NOTES ON PSI CLASSES

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Contents

Introduction

intro

This is about psi classes on moduli spaces of stable n-pointed curves, and more generally, of stable maps. The first section contains the definition and the basics: How psi classes restrict to the boundary; how they pull back along forgetful morphisms, and the string equation, which is the central basic property of psi classes.

Sections 2 and 3 are independent of each other and of Part II. Section 2 compares the intersection theory of psi classes on $\overline{M}_{g,n}$ with the intersection theory of kappa classes on $\overline{M}_{g,0}$ (as studied by Mumford). The result is that the two theories imply each other.

The third section gives an elementary introduction to Witten's conjecture (now Kontsevich's theorem) which states that the generating function of the top products of psi classes obeys the KdV hierarchy of differential equations. This determines all the top products. We make no attempt to get into Kontsevich's proof, but give some examples of its consequences.

In Part II we leave the realm of stable curves and enter that of stable maps. In Section 4 we recall the definition and basic properties about stable maps, evaluation classes (classes pulled back from the target space) and define psi classes. Most of the basic results on psi classes in the setting of stable maps are easy generalisations of the corresponding results for curves. Top products of psi classes and evaluation classes are called gravitational descendants, or just Gromov-Witten invariants. Section 5 is devoted to the part of the theory particular to genus zero: the WDVV equations and topological recursion, which determines all the descendent integrals from the Gromov-Witten invariants... In Section 6 we describe some properties of the virtual fundamental class...

Except for the techniques of the virtual fundamental class, all the ideas and most of the results covered in these notes go back to the paper of Edward Witten, *Two-dimensional gravity and intersection theory on moduli space* [29], which is highly recommended reading.

There are several other places to go, for continuation of these notes at a more advances level: from Section 2, the reader can have a look at Arbarello-Cornalba [1], (the cohomological field theories constructed by Kabanov-Kimura [17] are also in direct continuation of Sections 2 and 5). The reader will also enjoy reading something about Faber's conjecture on the tautological ring of M_g.

From Section 5 the reader might jump to the theory of Frobenius manifolds, but that is not so much focused on psi classes! Otherwise look at relative Gromov-Witten invariants à la Gathmann [7], or enumerative geometry and characteristic numbers, cf. Graber-Kock-Pandharipande [13].

From Section 6 it is natural to look into the Virasoro conjecture, which says that the generating function of the descendants is annihilated by certain differential operators that form (half of) the Virasoro algebra. The best place to start is probably Getzler's survey [11], (or maybe have a look at Getzler [10] first). Other directions: Hodge integrals, equivariant quantum cohomology (see [27])...

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Part I

Psi classes on moduli of pointed curves

1 Basic properties and the string equation

basic

1.1 Generalities and definitions

Consider a family of curves, with a section

$$\begin{tikzcd} \mathfrak{X} \arrow[d, shift left, "\pi"] \\ B \arrow[u, shift left, "\sigma"] \end{tikzcd}$$

Then σ is a regular embedding if and only if π is smooth in a neighbourhood of σ. In that case there is a natural isomorphism

$$N_\sigma \simeq \sigma^* T_\pi,$$

cf. Fulton [5] B.7.2 – B.7.3. The condition is equivalent to requiring that the section do not pass through any singular point of any fibre, i.e. in each of the curves $\mathfrak{X}_b$, the point singled out by the section is a smooth one.

Definition. In this situation, the *cotangent line bundle* is

$$\mathbb{L} := \sigma^* \omega_\pi.$$

It's first Chern class is called the *psi class*: $\boldsymbol{\psi} := c_1(\mathbb{L}) \in A^1(B)$.

Remark: in general the dualising sheaf need not be locally free, but in a neighbourhood of σ it is, so $\mathbb{L}$ is really a line bundle.

If B is just a point, we've got a curve C with a marked point p. Then $\mathbb{L} \simeq k(p) \otimes \omega_C \simeq \mathfrak{m}_p/\mathfrak{m}_p^2$, the cotangent line. Note that formation of the cotangent line bundle commutes with base change, due to the universal property of the dualising sheaf. So given a point $b \in B$, and let p denote the point $\sigma(b) \in \mathfrak{X}_b$, then the fibre of $\mathbb{L}$ at b is the cotangent line $\mathfrak{m}_p/\mathfrak{m}_p^2$ of the curve $\mathfrak{X}_b$. Indeed, this is just a special case of base change:

$$\begin{tikzcd} \mathfrak{X}_b \arrow[r, hook] \arrow[d] & \mathfrak{X} \arrow[d, "\pi"] \\ b \arrow[r, hook, "\iota"'] & B \end{tikzcd}$$

We can also describe the cotangent line bundle as

$$\mathbb{L} \simeq \sigma^* T_\pi^\vee \simeq N_\sigma^\vee .$$

D2 **1.1.1 Self-intersection formula.** Let D denote the image of σ. Then we have the following self-intersection formula. In terms of line bundles,

$$\sigma^* \mathcal{O}(D) \simeq N_\sigma \simeq \mathbb{L}^\vee,$$

or in terms of divisors,

$$\pi_*(D^2) = \sigma^* D = c_1(N_\sigma) \cap [B] = -\boldsymbol{\psi} \cap [B].$$

(In general, intersecting with D and pushing down in B is the same as pulling back along σ.) Interpreting the same formula as an identity in $A^2(\mathfrak{X})$, we get $D^2 = -\pi^* \boldsymbol{\psi} \cdot [D]$, and more generally

$$\boxed{D^{a+1} = (-\pi^* \boldsymbol{\psi})^a \cdot D}$$

1.1.2 Exercise. On any trivial family, the cotangent line bundle is trivial... ?

1.2 Stable n-pointed curves

1.2.1 Stable n-pointed curves. A stable n-pointed curve of genus g is a connected curve of arithmetic genus g with simple nodes as only singularities, n

marked points, which are distinct smooth points, and satisfying a stability condition: if a twig has genus 0 then it must have at least three special points. If a twig has genus one then it must have at least one special point. (Here “special point” means either a node or a marked point.)

Two n-pointed curves are isomorphic if there is an isomorphism between the curves sending the marks to the marks in the right order. The stability condition is equivalent to requiring that the curves have only a finite number of automorphisms.

Recall that there is a coarse moduli space $\overline{M}_{g,n}$, which is a normal variety of dimension $3g-3+n$. It contains as a dense open set, the locus of smooth n-pointed curves. The complement is called the boundary. (References: Knudsen [18], or Harris-Morrison [14].)

1.2.2 Universal family. We want to define psi classes on $\overline{M}_{g,n}$, so we need a universal family over it. Unfortunately that does not exist, since $\overline{M}_{g,n}$ is only a coarse moduli space. The best we have is the *forgetful morphism* $\pi_0 : \overline{M}_{g,n+1} \to \overline{M}_{g,n}$ which forgets the extra mark p_0 (and stabilises if necessary). (We will often call the extra mark p_0 instead of p_{n+1}, but that is just to simplify the indices.) Over moduli points parametrising curves with no non-trivial automorphisms, this is a tautological family, but over a moduli point corresponding to a curve C with automorphism group G, the fibre is the quotient C/G.

For this reason, it is necessary to consider $\overline{M}_{g,n}$ as a stack; it is then a smooth Deligne-Mumford stack, and $\pi_0 : \overline{M}_{g,n+1} \to \overline{M}_{g,n}$ is the universal family. There is a relative dualising sheaf ω_π which is locally free of rank 1. Now the n canonical sections $\sigma_i : \overline{M}_{0,n} \to \overline{M}_{0,n+1}$ define n cotangent line bundles and their corresponding psi classes:

$$\mathbb{L}_i := \sigma_i^* \omega_\pi \qquad \boldsymbol{\psi}_i := c_1(\mathbb{L}_i).$$

M03 **1.2.3 Example.** Clearly $\overline{M}_{0,3}$ is just a point. The next case, $\overline{M}_{0,4}$ is the space of cross-ratios, isomorphic to $\mathbb{P}^1$. Next $\overline{M}_{0,5}$ is isomorphic to the del Pezzo surface obtained by blowing up four points in the plane. In general, $\overline{M}_{0,n}$ is a smooth variety, and a fine moduli space, so in this case it coincides with the stack. In higher genus, $\overline{M}_{g,n}$ is only a normal variety, but the stack is smooth...

The space $\overline{M}_{1,1}$ of pointed elliptic curves another basic one: here every curve has an involution, so there is a big difference between the stack and the coarse moduli space... a factor 2...

boundary0 **1.2.4 The boundary.** The boundary of $\overline{M}_{g,n}$ is a normal crossings divisor. Its irreducible components are described as follows. Let $S = \{p_1, \ldots, p_n\}$ be the set of marks. For each partition of marks and genus, $S' \cup S'' = S$ and $g' + g'' = g$, there is an irreducible boundary divisor denoted $D(S', g' \mid S'', g'')$. There is also a boundary divisor consisting of irreducible uninodal curves, but it will not play a significant rôle in the sequel.

Each twig of $D = D(S', g' \mid S'', g'')$ corresponds to a moduli space of lower dimension, $\overline{M}' := \overline{M}_{g', S' \cup \{x'\}}$ and $\overline{M}'' := \overline{M}_{g, S'' \cup \{x''\}}$, more precisely, D is the isomorphic image of a morphism

$$\rho_D : \overline{M}' \times \overline{M}'' \to \overline{M}_{g,n}$$

from the product. Let $\tau' : \overline{M}' \times \overline{M}'' \to \overline{M}'$ denote the first projection (and let τ'' be the second). This set-up and notation is used throughout — summarised in the following diagram:

$$\begin{tikzcd} & D \subset \overline{M}_{g,n} & \\ \overline{M}' & \overline{M}' \times \overline{M}'' \arrow[l, "\tau'"'] \arrow[r, "\tau''"'] \arrow[u, "\rho_D"] & \overline{M}''. \end{tikzcd}$$

DD0 **1.2.5 The soft boundary.** The image of σ_i in $\overline{M}_{g,n+1}$ is the boundary divisor $D_{i,0}$ whose general point represents a curve with the two marks p_i and p_0 on a twig of genus zero, and all the other marks on the other twig; let $\mathcal{O}(D_{i,0})$ denote the corresponding line bundle.

Note that since the sections are disjoint, we have

$$\boxed{D_{i,0} \cdot D_{j,0} = 0} \qquad \text{for } i \neq j.$$

rhosigma

The boundary divisor $D_{0,i}$ is the isomorphic image of $\overline{M}_{0,\{p_i,p_0,x'\}} \times \overline{M}''$. But the one-primed space has got only three marks, so it is isomorphic to a point, cf.1.2.3. So in this case, $D_{i,0}$ is the isomorphic image of $\overline{M}''$ which is naturally identified with $\overline{M}_{g,n}$ (relabelling so x'' is called p_i). Under this identification, $\rho_{D_{i,0}}$ is just the section σ_i.

restr-psi **1.2.6 Lemma.** *Continuing the same notation we have*

$$\rho_D^* \mathbb{L}_i = \begin{cases} \tau'^* \mathbb{L}_i & \textit{when } p_i \in S' \\ \tau''^* \mathbb{L}_i & \textit{when } p_i \in S''. \end{cases}$$

In other words,

$$\rho_D^* \boldsymbol{\psi}_i = \begin{cases} \tau'^* \boldsymbol{\psi}_i & \text{when } p_i \in S' \\ \tau''^* \boldsymbol{\psi}_i & \text{when } p_i \in S''. \end{cases}$$

In particular, $\sigma_j^* \boldsymbol{\psi}_i = \boldsymbol{\psi}_i$, *for* $j \neq i$.

Proof. Let $C' \to \overline{M}'$ denote the "universal curve" over $\overline{M}'$. Consider the diagram of cartesian squares

$$\begin{tikzcd}
 & & & \tau'^*C' \underset{x}{\cup} \tau''^*C'' \arrow[dd] & & & \\
 & & \tau'^*C' \arrow[dll] \arrow[dr, "\bar{\pi}'"'] & & \tau''^*C'' \arrow[dl, "\bar{\pi}''"] \arrow[drr] & & \\
C' \arrow[dr, "\pi'"'] & & & \overline{M}' \times \overline{M}'' \arrow[dll, "\tau'"] \arrow[drr, "\tau''"'] & & & C'' \arrow[dl, "\pi''"] \\
 & \overline{M}' & & & & \overline{M}'' &
\end{tikzcd}$$

where $\tau'^*C' \underset{x}{\cup} \tau''^*C''$ denotes the gluing of the two families along the sections $\sigma_{x'}$ and $\sigma_{x''}$. Now compare with the universal family — you can convince yourself that the glued family is a tautological one so it is the pull-back of the universal family:

$$\begin{tikzcd}
\tau'^*C' \underset{x}{\cup} \tau''^*C'' \arrow[r, hook] \arrow[d] & \overline{M}_{g,n+1} \arrow[d, "\pi"] \\
\overline{M}' \times \overline{M}'' \arrow[r, hook, "\rho_D"'] & \overline{M}_{g,n}
\end{tikzcd}$$

Now we can compare: the cotangent line bundle of this family is just the pull-back of the cotangent line bundle of the universal family, but on the other hand: it only takes notice of what happens close to the section, so it is also pulled back from $\overline{M}'$ (in case $p_i \in S'$). $\square$

psiD0 **1.2.7 Corollary.** In particular, if the boundary divisor is just $D_{i,0}$, then (cf. 1.2.5) we get $\sigma_i^* \boldsymbol{\psi}_i = \rho_{D_{i,0}}^* \boldsymbol{\psi}_i = 0$, since on the space with only three marks we have $\boldsymbol{\psi}_i = 0$. Now pulling back along σ_i is the same as intersecting with $D_{i,0}$ and then pushing down, so $\pi_{0*}(\boldsymbol{\psi}_i \cap [D_{i,0}]) = 0$. Now since π_0 restricted to $D_{i,0}$ is an isomorphism we conclude the following identity in $A_*(\overline{M}_{g,1})$:

$$\boxed{\boldsymbol{\psi}_i \cdot D_{i,0} = 0}$$

1.3 The important comparison result

pull-psi **1.3.1 Lemma.** *Let $\pi_0 : \overline{M}_{g,n+1} \to \overline{M}_{g,n}$ be the forgetful morphism that forgets p_0. Then the following identity holds in $A^1(\overline{M}_{g,n+1})$:*

$$\boxed{\boldsymbol{\psi}_i = \pi_0^* \boldsymbol{\psi}_i + D_{i,0}}$$

where the psi class on the right-hand side lives on $\overline{M}_{g,n}$.

Proof. In terms of line bundles, looking in the fibres note that $\mathbb{L}_i$ is equal to the pull-back from below at all points off $D := D_{i,0}$. So we conclude that $\mathbb{L}_i \simeq \pi_0^* \mathbb{L}_i \otimes \mathcal{O}(rD)$ for some integer r. Now pull back along σ_i:

$$\begin{aligned} \mathcal{O} = \sigma_i^* \mathbb{L}_i &= \sigma_i^* \pi_0^* \mathbb{L}_i \otimes \sigma_i^* \mathcal{O}(D)^{\otimes r} \\ &= \mathbb{L}_i \otimes (\mathbb{L}_i^\vee)^{\otimes r} \end{aligned}$$

so we conclude that $r = 1$. □

selfint **1.3.2 Remark.** Keeping the same notation, we have (cf. 1.1.1)

$$\boxed{D_{i,0}^2 = -\pi_0^* \boldsymbol{\psi}_i \cdot D_{i,0}}$$

This follows immediately from the comparison result: $-\pi_0^* \boldsymbol{\psi}_i \cdot D_{i,0} = (-\boldsymbol{\psi}_i + D_{i,0}) \cdot D_{i,0} = D_{i,0}^2$, since $-\boldsymbol{\psi}_i \cdot D_{i,0} = 0$.

psi-a **1.3.3 Corollary.** The following two identities for a power of a psi class are easily established via induction:

$$\begin{aligned} \boldsymbol{\psi}_i^a &= \pi_0^* \boldsymbol{\psi}_i^a + \pi_0^* \boldsymbol{\psi}_i^{a-1} D_{i,0} \\ \boldsymbol{\psi}_i^a &= \pi_0^* \boldsymbol{\psi}_i^a + (-1)^{a-1} D_{i,0}^a \end{aligned}$$

These formulae will be useful in the sequel.

1.3.4 1-parameter families. There is another perspective that gives insight in the questions. Instead of looking at the big forgetful morphism between the two moduli spaces, look just at the little one between two 1-parameter families. Let $\widetilde{\pi} : \widetilde{\mathfrak{X}} \to B$ be a 1-parameter family of stable $(n+1)$-pointed curves, and let $D_{j,0}$ denote the divisor consisting of all points $b \in B$ such that $\widetilde{\mathfrak{X}}_b$ is a curve with the two marks p_j and p_0 alone on a twig of genus 0.

Now forget the last mark. We know there is a stabilised family $\pi : \mathfrak{X} \to B$ and a morphism $\varepsilon : \widetilde{\mathfrak{X}} \to \mathfrak{X}$ such that the following diagram commutes (with sections)

$$\begin{tikzcd} \widetilde{\mathfrak{X}} \arrow[r, "\varepsilon"] \arrow[dr, "\widetilde{\pi}"'] & \mathfrak{X} \arrow[d, "\pi"] \\ & B \end{tikzcd}$$

The morphism ε is the blow-down of all components of a fibre having p_0 and some p_j alone on a twig; let E denote the union of all the exceptional divisors.

Now compare the psi class $\widetilde{\boldsymbol{\psi}}_i$ of $\widetilde{\pi}$ with that of π.

$$\begin{aligned} \widetilde{\boldsymbol{\psi}}_i &= \widetilde{\sigma}_i^* \omega_{\widetilde{\pi}} \\ &= \widetilde{\sigma}_i^* \big(\varepsilon^* \omega_\pi + E \big) \\ &= \sigma_i^* \omega_\pi + \widetilde{\sigma}_i^* E \\ &= \boldsymbol{\psi}_i + D_{i,0}. \end{aligned}$$

This proves the pull-back formula for a 1-parameter family.

1.4 The string equation

Definition. Let us adopt Witten's notation and terminology (cf. [29]), introducing the *correlation functions*

$$\langle \tau_{a_1} \cdots \tau_{a_n} \rangle_g := \int \boldsymbol{\psi}_1^{a_1} \ \cdots \ \boldsymbol{\psi}_n^{a_n} \ \cap [\overline{M}_{g,n}],$$

(with the convention that the product is zero if it involves a negative a_i). Note that to get a non-zero integral we need $\sum a_i = 3g - 3 + n$.

Warning: the indices on the taus don't refer to particular marks, so the symbol τ_4 has nothing to do with a particular mark p_4. It means that there is one of the marks q such that $\boldsymbol{\psi}_q$ appears with exponent 4.

1.4.1 Example. Since $\overline{M}_{0,3}$ is just a point, it is immediate that

$$\langle\, \tau_0\tau_0\tau_0 \,\rangle_0 = \int [\overline{M}_{0,3}] = 1.$$

stringtau **1.4.2 Lemma.** *(The string equation — see also 3.2.1.) Except for the case of the previous example ($n = 2$ and $a_1 = a_2 = 0$), we have*

$$\boxed{\langle\ \tau_0 \cdot \prod_{i=1}^{n} \tau_{a_i}\ \rangle_g = \sum_{j=1}^{n} \langle\ \tau_{a_j-1} \prod_{i\neq j} \tau_{a_i}\ \rangle_g}$$

In the formula, the left-hand side is on a space with one extra mark.

Proof. As in the previous, let us call the extra mark p_0. The factor τ_0 means that there aren't any classes at this mark. We are going to forget this mark, pushing down along π_0. Prior to the push-down, we use the pull-back formula 1.3.1 to write each psi class as a pull-back plus a boundary divisor, and then use the projection formula. By definition, $\langle\ \tau_0 \cdot \prod_{i=1}^{n} \tau_{a_i}\ \rangle_g = \int \boldsymbol{\psi}_1^{a_1} \cdots \boldsymbol{\psi}_n^{a_n} \cap [\overline{M}_{g,n+1}]$. Now expand each factor using the comparison 1.3.1 (and in particular the corollary 1.3.3), getting

$$\int_{\overline{M}_{g,n+1}} \prod_{i=1}^{n} \big(\pi_0^*\boldsymbol{\psi}_i^{a_i} + \pi_0^*\boldsymbol{\psi}_i^{a_i-1} D_{i,0}\big).$$

Now since $D_{i,0} \cdot D_{j,0} = 0$ for $i \neq j$ (cf. 1.2.5), only $n+1$ terms survive in the expansion of that product:

$$\pi_0^*\big(\boldsymbol{\psi}_1^{a_1} \cdots \boldsymbol{\psi}_n^{a_n}\big) + \sum_{j=1}^{n} \pi_0^*\big(\boldsymbol{\psi}_j^{a_j-1} \prod_{i\neq j} \boldsymbol{\psi}_i^{a_i}\big) \cdot D_{j,0}.$$

Now push down, forgetting p_0, and use the projection formula to get the result: the first term has zero push-down for dimension reasons; for the other terms, note that $D_{j,0}$ pushes down to the fundamental class, since it is a section. □

1.5 Genus 0

The initial condition $\langle\, \tau_0\tau_0\tau_0 \,\rangle_0 = 1$ (on $\overline{M}_{0,3}$) together with the string equation is sufficient to determine all products in genus 0. Indeed, if not $\sum a_i = n - 3$, then the product is zero for dimension reasons; otherwise, at least one of the a_i's is equal to zero, so we can use the string equation to get fewer marks. Eventually we come down to the initial condition. In fact, pascal **1.5.1 Lemma.** *Suppose* $\sum a_i = n-3$. *Then*

$$\langle\, \tau_{a_1} \cdots \tau_{a_n} \,\rangle_0 = \frac{(n-3)!}{a_1! \cdots a_n!}$$

The proof is just a combinatorial identity. Recall the summation rule of Pascal's triangle $\binom{p}{q} = \binom{p-1}{q-1} + \binom{p-1}{q}$. In other words, when $q_1 + q_2 = p$ we have

$$\frac{p!}{q_1! \cdot q_2!} = \frac{(p-1)!}{(q_1-1)! \cdot q_2!} + \frac{(p-1)!}{q_1! \cdot (q_2-1)!}.$$

(It is the easy argument: put aside one of the p elements, and consider first all choices including it, and then all choices not including it.) The multinomial analogue is this (assuming $q_1 + q_2 + \cdots + q_n = p$):

$$\frac{p!}{q_1! \cdots q_n!} = \sum_{j=1}^{n} \frac{(p-1)!}{(q_j-1)! \prod_{i \neq j} q_i!}$$

Proof of the lemma. The formula holds in case of three marks: $\langle\, \tau_0\tau_0\tau_0 \,\rangle_0 = 0!/(0!0!0!) = 1$. Suppose it holds for n marks and consider a product with $n+1$ factors. As noticed, one of the a_i's must be zero, so by the string equation

$$\langle\ \tau_0 \cdot \prod_{i=1}^{n} \tau_{a_i}\ \rangle_0 = \sum_{j=1}^{n} \langle\ \tau_{a_j - 1} \prod_{i \neq j} \tau_{a_i}\ \rangle_0$$

and then by induction,

$$= \sum_{j=1}^{n} \frac{(n-3)!}{(a_j-1)! \prod_{i \neq j} a_i!}$$

and the result follows from the multinomial identity. $\square$

0123 **1.5.2 The psi class in terms of boundary divisors.** The following is really a feature specific to genus zero. In $\overline{M}_{0,n}$, let $(p_1|p_2,p_3)$ denote the sum of all boundary divisors having p_1 on one twig and p_2 and p_3 on the other. Now we claim that

$$\boxed{\psi_1 = (p_1|p_2,p_3)}$$

The claim holds trivially in the case $\overline{M}_{0,3}$. Now by induction the result follows in all other spaces. Indeed,

$$\begin{aligned}\boldsymbol{\psi}_1 &= \pi_0^* \boldsymbol{\psi}_1 + D_{1,0} \\ &= \pi_0^*(p_1|p_2,p_3) + D_{1,0} \\ &= (p_1|p_2,p_3).\end{aligned}$$

1.6 The push-down formula and the dilaton equation

Note first of all that even though the spaces $\overline{M}_{g,n}$ are not smooth, they are sufficiently nice so that Poincaré duality holds. Therefore we can talk about push-forth (or as we shall put it: push-down) of cohomology classes. Take the Poincaré dual, push it down, and take its Poincaré dual again.

First let us see how boundary divisors push down along the forgetful morphism. Any boundary divisor of type $D_{i,0}$ is the image of a section, so clearly $\pi_{0*}D_{i,0} = 1 \in A^0(\overline{M}_{g,n})$. (For any other boundary divisor, the image under π_0 is again a boundary divisor, in particular the dimension drops, so the direct image is zero.)

push-psi **1.6.1 Lemma.** *Let $\pi_0 : \overline{M}_{g,n+1} \to \overline{M}_{g,n}$ be the forgetful morphism that forgets the extra mark p_0. Then*

$$\boxed{\pi_{0*}\boldsymbol{\psi}_0 = 2g - 2 + n} \qquad \textit{in } A^0(\overline{M}_{g,n}).$$

(Or in terms of cycle classes: $\pi_{0*}\big(\boldsymbol{\psi}_0 \cap [\overline{M}_{g,n+1}]\big) = (2g-2+n)[\overline{M}_{g,n}]$.)

Proof. We defer the proof to 2.3.1 in the next section, where it will follow easily from a more general treatment of push-down. □

Proof in the case $g = 0$. Use 1.5.2 to write $\boldsymbol{\psi}_i = (p_i|p_j,p_k)$. Among all those boundary divisors, there are exactly $n-2$ that give non-zero push-down, namely the $n-2$ ways to put a single mark together with p_i. □

dilatontau **1.6.2 Dilaton equation.** Except for the case $\langle\, \tau_1 \,\rangle_1 = 1/24$ explained below, we have

$$\langle\, \tau_1 \cdot \prod_{i=1}^{n} \tau_{a_i} \,\rangle_g = (2g-2+n) \cdot \langle\, \prod_{i=1}^{n} \tau_{a_i} \,\rangle_g$$

Proof. As in the proof of the string equation, expand the product using the comparison result 1.3.3. Now for each i we have $\boldsymbol{\psi}_0 \cdot D_{i,0} = 0$ (cf.1.2.7), and since our product has a factor ψ_0, we can omit all the boundary divisors in the expansion. We get

$$= \int \psi_0 \cdot \pi_0^*(\psi_1^{a_1} \cdots \psi_n^{a_n}).$$

Now push down along π_0 (forgetting p_0), using the projection formula. The result follows from the push-down formula $\pi_{0*}\psi_0 = 2g - 2 + n$ of Lemma 1.6.1. (The case $\langle \tau_1 \rangle_1$ is not covered by this argument since by stability there is no forgetful morphism!) □

1.7 Genus 1

The one-dimensional space $\overline{M}_{1,1}$ is the bottom one: there is no forgetful morphism out of it, so it's a special case. Also, since every curve in this space has got a non-trivial involution, there is a difference between the stack and the coarse moduli space. . .

Definition. On any family of curves $\pi : \mathfrak{X} \to B$, define the *Hodge bundle* $\mathbb{E}$ to be the direct image sheaf of ω_π. That is, the fibre of $\mathbb{E}$ at a general point $[C]$ is the vector space $H^0(C, \omega_C)$. If the family is a flat family of curves of genus g, then $\mathbb{E}$ is locally free of rank g. The Chern classes of the Hodge bundle are called *lambda classes* (or *Hodge classes)*: $\lambda_i := c_i(\mathbb{E})$.

1.7.1 Remark. In particular for $g = 1$, the Hodge bundle is a line bundle. In this case, (i.e. on $\overline{M}_{1,1}$) we have $\lambda_1 = \psi_1 \ldots$

1.7.2 Lemma. *On $\overline{M}_{1,1}$ we have $\lambda_1 = \frac{1}{12}D$, where D denotes the boundary divisor consisting of irreducible uni-nodal curves. CAUTION: THIS IS PERHAPS A COARSE MODULI SPACE IDENTITY XXXX???*

Proof. This formula can either be proved using Grothendieck-Riemann-Roch, or it can proved explicitly: there is a holomorphic section (the discriminant) of the bundle $\mathbb{E}^{\otimes 12}$ which has a simple pole at the divisor D. . . (XXXX See if there isn't an account of this argument in Gatto [8], or in Harris-Morrison [14]. . .) □

So what we really are interested in is the fact

D/12 **1.7.3 Corollary.** *On $\overline{M}_{1,1}$ we have*

$$\psi_1 = \frac{1}{12}D.$$

1/24 **1.7.4 Corollary.** *We have* $\langle\, \tau_1 \,\rangle_1 = \frac{1}{24}$. *(This is a sort of a stack integral rather than an integral over the coarse moduli space. There are automorphisms all over the place — this accounts for the factor* $\frac{1}{2}$*...*

1.7.5 Lemma. *The initial condition* $\langle\, \tau_1 \,\rangle_1 = 1/24$ *together with string and dilaton equations determine all the genus* 1 *numbers.*

Proof. Given a product $\langle\, \tau_{a_1} \cdots \tau_{a_n} \,\rangle_1$, we must have $n = \sum a_i$ to get any contribution. Therefore not all the a_i can be greater than 1. So as long as $n \geq 2$ we can use the string or the dilaton equation to reduce the number of marks, until we get down to $n = 1$ and the initial condition. □

psiingenus1 **1.7.6 Lemma.** *On any* $\overline{M}_{1,n}$ *let* B_i *denote the sum of all boundary divisors having* p_i *on a rational twig. Then we have the expression*

$$\boldsymbol{\psi}_i = \tfrac{1}{12} D + B_i.$$

Proof. This is an analogue of Lemma 5.1.8. Like in that case, the result follows from the comparison result. First observe that for $n = 1$, the divisor B_1 is zero (by stability), so this case is just lemma 1.7.3. Next observe that D pulls back to give D again.

Finally it is easy to see from a set-theoretic description that $\pi_0^* B_i = B_i - D_{0,i}$. Now by induction, this completes the proof □

2 Psi classes and kappa classes

kappa

A nice reference for this section is Arbarello-Cornalba [1].

2.1 Psi classes described "from below"

Check out some similar arguments in Manin [25], p.259–260.

In the first section, we defined the psi class $\boldsymbol{\psi}_i$ on $\overline{M}_{g,n}$ as $\sigma_i^*(c_1(\boldsymbol{\omega}_{\pi_0}))$, where π_0 is the forgetful morphism coming from the space with one more mark p_0, i.e. the space one level *above*. Now in the hierarchy of moduli spaces, there are also morphisms downwards from $\overline{M}_{g,n}$ to the spaces *below* (i.e.with fewer marks), namely for each of the n marks p_i there is a morphism to a space with one mark fewer, which forgets the mark p_i.

In this subsection we're going to describe the class $\boldsymbol{\psi}_i$ in terms of the morphism that forgets p_i.

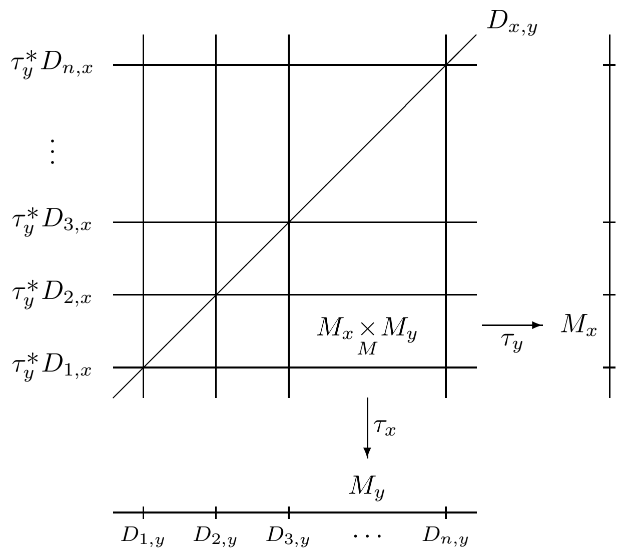
There are n ways in which $\overline{M}_{g,n}$ can be a universal curve over a space below, namely corresponding to the n ways of forgetting a mark. These n morphisms downwards will be denoted π_i, in such a way that the index always indicate which mark is forgotten. Each of these morphisms give rise to a relative dualising sheaf on $\overline{M}_{g,n}$, which we'll denote ω_{π_i} or simply ω_i.

Our first concern is to see how these classes pull back along other forgetful morphisms. We place ourselves somewhere in the system of moduli spaces and set $M := \overline{M}_{g,n}$. Over this space we consider two other moduli spaces, one space M_x having an extra mark x and the other M_y having an extra mark y. (It's only to save ink we do not put the bar over these symbols as we really should since they are compact...) Consider the fibre square

$$\begin{tikzcd}
M_x \times_M M_y \arrow[r, "\tau_y"] \arrow[d, "\tau_x"'] & M_x \arrow[d, "\pi_x"] \\
M_y \arrow[r, "\pi_y"'] & M
\end{tikzcd}$$

Here the morphisms denoted τ_x and τ_y are just the projections. Note that τ_y projects to M_x. This is to keep in sync with the convention that an index on the symbol for a morphism refers to the mark forgotten by the morphism.

The product space parametrises n-pointed stable curves with two distinguished points that can be any point: they are allowed to be equal and they can coincide with the marks. The most important divisors on $M_x \times_M M_y$ are the pull-backs of the soft boundary divisors on M_x and M_y, together with the diagonal divisor $D_{x,y}$ corresponding to all the curves whose two distinguished points are equal. To be more precise, on M_x we have the n divisors $D_{i,x}$, and on M_y we have $D_{i,y}$. Here is a picture of those $2n+1$ important divisors on the product:



Now recall how the space $M_{x,y}$ is constructed (cf. [18]): It is the blow-up of $M_x \times_M M_y$ along each of the (disjoint) intersections of the diagonal with the other divisors mentioned. The strict transforms of each of the $2n+1$ divisors are again boundary divisors in $M_{x,y}$. The n exceptional divisors E_i are the boundary divisors corresponding to the partition having x, y, i on a genus zero twig. A point in a blow-up centre corresponds to the case where three "marks" coincide. The exceptional fibre over such a point is a $\mathbb{P}^1$ parametrising all the ways of distributing the three marks (and the attachment point) on the twig which is a rational curve.

Now let $\varepsilon : M_{x,y} \to M_x \times_M M_y$ denote the blow-up morphism, and set $\tilde{\pi}_y := \tau_y \circ \varepsilon$. In other words, $\tilde{\pi}_y : M_{x,y} \to M_x$ is the forgetful morphism that forgets y. We have put the tilde over the symbol just to distinguish it from the morphism $\pi_y : M_y \to M$. We have the following commutative square which is central to all the following arguments:

$$\begin{tikzcd} M_{x,y} \arrow[r, "\tilde{\pi}_y"] \arrow[d, "\tilde{\pi}_x"'] & M_x \arrow[d, "\pi_x"] \\ M_y \arrow[r, "\pi_y"'] & M \end{tikzcd}$$

2.1.1 Example. From this description we see that $\tilde{\pi}_y^* D_{i,x} = \widetilde{D}_{i,x} + E_i$. Also, recall that there is the following description of the relative canonical class of a blow-up (cf. [15], II, ex. 8.5.):

$$K_{\tilde{\pi}_x} = \varepsilon^* K_{\tau_x} + \sum_{i=1}^{n} E_i$$

(At least this holds in the smooth case. Why is it true here?) Now since our square is cartesian we also have $K_{\tau_x} = \tau_y^* K_{\pi_x}$. So we conclude

$$K_{\tilde{\pi}_x} = \pi_y^* K_{\pi_x} + \sum_{i=1}^{n} E_i$$

2.1.2 Twisted canonical class. In a sense, when talking about pointed curves, there is the following variant of the dualising sheaf which often is more important than the original one, namely the twist

$$\omega_{\pi_x}(\sum_{i=1}^{n} D_{i,x}).$$

For example, one characterisation of an n-pointed curve $(C, p_1, \ldots, p_n)$ being stable is that $\omega_C(\sum p_i)$ is ample.

2.1.3 Lemma. *We have the following pull-back formula:*

$$\pi_y^*(K_{\pi_x} + \sum D_{i,x}) = (K_{\pi_x} + \sum D_{i,x}) + D_{x,y}.$$

This follows immediately from the example above.

2.1.4 Corollary. *We have*

$$\boxed{\boldsymbol{\psi}_i = K_i + \sum_{j \neq i} D_{ij}}$$

Proof. We just prove it in the case $n = 1$. This follows from the fact XXXX???? that $\overline{M}_{g,2} \simeq \overline{M}_{g,1} \times_{\overline{M}_{g,0}} \overline{M}_{g,1}$. For $n \geq 2$, the result follows easily from the pull-back formula for $K + \sum D$ and the formula for pull-back of $\boldsymbol{\psi}$.

□

2.2 Definition and basic properties of kappa classes

Let $\pi : \mathfrak{X} \to B$ be a family of curves and let ω_π be the relative dualising sheaf. Mumford [26] defined the kappa classes as

$$\boldsymbol{\kappa}_a := \pi_*(c_1(\omega_\pi)^{a+1}).$$

XXXX some history about kappa classes, conjectures, no relations. Now this definition was designed for stable curves without marks. For curves with marks, the definition has some disadvantages, due to the fact that the curves are not necessarily stable as abstract curves, but only as marked curves. In this case it is better to use the twisted canonical class mentioned above and set (cf. Arbarello and Cornalba [1])

$$\boldsymbol{\kappa}_a := \pi_*(c_1(\omega_\pi(D))^{a+1}) = \pi_*(\boldsymbol{\psi}_0^{a+1}).$$

Here D denotes the sum of all the divisors $D_{i,0}$. Of course the two definitions agree in case $n = 0$. (These classes were called *Mumford classes* by Arbarello-Cornalba [1] and in case $n = 0$ they also appear in Witten [29] under the name *Mumford-Morita-Miller classes.*)

2.2.1 Remark. Note that $\boldsymbol{\kappa}_0$ is in codimension zero. It is

$$\boldsymbol{\kappa}_0 = (2g - 2 + n)[\overline{M}_{g,n}].$$

(Just the degree of $K_\pi + \sum D_i$ along a fibre…)

2.2.2 Lemma. *There is the following comparison between the kappa classes in the sense of Arbarello-Cornalba and the naïve ones* $\pi_*(K_\pi^{a+1})$

$$\boldsymbol{\kappa}_a = \pi_*(K_\pi^{a+1}) + \sum_{i=1}^{n} \boldsymbol{\psi}_i^a.$$

Proof. Note that $c_1(\omega_\pi)^a \cap [D_{i,0}] = \sigma_i^* c_1(\omega_\pi)^a = \boldsymbol{\psi}_i^a$. Interpreted as a class in D, we get $c_1(\omega_\pi)^a \cap [D_{i,0}] = \pi^*\boldsymbol{\psi}_i^a \cdot D_{i,0} = (-1)^a D_{i,0}^{a+1}$.

$$\begin{aligned}
\boldsymbol{\kappa}_a = \pi_*\big((K_\pi + \textstyle\sum D_{i,0})^{a+1}\big) &= \pi_*(K_\pi^{a+1}) + \sum_{j=0}^{a} \tbinom{a+1}{j} \sum_{i=1}^{n} \pi_*(K_\pi^j \cdot D_{i,0}^{a-j+1}) \\
&= \pi_*(K_\pi^{a+1}) + \sum_{j=0}^{a} \tbinom{a+1}{j} \sum_{i=1}^{n} \sigma_i^*(K_\pi^j \cdot D_{i,0}^{a-j}) \\
&= \pi_*(K_\pi^{a+1}) + \sum_{j=0}^{a} \tbinom{a+1}{j} \sum_{i=1}^{n} (-1)^{a-j} \sigma_i^*(K_\pi^a) \\
&= \pi_*(K_\pi^{a+1}) + \sum_{j=0}^{a} \tbinom{a+1}{j} (-1)^{a-j} \sum_{i=1}^{n} \boldsymbol{\psi}_i^a \\
&= \pi_*(K_\pi^{a+1}) + \sum_{i=1}^{n} \boldsymbol{\psi}_i^a .
\end{aligned}$$

□

pull-kappa **2.2.3 Lemma.** *(Pull-back formula for kappa classes.) We have (with the notation of the preceding) the following identity on M_y*

$$\boxed{\boldsymbol{\kappa}_a = \pi_y^* \boldsymbol{\kappa}_a + \boldsymbol{\psi}_y^a}$$

This is a pushed-down version of the important pull-back formula 1.3.1.

Proof. Recall the fundamental pull-back formula 1.3.1: in our setting it reads

$$\boldsymbol{\psi}_x = \tilde{\pi}_y^* \boldsymbol{\psi}_x + D_{x,y},$$

which is an identity on $M_{x,y}$. Now take $(a+1)$'th powers on both sides of this identity (using 1.3.3) getting

$$\boldsymbol{\psi}_x^{a+1} = \tilde{\pi}_y^* \boldsymbol{\psi}_x^{a+1} + (-1)^a D_{x,y}^{a+1}.$$

Now push down via $\tilde{\pi}_x$ to get

$$\begin{aligned}
\boldsymbol{\kappa}_a = \tilde{\pi}_{x*}(\boldsymbol{\psi}_x^{a+1}) &= \tilde{\pi}_{x*}\big(\tilde{\pi}_y^* \boldsymbol{\psi}_x^{a+1}\big) + \tilde{\pi}_{x*}\big((-1)^a D_{x,y}^{a+1}\big) \\
&= \pi_y^*\big(\pi_{x*}(\boldsymbol{\psi}_x^{a+1})\big) + (-1)^a \sigma_y^* D_{x,y}^a \\
&= \pi_y^* \boldsymbol{\kappa}_a + \boldsymbol{\psi}_y^a,
\end{aligned}$$

where under way we swapped pull and push — this is allowed since the diagram is a blow-up of a fibre square (cf. Fulton [5], 6.2)... □

2.3 Push-down of psi classes in terms of kappa classes

Let π_0 be the morphism that forgets p_0. By definition, $\pi_{0*}(\boldsymbol{\psi}_0^{a_0+1}) = \boldsymbol{\kappa}_{a_0}$. This generalises as follows.

push-one-psi **2.3.1 Lemma.**

$$\pi_{0*}\big(\boldsymbol{\psi}_0^{a_0+1} \cdot (\boldsymbol{\psi}_1^{a_1} \cdots \boldsymbol{\psi}_n^{a_n})\big) = \boldsymbol{\kappa}_{a_0} \cdot (\boldsymbol{\psi}_1^{a_1} \cdots \boldsymbol{\psi}_n^{a_n})$$

Proof. This follows easily from the pull-back formula for psi classes, using induction. Write $\boldsymbol{\psi}_n = \pi_0^* \boldsymbol{\psi}_n + D_{0,n}$ and recall the formula $(\pi_0^*\boldsymbol{\psi}_n + D_{0,n})^{a_n} = \pi_0^*\boldsymbol{\psi}_n^{a_n} + (-1)^{a_n-1} D_{0,n}^{a_n}$. Therefore,

$$\pi_{0*}\big(\boldsymbol{\psi}_0^{a_0+1} \cdot (\boldsymbol{\psi}_1^{a_1} \cdots \boldsymbol{\psi}_n^{a_n})\big) = \pi_{0*}\big(\boldsymbol{\psi}_0^{a_0+1} \cdot (\boldsymbol{\psi}_1^{a_1} \cdots \boldsymbol{\psi}_{n-1}^{a_{n-1}})(\pi_0^*\boldsymbol{\psi}_n^{a_n} + (-1)^{a_n-1} D_{0,n}^{a_n})\big)$$

Now recall from 1.2.7 that $\boldsymbol{\psi}_0 D_{0,n} = 0$, so the last term is zero. Now the projection formula gives

$$= \pi_{0*}\big(\boldsymbol{\psi}_0^{a_0+1} \cdot (\boldsymbol{\psi}_1^{a_1} \cdots \boldsymbol{\psi}_{n-1}^{a_{n-1}})\big) \cdot \boldsymbol{\psi}_n^{a_n}$$

and the result follows by induction. □

dilatonkappa **2.3.2 Corollary.** *(Dilaton equation — cf. 1.6.2.) Since $\boldsymbol{\kappa}_0 = (2g-2+n)[\overline{M}_{g,n}]$ we get*

$$\int_{\overline{M}_{g,n+1}} \boldsymbol{\psi}_0 \cdot (\boldsymbol{\psi}_1^{a_1} \cdots \boldsymbol{\psi}_n^{a_n}) = (2g-2+n) \int_{\overline{M}_{g,n}} \boldsymbol{\psi}_1^{a_1} \cdots \boldsymbol{\psi}_n^{a_n}.$$

Let now S be a fixed set of marks, and let $W := \prod_{s \in S} \boldsymbol{\psi}_s^{a_s+1}$ be a product of psi classes corresponding to these marks.

2.3.3 Example. Now suppose there are yet another two marks p_1 and p_2, then we can perform a two-step push-down, forgetting these two marks, simply by applying the pull-back formula 2.2.3 between the steps, and using the projection formula. We get the following push-down identity.

$$\begin{aligned}
(\pi_2 \circ \pi_1)_*\big(\boldsymbol{\psi}_1^{a_1+1}\boldsymbol{\psi}_2^{a_2+1} \cdot W\big) &= \pi_{2*}\big(\boldsymbol{\kappa}_{a_1}\boldsymbol{\psi}_2^{a_2+1} \cdot W\big) \\
&= \pi_{2*}\big((\pi_2^*\boldsymbol{\kappa}_{a_1} + \boldsymbol{\psi}_2^{a_1})\boldsymbol{\psi}_2^{a_2+1} \cdot W\big) \\
&= (\boldsymbol{\kappa}_{a_1}\boldsymbol{\kappa}_{a_2} + \boldsymbol{\kappa}_{a_1+a_2}) \cdot W.
\end{aligned}$$

Similarly we find, when there are three marks to forget (omitting W from the notation):

$$(\pi_3 \circ \pi_2 \circ \pi_1)_*\big(\boldsymbol{\psi}_1^{a_1+1}\boldsymbol{\psi}_2^{a_2+1}\boldsymbol{\psi}_3^{a_3+1}\big) = \boldsymbol{\kappa}_{a_1}\boldsymbol{\kappa}_{a_2}\boldsymbol{\kappa}_{a_3} + \boldsymbol{\kappa}_{a_1+a_2}\boldsymbol{\kappa}_{a_3} + \boldsymbol{\kappa}_{a_1+a_3}\boldsymbol{\kappa}_{a_2} + \boldsymbol{\kappa}_{a_1}\boldsymbol{\kappa}_{a_2+a_3} + 2\boldsymbol{\kappa}_{a_1+a_2+a_3}.$$

The general result is:

Rkappa **2.3.4 Proposition.** *Let there be n marks in addition to S. Then*

$$(\pi_n \circ \cdots \circ \pi_2 \circ \pi_1)_* \big(\boldsymbol{\psi}_1^{a_1+1} \boldsymbol{\psi}_2^{a_2+1} \cdots \boldsymbol{\psi}_n^{a_n+1} \cdot W \big) = R_{a_1,\ldots,a_n}(\boldsymbol{\kappa}) \cdot W,$$

where $R_{a_1,\ldots,a_n}(\boldsymbol{\kappa})$ is an explicit polynomial in kappa classes which we now describe: Let $A := (a_1, \ldots, a_n)$ and let $\mathfrak{S}(A)$ be the symmetric group of permutations of A. Write each permutation $\sigma \in \mathfrak{S}(A)$ as a product of disjoint cycles $\sigma = \prod \alpha$ (including 1-cycles) and let $|\alpha|$ denote the sum of the elements a_i belonging to a cycle α. Setting $\boldsymbol{\kappa}_\sigma := \prod \boldsymbol{\kappa}_{|\alpha|}$, we can describe the polynomial R as

$$R_{a_1,\ldots,a_n} = \sum_{\sigma \in \mathfrak{S}(A)} \boldsymbol{\kappa}_\sigma .$$

Proof. Pure combinatorics — see the appendix. □

Let us content ourselves with two examples:

2.3.5 Example. Suppose there are just two marks involved in the push-down, so that we have $A = (a_1, a_2)$. The only two permutations are

$$(a_1)(a_2), \quad (a_1, a_2),$$

so we find $R_{a_1,a_2} = \boldsymbol{\kappa}_{a_1}\boldsymbol{\kappa}_{a_2} + \boldsymbol{\kappa}_{a_1+a_2}$, in accordance with the example above.

Now for the case of three marks; $A = (a_1, a_2, a_3)$, and there are six permutations:

$$(a_1)(a_2)(a_3),\ (a_1, a_2)(a_3),\ (a_1, a_3)(a_2),\ (a_1)(a_2, a_3),\ (a_1, a_2, a_3),\ (a_1, a_3, a_2),$$

so the corresponding polynomial is

$$R_{a_1,a_2,a_3} = \boldsymbol{\kappa}_{a_1}\boldsymbol{\kappa}_{a_2}\boldsymbol{\kappa}_{a_3} + \boldsymbol{\kappa}_{a_1+a_2}\boldsymbol{\kappa}_{a_3} + \boldsymbol{\kappa}_{a_1+a_3}\boldsymbol{\kappa}_{a_2} + \boldsymbol{\kappa}_{a_1}\boldsymbol{\kappa}_{a_2+a_3} + 2\boldsymbol{\kappa}_{a_1+a_2+a_3},$$

just as we found in the direct computation in the example.

2.3.6 Proposition. *Intersection theory involving psi classes and kappa classes on a given $\overline{M}_{g,m}$ is equivalent to the intersection theory on all $\overline{M}_{g,n+m}$ involving only psi classes. In particular, intersection theory of kappa classes on $\overline{M}_{g,0}$ is equivalent to intersection theory of psi classes on all $\overline{M}_{g,n}$.*

Proof. Let there be given an intersection product of psi classes. If there are any zero exponents use the string equation, until we've got a sum of products all of whose exponents are positive. Then use the proposition to write each product as a polynomial R of kappa classes.

The converse is only slightly more tricky. Induction on the number of kappa classes: for products with only one kappa class $\boldsymbol{\kappa}_{a_1} W$ we can just use lemma 2.3.1 to get $\boldsymbol{\psi}_1^{a_1+1} \cdot W$. Otherwise, consider a product $\boldsymbol{\kappa}_{a_1} \cdots \boldsymbol{\kappa}_{a_n} \cdot W$ and note that the polynomial $R_{a_1,\ldots,a_n}$ has $\boldsymbol{\kappa}_{a_1} \cdots \boldsymbol{\kappa}_{a_n}$ as its only term of that degree (corresponding to the fact that there is only one partition of $(a_1, \ldots, a_n)$ into n parts). So the product $\boldsymbol{\kappa}_{a_1} \cdots \boldsymbol{\kappa}_{a_n} \cdot W$ can be written $\boldsymbol{\psi}_1^{a_1+1} \cdot \boldsymbol{\psi}_n^{a_n+1} \cdot W$ minus lower terms of $R_{a_1,\ldots,a_n}$, which are handled by induction. □

XXXX work out the converse formula: kappa in terms of psi we have a conjectured formula that should be checked... XXXX

3 Witten's conjecture (theorem of Kontsevich)

KdV

3.1 Notation

tau-notation

3.1.1 Notation. Recall that the correlation functions are defined as intersection numbers on the moduli space of stable n-pointed curves as

$$\langle\, \tau_{k_1} \cdots \tau_{k_n} \,\rangle_g := \int \boldsymbol{\psi}_1^{k_1} \ \cdots \ \boldsymbol{\psi}_n^{k_n} \ \cap [\overline{M}_{g,n}],$$

which is non-zero only when $\sum k_i = 3g - 3 + n$. Witten's conjecture (which has been proved by Kontsevich) determines all the correlation functions via a complicated system of recursions. To organise all the relations it is convenient to adopt the formalism of generating functions.

First of all, since the symmetric group acts on the moduli spaces there is no reason to keep track of which tau belongs to which mark. So let us collect all the marks of equal exponent and adopt the notation

$$\langle\, \tau_0^{s_0} \tau_1^{s_1} \tau_2^{s_2} \cdots \tau_m^{s_m} \,\rangle \;=\; \langle\, \underbrace{\tau_0 \cdots \tau_0}_{s_0 \text{ factors}} \cdot \underbrace{\tau_1 \cdots \tau_1}_{s_1 \text{ factors}} \cdot \underbrace{\tau_2 \cdots \tau_2}_{s_2 \text{ factors}} \cdots \underbrace{\tau_m \cdots \tau_m}_{s_m \text{ factors}} \,\rangle\,.$$

(Warning: the indices on the taus don't refer to particular marks, so the symbol τ_4 has nothing to do with a particular mark p_4. It means that there is one of the marks q such that $\boldsymbol{\psi}_q$ appears with exponent 4.)

So the set of all possible correlation functions is indexed by the set of all infinite sequences of non-negative integers such that only a finite number of entries are non-zero. It is worth adopting the multi-index notation. Throughout, set

$$\mathbf{s} = (s_0, s_1, s_2, \dots);$$

set $|\mathbf{s}| := \sum s_i$; it is then the number of marks. For each such sequence there is a
fattau correlation function which we denote

$$\langle\, \boldsymbol{\tau}^{\mathbf{s}} \,\rangle := \langle\, \tau_0^{s_0} \tau_1^{s_1} \tau_2^{s_2} \cdots \rangle.$$

We suppress reference to the genus. This is harmless since for each sequence $\mathbf{s}$ there is at most one value of g giving a non-zero correlator $\langle\, \boldsymbol{\tau}^{\mathbf{s}} \,\rangle$. Indeed, the total codimension of $\langle\, \boldsymbol{\tau}^{\mathbf{s}} \,\rangle$ is $\sum i s_i$, so the equation $3g - 3 + |\mathbf{s}| = \sum i s_i$ determines g.

Some basic sequences deserve special names. For each $i \geq 0$ we let $\mathbf{e}^{(i)}$ denote the sequence $(0, \dots, 0, 1, 0, \dots)$ all of whose entries are zero except $e_i^{(i)} = 1$. Also we will have good use of the sequence $\mathbf{w}^{(i)}$ defined as $(0, \dots, 0, 1, -1, 0, \dots)$ all of whose entries are zero except the i'th which is equal to 1, and the $(i+1)$'th which is equal to -1. In other words, $\mathbf{w}^{(i)} = \mathbf{e}^{(i)} - \mathbf{e}^{(i+1)}$.

3.1.2 Example. Consider first all the sequences such that $s_i = 0$ for $i \geq 1$, (that is, $\mathbf{s} = s_0 \mathbf{e}^{(0)}$). There is only one of all these sequences giving non-zero correlation function: namely the sequence $(3, 0, 0, \dots)$ corresponding to the correlator $\langle\, \tau_0^3 \,\rangle_0$ (which incidentally we know to be equal to 1).

Next, consider all the sequences $\mathbf{s}$ such that $s_i = 0$ for $i \geq 2$. (I.e. sequences that are linear combinations of $\mathbf{e}^{(0)}$ and $\mathbf{e}^{(1)}$). The only ones giving contribution are the following: $\mathbf{s} = (3, s_1, 0, 0, \dots)$, giving $\langle\, \tau_0^3 \tau_1^{s_1} \,\rangle_0$ (which is easily seen to be equal to $s_1!$), and also $\mathbf{s} = (0, s_1, 0, 0, \dots)$, giving $\langle\, \tau_1^{s_1} \,\rangle_1$ (including the special case $\langle\, \tau_1 \,\rangle_1 = 1/24$, according to 1.7.4). Proof: the total codimension is s_1. Since
tau0taus0 s_0 is non-negative, the equation $3g - 3 + s_0 + s_1 = s_1$ forces $g = 0$ or $g = 1$.

Finally, let us note that the string equation (cf. 1.4.2) gives a formula for $\langle\, \tau_k \tau_0^{s_0} \,\rangle$.

$$\langle\, \tau_k \tau_0^{s_0} \,\rangle = \begin{cases} \langle\, \tau_0^{s_0 - k} \,\rangle & \text{for } k < s_0 \\ \langle\, \tau_{k - s_0} \,\rangle & \text{for } k > s_0 \end{cases}$$

Let us write the string equation in this new way:

$$\begin{aligned}\langle\, \boldsymbol{\tau}^{\mathbf{s}+\mathbf{e}^{(0)}} \,\rangle &= \sum_{j=1}^{\infty} s_j \,\langle\, \boldsymbol{\tau}^{\mathbf{s}+\mathbf{w}^{(j-1)}} \,\rangle \\ &= \sum_{j=0}^{\infty} s_{j+1} \,\langle\, \boldsymbol{\tau}^{\mathbf{s}+\mathbf{w}^{(j)}} \,\rangle,\end{aligned}$$

valid except for the case $\mathbf{s} = (2, 0, 0, \dots)$. Indeed, for each $j \geq 1$, there are s_j psi factors all of which can drop exponent and consequently the preceding one gains a factor.

3.2 The generating function

Now we collect all the correlation functions and use them as coefficients in a formal power series. Let $\mathbf{t} = (t_0, t_1, t_2, \dots)$ be an infinite sequence of formal variables. We adopt the convention that $\mathbf{t}^{\mathbf{s}} = \prod_{i=0}^{\infty} t_i^{s_i}$, and also the notation $\mathbf{s}! = \prod_{i=0}^{\infty} s_i!$. Now set

$$\begin{aligned}F(\mathbf{t}) &:= \sum_{\mathbf{s}} \frac{\mathbf{t}^{\mathbf{s}}}{\mathbf{s}!} \,\langle\, \boldsymbol{\tau}^{\mathbf{s}} \,\rangle \\ &= \sum_{\mathbf{s}} \prod_{i=0}^{\infty} \frac{t_i^{s_i}}{s_i!} \,\langle\, \tau_0^{s_0} \tau_1^{s_1} \tau_2^{s_2} \cdots \,\rangle.\end{aligned}$$

(In topological gravity, this series is called *the total free energy*.) The sum is over all sequences $\mathbf{s}$. This expression can also be thought of as the formal expansion of $\langle\, \exp(\sum t_i \tau_i) \,\rangle$. Indeed, $\exp(t_0\tau_0 + t_1\tau_1 + t_2\tau_2 + \cdots) = \sum_{m=0}^{\infty} (\sum t_i\tau_i)^m/m!$. Now in the expression $(\sum t_i\tau_i)^m$ we find all possible monomials $\mathbf{t}^{\mathbf{s}} \,\langle\, \boldsymbol{\tau}^{\mathbf{s}} \,\rangle$, and in each case the coefficient is the multinomial coefficient...

(Note: we will not bother about whether or where this is a convergent series and a true function; we only think of it as a formal expression.)

`stringF` **3.2.1 Lemma.** *(The string equation — cf. 1.4.2.) The string equation (as it appears in string theory) is the differential equation*

$$\boxed{\frac{\partial}{\partial t_0} F = \frac{t_0^2}{2} + \sum_{i=0}^{\infty} t_{i+1} \frac{\partial}{\partial t_i} F}$$

This form includes all possible manifestations of the string equation 1.4.2, and also the initial condition $\langle \tau_0^3 \rangle = 1$.

Proof. Let us start noting the effect of the differential operators on a monomial

$$\frac{\partial}{\partial t_i}\frac{\mathbf{t}^{\mathbf{s}}}{\mathbf{s}!} = \frac{\mathbf{t}^{\mathbf{s}-\mathbf{e}^{(i)}}}{(\mathbf{s}-\mathbf{e}^{(i)})!}$$

In particular, the effect of the operator $\sum t_{i+1}\frac{\partial}{\partial t_i}$ is

$$\sum t_{i+1}\frac{\partial}{\partial t_i}\frac{\mathbf{t}^{\mathbf{s}}}{\mathbf{s}!} = \sum \frac{\mathbf{t}^{\mathbf{s}-\mathbf{w}^{(i)}}}{(\mathbf{s}-\mathbf{w}^{(i)})!}(s_{i+1}+1).$$

Now it is just a matter of collecting all the string equations (keeping track of which are which by the bookkeeping symbol $\mathbf{t}$). The collected string equation is

$$\sum_{\mathbf{s}} \frac{\mathbf{t}^{\mathbf{s}}}{\mathbf{s}!}\langle \boldsymbol{\tau}^{\mathbf{s}+\mathbf{e}^{(0)}} \rangle = \frac{t_0^2}{2} + \sum_{\mathbf{s}}\sum_{i=0}^{\infty} \frac{\mathbf{t}^{\mathbf{s}}}{\mathbf{s}!} s_{i+1} \langle \boldsymbol{\tau}^{\mathbf{s}+\mathbf{w}^{(i)}} \rangle .$$

Note that the term $\frac{t_0^2}{2}$ comes from $\langle \tau_0^3 \rangle = 1$. Now make an index shift on each side: on the left-hand side we get

$$\sum_{\mathbf{s}} \frac{\mathbf{t}^{\mathbf{s}-\mathbf{e}^{(0)}}}{(\mathbf{s}-\mathbf{e}^{(0)})!}\langle \boldsymbol{\tau}^{\mathbf{s}} \rangle = \frac{\partial}{\partial t_0}F.$$

And the right-hand side gives (after changing the order of summation)

$$\frac{t_0^2}{2} + \sum_{i=0}^{\infty}\sum_{\mathbf{s}} \frac{\mathbf{t}^{\mathbf{s}-\mathbf{w}^{(i)}}}{(\mathbf{s}-\mathbf{w}^{(i)})!}(s_{i+1}+1)\langle \boldsymbol{\tau}^{\mathbf{s}} \rangle = \frac{t_0^2}{2} + \sum_{i=0}^{\infty} t_{i+1}\frac{\partial}{\partial t_i}F.$$

□

3.3 Witten's conjecture

3.3.1 KdV equations. The KdV equation (after Korteweg and de Vries) dates back to the 19th century and arises naturally in a lot of situations in physics: let $q(t,x)$ be a function in two variables, then the KdV equation is $q_t = qq_x + \frac{1}{12}q_{xxx}$ (or similar expressions with other constants). The equation is just the first of a whole hierarchy of equations known as the KdV hierarchy. Let us present it in Witten's notation. Let U be a function in infinitely many variables $\mathbf{t} = (t_0, t_1, t_2, \dots)$. We let dots over a function denote its derivatives with respect to t_0. That is, $\dot{U} = \frac{\partial U}{\partial t_0}$. Now the hierarchy of KdV equations is

$$\frac{\partial}{\partial t_i} U = \frac{\partial}{\partial t_0} R_{i+1}(U, \dot{U}, \ddot{U}, \dots)$$

where R_i are polynomials in the partial derivatives of U. These polynomials R_i are defined recursively to satisfy the following relation.

$$\begin{aligned} R_1 &= U \\ \dot{R}_{i+1} &= \frac{1}{2i+1}\big(R_i\dot{U} + 2\dot{R}_i U + \tfrac{1}{4}\dddot{R}_i\big). \end{aligned}$$

So the first equation is

$$\frac{\partial}{\partial t_1} U = \frac{1}{3}\big(R_1\dot{U} + 2\dot{R}_1 U + \tfrac{1}{4}\dddot{R}_1\big) = U\dot{U} + \tfrac{1}{12}\dddot{U}.$$

Note that for $i \geq 3$, R_i is not completely determined alone from U and the recursive description of the R_i, because we only have a description of $\dot{R}_{i+1}$, while on the right-hand side there is an occurrence of R_i itself. So to determine the R_i from U we need the main equation as well.

3.3.2 Notation. Following Witten, we set

$$\langle\!\langle\, \tau_{k_1} \cdots \tau_{k_n} \,\rangle\!\rangle := \frac{\partial}{\partial t_{k_1}} \cdots \frac{\partial}{\partial t_{k_n}} F.$$

Observe that the value of this function at $\mathbf{t} = \mathbf{0}$ is exactly $\langle\, \tau_{k_1} \cdots \tau_{k_n} \,\rangle$. Note further that $\langle\!\langle\, 1 \,\rangle\!\rangle = F$ encodes all correlators, while for example $\langle\!\langle\, \tau_{k_1} \cdots \tau_{k_n} \,\rangle\!\rangle$ encodes all correlators having a factor $\tau_{k_1} \cdots \tau_{k_n}$.

Just to get familiar with the notation, let us write the string equation again:

$$\langle\!\langle\, \tau_0 \,\rangle\!\rangle = \frac{t_0^2}{2} + \sum_{i=0}^{n} t_{i+1} \langle\!\langle\, \tau_i \,\rangle\!\rangle.$$

3.3.3 Witten's conjecture. *(Kontsevich's theorem) The object* $U := \frac{\partial^2}{\partial t_0^2} F = \langle\!\langle\, \tau_0^2 \,\rangle\!\rangle$ *obeys the KdV equations, with* $R_{i+1} = \langle\!\langle\, \tau_i \tau_0 \,\rangle\!\rangle$. *In other words,*

$$\langle\!\langle\, \tau_i \tau_0^2 \,\rangle\!\rangle = \frac{\langle\!\langle\, \tau_{i-1}\tau_0 \,\rangle\!\rangle \langle\!\langle\, \tau_0^3 \,\rangle\!\rangle + 2 \langle\!\langle\, \tau_{i-1}\tau_0^2 \,\rangle\!\rangle \langle\!\langle\, \tau_0^2 \,\rangle\!\rangle + \frac{1}{4} \langle\!\langle\, \tau_{i-1}\tau_0^4 \,\rangle\!\rangle}{2i+1}.$$

□

The conjecture has been proved by Kontsevich [22], with analytic methods. He translates everything into matrix integrals... No algebro geometric proof is known. We are not going to make any attempt to understand the ideas of Kontsevich's proof — Harris and Morrison [14] recommend Looijenga's survey [24] for an introduction to the proof. We will just explore some consequences of the theorem.

The reason leading Witten to make the conjecture is roughly this (see his paper [29]): There were two mathematical models that described the physical theory of two-dimensional gravity. One was a matrix model, with some Lagrangian, in which some flows were naturally governed by the KdV equations. The other model was the theory of intersection theory on $\overline{M}_{g,n}$. Now since the two theories described the same thing, Witten translated the crucial results of one theory over to the other. (And of course he made a lot of checks in low genus and so on.) Kontsevich's proof seems to follow the same ideas, formalising the dictionary between intersection theory and matrix integrals...

3.3.4 Proposition. *The KdV equations for U, together with the string equation, completely determine F. That is, all the intersection products $\langle\, \tau_{k_1} \cdots \tau_{k_n} \,\rangle$ are known.*

Before we give the proof, let us give some examples.

one-pointed **3.3.5 Example.** *One-pointed integrals.* Here we are concerned with all correlation functions of type $\langle\, \tau_i \,\rangle$ (that is, those corresponding to the sequences $\mathbf{e}^{(i)}$). The total codimension is i, and since there is only one mark, the dimension of the space is $3g-2$. So we want to compute $\langle\, \tau_{3g-2} \,\rangle_g$. We have already noticed that $\langle\, \tau_i \,\rangle = \langle\, \tau_{i+2}\tau_0^2 \,\rangle$, so we can read off a recursion directly from the KdV equation (evaluating all functions at $\mathbf{t} = \mathbf{0}$). Recalling that $\langle\, \tau_0^2 \,\rangle = 0$ and $\langle\, \tau_0^3 \,\rangle = 1$, we find

$$\begin{aligned} \langle\, \tau_i \,\rangle = \langle\, \tau_{i+2}\tau_0^2 \,\rangle &= \frac{\langle\, \tau_{i+1}\tau_0 \,\rangle + \frac{1}{4}\langle\, \tau_{i+1}\tau_0^4 \,\rangle}{2i+5} \\ &= \frac{\langle\, \tau_i \,\rangle + \frac{1}{4}\langle\, \tau_{i-3} \,\rangle}{2i+5}. \end{aligned}$$

Arranging the terms, we find the recursion

$$\langle\, \tau_i \,\rangle = \frac{\langle\, \tau_{i-3} \,\rangle}{8i+16}.$$

The denominator $8i+16$ is nicer when we substitute $3g-2$ for i. Then it becomes $8(3g-2)+16 = 24g$. Now it follows easily by induction that

$$\langle\, \tau_{3g-2} \,\rangle_g = \frac{1}{24^g \cdot g!}.$$

The general idea is the same as we saw in this example: to use the KdV equations whenever possible, and when it is not possible due to lack of tau-zero, use the string equation "back-wards". Let us have a look at the next case, that of two-pointed integrals.

3.3.6 Example. *Two-pointed integrals.* Since this is just an example anyway, let us look at the concrete problem of computing $\langle\, \tau_3\tau_5 \,\rangle$. What is worth noting is that there is one small tau class and one big. We're going to express this correlator in terms of other two-point correlators such that the big class is bigger and the small one is smaller. Note that the total codimension is $3+5=8$, and the relevant space has dimension $3g-3+2$, so the only contribution comes from $g=3$. We cannot apply KdV because there is no factor τ_0^2, so first we use the string equation in the following backwards way to reduce the problem to the computation of $\langle\, \tau_0^2\tau_3\tau_7 \,\rangle_3$ (introducing two tau-zeros and incrementing the biggest tau by two). The string equation (applied twice) compares these two correlators:

$$\begin{aligned}\langle\, \tau_0^2\tau_3\tau_7 \,\rangle &= \langle\, \tau_0\tau_2\tau_7 \,\rangle + \langle\, \tau_0\tau_3\tau_6 \,\rangle \\ &= \langle\, \tau_1\tau_7 \,\rangle + 2\,\langle\, \tau_2\tau_6 \,\rangle + \langle\, \tau_3\tau_5 \,\rangle\end{aligned}$$

(Note that all these correlators are in genus 3.) So this expresses our initial problem $\langle\, \tau_3\tau_5 \,\rangle$ in terms of other integrals which we should now argue why are better or easier to compute. As to the two new integrals appearing on the right-hand side of the string equation $\langle\, \tau_1\tau_7 \,\rangle$ and $\langle\, \tau_2\tau_6 \,\rangle$, they are better because they are "less equilibrated" in the sense that the biggest tau has become bigger and the smallest has become smaller. Repeating this procedure these terms eventually are substituted by $\langle\, \tau_0\tau_8 \,\rangle$ which we know how to compute by 3.3.5 (and 3.1.2). As to the term $\langle\, \tau_0^2\tau_3\tau_7 \,\rangle$ appearing on the left-hand side of the string equation, it is better than the original one because it has a factor τ_0^2 which allows us to use the KdV, as we now proceed to make precise.

Take the derivative $\frac{\partial}{\partial t_7}$ on the KdV equation with $i=3$ to get $\langle\!\langle\, \tau_3\tau_7\tau_0^2 \,\rangle\!\rangle$ expressed as

$$\frac{\langle\!\langle\, \tau_2\tau_7\tau_0 \,\rangle\!\rangle \langle\!\langle\, \tau_0^3 \,\rangle\!\rangle + \langle\!\langle\, \tau_2\tau_0 \,\rangle\!\rangle \langle\!\langle\, \tau_7\tau_0^3 \,\rangle\!\rangle + 2\,\langle\!\langle\, \tau_2\tau_7\tau_0^2 \,\rangle\!\rangle \langle\!\langle\, \tau_0^2 \,\rangle\!\rangle + 2\,\langle\!\langle\, \tau_2\tau_0^2 \,\rangle\!\rangle \langle\!\langle\, \tau_7\tau_0^2 \,\rangle\!\rangle + \frac{1}{4}\,\langle\!\langle\, \tau_2\tau_7\tau_0^4 \,\rangle\!\rangle}{2\cdot 3+1}$$

Now set $\mathbf{t} = \mathbf{0}$. Among all the integrals appearing in this expression, only the first one is not obviously better than the original one. But this first one, $\langle\, \tau_2 \tau_7 \tau_0 \,\rangle$ we can send back to the string equation to express it as $\langle\, \tau_2 \tau_8 \tau_0^2 \,\rangle - \langle\, \tau_1 \tau_8 \tau_0 \,\rangle$ which is better.

These arguments you can piece together and generalise to get an algorithm that can compute any correlator.

3.4 The dilaton equation, derived from KdV

3.4.1 Lemma. *(The dilaton equation in terms of the potential.)*

$$\boxed{\frac{\partial}{\partial t_1} F = \frac{1}{24} + \frac{1}{3}\left(\sum_{i=0}^{\infty} (2i+1)\, t_i\, \frac{\partial}{\partial t_i} F \right)}$$

This follows from and encodes all the manifestation of the dilaton equation as stated in 1.6.2,

$$\langle\, \tau_1\ \boldsymbol{\tau}^{\mathbf{s}} \,\rangle = (2g - 2 + |\mathbf{s}|) \cdot \langle\, \boldsymbol{\tau}^{\mathbf{s}} \,\rangle\,, \qquad (3.4.1.1)$$

dil

and the special case $\langle\, \tau_1 \,\rangle = 1/24$ which accounts for the constant term in the formula. To derive the differential equation, note first that in order to get any contribution from $\langle\, \boldsymbol{\tau}^{\mathbf{s}} \,\rangle$ we must have $3g - 3 + |\mathbf{s}| = \sum i s_i$. This is equivalent to

$$2g - 2 + |\mathbf{s}| = \tfrac{1}{3} \textstyle\sum (2i+1) s_i.$$

Now multiply both sides of (3.4.1.1) with $\mathbf{t}^{\mathbf{s}}/\mathbf{s}!$ and sum them all (as we did for the string equation). We get

$$\sum_{\mathbf{s}} \frac{\mathbf{t}^{\mathbf{s}}}{\mathbf{s}!} \langle\, \tau_1\ \boldsymbol{\tau}^{\mathbf{s}} \,\rangle = \frac{1}{24} + \sum_{\mathbf{s}} \frac{\mathbf{t}^{\mathbf{s}}}{\mathbf{s}!} \tfrac{1}{3} \textstyle\sum (2i+1) s_i \,\langle\, \boldsymbol{\tau}^{\mathbf{s}} \,\rangle$$

whence the result (recalling that multiplication by s_i corresponds to the differential operator $t_i \frac{\partial}{\partial t_i}$, as we observed in the proof of 3.2.1).

Variant af ovenstående.

We have

$$\langle\, \tau_1\ \boldsymbol{\tau}^{\mathbf{s}} \,\rangle = (2g - 2 + |\mathbf{s}|) \cdot \langle\, \boldsymbol{\tau}^{\mathbf{s}} \,\rangle\,, \qquad (3.4.1.2)$$

and also the special case. Now sum them all (with factors $\frac{\mathbf{t}^{\mathbf{s}}}{\mathbf{s}!} \lambda^{2g-2}$) getting

$$\begin{aligned} \langle\!\langle\, \tau_1 \,\rangle\!\rangle = \sum_{\mathbf{s},\lambda} \frac{\mathbf{t}^{\mathbf{s}}}{\mathbf{s}!} \lambda^{2g-2} \langle\, \tau_1\ \boldsymbol{\tau}^{\mathbf{s}} \,\rangle_g &= \frac{1}{24}\lambda^0 + \sum_{\mathbf{s},\lambda} \frac{\mathbf{t}^{\mathbf{s}}}{\mathbf{s}!} \lambda^{2g-2} (2g - 2 + |\mathbf{s}|) \,\langle\, \boldsymbol{\tau}^{\mathbf{s}} \,\rangle_g \\ &= \frac{1}{24} + \lambda \frac{\partial}{\partial \lambda} \langle\!\langle\, 1 \,\rangle\!\rangle + \sum_{i=0}^{\infty} t_i \,\langle\!\langle\, \tau_i \,\rangle\!\rangle\,. \end{aligned}$$

Now use as above that since by the selection rule we have $2g-2 = \frac{2}{3}\sum(i-1)s_i$, we get

$$\lambda \frac{\partial}{\partial \lambda} = \frac{2}{3}\sum(i-1)t_i\partial_i$$

this point of view was mostly to get a chance to play a little with λ...

3.4.2 Lemma. *The KdV equations and the string equation together imply the dilaton equation.*

Proof. First we prove a second derivative of the dilaton equation. Afterwards we see if there isn't a way to conclude the equation itself... Start out with this special case of the KdV equation:

$$\langle\!\langle \tau_1\tau_0^2 \rangle\!\rangle = \tfrac{1}{3}\underbrace{\langle\!\langle \tau_0\tau_0 \rangle\!\rangle}\langle\!\langle \tau_0^3 \rangle\!\rangle + \tfrac{2}{3}\underbrace{\langle\!\langle \tau_0\tau_0^2 \rangle\!\rangle}\langle\!\langle \tau_0^2 \rangle\!\rangle + \tfrac{1}{12}\underbrace{\langle\!\langle \tau_0\tau_0^4 \rangle\!\rangle} \qquad (3.4.2.1)$$

kdvspecial

Now on each of the underbraced factors we're going to use the string equation (and its derivatives):

$$\begin{aligned}
\langle\!\langle \tau_0^2 \rangle\!\rangle &= t_0 + \sum_{i=0}^{\infty} t_{i+1}\langle\!\langle \tau_0\tau_i \rangle\!\rangle \\
\langle\!\langle \tau_0^3 \rangle\!\rangle &= 1 + \sum_{i=0}^{\infty} t_{i+1}\langle\!\langle \tau_0^2\tau_i \rangle\!\rangle \\
\langle\!\langle \tau_0^5 \rangle\!\rangle &= \sum_{i=0}^{\infty} t_{i+1}\langle\!\langle \tau_0^4\tau_i \rangle\!\rangle .
\end{aligned}$$

Plugging these string equations into (3.4.2.1) we get

$$\begin{aligned}
\langle\!\langle \tau_1\tau_0^2 \rangle\!\rangle &= \tfrac{1}{3}\big(t_0 + \sum_{i=0}^{\infty} t_{i+1}\langle\!\langle \tau_i\tau_0 \rangle\!\rangle\big)\langle\!\langle \tau_0^3 \rangle\!\rangle + \tfrac{2}{3}\big(1 + \sum_{i=0}^{\infty} t_{i+1}\langle\!\langle \tau_i\tau_0^2 \rangle\!\rangle\big)\langle\!\langle \tau_0^2 \rangle\!\rangle + \tfrac{1}{12}\sum_{i=0}^{\infty} t_{i+1}\langle\!\langle \tau_i\tau_0^4 \rangle\!\rangle \\
&= \tfrac{1}{3}t_0\langle\!\langle \tau_0^3 \rangle\!\rangle + \tfrac{2}{3}\langle\!\langle \tau_0^2 \rangle\!\rangle + \frac{1}{3}\sum_{i=0}^{\infty} t_{i+1}\left(\langle\!\langle \tau_i\tau_0 \rangle\!\rangle\langle\!\langle \tau_0^3 \rangle\!\rangle + 2\langle\!\langle \tau_i\tau_0^2 \rangle\!\rangle\langle\!\langle \tau_0^2 \rangle\!\rangle + \tfrac{1}{4}\langle\!\langle \tau_i\tau_0^4 \rangle\!\rangle\right) \\
&= \tfrac{1}{3}t_0\langle\!\langle \tau_0^3 \rangle\!\rangle + \tfrac{2}{3}\langle\!\langle \tau_0^2 \rangle\!\rangle + \frac{1}{3}\sum_{i=1}^{\infty} t_i\left(\langle\!\langle \tau_{i-1}\tau_0 \rangle\!\rangle\langle\!\langle \tau_0^3 \rangle\!\rangle + 2\langle\!\langle \tau_{i-1}\tau_0^2 \rangle\!\rangle\langle\!\langle \tau_0^2 \rangle\!\rangle + \tfrac{1}{4}\langle\!\langle \tau_{i-1}\tau_0^4 \rangle\!\rangle\right) \\
&= \tfrac{1}{3}t_0\langle\!\langle \tau_0^3 \rangle\!\rangle + \tfrac{2}{3}\langle\!\langle \tau_0^2 \rangle\!\rangle + \frac{1}{3}\sum_{i=1}^{\infty} t_i\,(2i+1)\,\langle\!\langle \tau_i\tau_0^2 \rangle\!\rangle
\end{aligned}$$

which is exactly the right-hand side of the second derivative of the dilaton equation.

Now we just need to integrate and argue XXXX why the integration constants are what they are... ∎

3.5 Genus zero: topological recursion

3.5.1 Separating the genus contributions. One can write $F = \sum_{g=0}^{\infty} F_g$, where F_g is the contribution from genus g. (Sometimes it is convenient to have a formal variable to control the genus expansion, something like a factor λ^{2g-2}... see Harris-Morrison [14]. On the other hand, Witten [29] hasn't got this parameter...)

Now look at the KdV equation: if we restrict to $\langle\!\langle\, \tau_i\tau_0^2 \,\rangle\!\rangle_g$ on the left-hand side, then on the right-hand side, the two quadratic terms take form as a sum over all partitions of g, so that for example the first quadratic term reads

$$\tfrac{1}{2i+1} \sum_{g'+g''=g} \langle\!\langle\, \tau_{i-1}\tau_0 \,\rangle\!\rangle_{g'} \, \langle\!\langle\, \tau_0^3 \,\rangle\!\rangle_{g''}.$$

The last term will get only contribution from genus $g-1$. (Check the dimension relation: in a product $\langle\, \tau_i\tau_0^2\, \boldsymbol{\tau}^{\mathbf{s}} \,\rangle_g$ we must have $3g = i + \sum k_i + n$, while the last term $\frac{1}{4(2i+1)} \langle\!\langle\, \tau_{i-1}\tau_0^4\, \boldsymbol{\tau}^{\mathbf{s}} \,\rangle\!\rangle_g$ requires $3g = i - 3 + \sum k_i + n$, thus the contribution comes from genus one less.)

So the genus g KdV equation reads

$$\langle\!\langle\, \tau_i\tau_0^2 \,\rangle\!\rangle_g = \frac{\sum_{g'+g''=g} \big(\langle\!\langle\, \tau_{i-1}\tau_0 \,\rangle\!\rangle_{g'} \langle\!\langle\, \tau_0^3 \,\rangle\!\rangle_{g''} + 2 \langle\!\langle\, \tau_{i-1}\tau_0^2 \,\rangle\!\rangle_{g'} \langle\!\langle\, \tau_0^2 \,\rangle\!\rangle_{g''} \big) + \frac{1}{4} \langle\!\langle\, \tau_{i-1}\tau_0^4 \,\rangle\!\rangle_{g-1}}{2i+1}.$$

This looks very much as if it were a relation coming from restriction to the boundary: the two quadratic terms would then correspond to the boundary divisors consisting of two twigs (and the possible way of distributing the genus over them), while the last term should correspond to the boundary divisor consisting of irreducible uninodal curves (which always have genus one less than the smooth curves around them). However, there is at present no explanation of this analogy, and also it seems very difficult to explain the factor $1/(2i+1)$ (depending on i) if it had to do with the boundary (see Witten [29], p. ??).

The genus separated KdV equations are particularly simple in genus zero, where there is only one way of genus partitioning, and the last term does not exist:

$$\langle\!\langle\, \tau_i\tau_0^2 \,\rangle\!\rangle_0 = \frac{\langle\!\langle\, \tau_{i-1}\tau_0 \,\rangle\!\rangle_0 \langle\!\langle\, \tau_0^3 \,\rangle\!\rangle_0 + 2 \langle\!\langle\, \tau_{i-1}\tau_0^2 \,\rangle\!\rangle_0 \langle\!\langle\, \tau_0^2 \,\rangle\!\rangle_0}{2i+1}.$$

This equation is very similar to the topological recursion relation described in 3.5.3 below.

3.5.2 Genus zero. Of course in genus zero, the closed formula 1.5.1 coming from the string equation says everything about the top products of psi classes. However, there is another way of determining the numbers which uses other equations, and it generalises immediately to the setting of genus zero stable maps (as we shall see in the next section) where the string equation is of less importance. Recall from 1.5.2 that in genus zero there is the linear equivalence $\boldsymbol{\psi}_1 = (p_1|p_2,p_3)$. Now each of the irreducible boundary divisor appearing in the sum $(p_1|p_2,p_3)$ is isomorphic to a product of moduli spaces of lower dimension, and by the formula 1.2.6 we know how to express products of psi classes over such a divisor in terms of psi classes on the two moduli spaces of the twigs. In this way, every correlation function can be determined recursively. It is called topological recursion because it depends on the boundary which is stratified topologically XXXX

toprec

3.5.3 Topological recursion (for curves). Let the marking set be $S \cup \{p_1,p_2,p_3\}$, and set for short $\overline{M} := \overline{M}_{0,S\cup\{p_1,p_2,p_3\}}$. Then the first form of the topological recursion relation reads:

$$\begin{aligned}\int_{[\overline{M}]} \boldsymbol{\psi}_1^{k_1+1}\boldsymbol{\psi}_2^{k_2}\boldsymbol{\psi}_3^{k_3}\prod_{i\in S}\boldsymbol{\psi}_i^{k_i} &= \int_{(p_1|p_2,p_3)} \boldsymbol{\psi}_1^{k_1}\boldsymbol{\psi}_2^{k_2}\boldsymbol{\psi}_3^{k_3}\prod_{i\in S}\boldsymbol{\psi}_i^{k_i} \\ &= \sum_{S'\cup S''=S}\Big(\int_{[\overline{M}']}\boldsymbol{\psi}_1^{k_1}\boldsymbol{\psi}_x^{0}\prod_{i\in S'}\boldsymbol{\psi}_i^{k_i}\Big)\Big(\int_{[\overline{M}'']}\boldsymbol{\psi}_2^{k_2}\boldsymbol{\psi}_3^{k_3}\boldsymbol{\psi}_x^{0}\prod_{i\in S''}\boldsymbol{\psi}_i^{k_i}\Big),\end{aligned}$$

where $\overline{M}' := \overline{M}_{0,S'\cup\{p_1,x\}}$ and $\overline{M}'' := \overline{M}_{0,S''\cup\{p_2,p_3,x\}}$. The mark x on each twig is the gluing mark — the factors $\boldsymbol{\psi}_x^0 = 1$ have been put in just to recall that the mark is there.

Now let us state the same equation in the notation introduced on page 27,

$$\langle\, \boldsymbol{\tau}^{\mathbf{s}} \,\rangle_0 := \langle\, \tau_0^{s_0}\tau_1^{s_1}\tau_2^{s_2}\cdots \,\rangle_0.$$

The formula above treats a product where there are at least three factors $\tau_{k_1+1}\tau_{k_2}\tau_{k_3}$, so translating into the new notation it reads (for any k_1,k_2,k_3, and any sequence $\mathbf{s}$):

$$\langle\, \tau_{k_1+1}\tau_{k_2}\tau_{k_3}\cdot\boldsymbol{\tau}^{\mathbf{s}} \,\rangle_0 = \sum_{\mathbf{s}'+\mathbf{s}''=\mathbf{s}} \binom{\mathbf{s}}{\mathbf{s}'} \langle\, \tau_{k_1}\tau_0\cdot\boldsymbol{\tau}^{\mathbf{s}'} \,\rangle_0 \langle\, \tau_{k_2}\tau_{k_3}\tau_0\cdot\boldsymbol{\tau}^{\mathbf{s}''} \,\rangle_0.$$

This time the big sum is over all possible partitions of the sequence $\mathbf{s}$, and for each partition, there are $\binom{\mathbf{s}}{\mathbf{s}'}$ ways to distribute the corresponding marks to the two parts.

Finally the third form of the topological recursion is a statement about (the genus zero part of) the generating function $F_0(\mathbf{t}) = \sum_{\mathbf{s}} \frac{\mathbf{t}^{\mathbf{s}}}{\mathbf{s}!} \langle \boldsymbol{\tau}^{\mathbf{s}} \rangle_0$. It reads,

$$\frac{\partial}{\partial t_{k_1+1}} \frac{\partial}{\partial t_{k_2}} \frac{\partial}{\partial t_{k_3}} F_0 = \frac{\partial}{\partial t_{k_1}} \frac{\partial}{\partial t_0} F_0 \cdot \frac{\partial}{\partial t_{k_2}} \frac{\partial}{\partial t_{k_3}} \frac{\partial}{\partial t_0} F_0.$$

Establishing this from the previous formula is just a question of comparing coefficients and multiplying power series as we did previously in this section. It looks nicer in Witten's notation:

$$\boxed{\langle\!\langle\, \tau_{k_1+1}\tau_{k_2}\tau_{k_3} \,\rangle\!\rangle_0 = \langle\!\langle\, \tau_{k_1}\tau_0 \,\rangle\!\rangle_0 \, \langle\!\langle\, \tau_{k_2}\tau_{k_3}\tau_0 \,\rangle\!\rangle_0}$$

The special case $k_2 = k_3 = 0$ gives

$$\langle\!\langle\, \tau_{k_1}\tau_0^2 \,\rangle\!\rangle_0 = \langle\!\langle\, \tau_{k_1-1}\tau_0 \,\rangle\!\rangle_0 \, \langle\!\langle\, \tau_0^3 \,\rangle\!\rangle_0$$

which is very similar to what comes out of the KdV equation in this case. However, the KdV equation has this factor $1/(2k_1 + 1)$ which makes the two expressions very different at the same time...

In fact, comparing KdV and topological recursion in this case yields the relation

$$k_1 \, \langle\!\langle\, \tau_{k_1-1}\tau_0 \,\rangle\!\rangle_0 \, \langle\!\langle\, \tau_0^3 \,\rangle\!\rangle_0 = \langle\!\langle\, \tau_{k_1-1}\tau_0^2 \,\rangle\!\rangle_0 \, \langle\!\langle\, \tau_0^2 \,\rangle\!\rangle_0.$$

3.5.4 Topological recursion in genus one XXXX Using the expression 1.7.6, we find a relation for the genus-1 correlators:

$$\langle\!\langle\, \tau_{k+1} \,\rangle\!\rangle_1 = \tfrac{1}{12} \, \langle\!\langle\, \tau_k\tau_0^2 \,\rangle\!\rangle_0 + \langle\!\langle\, \tau_k\tau_0 \,\rangle\!\rangle_0 \, \langle\!\langle\, \tau_0 \,\rangle\!\rangle_1$$

check if it is not $\frac{1}{24}$...

Appendix: Combinatorics

3.6 Proof of Proposition 2.3.4

3.6.1 Permutations and partitions. Let A be a finite set of n elements. Let $\mathfrak{S}(A)$ denote the symmetric group of all permutations of the elements of A; $\mathfrak{S}(A)$ has order $n!$. Let $\mathfrak{P}(A)$ denote the set of all partitions of A. Each permutation has a unique decomposition into disjoint cycle (including 1-cycles) and as such determines a partition of A. So there is a map $\mathfrak{S}(A) \to \mathfrak{P}(A)$. Let π be a partition of A, then the *length* $\ell(\pi)$ is defined to be the number of parts (so it can be any integer in the range $1, \dots, n$). Let the *order* of π be defined as the number of pre-images in $\mathfrak{S}(A)$. That is, $\mathrm{ord}(\pi)$ is the number of permutations of A which map each part of π to itself. Clearly, if the partition π has k parts $\pi_1, \dots, \pi_k$ of cardinality $c_1, \dots, c_k$ resp. then

$$\mathrm{ord}(\pi) = \prod_{i=1}^{k} (c_i - 1)!.$$

3.6.2 Children and parents. Now let $A' := A \cup \{x\}$ denote the set with one extra element x. A *child* of a partition $\pi \in \mathfrak{P}(A)$ is a partition of A' obtained as follows: For each part π_i of π, decide whether x joins π_i or not. Clearly this gives $2^{\ell(\pi)}$ choices. In many of these choices x will have joined more than one part; in that case all the parts joined by x are concatenated together. (Maybe you find it enlightening to think like this: the elements are the vertices of a graph; the parts are the connected components of the graph; now introduce a new vertex x and look at all the graphs obtained by drawing edges from x to some of the connected components; if a edge is drawn to more than one connected component, these component have been connected via $x \dots$). Surely all the partitions obtained like that are distinct, so a given partition π has got $2^{\ell(\pi)}$ children.

Now let π' be a partition of A'. Clearly every partition of A' arises as child of some partition of A, then called a parent of π'. How many parents has π' got? Well, let π'_x be the part of π' containing x and suppose this part has r other elements in addition to x. Suppose π is a parent, then since only x can join parts, all the parts in π' distinct from π'_x must also be parts of π. Therefore the parents are characterised by the way π'_x is partitioned. So there is a 1–1 correspondence between the parents of π' and the partitions of π'_x. We let $\mathfrak{C}(\pi)$ denote the set of all children of π; and $\mathfrak{F}(\pi')$ denotes the set of all parents of π'.

We will be concerned with formal sums of type

$$\sum_{\pi \in \mathfrak{P}(A)} \mathrm{ord}(\pi)[\pi]$$

Clearly there are $n!$ terms in this sum (counted with multiplicity).

3.6.3 Lemma. *Let π' be a partition of A'. Then the order of π' is the sum of the orders of its parents:*

$$\mathrm{ord}(\pi') = \sum_{\pi \in \mathfrak{F}(\pi')} \mathrm{ord}(\pi).$$

(In other words, not only do all partitions of A' arise as children of partitions of A: their order is also inherited from their parents' orders.)

Proof. Let $P \cup \{x\}$ be the part containing x, and suppose $\#P = k$. Let the other parts be Q_i. Now

$$\begin{aligned}
\mathrm{ord}(\pi') &= \big(\prod_i (\#Q_i - 1)!\big) \cdot k! \\
&= \big(\prod_i (\#Q_i - 1)!\big) \cdot \big(\sum_{\mu \in \mathfrak{P}(P)} \mathrm{ord}(\mu)\big) \\
&= \sum_{\mu \in \mathfrak{P}(P)} \prod_i (\#Q - 1)!\big) \cdot \mathrm{ord}(\mu) \\
&= \sum_{\pi \in \mathfrak{F}(\pi')} \mathrm{ord}(\pi)
\end{aligned}$$

by the 1–1 correspondence between parents and partitions of the marked part. □

3.6.4 Proposition. *With notation as above we have the following identity*

$$\sum_{\pi \in \mathfrak{P}(A)} \mathrm{ord}(\pi) \sum_{\pi' \in \mathfrak{C}(\pi)} [\pi'] = \sum_{\pi' \in \mathfrak{P}(A')} \mathrm{ord}(\pi')[\pi'].$$

Proof. Clearly both sides are sums over all the partitions of A'. On the left-hand side, each partition π' appears $\mathrm{ord}(\pi)$ times for each parent π. But then the lemma applies. □

3.6.5 Corollary.

$$(n+1)! = \sum_{\pi \in \mathfrak{P}(A)} \mathrm{ord}(\pi) 2^{\ell(\pi)}.$$

3.6.6 Proof of Proposition 2.3.4. By induction the statement is true for n kappa classes, so we are in a situation $\pi_{n+1*}\big(\boldsymbol{\psi}_{n+1}^{a_{n+1}+1} R_{a_1,\dots,a_n}(\boldsymbol{\kappa}) \cdot W\big)$. Each term in R corresponds to a partition of $(a_1, \dots, a_n)$; now use the pull-back formula to substitute each kappa class by its pull-back plus a psi term. Expand: this corresponds to making all the children of the partition corresponding to the monomial. Indeed, for each variable in the monomial, we've got the choice of either taking the psi or not. So the result follows from the combinatorial Proposition above.

Part II

Psi classes on moduli of stable maps

4 Gromov-Witten invariants

GW

4.1 Stable maps

Now we turn our attention to the space of stable maps. Stable maps are a generalisation of the notion of stable curve, conceived by Edward Witten [29] as a mathematical model for "introducing gravity into topological sigma models" (to which words we are not going to associate any further meaning). The precise notion is due to Kontsevich.

Let X be a fixed smooth projective variety over the complex numbers. Instead of just studying abstract curves, we are going to study curves together with a map from the curve to X.

4.1.1 Stable maps. A *stable map* is a morphism from an n-pointed curve C to a variety X subject to the following stability condition. *If an irreducible rational component of C is mapped to a point in X then it must have at least three special points,* and if a irreducible component of genus 1 is mapped to a point, it must have at least one special point. (The second condition is automatically satisfied except in the case where $\beta = 0$ and $n = 0$, in which case there are no such maps, so in practice this condition is of no importance.)

An isomorphism of stable maps is an isomorphism of the source curves preserving all the structure. The stability condition is equivalent to saying that the stable map has got no non-trivial automorphisms.

4.1.2 Stacks of stable maps. When X is a smooth projective variety, there is a moduli stack $\overline{M}_{g,n}(X,\beta)$ of n-pointed stable maps $\mu : C \to X$ of genus g such that $\mu_*[C] = \beta \in H_2(X,\mathbb{Q})$. The stack $\overline{M}_{g,n}(X,\beta)$ is projective but not in general smooth. In fact, in general it will not be irreducible nor equidimensional. Example 6.1.2 below describes the instructive example of $\overline{M}_{1,0}(\mathbb{P}^2,3)$: there is a good component (closure of $M_{1,0}(\mathbb{P}^2,3)$) which corresponds to plane cubics, which is of dimension 9 as expected, but there is also a 'boundary divisor' of dimension 10. . .

This makes it difficult to do intersection theory on it. The solution to this problem is one of the features of Gromov-Witten theory: there is a virtual fundamental class which stands in for the topological fundamental class, and it lives in the dimension the space is expected to have. This is the central subject of Section 6.

In this section and the next, we will just accept it without knowing what it is like.

Important: for $g = 0$ and when X is a homogeneous variety (for example projective space), then the virtual fundamental class coincides with the topological one.

Example: For $X = \mathbb{P}^r$, we can make a quick dimension count and see that

$$\dim \overline{M}_{0,n}(\mathbb{P}^r, d) = rd + r + d + n - 3$$

4.1.3 Structure morphisms. For each mark p_i, there is the evaluation morphism $\nu_i : \overline{M}_{g,n}(X,\beta) \to X$ which takes the class of a map $\mu : C \to X$ to $\mu(p_i)$, the image of p_i in X. As in the case of moduli of curves, there is also a forgetful morphism for each mark, consisting in forgetting that mark, and stabilising if necessary. Let the mark to be forgotten be denoted p_0, and let the forgetful morphism be π_0. This morphism, together with the evaluation morphism,

$$\begin{tikzcd} \overline{M}_{g,n+1}(X,\beta) \arrow[r, "\nu_0"] \arrow[d, shift left, "\pi_0"] & X \\ \overline{M}_{g,n}(X,\beta) \arrow[u, shift left, "\sigma_i"] & \end{tikzcd}$$

plays the rôle of a universal family (in the setting of stacks, it is a true universal family). This means that the restriction of ν_0 to the fibre over a moduli point $[\mu] \in \overline{M}_{g,n}(X,\beta)$ is a stable map isomorphic to μ.

Here σ_i are the n sections corresponding to the marks; they single out the marks in each fibre. The image of σ_i in $\overline{M}_{g,n}(X,\beta)$ is the boundary divisor $D_{i,0}$ consisting of maps with reducible source curve, one of whose two twigs is a rational curve carrying just the two marks p_i and p_0, and mapping to a point in X.

4.1.4 Example. If X is a point and $\beta = 0$ then the stability condition is equivalent to stability of the source curve. In fact, it is easy to see that there is a natural isomorphism $\overline{M}_{g,n}(\mathrm{pt}, 0) \simeq \overline{M}_{g,n}$.

4.1.5 Example. More generally, there is a natural isomorphism

$$\overline{M}_{g,n}(X,0) \xrightarrow{\sim} \overline{M}_{g,n} \times X.$$

Indeed, since the whole map goes to the same point in X, all the evaluation morphisms coincide, and thus provide a morphism to X. There is also the forgetful morphism to $\overline{M}_{g,n}$. So there is a morphism to the product as claimed. We leave it as an exercise to check that it's an isomorphism.

4.2 Primary Gromov-Witten invariants

4.2.1 Gromov-Witten invariants. The evaluation morphisms relate the properties of the moduli space with those of X. For each cohomology class $\gamma \in H^*(X)$, the pull-backs $\nu_i^*(\gamma)$ are referred to as *evaluation classes*. The top products of evaluation classes are called *Gromov-Witten invariants*. They are interesting for at least two reasons: in genus zero, and when X is a homogeneous space, the Gromov-Witten invariants can be interpreted as the number of rational curves incident to subvarieties of class (Poincaré dual to) the γ's involved. Second, in genus zero, they are governed by the WDVV equations (named after Witten, Dijkgraaf, and the Verlinde brothers), thus providing the cohomology space of X with a rich structure: a new associative multiplication, the quantum product. Often the WDVV equations are sufficient to determine all the numbers from a small set of initial values. (For example, in the case $X = \mathbb{P}^r$, all the Gromov-Witten invariants are determined by the initial condition that through two distinct points there is a unique straight line.) The canonical reference for quantum cohomology is Fulton-Pandharipande [6].

We will not go into detail with these questions in this setting. In a moment we are going to include the psi classes, and state the WDVV equations in this greater generality. It has become common to refer to these more general top products simply as Gromov-Witten invariants, calling then the integrals with no psi classes for *primary* Gromov-Witten invariants.

4.3 Psi classes and descendant Gromov-Witten invariants

The psi classes are defined as in the curve case; their definition does not involve the map but only its source curve. Let π_0 be the projection from the universal map (i.e., the forgetful morphism $\overline{M}_{g,n+1}(X,\beta) \to \overline{M}_{g,n}(X,\beta)$), and let ω_{π_0} be the relative dualising sheaf. Then

$$\boldsymbol{\psi}_i := c_1(\sigma_i^* \omega_{\pi_0}).$$

All the basic facts about psi classes carry over without modification to the space $\overline{M}_{g,n}(X,\beta)$. We will state the results here, and defer their proofs to Section 6.

Definition. Top intersections of psi classes and evaluation classes (integrated against the virtual fundamental class) are called *gravitational descendants*, or descendent Gromov-Witten invariants (and it is also common plainly to call them Gromov-Witten invariants, qualifying then the products without psi classes as primary Gromov-Witten invariants). The following notation is similar to what has become standard. Let

$$\langle\ \tau_{k_1}(\gamma_1)\cdots\tau_{k_n}(\gamma_n)\ \rangle^X_{g,\beta} := \int \boldsymbol{\psi}_1^{k_1}\nu_1^*(\gamma_1)\cdots\boldsymbol{\psi}_n^{k_n}\nu_n^*(\gamma_n)\cap[\overline{M}_{g,n}(X,\beta)]^{\mathrm{virt}},$$

where $\gamma_1,\ldots,\gamma_n$ are cohomology classes on X.

`string(maps)` `dilaton`

4.3.1 The string and dilaton equations in the setting of stable maps are analogue to the equations for stable curves (cf. 1.4.2 and 1.6.2). The important thing to note is that the evaluation classes are compatible with pull-back along the forgetful morphism, cf. 5.1.4.
String equation:

$$\boxed{\langle\ \tau_0(0)\cdot\prod_{i=1}^n \tau_{k_i}(\gamma_i)\ \rangle^X_{g,\beta} = \sum_{j=1}^n \langle\ \tau_{k_j-1}(\gamma_j)\prod_{i\neq j}\tau_{k_i}(\gamma_i)\ \rangle^X_{g,\beta}}$$

There is a special case namely

$$\langle\ \tau_0(T_0)\tau_0(\gamma_i)\tau_0(\gamma_j)\ \rangle^X_{0,0} = \int_X \gamma_i\cup\gamma_j.$$

Dilaton equation:

$$\boxed{\langle\ \tau_1(T_0)\cdot\prod_{i=1}^n \tau_{k_i}(\gamma_i)\ \rangle^X_{g,\beta} = (2g-2+n)\cdot\langle\ \prod_{i=1}^n \tau_{k_i}(\gamma_i)\ \rangle^X_{g,\beta}}$$

In each formula, the left-hand side is on a space with one extra mark, and products involving a negative exponent on a psi class are declared to be zero.

For the dilaton equation there is also a special case (analogue to the special case in the point case), namely

$$\langle\ \tau_1(T_0)\ \rangle^X_{1,0} = \frac{\chi(X)}{24}$$

4.3.2 The divisor equation. There is also a similar formula for the case where the extra mark carries only a divisor evaluation class $\nu_0^*(D)$:

$$\boxed{\langle\ \tau_0(D)\cdot\prod_{i=1}^n \tau_{k_i}(\gamma_i)\ \rangle^X_{g,\beta}\ =\ d\cdot\langle\ \prod_{i=1}^n \tau_{k_i}(\gamma_i)\ \rangle^X_{g,\beta}\ +\ \sum_{j=1}^n \langle\ \tau_{k_i-1}(\gamma_i\cup D)\prod_{i\neq j}\tau_{k_i}(\gamma_i)\ \rangle^X_{g,\beta}}$$

where $d := \int_\beta D$.

Again there are special cases in genus zero and degree zero, namely

$$\langle\ \tau_0(D)\cdot\tau_0(\gamma_i)\tau_0(\gamma_j)\ \rangle^X_{0,0}\ = \int_X D\cup\gamma_i\cup\gamma_j$$

Proof. This is similar to the other proofs, taking 5.1.6 into account. □

4.4 The Gromov-Witten potential

4.4.1 Notation. As we did for the correlators in the curve case, we will tighten up the notation in order to state the results in a neater way. Two remarks account for this notation: First, since pull-back as well as integration is linear in its arguments, the Gromov-Witten invariants are linear in the cohomology classes γ. So we may express everything in the our basis $T_0,\dots,T_r$, and need only consider the basis elements. Second, as we already noted and exploited in the curve case (cf. 3.1.1), the symmetric group acts on the moduli space by permuting the names of the marks, and the integrals are clearly invariant under such a permutation. In other words, the names of the marks are irrelevant for the Gromov-Witten invariants. So let us collect all marks "of equal type", and use this information directly instead of referring to the marks themselves:

For each pair (k,a) where $k\in\mathbb{N}$ and $a=0,1,\dots,r$, we will let the symbol $\tau_{k,a}$ denote the cohomology class $\boldsymbol{\psi}_i^k\cup\nu_i^*(T_a)$ for any fixed mark p_i. The name of the mark is immaterial, but of course it is important that we only use this mark once, so that no other classes belong to this mark. Now let $s_{k,a}$ denote the number of marks carrying a class of this type, and let $\tau_{k,a}^{s_{k,a}}$ denote the product of all these classes (so it's a product involving exactly $s_{k,a}$ marks).

All these exponents $s_{k,a}$ fit together in a huge array $\mathbf{s}$ indexed by k and a, so it's an array consisting of $r+1$ infinite sequences. Note that the first entry in the array is $s_{0,0}$. Of course we will only be interested in arrays having only a finite number of non-zero entries — this will be assumed throughout. We adopt multi-index notation in a obvious way and set

$$\boldsymbol{\tau}^{\mathbf{s}} := \prod_{k,a} \tau_{k,a}^{s_{k,a}}.$$

When $\mathbf{s}$ runs through all possible arrays (with only a finite number of non-zero entries), $\boldsymbol{\tau}^{\mathbf{s}}$ runs through all possible monomials in psi classes and pull-back of the basis elements T_i.

Of course we are mostly interested in the integrals of these monomial, and the following notation should be clear by now,

$$\langle\, \boldsymbol{\tau}^{\mathbf{s}} \,\rangle_{g,\beta}^{X} = \int \boldsymbol{\tau}^{\mathbf{s}} \cap [\overline{M}_{g,n}(X,\beta)]^{\mathrm{virt}},$$

where $n = \sum\limits_{k,a} s_{k,a}$ is the numbers of marks. Note that this integral is zero unless there is equality between the total codimension of all these classes and the virtual dimension of the space $\overline{M}_{0,n}(X,\beta)$.

4.4.2 Example. Let X be the projective plane with hyperplane class h, and cohomology basis $T_a = h^a$, $a = 0,1,2$. Let us take β to be the class of conics; the space $\overline{M}_{0,n}(\mathbb{P}^2, 2)$ is of dimension $5+n$, according to 5.0.1. The codimension of $\nu^*(T_a)$ is just a in this case (since for $\mathbb{P}^2$, there is exactly one basis element for each codimension), so a factor $\tau_{k,a}$ is a cohomology class of codimension $k+a$. Since $\tau_{k,a}^{s_{k,a}}$ is then a class of codimension $s_{k,a}(k+a)$ we see that the total codimension of $\boldsymbol{\tau}^{\mathbf{s}}$ is equal to $\sum_{k,a} s_{k,a}(k+a)$. To get a non-zero integral, this number must be equal to $5+n$, where $n = \sum_{k,a} s_{k,a}$ is the number of marks. From this we finally conclude that to get a non-zero integral we only need to consider arrays $\mathbf{s}$ such that

$$\sum_{k,a} s_{k,a}(k+a-1) = 5.$$

Here is an example of such an array.

$$\begin{pmatrix} 0 & 2 & 0 \\ 3 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \end{pmatrix}$$

Let us write out the corresponding integral $\langle\, \boldsymbol{\tau}^{\mathbf{s}} \,\rangle_2$. The number of marks is $8 = 2+3+1+1+1$. The only non-zero entry in row 0 corresponds to two factors of type $\tau_{0,1}$; let us use the marks p_1 and p_2 for this, getting a factor $\nu_1^*(T_1) \cup \nu_2^*(T_1)$. Continuing like this we can write the integral as

$$\langle\, \boldsymbol{\tau}^{\mathbf{s}} \,\rangle_2 = \int \underbrace{\nu_1^*(T_1)\, \nu_2^*(T_1)}_{2} \cup \underbrace{\boldsymbol{\psi}_3\, \boldsymbol{\psi}_4\, \boldsymbol{\psi}_5}_{3} \cup \underbrace{\boldsymbol{\psi}_6\, \nu_6^*(T_1)}_{1} \cup \underbrace{\boldsymbol{\psi}_7^2}_{1} \cup \underbrace{\boldsymbol{\psi}_8^2\, \nu_8^*(T_2)}_{1} \cap [\overline{M}_{0,8}(\mathbb{P}^2, 2)].$$

(Here we have suppressed factors of type $\boldsymbol{\psi}^0$ and $\nu^*(T_0)$.)

4.4.3 The Gromov-Witten potential. As we did for the correlators in the curve case, we will construct the generating function of all the possible Gromov-Witten invariants. Introduce formal variables $\mathbf{t} = (t_{k,a})$ (with $k \in \mathbb{N}$ and $a = 0, 1, \ldots, r$) and let as expected $\mathbf{t}^{\mathbf{s}}$ be multi-index notation for the infinite product $\prod\limits_{k,a} t_{k,a}^{s_{k,a}}$.

We also need to sum over all the possible degrees $\beta \in A_1^+(X, \mathbb{Z})$. A priori, for a given array $\mathbf{s}$ there may be several choices of β such that the dimension compatibility condition is satisfied, so the symbol $\langle\, \boldsymbol{\tau}^{\mathbf{s}} \,\rangle$ is not well-defined unless we specify the degree. So we sum over all possible degrees. There are two reasons for introducing a parameter for this purpose: first of all, there might be infinitely many β giving contribution(??) so in order to obtain formal convergence we need the parameter. Second, if we want to be able to keep track of how much each individual integral contributes to the generating function, so that we can extract the individual invariants, we need a set of parameters. Suppose the basis is ordered in such a way that $T_1, \ldots, T_p$ constitute a basis for $A^1(X)$ (so here p is the Picard number of X). For fixed β, consider the vector $\mathbf{d} = (d_1, \ldots, d_p)$ of "directional degrees" $d_i := \int_\beta T_i$. Now introduce formal variables $\mathbf{q} = (q_1, \ldots, q_p)$, and use the expansion $\mathbf{q}^{\mathbf{d}}$ to keep track of the degree.

The symbols $\mathbf{q}^{\mathbf{d}}$ belong to the Novikov ring (see Getzler [9]).

Now define

$$\langle\, \boldsymbol{\tau}^{\mathbf{s}} \,\rangle := \sum_{\beta} \mathbf{q}^{\mathbf{d}} \,\langle\, \boldsymbol{\tau}^{\mathbf{s}} \,\rangle_{\beta}$$

Now the Gromov-Witten potential is the generating function

$$\begin{aligned}\Phi(\mathbf{t}) &:= \sum_{\mathbf{s}} \frac{\mathbf{t}^{\mathbf{s}}}{\mathbf{s}!} \,\langle\, \boldsymbol{\tau}^{\mathbf{s}} \,\rangle \\ &= \sum_{\beta} \mathbf{q}^{\mathbf{d}} \sum_{\mathbf{s}} \frac{\mathbf{t}^{\mathbf{s}}}{\mathbf{s}!} \,\langle\, \boldsymbol{\tau}^{\mathbf{s}} \,\rangle_{\beta}.\end{aligned}$$

As in the curve case, we will employ also the Witten notation

$$\langle\!\langle\, 1 \,\rangle\!\rangle := \Phi(\mathbf{t}, \mathbf{q}),$$

which is convenient for partial derivatives: we set

$$\langle\!\langle\, \tau_{k_1,a_1} \cdots \tau_{k_n,a_n} \,\rangle\!\rangle := \frac{\partial}{\partial t_{k_1,a_1}} \cdots \frac{\partial}{\partial t_{k_n,a_n}} \Phi(\mathbf{t}, \mathbf{q}).$$

In this notation the symbols $\tau_{k,a}$ acquire meaning as differential operators or *vector fields* on the large phase space.

The advantage of this notation is the following observation: $\langle\!\langle\, \tau_{k_1,a_1} \cdots \tau_{k_n,a_n} \,\rangle\!\rangle$ is just the generating function for the correlators $\langle\, \tau_{k_1,a_1} \cdots \tau_{k_n,a_n} \cdot \boldsymbol{\tau}^{\mathbf{s}} \,\rangle_{\beta}$, in the sense that the coefficient of $\mathbf{q}^{\beta} \frac{\mathbf{t}^{\mathbf{s}}}{\mathbf{s}!}$ in the series $\langle\!\langle\, \tau_{k_1,a_1} \cdots \tau_{k_n,a_n} \,\rangle\!\rangle$ is exactly $\langle\, \tau_{k_1,a_1} \cdots \tau_{k_n,a_n} \cdot \boldsymbol{\tau}^{\mathbf{s}} \,\rangle$. This is,

$$\langle\!\langle\, \tau_{k_1,a_1} \cdots \tau_{k_n,a_n} \,\rangle\!\rangle = \sum_{\beta} \mathbf{q}^{\mathbf{d}} \sum_{\mathbf{s}} \frac{\mathbf{t}^{\mathbf{s}}}{\mathbf{s}!} \,\langle\, \tau_{k_1,a_1} \cdots \tau_{k_n,a_n} \cdot \boldsymbol{\tau}^{\mathbf{s}} \,\rangle_{\beta}.$$

5 Gromov-Witten theory in genus zero

WDVV

In this section we restrict attention to the case of genus zero.

From now on we restrict attention to genus zero

dim

5.0.1 The nice cases. In this section our primary concern will be with the case where $g = 0$ and the target space X is a projective homogeneous variety. In this case, the stack $\overline{M}_{0,n}(X, \beta)$ is an irreducible smooth Deligne-Mumford stack, and its virtual fundamental class coincides with the topological one. We will allow ourselves to confuse the stack with the corresponding coarse moduli space which is a normal projective variety of the expected dimension. In particular, for $X = \mathbb{P}^r$ we have

$$\dim \overline{M}_{0,n}(\mathbb{P}^r, d) = rd + r + d + n - 3.$$

5.1 The boundary

boundary **5.1.1 The boundary divisors.** The boundary set-up is slightly more complicated for maps than the one for curves described in 1.2.4, because in order to glue two maps, they must map the gluing marks to the same point in X. Let $S = \{p_1, \ldots, p_n\}$ denote the set of marks. For each partition $S' \cup S'' = S$ and $\beta' + \beta'' = \beta$ there is an irreducible boundary divisor $D(S', \beta' | S'', \beta'')$, consisting generically of maps with a two-twig source curve, one twig of which carries the marks S' and maps with degree β', and the other twig carrying S'' and of degree β''. Each twig corresponds to a moduli space of lower dimension, $\overline{M}' := \overline{M}_{0,S' \cup \{x'\}}(X, \beta')$ and $\overline{M}'' := \overline{M}_{0,S'' \cup \{x''\}}(X, \beta'')$. More precisely, $D := D(S'\beta' \mid S''\beta'')$ is the image of a finite morphism $\rho_D : \overline{M}' \times_X \overline{M}'' \to \overline{M}_{0,n}(X, \beta)$. The fibred product is over the two evaluation morphisms $\nu_{x'} : \overline{M}' \to X$ and $\nu_{x''} : \overline{M}'' \to X$, reflecting the fact that in order to glue, the two maps must agree at the mark. The fibred product is a subvariety in the cartesian product $\overline{M}' \times \overline{M}''$; let $\jmath_D$ denote the inclusion. This set-up and notation is used throughout — summarised in the following diagram:

$$\begin{tikzcd}
\overline{M}_{0,n}(X,\beta) & \\
\overline{M}' \times_X \overline{M}'' \arrow[u, "\rho_D"] \arrow[r, hook, "\jmath_D"'] & \overline{M}' \times \overline{M}''.
\end{tikzcd}$$

The morphism ρ_D is an isomorphism if either there are marks on both twigs or if one of the degrees is zero. These are the only cases we will be concerned with.

Again (cf. 1.2.5) note that for the boundary divisor $D = D_{i,0}$, the morphism ρ_D is naturally identified with the section σ_i itself (since $\overline{M}_{0,3} \simeq \mathrm{pt}$).

5.1.2 Caution. In general, for $g > 0$ and X not homogeneous, these boundary divisors need not be irreducible, and being a divisor means virtual divisor in the sense that they are of codimension 1 inside each individual component of the moduli stack... see 6.2

5.1.3 Restricting evaluation classes to the boundary. Let p_i be a mark, say in S'. In the above diagram one can draw in the evaluation morphisms of the spaces $\overline{M}_{0,n}(X, \beta)$ and $\overline{M}'$ respectively, and note that the resulting diagram commutes. This shows that the evaluation classes $\boldsymbol{\eta}_i := \nu_i^*(\gamma)$ restrict to the boundary in this straightforward way:

$$\rho_D^* \boldsymbol{\eta}_i = \jmath_D^* \boldsymbol{\eta}_i \tag{5.1.3.1}$$ restr-eta

where it is understood that the eta class on the right-hand side is pulled back from $\overline{M}'$.

pull-eta **5.1.4 Compatibility with the forgetful morphism.** The evaluation classes are compatible with the forgetful morphism $\pi_0 : \overline{M}_{0,n+1}(X,\beta) \to \overline{M}_{0,n}(X,\beta)$, in the following sense. If $\boldsymbol{\eta}_i$ denotes the i'th evaluation class on $\overline{M}_{0,n}(X,\beta)$ and $\widetilde{\boldsymbol{\eta}}_i$ denotes the i'th evaluation class on $\overline{M}_{0,n+1}(X,\beta)$ then

$$\pi_0^* \boldsymbol{\eta}_i = \widetilde{\boldsymbol{\eta}}_i.$$

push-divisor **5.1.5 Lemma.** *For a divisor class $\gamma H^2(X)$, set $d := \int_\beta \gamma$, and set $\boldsymbol{\eta}_0 := \nu_0^*(\gamma)$. Then*

$$\pi_{0*} \boldsymbol{\eta}_0 = d \qquad \text{in } A^0(\overline{M}_{0,n}(X,\beta)).$$

Proof. Let Γ be a general divisor of class (Poincaré dual to) γ; then $\nu_0^{-1}(\Gamma)$ is of class (Poincaré dual to) $\boldsymbol{\eta}_0$. Look at the restriction of the forgetful morphism

$$\pi_{0|\nu_0^{-1}(\Gamma)} : \nu_0^{-1}(\Gamma) \to \overline{M}_{0,n}(X,\beta).$$

Since a general map of degree β meets Γ in d points, the above morphism is generically finite of degree d, whence the result. $\square$

swap **5.1.6 Remark.** The following identity holds in $A^*(\overline{M}_{0,n}(X,\beta))$.

$$\boldsymbol{\eta}_i \cdot D_{ij} = \boldsymbol{\eta}_j \cdot D_{ij}.$$

In particular, there is the following push-down formula

$$\pi_{0*}(\boldsymbol{\eta}_0 D_{j0}) = \boldsymbol{\eta}_j.$$

5.1.7 Formulae. The psi classes restrict to a boundary divisor $D = D(S',\beta' \mid S'',\beta'')$ exactly as in the case of stable curves, cf. 1.2.6.

$$\rho_D^* \boldsymbol{\psi}_i = \jmath_D^* \boldsymbol{\psi}_i \tag{5.1.7.1}$$ restr-psi+

where it is understood that the psi class on the right-hand side is pulled back along the projection $\tau' : \overline{M}' \times M'' \to \overline{M}'$ if $p_i \in S'$ (and the other projection if $p_i \in S''$).

Let $D_{i,0}$ denote the image of the section σ_i; it is the boundary divisor having p_i and p_0 as only marks on a twig of degree zero (and genus zero). The following formulae are immediate from the arguments of 1.2.5, 1.3.2, and 1.2.7.

$$\begin{aligned} D_{i,0} \cdot D_{j,0} &= 0 \quad \text{for } i \neq j \\ D_{i,0}^2 &= -\pi_0^* \boldsymbol{\psi}_i \cdot D_{i,0} \\ \boldsymbol{\psi}_i \cdot D_{i,0} &= 0. \end{aligned}$$

The important comparison formula is also identical to the corresponding one for stable curves (cf. 1.3.1):

$$\boxed{\boldsymbol{\psi}_i = \pi_0^* \boldsymbol{\psi}_i + D_{i,0}} \tag{5.1.7.2}$$ compare(maps)

XXXX state this for any genus ??? XXXX Finally, the push-down formula is also identical to 1.6.1:

$$\boxed{\pi_{0*}\boldsymbol{\psi}_0 = 2g - 2 + n} \qquad \text{times } [\overline{M}_{0,n}(X,\beta)]^{\text{virt}}. \tag{5.1.7.3}$$ pushpsi(maps)

The first result that requires a new argument is the analogue of 1.5.2:

123gw **5.1.8 Proposition.** *Let $g = 0$. In case $n \geq 3$, there is the following expression for the psi class:*

$$\boxed{\boldsymbol{\psi}_1 = (p_1 | p_2, p_3),}$$

where $(p_1|p_2,p_3)$ denotes the sum of boundary divisors having p_1 on one twig and p_2 and p_3 on the other twig.

As in the proof of 1.5.2, the idea is to compare how the two classes pull back along forgetful morphisms down to $\overline{M}_{0,3}$. This forgetful morphism is composed by forgetful morphisms $\overline{M}_{0,n+1} \to \overline{M}_{0,n}$ (along which we have already seen that $\boldsymbol{\psi}_1$ and $(p_1|p_2,p_3)$ pull back in the same way), and a forgetful morphism $\overline{M}_{0,n}(X,\beta) \to \overline{M}_{0,n}$ (for $n \geq 3$) which we treat in the following lemma.

epsilon **5.1.9 Lemma.** *For $n \geq 3$, let $\varepsilon : \overline{M}_{0,n}(X,\beta) \to \overline{M}_{0,n}$ denote the morphism that forgets the X-structure and stabilises. Let E_1 denote the sum of all boundary divisors in $\overline{M}_{0,n}(X,\beta)$ such that p_1 is alone on a twig. Then*

$$\varepsilon^* \boldsymbol{\psi}_1 = \boldsymbol{\psi}_1 - E_1 \qquad \text{and} \qquad \varepsilon^*(p_1|p_2,p_3) = (p_1|p_2,p_3) - E_1.$$

In particular, the preceding proposition follows.

Proof. We perform the proof in the baby model: let $\widetilde{\pi} : \widetilde{\mathfrak{X}} \to B$ be a family of n-pointed stable maps (with map $\mu : \widetilde{\mathfrak{X}} \to X$). Then the stabilised family of n-pointed curves $\pi : \mathfrak{X} \to B$ is obtained blowing down unstable twigs E of the fibres. These unstable twigs occur exactly over the divisor $\sum E_i \subset B$. (In other words, $\varepsilon : \widetilde{\mathfrak{X}} \to \mathfrak{X}$ is the blow-up along the base points of the linear system corresponding to the map μ. These base points occur exactly over the divisor $\sum E_i$, and a base point lying over E_i is then a point on the section σ_i.

Now compare the psi classes:

$$\begin{aligned}
\widetilde{\boldsymbol{\psi}}_1 &= \widetilde{\sigma}_1^* \omega_{\widetilde{\pi}} \\
&= \widetilde{\sigma}_1^* \left(\varepsilon^* \omega_\pi + E \right) \\
&= \sigma_1^* \omega_\pi + \widetilde{\sigma}_1^* E \\
&= \boldsymbol{\psi}_1 + E_1
\end{aligned}$$

since the section $\widetilde{\sigma}_1$ intersects just the exceptional divisor corresponding to having p_1 alone on a twig.

As to the assertion on the pull-back of $(p_1|p_2,p_3)$: it is clear from a set-theoretic argument: when pulling back this divisor, we get all the divisors in $\overline{M}_{0,n}(X,\beta)$ of that type, except those where p_1 is alone on a twig. $\square$

5.2 The splitting lemma

The following lemma is the engine behind the two main recursions known in genus zero. It is a manifestation of the recursive structure of the boundary. To treat it in a reasonable way, it is convenient to introduce coordinates on $A^*(X)$, in order to reduce the possibilities of cohomology classes: the observation is simply that the Gromov-Witten invariants are linear in the cohomology classes of X, so it suffices to treat the elements of a basis. So let $T_0, \ldots, T_r$ denote the elements of a homogeneous basis of $A^*(X)$. (We always order the elements such that T_0 is the fundamental class, and T_r is the class of a point.) Let the matrix (g_{ef}) be defined as

$$g_{ef} := \int_X T_e \cup T_f.$$

It is an invertible matrix, and we let g^{ef} denote the entries of the inverse matrix. There is another characterisation of this matrix which is the reason for its importance in this context: if $\Delta \subset X \times X$ is the diagonal, then the class of Δ is given by the Künneth decomposition formula

$$[\Delta] = \sum_{e,f} \pi'^{*}T_e \, g^{ef} \, \pi''^{*}T_f,$$

where π' and π'' are the two projections. XXXX use $\boxtimes$ notation XXXX

Now the recursive structure of the boundary consists in the fact that each boundary divisor is the image of a fibred product of moduli spaces of lower dimension. The fibred product $\overline{M}' \times_X \overline{M}''$ is a subvariety in $\overline{M}' \times \overline{M}''$ (see 5.1.1 for notation), and it coincides with the pull-back of $\Delta \subset X \times X$ to $\overline{M}' \times \overline{M}''$ along the product of the evaluation morphisms $\nu_{x'}$ and $\nu_{x''}$ of the glueing marks. Precisely, there is a fibre square

$$\begin{tikzcd}
\overline{M}' \times_X \overline{M}'' \arrow[r, hook, "\jmath"] \arrow[d] & \overline{M}' \times \overline{M}'' \arrow[d, "\nu_{x'} \times \nu_{x''}"] \\
\Delta \arrow[r, hook] & X \times X.
\end{tikzcd}$$

Therefore, the class of fibred product $\overline{M}' \times_X \overline{M}''$ in $\overline{M}' \times \overline{M}''$ is given as

$$\sum_{e,f} \nu^*_{x'}(T_e) \, g^{ef} \, \nu^*_{x''}(T_f).$$

Here, as always, we have suppressed the symbols for pull-back along the projections from $\overline{M}' \times \overline{M}''$ to its factors. This gives a formula for $\jmath^*$ which appears in the restriction formulae 5.1.3.1 and 5.1.7.1.

$\boxed{\texttt{splittinglemma}}$ **5.2.1 Lemma.** SPLITTING LEMMA. *On a space $\overline{M}_{0,S}(X,\beta)$, consider a boundary divisor $D = D(S',\beta' \mid S'',\beta'')$, and let $\langle\, D \cdot \alpha \,\rangle_\beta$ denote the integral of a cohomology class α over D. Then*

$$\langle\, D \cdot \prod_{p_i \in S} \tau_{k_i}(\gamma_i) \,\rangle_\beta = \sum_{e,f} \langle \prod_{p_i \in S'} \tau_{k_i}(\gamma_i) \; \tau_0(T_e) \,\rangle_{\beta'} \; g^{ef} \;\; \langle\, \tau_0(T_f) \prod_{p_i \in S''} \tau_{k_i}(\gamma_i) \,\rangle_{\beta''}.$$

This follows readily from the restriction formulae 5.1.3.1 and 5.1.7.1 together with the formula for pull-back along $\jmath$ we just described.

perhaps a complete proof is in place here...

5.3 WDVV equations

Named after Witten, Dijkgraaf, Verlinde and Verlinde... XXXX associativity of quantum product...

5.3.1 Fundamental linear equivalence. Genus zero has some special features. One stems from the fact that when there are at least four marks, there is a forgetful morphism down to $\overline{M}_{0,4} \simeq \mathbb{P}^1$. Let $(p_1, p_2 \mid p_3, p_4)$ denote the divisor in $\overline{M}_{0,n+4}(X, \beta)$ which is the sum of all boundary divisors having p_1 and p_2 on one twig while p_3 and p_4 are on the other twig. Such a sum of divisors is sometimes called a special boundary divisor. The irreducible components of $(p_1, p_2 \mid p_3, p_4)$ correspond to all ways of distributing the remaining n marks on the two twigs, and all (non-negative) degree partitions $\beta = \beta' + \beta''$. The important thing to note, however, is that $(p_1, p_2 \mid p_3, p_4)$ is the pull-back of the irreducible divisor $(p_1, p_2 \mid p_3, p_4)$ of $\overline{M}_{0,4}$ along the forgetful morphism.

Now since every two divisors on $\overline{M}_{0,4} \simeq \mathbb{P}^1$ are linearly equivalent, (and since such an equivalence is preserved under pull-back), we get the following identity in $A^1\big(\overline{M}_{0,n+4}(X, \beta)\big)$:

$$(p_1, p_2 \mid p_3, p_4) = (p_2, p_3 \mid p_1, p_4) \qquad \text{in } A^1(\overline{M}_{0,4}).$$

5.3.2 Integrating over a special boundary divisor. Now let us integrate a product $\tau_{k_1,a_1}\tau_{k_2,a_2}\tau_{k_3,a_3}\tau_{k_4,a_4}\ \boldsymbol{\tau}^{\mathbf{s}}$ over the divisor $(p_1, p_2 \mid p_3, p_4)$. We can simply apply the splitting lemma 5.2.1 to each of the irreducible components of the divisor. The irreducible components correspond to all the ways of distributing the remaining marks (those corresponding to the product $\boldsymbol{\tau}^{\mathbf{s}}$) and the degree. The ways of distributing marks corresponds to choosing two arrays $\mathbf{s}'$ and $\mathbf{s}''$ such that $\mathbf{s}' + \mathbf{s}'' = \mathbf{s}$, namely such that the marks corresponding to $\mathbf{s}'$ belong to the left-hand twig, and the marks of $\mathbf{s}''$ belong to the right-hand twig. But for each such partition, there are $\binom{\mathbf{s}}{\mathbf{s}'}$ ways of actually distributing the marks. We get

$$\sum_{\beta'+\beta''=\beta} \sum_{\mathbf{s}'+\mathbf{s}''=\mathbf{s}} \binom{\mathbf{s}}{\mathbf{s}'} \sum_{e,f} \big\langle\, \tau_{k_1,a_1}\tau_{k_2,a_2}\tau_{0,e}\, \boldsymbol{\tau}^{\mathbf{s}'} \,\big\rangle_{\beta'}\ g^{ef}\ \big\langle\, \tau_{k_3,a_3}\tau_{k_4,a_4}\tau_{0,f}\, \boldsymbol{\tau}^{\mathbf{s}''} \,\big\rangle_{\beta''}.$$

The summation over e and f comes of course from the splitting lemma, which also accounts for the appearance of the factors $\tau_{0,e}$ and $\tau_{0,f}$.

5.3.3 WDVV equation for a particular product. Now doing the same for the equivalent special boundary divisor $(p_2, p_3 \mid p_1, p_4)$ we get a quadratic relation among the Gromov-Witten invariants, namely

$$\sum_{\beta'+\beta''=\beta} \sum_{\mathbf{s}'+\mathbf{s}''=\mathbf{s}} \binom{\mathbf{s}}{\mathbf{s}'} \sum_{e,f} \langle\, \tau_{k_1,a_1}\tau_{k_2,a_2}\tau_{0,e}\, \boldsymbol{\tau}^{\mathbf{s}'} \,\rangle_{\beta'}\; g^{ef}\; \langle\, \tau_{k_3,a_3}\tau_{k_4,a_4}\tau_{0,f}\, \boldsymbol{\tau}^{\mathbf{s}''} \,\rangle_{\beta''} =$$
$$\sum_{\beta'+\beta''=\beta} \sum_{\mathbf{s}'+\mathbf{s}''=\mathbf{s}} \binom{\mathbf{s}}{\mathbf{s}'} \sum_{e,f} \langle\, \tau_{k_2,a_2}\tau_{k_3,a_3}\tau_{0,e}\, \boldsymbol{\tau}^{\mathbf{s}'} \,\rangle_{\beta'}\; g^{ef}\; \langle\, \tau_{k_1,a_1}\tau_{k_4,a_4}\tau_{0,f}\, \boldsymbol{\tau}^{\mathbf{s}''} \,\rangle_{\beta''}.$$

Now this is not very nice to look at, but when formulated in terms of the Gromov-Witten potential, it looks like this, which of course is much better, and should suffice to appreciate the concept of generating functions:

5.3.4 Proposition. (WDVV equations.) *The (genus zero) Gromov-Witten potential* $\langle\!\langle\, 1 \,\rangle\!\rangle$ *satisfies the following partial differential equation*

$$\sum_{e,f} \langle\!\langle\, \tau_{k_1,a_1}\tau_{k_2,a_2}\tau_{0,e} \,\rangle\!\rangle\; g^{ef}\; \langle\!\langle\, \tau_{k_3,a_3}\tau_{k_4,a_4}\tau_{0,f} \,\rangle\!\rangle = \sum_{e,f} \langle\!\langle\, \tau_{k_2,a_2}\tau_{k_3,a_3}\tau_{0,e} \,\rangle\!\rangle\; g^{ef}\; \langle\!\langle\, \tau_{k_1,a_1}\tau_{k_4,a_4}\tau_{0,f} \,\rangle\!\rangle\,.$$

Proof. It is just a rewrite of the previous statement; the proof amounts to (on each side of the equation) multiplying two formal series, extracting the coefficients, and comparing them with the coefficients of the other side of the equation. For each coefficient, the equation is just the one of the previous lemma. Let us write out the left-hand side:

$$\sum_{e,f} \langle\!\langle\, \tau_{k_1,a_1}\tau_{k_2,a_2}\tau_{0,e} \,\rangle\!\rangle\; g^{ef}\; \langle\!\langle\, \tau_{k_3,a_3}\tau_{k_4,a_4}\tau_{0,f} \,\rangle\!\rangle =$$
$$\sum_{e,f}\Big(\sum_{\beta'}\mathbf{q}^{\beta'}\sum_{\mathbf{s}'}\frac{\mathbf{t}^{\mathbf{s}'}}{\mathbf{s}'!}\langle\, \tau_{k_1,a_1}\tau_{k_2,a_2}\tau_{0,e}\cdot \boldsymbol{\tau}^{\mathbf{s}'} \,\rangle_{\beta'}\Big) g^{ef}\Big(\sum_{\beta''}\mathbf{q}^{\beta''}\sum_{\mathbf{s}''}\frac{\mathbf{t}^{\mathbf{s}''}}{\mathbf{s}''!}\langle\, \tau_{k_3,a_3}\tau_{k_4,a_4}\tau_{0,f}\cdot \boldsymbol{\tau}^{\mathbf{s}''} \,\rangle_{\beta''}\Big)$$
$$= \sum_{e,f} g^{ef} \sum_{\beta'+\beta''} \mathbf{q}^{\beta'+\beta''} \sum_{\mathbf{s}'+\mathbf{s}''} \frac{\mathbf{t}^{\mathbf{s}'+\mathbf{s}''}}{\mathbf{s}'!\mathbf{s}''!} \langle\, \tau_{k_1,a_1}\tau_{k_2,a_2}\tau_{0,e}\cdot \boldsymbol{\tau}^{\mathbf{s}'} \,\rangle_{\beta'} \langle\, \tau_{k_3,a_3}\tau_{k_4,a_4}\tau_{0,f}\cdot \boldsymbol{\tau}^{\mathbf{s}''} \,\rangle_{\beta''}$$

so the coefficient of $\mathbf{q}^{\beta}\frac{\mathbf{t}^{\mathbf{s}}}{\mathbf{s}!}$ is

$$\sum_{\beta'+\beta''=\beta} \sum_{\mathbf{s}'+\mathbf{s}''=\mathbf{s}} \binom{\mathbf{s}}{\mathbf{s}'} \sum_{e,f} \langle\, \tau_{k_1,a_1}\tau_{k_2,a_2}\tau_{0,e}\, \boldsymbol{\tau}^{\mathbf{s}'} \,\rangle_{\beta'}\; g^{ef}\; \langle\, \tau_{k_3,a_3}\tau_{k_4,a_4}\tau_{0,f}\, \boldsymbol{\tau}^{\mathbf{s}''} \,\rangle_{\beta''},$$

which is just the left-hand side of the previous lemma. Similarly, the corresponding coefficient on the right-hand side of the claimed identity is just the right-hand side of the previous lemma. □

5.4 Topological recursion

The second important set of recursion relations satisfied by the Gromov-Witten potential is topological recursion. It is a simple generalisation of the topological recursion relations we established for stable curves in 3.5.3: The point is simply that (assuming $n \geq 3$) each time there is a psi class in a top product, we can write it as a sum of boundary divisor (cf. 5.1.8), and then compute the integral by restricting the remaining factors to each of the boundary divisors, using the splitting lemma.

5.4.1 Topological recursion relation. (for any k_1, k_2, k_3, and any sequence $\mathbf{s}$):

$$\langle\, \tau_{k_1+1,a_1}\tau_{k_2,a_2}\tau_{k_3,a_3}\cdot \boldsymbol{\tau}^{\mathbf{s}}\,\rangle_\beta = \sum_{\beta'+\beta''=\beta}\ \sum_{\mathbf{s}'+\mathbf{s}''=\mathbf{s}} \binom{\mathbf{s}}{\mathbf{s}'} \sum_{e,f} \langle\, \tau_{k_1,a_1}\tau_{0,e}\cdot \boldsymbol{\tau}^{\mathbf{s}'}\,\rangle_{\beta'}\, g^{ef}\ \langle\, \tau_{k_2,a_2}\tau_{k_3,a_3}\tau_{0,f}\cdot \boldsymbol{\tau}^{\mathbf{s}''}\,\rangle_{\beta''}.$$

The big sum over all possible partitions of the sequence $\mathbf{s}$, corresponds to all the possible ways of distributing the remaining marks, and for each partition there are $\binom{\mathbf{s}}{\mathbf{s}'}$ ways to distribute the corresponding marks to the two parts.

Proof. Write the first psi class as $(p_1|p_2,p_3)$ (cf. 5.1.8). Each term in this expression corresponds to a term in the big sum of the theorem. Now restrict the rest to each of these boundary divisors. □

Now that we are getting experienced with generating functions in general, and the Gromov-Witten potential in particular, we can easily write this recursion relation in terms of a partial differential equation for $\langle\!\langle\, 1\,\rangle\!\rangle$:

$$\langle\!\langle\, \tau_{k_1+1,a_1}\tau_{k_2,a_2}\tau_{k_3,a_3}\,\rangle\!\rangle = \sum_{e,f} \langle\!\langle\, \tau_{k_1,a_1}\tau_{0,e}\,\rangle\!\rangle\ g^{ef}\ \langle\!\langle\, \tau_{k_2,a_2}\tau_{k_3,a_3}\tau_{0,f}\,\rangle\!\rangle\,.$$

The moral is that if we have a recursion among $\langle\ \ \rangle$ involving sums over all partitions, and where the binomial coefficients occur at the right place then we can translate it into a relation among the partial derivatives of $\langle\!\langle\, 1\,\rangle\!\rangle$ and thus avoid all the summations...

`reconstruction` **5.4.2 Reconstruction for descendants.** Together, these equations are sufficient to reduce any gravitational descendant to a Gromov-Witten invariant: Induction on the number of psi classes: if there are three or more marks, use the topological recursion relation to reduce the number of psi classes. Otherwise, use first the divisor equation (backwards) to introduce more marks. (Note that this step may introduce rational coefficients.)

XXXX Small and big phase space... ?

6 Gromov-Witten theory in higher genus

higher

That case requires the construction of a *virtual fundamental class*, a certain homology class living in the expected dimension, which plays the rôle of the usual topological fundamental class. The expected dimension of $\overline{M}_{g,n}(X,\beta)$ is

$$(3 - \dim X)(g-1) + \int_\beta c_1(T_X) + n.$$

These mapping to a point spaces are the simplest examples of how a space can fail to have the expected dimension: $\overline{M}_{g,n} \times X$ is a irreducible space of dimension $3g - 3 + n + \dim X$, while the expected dimension of $\overline{M}_{g,n}(X,0)$ is only $(3 - \dim X)(g-1) + n$. So the expected dimension is $g \cdot \dim X$ less than the actual dimension!

As a concrete example, consider the case $g = 1, n = 1$, which is one of the minimal stable spaces. In this case the actual dimension of $\overline{M}_{1,1}(X,0)$ is $1+\dim X$, while the expected dimension, the virtual dimension is 1.

We will return to a discussion of this excessive dimension, and of the virtual classXXX.

On the other hand we see that in genus 0, there is no problem with these constant maps. This is one indication that everything is nicer in genus 0. In fact:

6.1 The virtual fundamental class

6.1.1 Spurious components of excessive dimension. XXXX With $g \geq 1$ occur phenomena similar to those in the case of a non-convex target space: the moduli spaces are in general reducible and have components of too high dimension.

Example $\overline{M}_{1,0}(\mathbb{P}^2, 3)$ which has a component of excessive dimension. The problem is when a twig of positive genus contracts, because then the moduli of that twig can vary freely. The following example will be the red thread of the exposition.

cubics

6.1.2 Example. Consider the space $\overline{M}_{1,0}(\mathbb{P}^2, 3)$. Its locus of irreducible maps is birational to the $\mathbb{P}^9$ of plane cubics, and thus of dimension 9. But in addition to this good component (the closure of this locus), there is a component of excessive dimension, namely the "boundary" component consisting of maps having a rational twig of degree 3 and an elliptic twig contracting to a point. The dimension of this component is

$$\dim \overline{M}_{0,1}(\mathbb{P}^2, 3) + \dim \overline{M}_{1,1} = 9 + 1 = 10.$$

(The marks are the gluing marks.)

XXXXIn fact these two components we have just described are in fact irreducible. For the first one, we just accept that as a fact, for the second one, note that it is the image of a map from this product, so it should be irr???

In the example above there is no doubt about what is meant by 'expected dimension': we know that the space of all cubics is of dimension 9, and that they generically are of genus 1.

6.1.3 The expected dimension of $\overline{M}_{g,n}(X,\beta)$, which is called the *virtual dimension* is

$$\operatorname{vdim} \overline{M}_{g,n}(X,\beta) = (\dim X - 3)(1-g) + \int_\beta c_1(T_X) + n$$

The explanation for this is the following. Surely, n marks contribute n to the dimension, so for simplicity, put $n = 0$. Let $\mu : C \to X$ be a general point of $\overline{M} = \overline{M}_{g,0}(X,\beta)$, so we assume that C is smooth and that μ is an immersion. The there is a well-behaved normal bundle of μ denoted N_μ, defined by the short exact sequence

$$0 \to T_C \to \mu^* T_X \to N_\mu \to 0. \qquad (6.1.3.1)$$ Nmu

Assuming furthermore that $\overline{M}$ is smooth at this point then the Kodaira-Spencer map provides an isomorphism between the tangent space of $\overline{M}$ at μ and $H^0(C, N_\mu)$, the space of first order infinitesimal deformations of μ. Now we get by Riemann-Roch (e.g. Fulton [5], Ex. 15.2.1):

$$\begin{aligned} h^0(C,N_\mu) - h^1(C,N_\mu) &= (\dim X - 1)(1-g) + \int_C c_1(N_\mu) \\ &= (\dim X - 1)(1-g) + \int_C c_1(\mu^* T_X) - \int_C c_1(T_C) \\ &= (\dim X - 1)(1-g) + \int_\beta c_1(T_X) - 2(1-g) \\ &= (\dim X - 3)(1-g) + \int_\beta c_1(T_X). \end{aligned}$$

Now we furthermore assume that the first order deformations are unobstructed, i.e., that $H^1(C, N_\mu)$ is zero. Then $h^0(C,N_\mu) = (\dim X - 3)(1-g) + \int_\beta c_1(T_X)$, which is therefore the dimension of $\overline{M}$ near μ, and that is what we call the 'expected dimension'.

In any case, we see that the number $h^1(C, N_\mu)$ is the difference between the actual dimension and the expected dimension at $[\mu]$.

6.1.4 Some more deformation theory. In the long exact sequence coming from (6.1.3.1),

$$\begin{array}{llll} 0 & \to H^0(C,T_C) & \to H^0(C,\mu^*T_X)\to & H^0(C,N_\mu)\to \\ \to & H^1(C,T_C) & \to H^1(C,\mu^*T_X)\to & H^1(C,N_\mu)\to \quad 0 \end{array} \tag{6.1.4.1}$$

long

all the spaces have a distinct interpretation in deformation theory. A good reference for these questions is Harris-Morrison [14], Section 3B. The two left-most spaces depend only on the source curve C and not on the map: $H^0(C, T_C)$ is the space of infinitesimal first order automorphisms of C, and $H^1(C, T_C)$ is space of first order deformations of C, i.e., the tangent space of $\overline{M}_{g,0}$ at C. The two middle spaces describe the deformations of the map that leave C fixed: $H^0(C, \mu^*T_X)$ is the space of such deformations, and $H^1(C, \mu^*T_X)$ are the obstructions to such deformations. Finally, the two most important spaces are, as we have already talked about: $H^0(C, N_\mu)$ is the space of deformations of the map μ, i.e., the tangent space of $\overline{M}_{g,0}(X, \beta)$ at μ, while $H^1(C, N_\mu)$ is the obstruction space.

6.1.5 Concerning the marks. In genus zero and with no marks, we see that the first space is non-zero, indeed $h^0(\mathbb{P}^1, T_{\mathbb{P}^1}) = 3$, corresponding to the three dimensions of automorphisms. This reflects that $\mathbb{P}^1$ is not a stable curve (indeed, one characterisation of stable curve is 'no global vector fields').

If we now put three marks on it then it becomes stable. How can we arrange things in order to reflect this in the deformations spaces just described? Note that for *pointed* curves, the characterisation of stableness is 'no global vector fields have zeros exactly at the marked points'; in other words, $h^0(\mathbb{P}^1, T_{\mathbb{P}^1}(-p_1-p_2-p_3)) = 0$. Similarly, if we want the various deformations spaces to be those that fix the marked points, we must twist the normal bundle sequence by $\mathcal{O}_C(-\sum p_i)$. Even though this certainly changes all??? (XXX what about the middle spaces?) the dimension of the spaces in the long exact sequence, the principle of it remains the same, and in the sequel for simplicity we will not write any twists

6.1.6 The obstruction bundle. Now instead of looking at all these tangent and obstruction spaces for each individual map, the right thing to do is of course do to it relative to the universal map, i.e., assemble all these spaces into bundles (or more precisely: sheaves) over $\overline{M}_{g,n}(X, \beta)$. Since the maps and their source curves can be ugly, the good notions are not tangent bundles and normal bundles — the 'normal bundle' is no longer necessarily locally free. So it's better to express things in terms of sheaves of differentials. After these modifications, the long sequence of cohomology *sheaves* still exists, and its third term is the tangent sheaf of $\overline{M}_{g,n}(X,\beta)$. The sixth and last sheaf is called the *obstruction bundle*, denoted $\mathcal{E}$, although it is not necessarily locally free (?)XXXX.

In particular, $\mathcal{E}$ is the quotient of the sheaf $R^1\pi_*\mu^*T_X$, so if this sheaf is zero, then the moduli space is of expected dimension.

6.1.7 Genus zero and convex varieties. As an example of this, we can now exhibit a large class of target spaces X for which the moduli space $\overline{M}_{0,n}(X,\beta)$ is of expected dimension. A variety X is called *convex* if for all maps $\mu : \mathbb{P}^1 \to X$ we have $H^1(\mathbb{P}^1, \mu^*T_X) = 0$. One can show that this condition implies $H^1(C, \mu^*T_X) = 0$ for *all* stable maps in genus zero. So in this case $R^1\pi_*\mu^*T_X$, and thus $\mathcal{E}$, is zero. So if X is convex, $\overline{M}_{0,n}(X,\beta)$ is of expected dimension.

A common reason to be convex is that the tangent bundle is generated by global sections. This is the case for example for homogeneous varieties.

In higher genus, there are always reducible maps $\mu : C \to X$ such that $H^1(C, \mu^*T_X) \neq 0$, for example maps that contracts a twig of positive genus, as illustrated in the guiding example.

6.1.8 The virtual fundamental class. Let s denote the virtual dimension of $\overline{M}_{g,n}(X,\beta)$. There is a well-defined homology class of dimension s, called the virtual fundamental class, which stands in for the topological fundamental class. In case the obstruction bundle vanishes, the virtual fundamental class coincides with the topological fundamental class. A good introduction to the virtual fundamental class is the survey of Behrend [2].

In general it is not so easy to describe the virtual class of a moduli space. In practice, what is really important is that it is known how it transforms under those operations that are central to the theory: forgetting marks, restricting to divisors, etc. In this way, knowledge of the virtual class is given indirectly by knowledge of spaces in lower genus or of lower degree. For convex varieties, one pillar for this recursive knowledge is the case of genus zero, since in that case the virtual class coincides with the topological one. Another pillar is the case of degree zero, which we now proceed to treat.

More generally, in the following case, there is a good hold of the virtual class.

obstr **6.1.9 Lemma.** *(Behrend) If the obstruction bundle $\mathcal{E}$ is locally trivial of rank e, then $\overline{M}$ is smooth of dimension $e + \operatorname{vdim} \overline{M}$, and in that case its virtual class is*

$$[\overline{M}]^{\text{virt}} = c_e(\mathcal{E}) \cap [\overline{M}]$$

6.1.10 Mapping to a point. As illustrated by the guiding example, the problem of excessive dimension often arises in connection with contracting twigs. Therefore it is very important to study the special case of degree zero carefully. At the same time it provides the easiest example of a virtual class computation.

It is easy to see that there is a natural isomorphism

$$\overline{M}_{g,n}(X, 0) \simeq \overline{M}_{g,n} \times X.$$

Indeed, since the whole map goes to the same point in X, all the evaluation morphisms coincide. So there is a map to X. There is also the forgetful morphisms to $\overline{M}_{g,n}$. So there is a morphism to the product as claimed. Now it should not be difficult to check that it's an isomorphism.

Let $d = \dim X$. Now $\overline{M}_{g,n} \times X$ is an irreducible space of dimension $3g-3+n+d$. On the other hand, the expected dimension of the space as a moduli stack is only $(d-3)(1-g)+n$. So the difference, $g \cdot d$ should be the rank of the obstruction bundle.

We will show that the obstruction bundle is

$$\mathcal{E} \simeq \mathbb{E}^\vee \boxtimes T_X.$$

locally trivial of rank $g \cdot d$, so by Lemma 6.1.9 the virtual class of $\overline{M}_{g,n}(X, 0)$ is

$$[\overline{M}_{g,n}(X, 0)]^{\text{virt}} = c_{gd}(\mathbb{E}^\vee \boxtimes T_X) \cap [\overline{M}_{g,n} \times X]$$

under the identification above.

Consider the long cohomology sequence. Note first that $H^0(C, T_C) = 0$ by stability (in degree 0, stability of the map is equivalent to stability of the source curve). And recall that stability is equivalent to this. Actually, if there are marked points on the curve we must use $T_C(-\sum p_i)$ throughout...

We first claim that $H^0(C, N_\mu) \to H^1(C, T_C)$ is surjective (for any map $\mu : C \to Q \in X$). Here the notation is awkward: the notion of normal bundle is dubious for constant maps... (?) But anyway, it is true that the space in question is the space of deformations of the map, and the next one is the space of deformations of the source curve alone. Now since $\overline{M}_{g,n}(X, 0)$ is just a product, its tangent space is just the product. In other words, the space of deformations is the direct product of the space of deformations of C and the space of deformations of the image point Q in X. In conclusion, the tangent space of $\overline{M}_{g,n}(X,0)$ is $H^1(C,T_C) \oplus T_Q X$. And the coboundary map $H^0(C,N_\mu) \to H^1(C,T_C)$ is just the projection and hence surjective. So the long exact sequence breaks into two sequences: $0 \to H^0(C,\mu^*T_X) \to H^0(C,N_\mu) \to H^1(C,T_C) \to 0$, and next (by exactness)

$$0 \to H^1(C,\mu^*T_X) \to H^1(C,N_\mu) \to 0.$$

So it remains to compute $H^1(C,\mu^*T_X)$. To this end, by Serre duality,

$$H^1(C,\mu^*T_X) \simeq \big(H^0(C,\omega_C \otimes \mu^*T_X^\vee)\big)^\vee$$

Now, μ^*T_X is just the trivial bundle $T_Q X$, so we can take it outside the global section functor, getting

$$\big(H^0(C,\omega_C) \boxtimes T_Q X^\vee\big)^\vee \simeq H^0(C,\omega_C)^\vee \boxtimes T_Q X.$$

Now if you are careful and skilful with these direct image sheaves, you should be able to carry out this same argument to show that in fact on the level of sheaves on $\overline{M}_{g,n}(X,0)$ we have

$$\mathcal{E} \simeq \mathbb{E}^\vee \boxtimes T_X.$$

Where $\mathbb{E} = \pi_*\omega_\pi$ is the Hodge bundle. (The bundle whose fibre over a moduli point $[C]$ is the space $H^0(C,\omega_C)$. Now by Serre duality, this space is isomorphic to $H^1(C,\mathcal{O}_C)$. So the Hodge bundle can also be described as $\mathbb{E} = R^1\pi_*\mathcal{O}_\pi$, where $\pi : C \to \overline{M}_{g,n}$ is the universal curve. So the Hodge bundle is understood to have been pulled back from $\overline{M}_{g,n}$ (and in fact from $\overline{M}_{g,0}$) while of course T_X is the pull-back of the tangent bundle of X.

6.1.11 Example. Consider the case $g = 1, n = 1$, which is one of the minimal stable spaces. Let d denote the dimension of X. In this case the actual dimension of $\overline{M}_{1,1}(X,0)$ is $1 + d$, while the expected dimension, the virtual dimension is 1. The obstruction bundle $\mathcal{E} = \mathbb{E}^\vee \boxtimes T_X$ is of rank d. So the virtual fundamental class of $\overline{M}_{1,1}(X,0)$ is

$$[\overline{M}_{1,1}(X,0)]^{\text{virt}} = c_d(\mathbb{E}^\vee \boxtimes T_X) \cap [\overline{M}_{1,1} \times X]$$

Now $\mathbb{E}^\vee$ has Chern polynomial $1 - \lambda_1 t$, so

$$c_d(\mathbb{E}^\vee \boxtimes T_X) = 1 \boxtimes c_d(T_X) - \lambda_1 \boxtimes c_{d-1}(T_X).$$

Now we can compute our first integral and prove the special case of the dilaton equation:

$$\begin{aligned}
\int_{[\overline{M}_{1,1}(X,0)]^{\mathrm{virt}}} \boldsymbol{\psi}_1 &= \int \boldsymbol{\psi}_1 \cup c_d(\mathbb{E}^\vee \boxtimes T_X) \cap [\overline{M}_{1,1} \times X] \\
&= \int_{[\overline{M}_{1,1}]} \boldsymbol{\psi}_1 \cdot \int_X c_d(T_X) \\
&= \frac{1}{24}\chi(X)
\end{aligned}$$

where $\chi(X) = \int_{[X]} c_d(T_X)$ is the *Euler class* of X.

Similarly, we get the special case of the divisor equation. Let h denote a divisor class on X; then

$$\begin{aligned}
\int_{[\overline{M}_{1,1}(X,0)]^{\mathrm{virt}}} \nu_1^*(h) &= \int \nu_1^*(h) \cup c_d(\mathbb{E}^\vee \boxtimes T_X) \cap [\overline{M}_{1,1} \times X] \\
&= \Big(\int_{[\overline{M}_{1,1}]} -\lambda_1 \Big) \Big(\int_X h \cup c_{d-1}(T_X) \Big) \\
&= -\frac{1}{24} \int_X h \cup c_{d-1}(T_X)
\end{aligned}$$

6.2 Virtual boundary divisors

virtualdiv

Now in general, we don't need to know the virtual class, since it is very rare that one actually computes an integral directly on the moduli space. In practice, integrals are always computed in indirect ways; by recursions which relate the integral to integrals over the boundary (and thus over smaller moduli spaces). Another way to compute integrals is by localisation techniques (which have not been touched upon in these notes: briefly, one need the action of a torus, and then the formula expresses the integral over the whole space as a sum of contributions from all the fixed point loci, so what you really need in this case is knowledge of how the obstruction bundle restricts to such loci, and here Lemma 6.1.9 seems to be central.

Here we are mostly concerned with recursions, i.e., we want to compute integrals over the boundary. The wonderful thing here is that there is a compatibility between the virtual class of an ambient moduli stack and the virtual classes of the substacks corresponding to boundary strata

Discussion of boundary divisors.

Now in this space there is a boundary divisor D_0 which consists of maps whose source curve is a uninodal irreducible curve (geometric genus 0, arithmetic genus 1). Now what does it mean that it is a divisor, in a space which is not equidimensional?

Now here is a better description: it is the image of the clutching map that takes a stable map of genus 0 and degree 3, and identifies the two inverse images of the node. To be more precise, we must put marks on this map, so consider the locus $N \subset \overline{M}_{0,2}(\mathbb{P}^2, 3)$ consisting of maps such that $\mu(p_1) = \mu(p_2)$. This ambient space is of dimension $8 + 2$, and we'd expect the locus N to be of codimension 2, because after all it is the inverse image of the diagonal in $\mathbb{P}^2 \times \mathbb{P}^2$ under the product of the two evaluation morphisms, and since the diagonal has codimension 2, we expect N to have codimension 2 as well. Now it happens that although the two evaluation morphisms each are flat, their product is not! And the inverse image of Δ is not of the expected dimension. In addition to the good locus of honest immersions whose two marks map to the same point, there is the locus of reducible maps, such that one twig is of degree 3 and maps as it pleases, while the other twig contracts to a point and carries the two marks. Then certainly the images of the two marks coincide, and therefore by definition we are in the inverse image of Δ and thus in N. Since we are just talking about one of the boundary divisors, surely this bad component has codimension 1. So the space N is not equidimensional: it has an expected dimension 8, but it comprises a spurious component of dimension 9.

This is a set-up we are familiar with from intersection theory

$$\begin{tikzcd} N \arrow[r, hook] \arrow[d] & \overline{M}_{0,2}(\mathbb{P}^2, 3) \arrow[d, "\nu_1 \times \nu_2"] \\ \mathbb{P}^2 \arrow[r, hook, "\Delta"'] & \mathbb{P}^2 \times \mathbb{P}^2 \end{tikzcd}$$

Then we know there is a refined intersection class, supported on the physical intersection N, and it is denoted $\Delta^! \overline{M}_{0,2}(\mathbb{P}^2, 3)$. In more concrete terms, the class is simply obtained by pulling back bundle that defined Δ in $\mathbb{P}^2 \times \mathbb{P}^2$. So the refined class — or the virtual class — of N is the class $\nu_1^*(h^0) \cup \nu_2^*(h^2) + \nu_1^*(h^1) \cup \nu_2^*(h^1) + \nu_1^*(h^2) \cup \nu_2^*(h^0)$. The pull-back of the diagonal, the Künneth splitting class, or whatever you prefer to call it.

OK, all this to say that a boundary divisor, in this case D_0, inside the strange space $\overline{M}_{1,0}(\mathbb{P}^2, 3)$ is in fact a codimension-1 thing inside each irreducible component. In the good component the maps in D_0 arise by tying together two points of a rational curve, an honest immersion. In the spurious component it is obtained by taking a fake map from N and simply tying the two marks on the contracting $\mathbb{P}^1$ together, such that it becomes a contracting nodal curve, and thus a contracting curve of arithmetic genus 1.

So in this way, inside the strange space (whose virtual class we don't know yet) we know a virtual divisor, and this divisor has a virtual class, and that is just the virtual class of N. This is an axiom: virtual classes of boundary divisors are just the virtual classes coming from their factors. In the case we just saw, there is only one factor, and as such it is a special example. But it is the most illustrative, since in a two-twig divisor we would state that the virtual class would be the virtual class of the factors, but then one of the twigs would be of genus 1, and then we would still not know how to compute the virtual class...

Now in general, virtual classes are indeed a kind of a generalisation of this concept: its the top Chern class of some bundle...

Should come before: Now we can compute our first integral.

$$\begin{aligned}
\int_{[\overline{M}_{1,1}(X,0)]^{\text{virt}}} \boldsymbol{\psi}_1 &= \int_{[\overline{M}_{1,1}(X,0)]^{\text{virt}}} \frac{1}{24} D_0 \\
&= \frac{1}{24}\int 1 \cap [D_0]^{\text{virt}} \\
&= \frac{1}{24}\int 1 \cap [N]^{\text{virt}} \\
&= \frac{1}{24}\int \nu_1^*(\Delta) \cap [\overline{M}_{0,3}(X,0)] \\
&= \frac{1}{24}\int \Delta^! \Delta \cap [X] \\
&= \frac{1}{24}\text{Euler class of } X \\
&= \frac{1}{24}\int_X c_{top}(T_X)
\end{aligned}$$

(Recall that $c_{top}(T_X)$ is called the Euler class of X.)

$$\begin{tikzcd} N \arrow[r, hook] \arrow[d] & \overline{M}_{0,3}(X,0) \arrow[d, "\nu_1 \times \nu_2"] \\ X \arrow[r, hook, "\Delta"'] & X \times X \end{tikzcd}$$

We need $\Delta^! \overline{M}_{0,3}(X,0)$. In more down to earth terms we can think of it as the pull-back of the diagonal via the product of the evaluation maps

$$(\nu_1 \times \nu_2)^* \Delta$$

Now in the identification of $\overline{M}_{0,3}(X,0)$ with $\overline{M}_{0,3} \times X \simeq X$, each evaluation map is just the projection to X, so the product is identified with the diagonal embedding itself. So in the end we are talking about the self-intersection of the diagonal. This is nothing but the Euler class.

Good references: Kontsevich-Manin [23] and Getzler-Pandharipande [12]

6.2.1 XXXX genus 1 Let $\varepsilon : \overline{M}_{g,n}(X,\beta) \to \overline{M}_{g,n}$ denote the forgetful morphism that forgets the X-structure. On either of these spaces, let B_i denote the sum of all boundary divisors such that p_i is on a rational twig. Let E_i denote the sum of all boundary divisors such that p_i is alone on a rational twig. Then (cf. 5.1.9)

$$\varepsilon^* \boldsymbol{\psi}_i = \boldsymbol{\psi}_i - E_i$$

and also

$$\varepsilon^* B_i = B_i - E_i.$$

Now in genus 1, we have $\boldsymbol{\psi}_i = \frac{1}{12} D + B_i$ on any $\overline{M}_{1,n}$, so the two comparison results imply

$$\boldsymbol{\psi}_i = \tfrac{1}{12} D + B_i$$

on $\overline{M}_{1,n}(X,\beta)$.

This yields the following topological recursion relation (cf. also Getzler [10]):

$$\langle\!\langle\, \tau_{k+1,a} \,\rangle\!\rangle_1 = \sum_{e,f} \langle\!\langle\, \tau_{k,a} \tau_{0,e} \,\rangle\!\rangle_0 g^{ef} \langle\!\langle\, \tau_{0,f} \,\rangle\!\rangle_1 + \frac{1}{24} \sum_{e,f} g^{ef} \langle\!\langle\, \tau_{k,a} \tau_{0,e} \tau_{0,f} \,\rangle\!\rangle_0$$

. . . sections that might come some day. . .

Psi classes and enumerative geometry

— plans for this section:

tangency conditions and psi classes (jet bundles)

my own work [13], [19], [20]

introduction to the work of Andreas Gathmann [7] on multiple contacts and relative Gromov-Witten invariants. . .

Psi classes in equivariant quantum cohomology

I hope to write something about this soon. . .

something about how gravitational descendants arise as fundamental solutions in Givental's quantum differential equation. . . and Gromov-Witten invariants of hypersurfaces in $\mathbb{P}^r$, and torus actions, and localisation formulae. . . The famous J-function. . .

reference: Pandharipande [27]

Virasoro constraints

as soon as I understand a little bit, I'll write something about the Virasoro conjecture, which states that (the exponential of) the generating function of the gravitational descendants (all genus) is annihilated by some differential operators that form (half of) the Virasoro algebra. . . Good reference: Getzler [11]. When X is a point, the Virasoro conjecture is equivalent to Witten's conjecture, so the proof of Kontsevich is also a proof of the Virasoro conjecture for a point. One can separate the genus contributions, like for Witten's conjecture, and thus speak of 'Virasoro conjecture up to genus g'. It has been shown that the conjecture hold up to genus 0 (Eguchi, Hori, Xiong, Dubrovin, Zhang?), and perhaps genus 1 is done too. Katz has shown the conjecture holds for Calabi-Yau threefolds. . . Even for simple varieties like $\mathbb{P}^1$, it is not known. . .

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