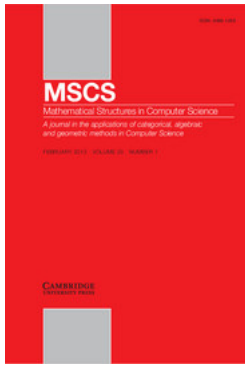

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Local fibred right adjoints are polynomial

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For any locally cartesian closed category $\mathscr{E}$, we prove that a local fibred right adjoint between slices of $\mathscr{E}$ is given by a polynomial. The slices in question are taken in a well-known fibred sense.

1. Introduction

In a locally cartesian closed category $\mathscr{E}$, a diagram of the form

$$I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} J \tag{1}$$

gives rise to a so-called *polynomial functor* $\mathscr{E}/I \to \mathscr{E}/J$, namely the composite functor

$$\mathscr{E}/I \xrightarrow{s^*} \mathscr{E}/E \xrightarrow{p_*} \mathscr{E}/B \xrightarrow{t_!} \mathscr{E}/J. \tag{2}$$

The use of polynomial functors as data type constructors goes back at least to the 1980s (*cf.* Manes and Arbib (1986)). Moerdijk and Palmgren (2000), and in a more general setting, Gambino and Hyland (2004), showed that initial algebras for polynomial endofunctors are precisely the W-types of Martin-Löf type theory. In a series of papers, Abbott, Altenkirch, Ghani and collaborators have further developed the theory of polynomial functors (called container functors in Abbott *et al.* (2003)) and their natural transformations as data type constructors and polymorphic functions, subsuming notions as shapely types, strictly positive types and general tree types – see Gambino and Kock (2009) for background on polynomial functors and an extensive bibliography, and also for pointers to the use of polynomial functors in logic, combinatorics, representation theory, topology and higher category theory.

In Gambino and Kock (2009), six different intrinsic characterisations of polynomial functors were listed for the case where $\mathscr{E} = \mathbf{Set}$, one of them being that a functor $P : \mathbf{Set}/I \to \mathbf{Set}/J$ is polynomial if and only if it is a local right adjoint, that is, the slice of P at the terminal object of $\mathbf{Set}/I$ is a right adjoint. However, this characterisation already fails when $\mathscr{E}$ is a presheaf topos, as pointed out in Weber (2007). His counter-example is a significant one: the free-category monad on the category of graphs is a local right adjoint, but it is not polynomial.

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In this paper we adjust the local-right-adjoint characterisation so that it is valid in every locally cartesian closed category. To do this, we pass to the setting of fibred slice categories and fibred functors. The *fibred slice* $\mathscr{E}|I$ is the fibred category over $\mathscr{E}$ whose fibre over an object K is the plain slice $\mathscr{E}/(I \times K)$. Diagram (1) can also be used to define a fibred polynomial functor

$$\mathscr{E}|I \xrightarrow{s^*} \mathscr{E}|E \xrightarrow{p_*} \mathscr{E}|B \xrightarrow{t_!} \mathscr{E}|J,$$

whose fibre over the terminal object $1 \in \mathscr{E}$ is the plain polynomial functor (2). Hence, any polynomial functor has a canonical extension to a fibred functor.

Our main theorem states that *a fibred functor $P : \mathscr{E}|I \to \mathscr{E}|J$ is polynomial if and only if it is a local fibred right adjoint.*

2. Fibred categories and fibred slices

The purpose of this section is mainly to fix terminology and notation – see Borceux (1994), Johnstone (2002) and Streicher (1999) for further background on fibred categories.

A category $\mathscr{E}$ is fixed throughout; for the main result, it is assumed to be a locally cartesian closed category (lccc) with a terminal object. For some of our considerations, it will be enough that it is a category with finite limits, as in Streicher (1999). We also assume that we have chosen pullbacks once and for all so that for each arrow $a : J \to I$, we have at our disposal two functors

$$\begin{aligned} a_! &: \mathscr{E}/J \longrightarrow \mathscr{E}/I \\ a^* &: \mathscr{E}/I \longrightarrow \mathscr{E}/J, \end{aligned}$$

and if $\mathscr{E}$ is also an lccc, the a^*s have right adjoints

$$a_* : \mathscr{E}/J \longrightarrow \mathscr{E}/I,$$

which we also assume are chosen once and for all; so

$$a_! \dashv a^* \dashv a_*.$$

Notion 2.1 (Fibred categories). We shall work with categories fibred over $\mathscr{E}$, henceforth just called fibred categories. These form a 2-category $\mathbf{Fib}_{\mathscr{E}}$ whose objects are fibred categories, whose 1-cells are fibred functors (that is, functors commuting with the structure functor to $\mathscr{E}$ and sending cartesian arrows to cartesian arrows), and whose 2-cells are fibred natural transformations (that is, natural transformations whose components are vertical arrows). If $\mathscr{F}$ is a fibred category, we use $\mathscr{F}^I$ to denote the (strict) fibre over an object $I \in \mathscr{E}$, and if $L : \mathscr{G} \to \mathscr{F}$ is a fibred functor, we write $L^I : \mathscr{G}^I \to \mathscr{F}^I$ for the induced functor between the I-fibres.

Notion 2.2 (Cleavage and base change). A *cleavage* of a fibred category $\mathscr{F}$ is a choice of cartesian lifts: for each arrow $a : J \to I$ in $\mathscr{E}$ and object $X \in \mathscr{F}^I$, we choose a cartesian arrow over a with codomain X. For each a, this assignment defines a functor $a^* : \mathscr{F}^I \to \mathscr{F}^J$ called the *base change* along a: the value of a^* on $X \in \mathscr{F}^I$ is taken to be the domain of the chosen cartesian lift of a with codomain X, and the value on morphisms is given by exploiting the universal property of cartesian arrows.

For a composable pair of arrows a_1 and a_2 in $\mathscr{E}$, there is a canonical isomorphism ${a_1}^* \circ {a_2}^* \xrightarrow{\sim} (a_2 \circ a_1)^*$, which is likewise derived from the universal property of cartesian arrows.

For a fibred functor $L : \mathscr{G} \to \mathscr{F}$, the functors induced on the fibres commute with base change functors up to canonical isomorphisms: for $a : J \to I$ in $\mathscr{E}$, the universal property of the chosen cartesian lifts of a in $\mathscr{F}$ gives rise to 2-cells:

$$\begin{tikzcd} \mathscr{G}^I \arrow[r, "L^I"] \arrow[d, "a^*"'] & \mathscr{F}^I \arrow[d, "a^*"] \\ \mathscr{G}^J \arrow[r, "L^J"'] \arrow[ur, phantom, "L^a \nearrow"] & \mathscr{F}^J \end{tikzcd} \tag{3}$$

The fact that the L^as are invertible is due to the fact that L preserves cartesian arrows. Again, because the isomorphisms arise from universal properties, coherence conditions can be deduced.

Notion 2.3 (Fibred adjunctions). An adjunction between two fibred functors in opposite directions is given by two fibred natural transformations $\eta : \mathrm{id} \Rightarrow R \circ L$ and $\varepsilon : L \circ R \Rightarrow \mathrm{id}$ satisfying the usual triangle identities. In other words, it is an adjunction in the 2-category $\mathbf{Fib}_{\mathscr{E}}$. It is clear that a fibred adjunction induces a fibre-wise adjunction for each fibre. Conversely (see, for example, Borceux (1994, 8.4.2)), if $R : \mathscr{F} \to \mathscr{G}$ is a fibred functor and for each $I \in \mathscr{E}$ there is a left adjoint $L^I \dashv R^I$, then these L^I assemble into a fibred left adjoint if for every arrow $a : J \to I$ in $\mathscr{E}$, the mate of the canonical invertible 2-cell

$$\begin{tikzcd} \mathscr{F}^I \arrow[r, "R^I"] \arrow[d, "a^*"'] & \mathscr{G}^I \arrow[d, "a^*"] \arrow[dl, phantom, "(R^a)^{-1} \swarrow"] \\ \mathscr{F}^J \arrow[r, "R^J"'] & \mathscr{G}^J \end{tikzcd}$$

namely

$$\begin{tikzcd} \mathscr{F}^I \arrow[d, "a^*"'] & \mathscr{G}^I \arrow[l, "L^I"'] \arrow[d, "a^*"] \\ \mathscr{F}^J \arrow[ur, phantom, "\nwarrow"] & \mathscr{G}^J \arrow[l, "L^J"] \end{tikzcd}$$

is again invertible, and hence turns the family L^I into a fibred functor.

Notion 2.4 (Bifibrations and $\mathscr{E}$-indexed sums). A fibred category $\mathscr{F}$ is said to be *bifibred* if the structure functor $\mathscr{F} \to \mathscr{E}$ is also an opfibration, that is, has all opcartesian lifts. We then assume the choice of an opcartesian lift for each arrow $a : J \to I$ and each object $T \in \mathscr{F}^J$. This defines cobase-change functors, which we denote by the suffix lowershriek, thus $a_! : \mathscr{F}^J \to \mathscr{F}^I$. Cobase change is left adjoint to base change: $a_! \dashv a^*$. Indeed, for $a : J \to I$ in $\mathscr{E}$, we have natural bijections

$$\mathscr{F}^J(T, a^*X) \cong \mathscr{F}^a(T, X) \cong \mathscr{F}^I(a_!T, X)$$

according to the universal properties of cartesian and opcartesian arrows (here $\mathscr{F}^a(T, X)$ denotes the set of arrows $T \to X$ lying over a).

A fibred functor is said to be *bifibred* if it also preserves opcartesian arrows. In terms of cobase-change functors, we can say that a fibred functor $L : \mathscr{G} \to \mathscr{F}$ is bifibred if for every arrow $a : J \to I$ in $\mathscr{E}$, the mate of the compatibility-with-base-change square (3)

$$\begin{tikzcd}
\mathscr{G}^I \arrow[r, "L^I"] & \mathscr{F}^I \\
\mathscr{G}^J \arrow[u, "a_!"] \arrow[r, "L^J"'] \arrow[ur, phantom, "\nwarrow \overline{L}^a"] & \mathscr{F}^J \arrow[u, "a_!"']
\end{tikzcd} \tag{4}$$

is invertible.

Bifibred categories are mostly interesting when they also satisfy the *Beck–Chevalley condition*. In terms of the chosen cartesian and opcartesian lifts, this condition says that for every pullback square in $\mathscr{E}$

$$\begin{tikzcd}
\cdot \arrow[r, "b"] \arrow[d, "u"'] \arrow[dr, phantom, "\lrcorner", very near start] & \cdot \arrow[d, "v"] \\
\cdot \arrow[r, "a"'] & \cdot
\end{tikzcd}$$

the fibred natural transformation

$$u_! b^* \Rightarrow a^* v_!$$

is invertible.

A fibred category $\mathscr{F}$ is said to have *$\mathscr{E}$-indexed sums* when $\mathscr{F} \to \mathscr{E}$ is bifibred, and the Beck–Chevalley condition holds. We use $\mathbf{Fib}^{\Sigma}_{\mathscr{E}}$ to denote the full subcategory of bifibred categories/bifibred functors thus determined.

Proposition 2.5. A fibred left adjoint between bifibred categories preserves cobase change (in particular, it preserves $\mathscr{E}$-indexed sums if the categories have such).

Proof. We just take left adjoints of all the arrows in the base-change compatibility square (3) for the right adjoint. □

Notion 2.6 (Fibred slices). The *fibred slice* $\mathscr{E}|I$ is the category whose objects are spans

$$I \xleftarrow{p} M \xrightarrow{q} K,$$

and whose morphisms are diagrams

$$\begin{tikzcd}[row sep=small]
 & M' \arrow[dl, "p'"'] \arrow[dd, "v"] \arrow[r, "q'"] & K' \arrow[dd, "w"] \\
I & & \\
 & M \arrow[ul, "p"] \arrow[r, "q"'] & K
\end{tikzcd}$$

The structural functor $\mathscr{E}|I \to \mathscr{E}$ that returns the right-most object, or arrow, is a bifibration. The cartesian arrows are the diagrams for which the square is a pullback, while the opcartesian arrows are those for which v is invertible, as is easy to verify. The vertical arrows are those for which w is an identity arrow.

More conceptually, the fibred slice is obtained from the plain slice as the following pullback, and the structural functor as the left-hand vertical composite:

$$\begin{tikzcd}
\mathscr{E}|I \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathscr{E}/I \arrow[d, "\mathrm{dom}"] \\
\mathrm{Ar}(\mathscr{E}) \arrow[r, "d"] \arrow[d, "c"'] & \mathscr{E} \\
\mathscr{E} &
\end{tikzcd}$$

(here $\mathrm{Ar}(\mathscr{E})$ is the category of arrows in $\mathscr{E}$, and d and c are the domain and codomain fibrations, respectively). This is to say that the fibred slice is the so-called family fibration of the fibration $\mathrm{dom} : \mathscr{E}/I \to \mathscr{E}$ (*cf.* Streicher (1999, 6.2) or Johnstone (2002, Proposition 1.4.16)).

For the K-fibre of $\mathscr{E}|I$, we have the canonical identification

$$(\mathscr{E}|I)^K \cong \mathscr{E}/(I \times K).$$

In particular, the plain slice $\mathscr{E}/I$ sits inside the fibred slice $\mathscr{E}|I$ as the fibre over the terminal object $1 \in \mathscr{E}$:

$$(\mathscr{E}|I)^1 \cong \mathscr{E}/I.$$

Note also that we have $\mathscr{E}|1 \cong \mathrm{Ar}(\mathscr{E})$.

In the I-fibre of $\mathscr{E}|I$ we have the canonical object given by the identity span

$$I \xleftarrow{\mathrm{id}} I \xrightarrow{\mathrm{id}} I,$$

which, in view of the interpretation in terms of plain slices, we denote by δ (for 'diagonal').

Note that $\mathscr{E}|I$ has $\mathscr{E}$-indexed sums: for $a : K' \to K$ in $\mathscr{E}$, the base-change functor $a^* : (\mathscr{E}|I)^K \to (\mathscr{E}|I)^{K'}$ is identified with $(\mathrm{id}_I \times a)^* : \mathscr{E}/(I \times K) \to \mathscr{E}/(I \times K')$, which has left adjoint $(\mathrm{id}_I \times a)_!$, and the Beck–Chevalley condition follows from the case of plain slices.

Notion 2.7 (Polynomial functors – fibred version). We now assume that $\mathscr{E}$ is an lccc. Each arrow $f : J \to I$ in $\mathscr{E}$ induces fibred functors

$$\begin{aligned} f_! &: \mathscr{E}|J \longrightarrow \mathscr{E}|I \\ f^* &: \mathscr{E}|I \longrightarrow \mathscr{E}|J \\ f_* &: \mathscr{E}|J \longrightarrow \mathscr{E}|I \end{aligned}$$

and fibred adjunctions

$$f_! \dashv f^* \dashv f_*.$$

These extend the basic functors on plain slices: for example, if $f^* : \mathscr{E}/I \to \mathscr{E}/J$ is the plain pullback, then the K-fibre of the fibred pullback functor f^* is

$$(f \times \mathrm{id}_K)^* : \mathscr{E}/(I \times K) \to \mathscr{E}/(J \times K).$$

A fibred functor of the form

$$\mathscr{E}|I \xrightarrow{\ s^*\ } \mathscr{E}|E \xrightarrow{\ p_*\ } \mathscr{E}|B \xrightarrow{\ t_!\ } \mathscr{E}|J$$

for a diagram in $\mathscr{E}$

$$I \xleftarrow{\ s\ } E \xrightarrow{\ p\ } B \xrightarrow{\ t\ } J$$

is called a (fibred) *polynomial functor*.

3. Fibred left adjoints

In this section, the base category $\mathscr{E}$ is just assumed to have finite limits.

Recall that $\delta \in \mathscr{E}|I$ denotes the identity span $I \xleftarrow{\ \mathrm{id}\ } I \xrightarrow{\ \mathrm{id}\ } I$. The following is the main lemma.

Lemma 3.1. Let $\mathscr{F}$ be a fibred category with $\mathscr{E}$-indexed sums. For each $I \in \mathscr{E}$, the functor

$$\begin{aligned} \mathrm{ev}_\delta : \mathbf{Fib}^{\Sigma}_{\mathscr{E}}(\mathscr{E}|I, \mathscr{F}) &\longrightarrow \mathscr{F}^I \\ L &\longmapsto L(\delta) \end{aligned}$$

is an equivalence of categories. A pseudo-inverse is given on objects by

$$\begin{aligned} h : \mathscr{F}^I &\longrightarrow \mathbf{Fib}^{\Sigma}_{\mathscr{E}}(\mathscr{E}|I, \mathscr{F}) \\ X &\longmapsto [\langle p, q\rangle \mapsto q_! p^* X], \end{aligned}$$

where $\langle p, q\rangle$ denotes a span $I \xleftarrow{\ p\ } M \xrightarrow{\ q\ } K$.

Proof. We have already noted that $\mathscr{E}|I$ is the family fibration of $\mathscr{E}/I \to \mathscr{E}$, which implies, see Johnstone (2002, Proposition 1.4.16 (ii)), that it is the $\mathscr{E}$-indexed-sum completion of $\mathscr{E}/I$. More precisely, if $\mathscr{F}$ has $\mathscr{E}$-indexed sums, then precomposition with the obvious 'diagonal' functor $\eta : \mathscr{E}/I \to \mathscr{E}|I$ provides an equivalence

$$\mathbf{Fib}^{\Sigma}_{\mathscr{E}}(\mathscr{E}|I, \mathscr{F}) \xrightarrow{\sim} \mathbf{Fib}_{\mathscr{E}}(\mathscr{E}/I, \mathscr{F}).$$

We also have the well-known fibred Yoneda Lemma (see, for example, Borceux (1994, Proposition 8.2.7)), which says that evaluation at id_I provides an equivalence

$$\mathbf{Fib}_{\mathscr{E}}(\mathscr{E}/I, \mathscr{F}) \xrightarrow{\sim} \mathscr{F}^I$$

that is valid for any fibred category $\mathscr{F}$. The composite of the two functors in the statement of the lemma is evaluation at δ, so we get an explicit pseudo-inverse for the composite by composing the pseudo-inverses for the two functors exhibited, and they are described in the two cited references (using a co-cleavage and a cleavage, respectively). Inspecting these descriptions gives, in particular, our (partial) description of the pseudo-inverse. □

Corollary 3.2. If $L : \mathscr{E}|I \to \mathscr{E}|J$ is a bifibred functor, it is isomorphic, as a fibred functor, to $q_! \circ p^*$ for the span $I \xleftarrow{p} M \xrightarrow{q} J$ obtained as $L(\delta)$.

Proof. Take $\mathscr{F} = \mathscr{E}|J$ in the Main Lemma, and let $\langle p, q\rangle := L(\delta)$. Then we also have $q_! p^*(\delta) = \langle p, q\rangle$, and $q_! p^*$ is bifibred by Proposition 2.5 since it is a fibred left adjoint. But the Main Lemma shows that bifibred functors $\mathscr{E}|I \to \mathscr{E}|J$ are determined up to isomorphism by their value on δ; hence $L \cong q_! \circ p^*$. □

Corollary 3.3. If $L : \mathscr{E}|I \to \mathscr{E}|J$ is a bifibred functor that preserves terminal objects, then it is isomorphic, as a fibred functor, to p^*, for some $p : J \to I$.

Proof. The previous corollary shows that $L \cong q_! \circ p^*$. Since * functors preserve terminal objects, we conclude that $q_!$ preserves the terminal object, but this is possible only if q is invertible. Hence $L \cong p^*$. □

We note the following special case of Corollary 3.2.

Theorem 3.4. If $L : \mathscr{E}|I \to \mathscr{E}|J$ is a fibred left adjoint, it is obtained from a span $I \xleftarrow{p} M \xrightarrow{q} J$, as $L \cong q_! \circ p^*$. And if $R : \mathscr{E}|J \to \mathscr{E}|I$ is a fibred right adjoint, it is obtained from such a span as $R \cong p_* \circ q^*$.

4. Local fibred right adjoints

Notion 4.1 (Local right adjoints). If $\mathscr{F}$ is a category with a terminal object $1_{\mathscr{F}}$, and $R : \mathscr{F} \to \mathscr{G}$ is a functor, there is a well-known canonical factorisation of R

$$\mathscr{F} \xrightarrow{\overline{R}} \mathscr{G}/R(1_{\mathscr{F}}) \xrightarrow{\mathrm{dom}} \mathscr{G} \tag{5}$$

where $\overline{R}$ takes $A \in \mathscr{F}$ to the value of R on $A \to 1_{\mathscr{F}}$. We say that R is a *local right adjoint* if $\overline{R}$ is a right adjoint. (This implies that all the evident functors $\mathscr{F}/X \to \mathscr{G}/R(X)$ are also right adjoints, but we shall not use this fact.)

Notion 4.2 (Local fibred right adjoints). Let $\mathscr{F}$ and $\mathscr{G}$ be fibred over $\mathscr{E}$ and assume that $\mathscr{F}$ has a terminal object $1_{\mathscr{F}}$. Let $R : \mathscr{F} \to \mathscr{G}$ be a fibred functor. Then the factorisation (5) is actually a factorisation of fibred functors, where $\mathscr{G}/R(1_{\mathscr{F}})$ is fibred over $\mathscr{E}$ through the fibration dom and the structural functor $\mathscr{G} \to \mathscr{E}$. If in this situation $\overline{R}$ is a fibred right adjoint, we say that R is a *local fibred right adjoint*. (In particular, R is then a (plain) local right adjoint.)

In the case where $\mathscr{G}$ is of the form $\mathscr{E}|J$,

$$\mathscr{F} \xrightarrow{\overline{R}} (\mathscr{E}|J)/R(1_{\mathscr{F}}) \xrightarrow{\mathrm{dom}} \mathscr{E}|J,$$

we shall see that the middle category $(\mathscr{E}|J)/R(1_{\mathscr{F}})$ is itself a fibred slice. To do this, we will need a little preparation.

Note 4.3 (Plain slices of fibred slices). In addition to the structural functor $\mathscr{E}|J \to \mathscr{E}$ (which given a span $J \leftarrow M \to K$ returns K), we also have the 'apex' functor

$$\begin{aligned} d : \mathscr{E}|J &\longrightarrow \mathscr{E} \\ [J \leftarrow M \to K] &\longmapsto M. \end{aligned}$$

For a fixed span $(J \xleftarrow{t} M \to K) = Q \in \mathscr{E}|J$, a forgetful fibred functor

$$(\mathscr{E}|J)/Q \longrightarrow \mathscr{E}|d(Q)$$

is induced; it sends an object

$$\begin{tikzcd} & Y \arrow[r] \arrow[dl] \arrow[d] & X \arrow[d] \\ J & M \arrow[l, "t"] \arrow[r] & K \end{tikzcd}$$

to the span

$$M \longleftarrow Y \longrightarrow X.$$

This functor is the first leg of a factorisation of the domain functor dom:

$$\begin{tikzcd} (\mathscr{E}|J)/Q \arrow[rr, "\mathrm{dom}"] \arrow[dr] & & \mathscr{E}|J \\ & \mathscr{E}|d(Q) \arrow[ur, "t_!"'] & \end{tikzcd}$$

Lemma 4.4. Using the above notation, when Q belongs to the 1-fibre for the structural fibration (that is, is of the form $J \leftarrow M \to 1$), the forgetful functor

$$(\mathscr{E}|J)/Q \longrightarrow \mathscr{E}|d(Q)$$

is an equivalence of fibred categories over $\mathscr{E}$.

We now return to the factorisation (5) as fibred functors, and take

$$\begin{aligned} \mathscr{F} &= \mathscr{E}|I \\ \mathscr{G} &= \mathscr{E}|J. \end{aligned}$$

Note that $\mathscr{E}|I$ has a terminal object 1_I, namely, the span $I \xleftarrow{\mathrm{id}_I} I \to 1$. It belongs to the 1-fibre, and hence so does $Q := R(1_I)$. Combining the above discussion with the lemma, we get the following corollary.

Corollary 4.5. Any local fibred right adjoint $R : \mathscr{E}|I \to \mathscr{E}|J$ has a canonical factorisation by fibred functors

$$\mathscr{E}|I \xrightarrow{\overline{R}} \mathscr{E}|B \xrightarrow{t_!} \mathscr{E}|J, \tag{6}$$

with $\overline{R}$ a fibred right adjoint. Here $(J \xleftarrow{t} B \to 1) := R^1(I \xleftarrow{\mathrm{id}_I} I \to 1)$.

We can now prove the theorem stated in the title of the paper.

Theorem 4.6. Let $\mathscr{E}$ be a locally cartesian closed category with a terminal object. If $R : \mathscr{E}|I \to \mathscr{E}|J$ is a local fibred right adjoint, then it is a polynomial functor.

Proof. According to the description in Notion 2.7, we need to construct the polynomial

$$I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} J \tag{7}$$

representing R, that is, such that $R \cong t_! \circ p_* \circ s^*$. By Corollary 4.5, we have $R = t_! \circ \overline{R}$, where $\overline{R} : \mathscr{E}|I \to \mathscr{E}|B$ has a fibred left adjoint L. Explicitly, B and t are determined by

$$(J \xleftarrow{t} B \to 1) \ := \ R^1(I \xleftarrow{\mathrm{id}_I} I \to 1).$$

By the main lemma (Lemma 3.1), or, more precisely, its corollary (Corollary 3.2), we can write $L \cong s_! \circ p^*$, hence $\overline{R} \cong p_* \circ s^*$. The maps s and p are given explicitly by

$$(I \xleftarrow{s} E \xrightarrow{p} B) \ := \ L^B(B \xleftarrow{\mathrm{id}_B} B \xrightarrow{\mathrm{id}_B} B).$$

Putting all this together, we have $R \cong t_! \circ p_* \circ s^*$ as claimed. $\square$

Remarks 4.7. The converse of the theorem is also true: (fibred) polynomial functors are always local fibred right adjoints. Indeed, if $P : \mathscr{E}|I \to \mathscr{E}|J$ is the fibred functor $P = t_! \circ p_* \circ s^*$, we have $\overline{P} = p_* \circ s^*$ with fibred left adjoint $s_! \circ p^*$.

It should also be noted that the diagram (7) 'representing' a local fibred right adjoint is essentially unique. This follows from Gambino and Kock (2009, Theorem 2.17), which establishes a biequivalence between a bicategory whose 1-cells are 'polynomials' (1) and a 2-category whose 1-cells are 'polynomial functors' (that is, functors that are isomorphic to one given by a polynomial, and hence have a canonical extension to a fibred functor). Theorem 4.6 gives an intrinsic characterisation of this essential image.

Example 4.8. If $\mathscr{E}$ is the category of sets, any category $\mathscr{F}$ can canonically be seen as the 1-fibre of a category fibred over $\mathscr{E}$, namely, the category whose I-fibre is the category of I-indexed families of objects in $\mathscr{F}$; and any functor $\mathscr{F} \to \mathscr{G}$ extends canonically to a fibred functor. One often expresses this by saying "any category $\mathscr{F}$ 'is' fibred over Sets, and any functor 'is' fibred". So, rather than considering local fibred right adjoints, we can just consider local right adjoints $\mathscr{E}/I \to \mathscr{E}/J$ and prove that they are given by a polynomial (1); see, for example, Gambino and Kock (2009).

If $\mathscr{E}$ is a more general topos, functors $\mathscr{E}/I \to \mathscr{E}/J$ need not 'be' fibred, not even for $I = J = 1$, that is, they may not be the 1-fibres of a fibred functor $\mathscr{E}|I \to \mathscr{E}|J$, as the following examples show.

Example 4.9 (Weber). Let $\mathscr{E}$ be the category of directed graphs, that is, the presheaf category of $(0 \rightrightarrows 1)$, and let $T : \mathscr{E} \to \mathscr{E}$ be the free-category monad. Weber (2007) observes that T is a local right adjoint but not a polynomial functor. The argument (given in detail in Weber (2007, Example 2.5)) amounts to showing that the left adjoint to $\overline{T}$ does not preserve monos. In contrast, for a polynomial functor $P = t_! p_* s^*$, the left adjoint to $\overline{P} = p_* s^*$ is $s_! p^*$, which does preserve monos (as does every polynomial functor). It follows *a posteriori* from our Main Theorem (Theorem 4.6) that T cannot be the 1-fibre of a local fibred right adjoint $\mathscr{E}|1 \to \mathscr{E}|1$.

Example 4.10. Let $\mathscr{E}$ be the category of G-sets, where G is a non-trivial group. The group homomorphism $p : G \to 1$ induces functors $p_! \dashv p^* \dashv p_*$, where p^* applied to a set S makes it into a G-set, with trivial G-action. (The functors $p_!$ and p_* may be seen as left- and right-Kan extensions, respectively.) The endofunctor $R : \mathscr{E} \to \mathscr{E}$ given by $p^* \circ p_*$ converts a G-set X to the subset consisting of the fixpoints for the action (and equipped with trivial action). Now R has a left adjoint, namely $p^* \circ p_!$. So R is a right adjoint $\mathscr{E} \to \mathscr{E}$ (or $\mathscr{E}/1 \to \mathscr{E}/1$), but it is not the 1-fibre of a fibred right adjoint $\mathscr{E}|1 \to \mathscr{E}|1$, since this would imply, by general theory (see Remark 4.12) that R could be equipped with a tensorial strength (in the sense of Kock (1972)) $I \times R(X) \to R(I \times X)$ that is natural in I and X (objects of $\mathscr{E}$). In particular, if we take $X = 1$, we have $I = I \times R(1) \to R(I)$; but if I is a non-empty G-set without stationary points, this is impossible, since then $R(I)$ is empty.

Example 4.11. The following is, essentially, from Yetter (1987) and Kock and Reyes (1999, Section 4). Let D be an atom in an lccc $\mathscr{E}$, meaning that the endofunctor $X \mapsto X^D$ has a right adjoint. Clearly, $(-)^D$ extends to a fibred functor $\mathscr{E}|1 \to \mathscr{E}|1$ since it is polynomial. By Yetter (1987), this extension has a right adjoint fibre by fibre, but there cannot be a fibred right adjoint, unless $D = 1$. Indeed, the strength of $(-)^D$ is given by the obvious $I \times X^D \to (I \times X)^D$, and if the functor $(-)^D$ is a fibred left adjoint, this implies that the strength is an isomorphism (see Remark 4.12). In particular, instantiating at $X = 1$, we get the 'diagonal': $I \to I^D$ (exponential transpose of the projection map $I \times D \to I$), which is then an isomorphism natural in I. By an easy application of the strong Yoneda Lemma (see, for example, Gambino and Kock (2009, Lemma 2.6)), we get that $D = 1$.

Remark 4.12 (From fibred functors to strength). If $\mathscr{F}$ is fibred over $\mathscr{E}$ and has $\mathscr{E}$-indexed sums, then $\mathscr{F}^1$ is tensored over $\mathscr{E}$: if S is in $\mathscr{E}$ and $X \in \mathscr{F}^1$, we have $S \otimes X = \mathrm{pr}_!(\mathrm{pr}^*(X))$, where pr denotes the unique map $S \to 1$. If $L : \mathscr{F} \to \mathscr{G}$ is a fibred functor (where $\mathscr{G}$ similarly has $\mathscr{E}$-indexed sums), we may rewrite $S \otimes L^1(X)$ as

$$S \otimes L^1(X) = \mathrm{pr}_!(\mathrm{pr}^*(L^1(X))) \cong \mathrm{pr}_!(L^S(\mathrm{pr}^*(X))).$$

On the other hand, the 2-cell exhibited in (4) in particular provides a map

$$\mathrm{pr}_!(L^S(\mathrm{pr}^*X) \to L^1(\mathrm{pr}_!\,\mathrm{pr}^*X) = L^1(S \otimes X).$$

So, by composition, we get a map

$$S \otimes L^1(X) \to L^1(S \otimes X),$$

which is a tensorial strength for the functor L^1. So, briefly, 'a fibring implies a (tensorial) strength' (see Johnstone (1997, Section 3) for the case $\mathscr{F} = \mathscr{G} = \mathscr{E}$). Note that if L is a fibred left adjoint, it commutes with lowershriek (cobase-change) functors, and therefore the strength is an isomorphism.

Remark 4.13 (Is strength sufficient?). There is a partial converse to 'fibring implies strength', due to Paré (*cf.* Johnstone (1997, Proposition 3.3)): *a pullback-preserving functor*

$\mathscr{E} \to \mathscr{E}$ with a strength extends uniquely to a fibred functor $\mathscr{E}|1 \to \mathscr{E}|1$. It is plausible that a similar result could hold also for functors of the form $\mathscr{E}/I \to \mathscr{E}/J$, and that in this situation strong natural transformation would correspond to fibred natural transformations, so the notions of strong adjunction and fibred adjunction would match up.

Gambino and Kock (2009) studied polynomial functors in the setting of functors equipped with a tensorial strength ('strong functors') and strong natural transformations; in a sense, when compared with the fibred setting, this is more economical since only plain slices are needed. Should the above speculation turn out to be true, it would seem to imply a strength version of our main theorem, namely that every local strong right adjoint $\mathscr{E}/I \to \mathscr{E}/J$ is polynomial.

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