Convexity in Complex Hyperbolic Space

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In the study of some problems in plane geometric probability appears the limit of the quotient between the area F of a convex set and the length L of its boundary. In the Euclidean plane, given a family of convex domains expanding over the whole plane, this quotient tends to infinity. But, as it was pointed out by L.A. Santaló and I. Yañez in 1972 in [5], this is no longer true in the hyperbolic plane. In fact they proved that for a certain class of convex sets in the hyperbolic plane, the horocyclic convex sets, the limit F/L is 1. Later it was shown in [4] that this limit can attain, in the hyperbolic plane, any value between 0 and 1.

These results were extended in hyperbolic space $\mathbb{H}^{n+1}$, [3]. It was studied the limit of the quotient $\mathrm{vol}(\Omega_t)/\mathrm{vol}(\partial\Omega_t)$ where $\{\Omega_t\}$ is a sequence of convex sets expanding over the whole hyperbolic space $\mathbb{H}^{n+1}$. In this case the limit of this quotient is less than $1/n$, and exactly $1/n$ when the sets considered are convex with respect to horocycles. In 2001, in [2], it was obtained the possible values for this quotient in any manifold of bounded negative curvature, Hadamard manifolds. As in hyperbolic space, the possible values for the quotient depend on the convexity of the domains. The notion used is called λ-convexity and gives some information on how the boundary bends.

In a Riemannian manifold a regular convex domain is said to be λ-convex if its normal curvature at each point is bigger or equal than λ.

We would like to present the results concerning on λ-convexity and the possible values for the quotient in complex hyperbolic space $\mathbb{CH}^n(-4k^2)$, a Hadamard manifold with constant holomorphic curvature equal to $-4k^2$, obtained in [1]. The two main results we prove are:

Theorem 1 *In complex hyperbolic space, $\mathbb{CH}^n(-4k^2)$, it can only exists families of compact convex domains piecewise $\mathscr{C}^2$ expanding over the whole $\mathbb{CH}^n(-4k^2)$ if they are λ-convex with $\lambda \leq k$.*

Theorem 2 *Let $\{\Omega_t\}_{t\in\mathbb{R}^+}$ be a family of compact λ-convex domains, $\lambda \leq k$, expanding over the whole space $\mathbb{CH}^n(-4k^2)$, $n \geq 2$. Then,*

$$\frac{\lambda}{4nk^2} \leq \liminf_{t\to\infty} \frac{\mathrm{vol}(\Omega_t)}{\mathrm{vol}(\partial\Omega_t)} \leq \limsup_{t\to\infty} \frac{\mathrm{vol}(\Omega_t)}{\mathrm{vol}(\partial\Omega_t)} \leq \frac{1}{2nk}.$$

Moreover, the upper bound is sharp.

References

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