}(S) := \bigcup_{f \in S} \mathrm{supp}(f) = \{\mathfrak{p} \mid \mathfrak{p} \not\supset S\} = \complement Z(S).$$

Given $Y \in \mathcal{P}(\mathrm{Spec}(\mathcal{K}))$, define a radical tensor ideal

$$\begin{aligned} K(Y) &:= \{f \in \mathcal{K} \mid \mathrm{supp}(f) \subset Y\} \\ &= \{f \in \mathcal{K} \mid \complement Z(f) \subset Y\} \\ &= \{f \in \mathcal{K} \mid Z(f) \supset \complement Y\} \\ &= I(\complement Y). \end{aligned}$$

Consider the set of subsets in $\mathrm{Spec}(\mathcal{K})$ of the form $\bigcup_{\lambda \in \Lambda} Y_\lambda$, where Y_λ is Balmer closed with quasi-compact open complement. These sets can also be characterised as the sets of the form $\complement \bigcap_{\lambda \in \Lambda} W_\lambda$ where W_λ are quasi-compact open sets in Balmer's topology. These sets are precisely the open sets of the Hochster dual of the Balmer topology; in other words they are precisely the Zariski open sets of $\mathrm{Spec}(\mathcal{K})$.

The above order-reversing bijection $Z \dashv I$ between radical tensor ideals and Zariski closed subsets can be restated as an order-preserving bijection $\mathrm{supp} \dashv K$ between radical tensor ideals and subsets of the form $\bigcup_{\lambda \in \Lambda} Y_\lambda$, where Y_λ is Balmer closed with quasi-compact complement.

7.5 Dual open supports. A *dual open support* (with values in a frame) is a pair (F, e) where F is a frame and $e : \mathcal{K} \to F$ is a map satisfying the axioms dual to those of 7.2, to wit:

$$\begin{aligned} e(0) &= 1 \\ e(1) &= 0 \\ e(f \oplus g) &= e(f) \wedge e(g) \\ e(f \otimes g) &= e(f) \vee e(g) \\ e(\Sigma f) &= e(f) \\ f \to g \to h \to \Sigma f \quad &\Rightarrow \quad e(f) \geq e(g) \wedge e(g) \end{aligned}$$

The Hochster dual of the frame $\mathrm{RadId}(\mathcal{K})$ *is the universal dual open support.* To see what the map $\mathcal{K} \to \mathrm{RadId}(\mathcal{K})^\vee$ is, notice that $\sqrt{(f)}$ is finitely generated hence determines a well-defined element in the Hochster dual frame. These arguments are clearer in the compact version: A dual open support with values in a distributive lattice is a map satisfying the above axioms, and $\mathrm{fgRadId}(\mathcal{K})^\vee$ is the universal such. At this level, the result is nothing but a dualisation of the universal distributive-lattice valued open support.

Now formulate the topological-space version of this result: a dual open support on $\mathcal{K}$ with values in a topological space is a pair (X, e) where X is a topological space and $e : \mathcal{K} \to \mathcal{O}(X)$ is a map as in the beginning of 7.5. The Balmer spectrum is the universal such dual open support (i.e. $\mathrm{Spec}(\mathcal{K})^\vee$ with the function $Z : \mathcal{K} \to \mathcal{O}(\mathrm{Spec}(\mathcal{K})^\vee)$ given in 7.1). (Note again that the Zariski-closed set $Z(f)$ has quasi-compact complement, hence defines an open set in the dual topology.) So far we are still talking about open sets, but we are treating a strange dual version of the axioms. The final step is to take complements:

7.6 Closed supports. *Definition.* A closed support is a pair (X, σ) where X is a topological space and $\sigma : \mathcal{K} \to \mathrm{Closed}(X)$ is a function satisfying

$$\begin{array}{rcl} \sigma(0) & = & \emptyset \\ \sigma(1) & = & X \\ \sigma(a \oplus b) & = & \sigma(a) \cup \sigma(b) \\ \sigma(a \otimes b) & = & \sigma(a) \cap \sigma(b) \\ \sigma(\Sigma a) & = & \sigma(a) \\ a \to b \to c \to \Sigma a & \Rightarrow & \sigma(a) \subset \sigma(b) \cup \sigma(c) \end{array}$$

Theorem. The Balmer spectrum $\mathrm{Spec}(\mathcal{K})^\vee$ *together with the function* $\mathrm{supp} := \complement Z$ *is the universal closed support.* In other words, for any closed support (X, σ) there is a unique continuous map $v : X \to \mathrm{Spec}(\mathcal{K})^\vee$ such that

$$\begin{tikzcd} \mathcal{K} \arrow[rr, "\sigma"] \arrow[dr, "\mathrm{supp}"'] & & \mathrm{Closed}(X) \\ & \mathrm{Closed}(\mathrm{Spec}(\mathcal{K})^\vee) \arrow[ur, "v^{-1}"'] & \end{tikzcd}$$

(Note that the passage from the setting of open sets to the setting of closed sets does not change the variance of the comparison map v^{-1}, so the natural viewpoint on the universal property is still that of frames, not that of locales, although we are talking about closed sets.)